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A new local function and a new compatibility type in ideal topological spaces

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Abstract: In this study, a ζ^*_Γ-local function is defined and its properties are examined. This newly defined local function is compared with the well-known local function and the local closure function according to the relation of being a subset. With the help of this new local function, the $\Psi_{\zeta^*_\Gamma}$ operator is defined and topologies are obtained. Moreover, alternative answers are given to an open question found in the literature. $\Psi_{\zeta^*_\Gamma}$-compatibility is defined and its properties are examined. $\Psi_{\zeta^*_\Gamma}$-compatibility is characterized with the help of the new operator. Finally, new spaces were defined and characterized.

Keywords: ideal topological space; local function; local closure function; closure operator; Hayashi-Samuel space; closure compatibility
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1. Introduction

The concepts of ideal and local function were first defined by Kuratowski in [1, 2]. In [3, 4], Vaidyanathaswamy studied the behavior of the local function of a set using many special ideals such as the ideal of meager sets, the ideal of nowhere dense sets, the ideal of finite sets, the ideal of countable sets. Jankovic and Hamlet developed several well-known results in [5]. If you choose the ideal as the minimal ideal i.e., $\{\emptyset\}$ in any ideal topological space, the local function of any subset will be equal to the closure of the subset. If you choose the ideal of finite sets (respectively the ideal of countable sets), the local function of any set will be equal to the set of ω-accumulation (resp. condensation) points of the subset. That is, the local function of a set can be thought of as a generalization of the set of closure, ω-accumulation and condensation points of a set [5]. With the help of the concept of compatibility in ideal topological space, Freud generalized [6] the Cantor-Bendixson theorem. More results on compatibility can also be found in [5].

In general topology, there are methods of obtaining new topologies from old topologies such as product topology, initial topology, final topology, quotient topology and subspace topology. Similarly, in ideal topological spaces, there is a method of obtaining a new topology from the old topology.

The union of a set and its local function in an ideal topological space creates a Kuratowski closure operator called star closure. So a new topology is obtained. With the help of ideal, local function and this star topology, many new space definitions such as $\mathcal{I}$-Baire spaces [7], $\mathcal{I}$-Alexandroff and $\mathcal{I}_g$-Alexandroff spaces [8], $\mathcal{I}$-Extremally disconnected spaces [9], $\mathcal{I}$-Resolvable spaces and $\mathcal{I}$-Hyperconnected spaces [10], $\mathcal{I}$-Rothberger spaces [11] have been given in the literature. These spaces are compared with the definitions given in general topology spaces.

Weak forms of open set such as α-open [12], semi-open [13], pre-open [14], β-open [15] are defined in general topological spaces. Based on a similar idea, many weak forms of open set such as $\mathcal{I}$-open [16], α-$\mathcal{I}$-open [17], pre-$\mathcal{I}$-open [18], semi-$\mathcal{I}$-open [17], β-$\mathcal{I}$-open [17] have been defined in ideal topological spaces. Moreover, the weak forms of the open set in ideal topological spaces and the weak forms of the open set in general topological spaces are compared. Many new continuity types were given using these new weak open forms and decompositions of the well-known continuity were obtained using these continuity types.

The local closure function and Ψ_Γ-operator were defined by Al-Omari and Noiri in [19]. They obtained the topologies σ and σ_0 with the help of Ψ_Γ-operator. They left it as an open question to show that the σ_0-topology is strictly thinner than σ-topology. Pavlovic answered [20] this question with an example. Moreover, he gave a useful theorem [20] showing when the local function and the local closure function coincide.

Al-Omari and Noiri gave the definition of closure compatibility. They emphasized that [19] every compatibility space is closure compatibility. However, the counterexample illustrating this situation was given by Njamcul and Pavlovic in [21]. They characterized closed sets according to the σ-topology. Moreover, the idempotency of local closure function has been discussed. In recent years, new local function types such as semi-local [22], semi-closure [23] and weak semi-local [24] functions have been defined besides the local closure function and the basic properties of these new types of local function are examined. In [24, 25], local function, local closure and weak semi-local were compared according to subset relation.

In this study, we define the new type of local function by using well-known local function in the sense of Kuratowski. We examine the basic properties of this local function and give the definition of $\Psi_{\zeta^*_\Gamma}$ operator. We obtain new topologies using this operator and compare ζ^*_Γ-compatibility with closure compatibility. We answer the question of when the concepts of compatibility, closure compatibility and α^*_Γ compatibilty coincide. Moreover, we give definitions $*$-nearly discrete space and τ^*-nearly discrete space. In [5], nearly discrete spaces are also characterized by the local function. With similar thinking, we characterize $*$-nearly discrete spaces with the help of ζ^*_Γ-local function.

2. Preliminaries

Let (U, τ) be a topological space. The family of all open neighborhoods of the point $x \in U$ is denoted by $\tau(x)$. We show the interior and the closure of subset M as $i(M)$ and $c(M)$, respectively. The family of all subsets of U is denoted by $\mathcal{P}(U)$. The set of natural and real numbers is denoted by $\mathbb{N}$ and $\mathbb{R}$, respectively. The set of natural numbers containing 0 is denoted by ω.

Definition 2.1. *([2]) Let U be a nonempty subset and $\mathcal{I} \subseteq \mathcal{P}(U)$. If the following are satisfied*

a) $\emptyset \in \mathcal{I}$.

b) If $M \in \mathcal{I}$ and $K \subseteq M$, then $K \in \mathcal{I}$.

c) If $M, K \in \mathcal{I}$, then $M \cup K \in \mathcal{I}$.

Then $\mathcal{I}$ is called an ideal on U.

The ideal of finite subsets of U is denoted by $\mathcal{I}_{fin}$. If (U, τ) is a topological space with an ideal $\mathcal{I}$ on U, this space is called an ideal topological space, is denoted by the trible $(U, \tau, \mathcal{I})$ or briefly $\mathcal{I}$-space. The subset M is called nowhere dense if $i(c(M)) = \emptyset$ in any topological space. In any (U, τ), the family of nowhere dense subsets forms an ideal on U. This ideal is denoted by $\mathcal{I}_{nw}$. A subset M is called discrete set if $M \cap M^d = \emptyset$ (where M^d is derived set of M). In any (U, τ), the family of closed and discrete subsets forms an ideal on U. This ideal is denoted by $\mathcal{I}_{cd}$.

Definition 2.2. *([2]) Let (U, τ) be an $\mathcal{I}$-space and $M \subseteq U$. The operator $(.)^* : \mathcal{P}(U) \to \mathcal{P}(U)$ is defined by $M^*(\mathcal{I}, \tau) = \{x \in U : (O \cap M) \notin \mathcal{I}$ for every $O \in \tau(x)\}$ is called local function of the subset M. $M^*(\mathcal{I})$ or M^* sometimes is written instead of $M^*(\mathcal{I}, \tau)$.*

Theorem 2.3. *([2–4]) Let (U, τ) be an $\mathcal{I}$-space and $M, K \subseteq U$.*

a) $M^* = c(M^*) \subseteq c(M)$

b) $(M \cap K)^* \subseteq M^* \cap K^*$

c) $M^*(\mathcal{I}_{nw}) = c(i(c(M)))$

d) If $\mathcal{I} \subseteq \mathcal{J}$, then $M^*(\mathcal{J}) \subseteq M^*(\mathcal{I})$.

Theorem 2.4. *([5]) Let (U, τ) be an $\mathcal{I}$-space and $M \subseteq U$. The following statements are equivalent:*

a) $U = U^*$.

b) $\tau \cap \mathcal{I} = \{\emptyset\}$.

c) If $M \in \mathcal{I}$, then $i(M) = \emptyset$.

d) For every $M \in \tau$, $M \subseteq M^*$.

The ideal space that satisfies any of the above statements is called the Hayashi-Samuel space ([26,27]).

Definition 2.5. *[19] Let (U, τ) be an $\mathcal{I}$-space and $M \subseteq U$. The operator $\Gamma(.) : \mathcal{P}(U) \to \mathcal{P}(U)$ is defined by $\Gamma(M)(\mathcal{I}, \tau) = \{x \in U : (c(O) \cap M) \notin \mathcal{I}$ for every $O \in \tau(x)\}$ is called local closure function of the subset M. $\Gamma(M)(\mathcal{I})$ or $\Gamma(M)$ sometimes is written instead of $\Gamma(M)(\mathcal{I}, \tau)$.*

Let (U, τ) be a topological space. A subset M of U is called θ-open [28], if each point of M has an open neighborhood O such that $c(O) \subseteq M$. The θ-closure [28] of a subset M in any topological space (U, τ) is defined by $c_\theta(M) = \{x \in U : c(O) \cap M \neq \emptyset$ for every $O \in \tau(x)\}$. The family of θ-open subsets forms the topology on U and is denoted by τ_θ. Since $\tau_\theta \subseteq \tau$, $c(M) \subseteq c_\theta(M)$ for every $M \subseteq U$.

Theorem 2.6. *[19] Let (U, τ) be an $\mathcal{I}$-space and $M \subseteq U$. Then, $\Gamma(M) = c(\Gamma(M)) \subseteq c_\theta(M)$.*

Theorem 2.7. *[19] Let (U, τ) be a topological space and $M \in \tau$. Then, $c(M) = c_\theta(M)$.*

Theorem 2.8. *[20] Let M be a subset in any ideal topological space. if $\mathcal{I}_{nw} \subseteq \mathcal{I}$, then $\Gamma(M)(\mathcal{I}) = M^*(\mathcal{I})$.*

In [19], Al-Omari and Noiri defined the operator Ψ_Γ as follows:

Definition 2.9. *[19] Let (U,τ) be an $\mathcal{I}$-space and $M \subseteq U$. An operator $\Psi_\Gamma : P(X) \to \tau$ is defined by:*

$$\Psi_{\zeta^*_\Gamma}(M) = \{x \in U : \textit{there exists } O \in \tau(x) \textit{ such that } c(O) \setminus M \in \mathcal{I}\} = U \setminus \Gamma(U \setminus M).$$

Using this operator, the following two topologies and Diagram I are obtained in [19]:

$$\sigma = \{M \subseteq U : M \subseteq \Psi_\Gamma(M)\} \text{ and } \sigma_0 = \{M \subseteq U : M \subseteq i(c(\Psi_\Gamma(M)))\}.$$

Elements of the topology σ are called σ-open set and elements of the topology σ_0 are called σ_0-open set.

$$\begin{array}{ccc} \theta - open & \longrightarrow & open \\ \downarrow & & \\ \sigma - open & \longrightarrow & \sigma_0 - open \end{array}$$

Diagram I.

3. ζ^*_Γ-local function

Definition 3.1. *Let M be a subset of an $\mathcal{I}$-space (U,τ). An operator $\zeta^*_\Gamma : \mathcal{P}(U) \to \mathcal{P}(U)$ is defined by*

$$\zeta^*_\Gamma(M)(\mathcal{I},\tau) = \{x \in U : (O^*(\mathcal{I},\tau) \cap M) \notin \mathcal{I} \textit{ for every } O \in \tau(x)\}$$

is called the ζ^*_Γ-local function of M with respect to an ideal $\mathcal{I}$ and a topology τ on U. We sometimes write $\zeta^*_\Gamma(M)(\mathcal{I})$ or $\zeta^*_\Gamma(M)$ instead of $\zeta^*_\Gamma(M)(\mathcal{I},\tau)$.

Theorem 3.2. *Let M be a subset of an $\mathcal{I}$-space (U,τ). Then, $\zeta^*_\Gamma(M)(\mathcal{I},\tau) \subseteq \Gamma(M)(\mathcal{I},\tau)$.*

Proof. Let $x \in \zeta^*_\Gamma(M)$. Then, $(O^* \cap M) \notin \mathcal{I}$ for every $O \in \tau(x)$. From Theorem 2.3-a), $O^* \cap M \subseteq (c(O) \cap M) \notin \mathcal{I}$. So $x \in \Gamma(M)$. □

The following examples show that the relation $\zeta^*_\Gamma(M)(\mathcal{I},\tau) \subseteq \Gamma(M)(\mathcal{I},\tau)$ is strictly holds.

Example 3.3. *Let $\tau = \{U, \emptyset, \{d\}, \{a,c\}, \{a,c,d\}\}$ and $\mathcal{I} = \{\emptyset, \{c\}, \{d\}, \{c,d\}\}$ be a topology and an ideal on $U = \{a,b,c,d\}$, respectively. For the subset $M = \{b\}$, $M^* = \{b\}$ and $\Gamma(M) = U$. The ζ^*_Γ-local function of M is $\zeta^*_\Gamma(M) = \{a,b,c\}$.*

Example 3.4. *Let $U = \omega + 1 = \omega \cup \{\omega\}$ and $\tau = P(\omega) \cup \{\{\omega\} \cup (\omega \setminus K) : K \subseteq \omega \textit{ and } K \textit{ is finite}\}$ with the ideal $\mathcal{I}_{fin}$. For the subset $M = \omega$, $\Gamma(M) = \{\omega\}$ ([20]) and $M^* = \{\omega\}$. For any $x \in U$ and for every $O \in \tau(x)$, $O^* = \emptyset$ or $O^* = \{\omega\}$. So, $(O^* \cap M) \in \mathcal{I}_{fin}$ for every $O \in \tau(x)$. Consequently $\zeta^*_\Gamma(M) = \emptyset$.*

In Examples 3.3 and 3.4, it is seen that the local function and ζ^*_Γ-local function are different from each other. That is, $M^* \subsetneq \zeta^*_\Gamma(M)$ and $\zeta^*_\Gamma(M) \subsetneq M^*$ in Examples 3.3 and 3.4, respectively.
Question 1: For a subset M in any $\mathcal{I}$-space, are local function and ζ^*_Γ-local function always comparable with respect to the subset relation? So is it always either $M^* \subseteq \zeta^*_\Gamma(M)$ or $\zeta^*_\Gamma(M) \subseteq M^*$?

Theorem 3.5. *Let (U,τ) be an $\mathcal{I}_{nw}$-space and $M \subseteq U$. Then, $\Gamma(M)(\mathcal{I}_{nw}) = M^*(\mathcal{I}_{nw}) = \zeta^*_\Gamma(M)(\mathcal{I}_{nw})$.*

Proof. For every $O \in \tau$, from Theorem 2.3-c),

$$\begin{aligned} O^*(\mathcal{I}_{nw}) &= c(i(c(O))) \\ &= c(i(c(i(O)))) \\ &= c(i(O)) \\ &= c(O). \end{aligned}$$

Therefore,

$$\begin{aligned} \zeta_\Gamma^*(M)(\mathcal{I}_{nw}) &= \{x \in U : (O^*(\mathcal{I}_{nw}) \cap M) \notin \mathcal{I}_{nw} \text{ for every } O \in \tau(x)\} \\ &= \{x \in U : (c(O) \cap M) \notin \mathcal{I}_{nw} \text{ for every } O \in \tau(x)\} \\ &= \Gamma(M)(\mathcal{I}_{nw}). \end{aligned}$$

From Theorem 2.8, $\Gamma(M)(\mathcal{I}_{nw}) = M^*(\mathcal{I}_{nw}) = \zeta_\Gamma^*(M)(\mathcal{I}_{nw})$. □

Theorem 3.6. *Let (U, τ) be a Hayashi-Samuel $\mathcal{I}$-space and $M \subseteq U$. Then, $M^*(\mathcal{I}, \tau) \subseteq \zeta_\Gamma^*(M)(\mathcal{I}, \tau)$.*

Proof. If $x \in M^*$, then $(O \cap M) \notin \mathcal{I}$ for every $O \in \tau(x)$. From Theorem 2.4-d), $(O \cap M) \subseteq (O^* \cap M) \notin \mathcal{I}$. Therefore $x \in \zeta_\Gamma^*(M)$ and we obtain $M^* \subseteq \zeta_\Gamma^*(M)$. □

Theorem 3.7. *Let (U, τ) be a topological space, $\mathcal{I}$, $\mathcal{J}$ be two ideals on U and M, K be two subsets of U. Then,*

a) If $M \subseteq K$, then $\zeta_\Gamma^*(M) \subseteq \zeta_\Gamma^*(K)$.

b) If $\mathcal{I} \subseteq \mathcal{J}$, then $\zeta_\Gamma^*(M)(\mathcal{J}) \subseteq \zeta_\Gamma^*(M)(\mathcal{I})$.

c) For every $M \subseteq U$, $\zeta_\Gamma^*(M) = c(\zeta_\Gamma^*(M)) \subseteq \Gamma(M) \subseteq c_\theta(M)$.

d) If $M \subseteq \zeta_\Gamma^*(M)$ and $\zeta_\Gamma^*(M)$ is open set, then $\zeta_\Gamma^*(M) = \Gamma(M) = c_\theta(M)$.

e) If $M \in \mathcal{I}$, then $\zeta_\Gamma^*(M) = \emptyset$.

f) $\zeta_\Gamma^*(\emptyset) = \emptyset$.

g) $\zeta_\Gamma^*(M \cup K) = \zeta_\Gamma^*(M) \cup \zeta_\Gamma^*(K)$.

Proof. a) Let $M \subseteq K$ and $x \in \zeta_\Gamma^*(M)$. Then, $(O^* \cap M) \notin \mathcal{I}$ for every $O \in \tau(x)$. Therefore, $(O^* \cap M) \subseteq (O^* \cap K) \notin \mathcal{I}$. Consequently, $x \in \zeta_\Gamma^*(K)$ and $\zeta_\Gamma^*(M) \subseteq \zeta_\Gamma^*(K)$.

b) Let $x \notin \zeta_\Gamma^*(M)(\mathcal{I})$. There exists an $O \in \tau(x)$ such that $(O^*(\mathcal{I}) \cap M) \in \mathcal{I} \subseteq \mathcal{J}$. From Theorem 2.3-d), $(O^*(\mathcal{J}) \cap M) \in \mathcal{I} \subseteq \mathcal{J}$. Consequently, $x \notin \zeta_\Gamma^*(M)(\mathcal{J})$ and $\zeta_\Gamma^*(M)(\mathcal{J}) \subseteq \zeta_\Gamma^*(M)(\mathcal{I})$.

c) We have $\zeta_\Gamma^*(M) \subseteq c(\zeta_\Gamma^*(M))$. We only prove that $c(\zeta_\Gamma^*(M)) \subseteq \zeta_\Gamma^*(M)$. Let $x \in c(\zeta_\Gamma^*(M))$. $(O \cap \zeta_\Gamma^*(M)) \neq \emptyset$ for every $O \in \tau(x)$. Let $y \in (O \cap \zeta_\Gamma^*(M))$. Then $y \in O$ and $y \in \zeta_\Gamma^*(M)$. Moreover, $O \in \tau(y)$. Since $y \in \zeta_\Gamma^*(M)$, $(O^* \cap M) \notin \mathcal{I}$. Consequently, $x \in \zeta_\Gamma^*(M)$ and $c(\zeta_\Gamma^*(M)) = \zeta_\Gamma^*(M)$. From Theorems 3.2 and 2.6, $\zeta_\Gamma^*(M) = c(\zeta_\Gamma^*(M)) \subseteq \Gamma(M) \subseteq c_\theta(M)$.

d) Since $M \subseteq \zeta^*_\Gamma(M)$, $c_\theta(M) \subseteq c_\theta(\zeta^*_\Gamma(M))$. From Theorems 2.6, 2.7 and the previous feature c),

$$\Gamma(M) \subseteq c_\theta(M) \subseteq c_\theta(\zeta^*_\Gamma(M)) = c(\zeta^*_\Gamma(M)) = \zeta^*_\Gamma(M) \subseteq \Gamma(M) \subseteq c_\theta(M).$$

Therefore, $\zeta^*_\Gamma(M) = \Gamma(M) = c_\theta(M)$.

e) Let $M \in \mathcal{I}$. Since $(O^* \cap M) \subseteq M$ for every $O \in \tau(x)$, $(O^* \cap M) \in \mathcal{I}$. So, $\zeta^*_\Gamma(M) = \emptyset$.

f) From e), it is obvious.

g) Since $M \subseteq M \cup K$ and $K \subseteq M \cup K$, by using a), $\zeta^*_\Gamma(M) \subseteq \zeta^*_\Gamma(M \cup K)$ and $\zeta^*_\Gamma(K) \subseteq \zeta^*_\Gamma(M \cup K)$. Therefore $\zeta^*_\Gamma(M) \cup \zeta^*_\Gamma(K) \subseteq \zeta^*_\Gamma(M \cup K)$.
Let $x \notin (\zeta^*_\Gamma(M) \cup \zeta^*_\Gamma(K))$. Then, $x \notin \zeta^*_\Gamma(M)$ and $x \notin \zeta^*_\Gamma(K)$. Therefore, x has open neigborhoods $O, V \in \tau(x)$ such that $(O^* \cap M) \in \mathcal{I}$ and $(V^* \cap K) \in \mathcal{I}$. From the definition of ideal, $(O^* \cap V^*) \cap M \in \mathcal{I}$ and $(O^* \cap V^*) \cap K \in \mathcal{I}$. Moreover $(O^* \cap V^*) \cap (M \cup K) \in \mathcal{I}$. Since $(O \cap V) \in \tau(x)$ and $(O \cap V)^* \subseteq (O^* \cap V^*)$ (Theorem 2.3-b)), $(O \cap V)^* \cap (M \cup K) \in \mathcal{I}$. So $x \notin \zeta^*_\Gamma(M \cup K)$ and the desired result is obtained.

□

Theorem 3.8. *Let (U, τ) be an $\mathcal{I}$-space and $O \in \tau_\theta$. Then, $(O \cap \zeta^*_\Gamma(M)) = (O \cap \zeta^*_\Gamma(O \cap M)) \subseteq \zeta^*_\Gamma(O \cap M)$.*

Proof. Let $x \in (O \cap \zeta^*_\Gamma(M))$. Then $x \in O$ and $x \in \zeta^*_\Gamma(M)$. Since $O \in \tau_\theta$, there exists a $W \in \tau$ such that $x \in W \subseteq c(W) \subseteq O$. Let V be any open neighborhood of x. Then $(V \cap W) \in \tau(x)$. Since $x \in \zeta^*_\Gamma(M)$, $[(V \cap W)^* \cap M] \notin \mathcal{I}$. From Theorem 2.3-b) and a),

$$\begin{aligned} [(V \cap W)^* \cap M] &\subseteq (V^* \cap W^*) \cap M \\ &\subseteq (V^* \cap c(W)) \cap M \\ &\subseteq [V^* \cap (O \cap M)] \notin \mathcal{I}. \end{aligned}$$

Therefore $x \in \zeta^*_\Gamma(O \cap M)$ and we obtain that $O \cap \zeta^*_\Gamma(M) \subseteq O \cap \zeta^*_\Gamma(O \cap M)$. From Theorem 3.7-a), $O \cap \zeta^*_\Gamma(O \cap M) \subseteq O \cap \zeta^*_\Gamma(M)$. Therefore $(O \cap \zeta^*_\Gamma(M)) = (O \cap \zeta^*_\Gamma(O \cap M)) \subseteq \zeta^*_\Gamma(O \cap M)$. □

Theorem 3.9. *Let (U, τ) be an $\mathcal{I}$-space and $M, K \subseteq U$. Then, $(\zeta^*_\Gamma(M) \setminus \zeta^*_\Gamma(K)) = \zeta^*_\Gamma(M \setminus K) \setminus \zeta^*_\Gamma(K)$.*

Proof. Since $M = [(M \setminus K) \cup (M \cap K)]$ and Theorem 3.7-g) and a) ,

$$\begin{aligned} \zeta^*_\Gamma(M) &= \zeta^*_\Gamma((M \setminus K) \cup (M \cap K)) \\ &= \zeta^*_\Gamma(M \setminus K) \cup \zeta^*_\Gamma(M \cap K) \\ &\subseteq \zeta^*_\Gamma(M \setminus K) \cup \zeta^*_\Gamma(K). \end{aligned}$$

Therefore $(\zeta^*_\Gamma(M) \setminus \zeta^*_\Gamma(K)) \subseteq \zeta^*_\Gamma(M \setminus K) \setminus \zeta^*_\Gamma(K)$. From Theorem 3.7-a), $\zeta^*_\Gamma(M \setminus K) \subseteq \zeta^*_\Gamma(M)$ and $\zeta^*_\Gamma(M \setminus K) \setminus \zeta^*_\Gamma(K) \subseteq (\zeta^*_\Gamma(M) \setminus \zeta^*_\Gamma(K))$. As a result $(\zeta^*_\Gamma(M) \setminus \zeta^*_\Gamma(K)) = \zeta^*_\Gamma(M \setminus K) \setminus \zeta^*_\Gamma(K)$.

□

Theorem 3.10. *Let (U, τ) be an $\mathcal{I}$-space and $M, K \subseteq U$. If $K \in \mathcal{I}$, then $\zeta^*_\Gamma(M \cup K) = \zeta^*_\Gamma(M) = \zeta^*_\Gamma(M \setminus K)$.*

Proof. Since $K \in \mathcal{I}$, $\zeta^*_\Gamma(K) = \emptyset$. From Theorem 3.9, $\zeta^*_\Gamma(M \setminus K) = \zeta^*_\Gamma(M)$. From Theorem 3.7-g),

$$\zeta^*_\Gamma(M \setminus K) = \zeta^*_\Gamma(M) = \zeta^*_\Gamma(M) \cup \zeta^*_\Gamma(K) = \zeta^*_\Gamma(M \cup K).$$

. □

4. $\Psi_{\zeta_\Gamma^*}$ operator

Definition 4.1. *Let (U, τ) be an $\mathcal{I}$-space. For any subset M of U, an operator $\Psi_{\zeta_\Gamma^*} : \mathcal{P}(U) \to \tau$ is defined by:*

$$\Psi_{\zeta_\Gamma^*}(M) = \{x \in U : \text{there exists } O \in \tau(x) \text{ such that } (O^*(\mathcal{I}, \tau) \setminus M) \in \mathcal{I}\}.$$

It is also obvious that $\Psi_{\zeta_\Gamma^*}(M) = U \setminus \zeta_\Gamma^*(U \setminus M)$.

Theorem 4.2. *Let (U, τ) be an $\mathcal{I}$-space. The operator $\Psi_{\zeta_\Gamma^*}$ satisfies the following properties:*

a) For $M \subseteq U$, $\Psi_{\zeta_\Gamma^*}(M)$ is open set.

b) If $M \subseteq K$, then $\Psi_{\zeta_\Gamma^*}(M) \subseteq \Psi_{\zeta_\Gamma^*}(K)$.

c) If $M, K \subseteq U$, then $\Psi_{\zeta_\Gamma^*}(M \cap K) = \Psi_{\zeta_\Gamma^*}(M) \cap \Psi_{\zeta_\Gamma^*}(K)$.

d) For $M \subseteq U$, $\Psi_{\zeta_\Gamma^*}(\Psi_{\zeta_\Gamma^*}(M)) = U \setminus \zeta_\Gamma^*(\zeta_\Gamma^*(U \setminus M))$.

e) For every subset $M \subseteq U$, $\Psi_{\zeta_\Gamma^*}(M) = \Psi_{\zeta_\Gamma^*}(\Psi_{\zeta_\Gamma^*}(M)) \Leftrightarrow \zeta_\Gamma^*(U \setminus M) = \zeta_\Gamma^*(\zeta_\Gamma^*(U \setminus M))$.

f) If $M \in \mathcal{I}$, then $\Psi_{\zeta_\Gamma^*}(M) = U \setminus \zeta_\Gamma^*(U)$.

g) If $M \subseteq U$ and $K \in \mathcal{I}$, then $\Psi_{\zeta_\Gamma^*}(M \setminus K) = \Psi_{\zeta_\Gamma^*}(M)$.

h) If $M \subseteq U$ and $K \in \mathcal{I}$, then $\Psi_{\zeta_\Gamma^*}(M \cup K) = \Psi_{\zeta_\Gamma^*}(M)$.

i) If $(M \setminus K) \cup (K \setminus M) \in \mathcal{I}$, then $\Psi_{\zeta_\Gamma^*}(M) = \Psi_{\zeta_\Gamma^*}(K)$.

Proof. a) Since $\zeta_\Gamma^*(U \setminus M)$ is a closed set, $\Psi_{\zeta_\Gamma^*}(M)$ is an open set.

b) From Theorem 3.7-a), it is obtained.

c) From Theorem 3.7-g),
$\Psi_{\zeta_\Gamma^*}(M \cap K) = U \setminus \zeta_\Gamma^*(U \setminus (M \cap K)) = U \setminus [\zeta_\Gamma^*(U \setminus M) \cup \zeta_\Gamma^*(U \setminus K)] = [U \setminus \zeta_\Gamma^*(U \setminus M)] \cap [U \setminus \zeta_\Gamma^*(U \setminus K)] = \Psi_{\zeta_\Gamma^*}(M) \cap \Psi_{\zeta_\Gamma^*}(K)$.

d) Using Definition 4.1,
$\Psi_{\zeta_\Gamma^*}(\Psi_{\zeta_\Gamma^*}(M)) = \Psi_{\zeta_\Gamma^*}(U \setminus \zeta_\Gamma^*(U \setminus M)) = U \setminus \zeta_\Gamma^*(U \setminus (U \setminus \zeta_\Gamma^*(U \setminus M)) = U \setminus \zeta_\Gamma^*(\zeta_\Gamma^*(U \setminus M))$.

e) Using previous proposition, $\Psi_{\zeta_\Gamma^*}(\Psi_{\zeta_\Gamma^*}(M)) = \Psi_{\zeta_\Gamma^*}(M) \Leftrightarrow U \setminus \zeta_\Gamma^*(\zeta_\Gamma^*(U \setminus M)) = U \setminus \zeta_\Gamma^*(U \setminus M) \Leftrightarrow \zeta_\Gamma^*(\zeta_\Gamma^*(U \setminus M)) = \zeta_\Gamma^*(U \setminus M)$.

f) Let $M \in \mathcal{I}$. Then, $U \setminus \zeta_\Gamma^*(U \setminus M) = U \setminus \zeta_\Gamma^*(U)$ from Theorem 3.10. So, $\Psi_{\zeta_\Gamma^*}(M) = U \setminus \zeta_\Gamma^*(U)$.

g) From Theorem 3.7-e) and g), $\Psi_{\zeta_\Gamma^*}(M \setminus K) = U \setminus \zeta_\Gamma^*(U \setminus (M \setminus K)) = U \setminus \zeta_\Gamma^*((U \setminus M) \cup K) = U \setminus [\zeta_\Gamma^*(U \setminus M) \cup \zeta_\Gamma^*(K)] = U \setminus \zeta_\Gamma^*(U \setminus M) = \Psi_{\zeta_\Gamma^*}(M)$.

h) Using Theorem 3.10, $\Psi_{\zeta_\Gamma^*}(M \cup K) = U \setminus \zeta_\Gamma^*(U \setminus (M \cup K)) = U \setminus \zeta_\Gamma^*((U \setminus M) \cap (U \setminus K)) = U \setminus \zeta_\Gamma^*((U \setminus M) \setminus K) = U \setminus \zeta_\Gamma^*(U \setminus M) = \Psi_{\zeta_\Gamma^*}(M)$.

i) Let $(M\setminus K)\cup(K\setminus M)\in\mathcal{I}$. From the definition of ideal, $(M\setminus K)\in\mathcal{I}$ and $(K\setminus M)\in\mathcal{I}$. Using g) and h), $\Psi_{\zeta_\Gamma^*}(M)=\Psi_{\zeta_\Gamma^*}(M\setminus(M\setminus K))=\Psi_{\zeta_\Gamma^*}((M\setminus(M\setminus K))\cup(K\setminus M))=\Psi_{\zeta_\Gamma^*}((M\cap K)\cup(K\setminus M))=\Psi_{\zeta_\Gamma^*}(K)$. □

Definition 4.3. *In any $\mathcal{I}$-space (U,τ), the subset M is called $\sigma_{\zeta_\Gamma^*}$-open if $M\subseteq\Psi_{\zeta_\Gamma^*}(M)$.*

Theorem 4.4. *In any $\mathcal{I}$-space (U,τ), the family of $\sigma_{\zeta_\Gamma^*}$-open sets forms a topology. That is, $\sigma_{\zeta_\Gamma^*}=\{M\subseteq U : M\subseteq\Psi_{\zeta_\Gamma^*}(M)\}$ is a topology on U.*

Proof. It is obvious that $\emptyset, U\in\sigma_{\zeta_\Gamma^*}$. Let $M,K\in\sigma_{\zeta_\Gamma^*}$. Since $M\subseteq\Psi_{\zeta_\Gamma^*}(M)$ and $K\subseteq\Psi_{\zeta_\Gamma^*}(K)$, $M\cap K\subseteq\Psi_{\zeta_\Gamma^*}(M)\ \cap\Psi_{\zeta_\Gamma^*}(K)=\Psi_{\zeta_\Gamma^*}(M\cap K)$ from Theorem 4.2-c). Therefore, $M\cap K\in\sigma_{\zeta_\Gamma^*}$. Let $\{M_\alpha\}_{\alpha\in I}$ be a family of subsets of $\sigma_{\zeta_\Gamma^*}$ for any index set I. Since $M_\alpha\subseteq\Psi_{\zeta_\Gamma^*}(M_\alpha)$ for every $\alpha\in I$, $M_\alpha\subseteq\Psi_{\zeta_\Gamma^*}(M_\alpha)\subseteq\Psi_{\zeta_\Gamma^*}(\cup_{\alpha\in I}M_\alpha)$. Then, $\cup_{\alpha\in I}M_\alpha\subseteq\Psi_{\zeta_\Gamma^*}(\cup_{\alpha\in I}M_\alpha)$. Hence $\cup_{\alpha\in I}M_\alpha\in\sigma_{\zeta_\Gamma^*}$. Consequently, $\sigma_{\zeta_\Gamma^*}$ is a topology on U. □

Lemma 4.5. *In any $\mathcal{I}$-space (U,τ), $\Psi_\Gamma(M)\subseteq\Psi_{\zeta_\Gamma^*}(M)$ for every subset M.*

Proof. From Theorem 3.2, we have $\zeta_\Gamma^*(U\setminus M)\subseteq\Gamma(U\setminus M)$. Hence, $\Psi_{\zeta_\Gamma^*}(M)=U\setminus\zeta_\Gamma^*(U\setminus M)\supseteq U\setminus\Gamma(U\setminus M)=\Psi_\Gamma(M)$. □

Theorem 4.6. *In any $\mathcal{I}$-space (U,τ), every σ-open subset is $\sigma_{\zeta_\Gamma^*}$-open.*

Proof. If M is a σ-open set, then $M\subseteq\Psi_\Gamma(M)$. From Lemma 4.5, $M\subseteq\Psi_\Gamma(M)\subseteq\Psi_{\zeta_\Gamma^*}(M)$. □

Definition 4.7. *In any $\mathcal{I}$-space (U,τ), the subset M is called $\sigma_{\zeta_\Gamma^*0}$-open if $M\subseteq i(c(\Psi_{\zeta_\Gamma^*}(M)))$.*

Lemma 4.8. *[19] Let (U,τ) be a topological space and let M, K be subsets of U. If either $M\in\tau$ or $K\in\tau$, then*

$$i(c(M\cap K))=i(c(M))\cap i(c(K)).$$

Theorem 4.9. *In any $\mathcal{I}$-space (U,τ), the family of $\sigma_{\zeta_\Gamma^*0}$-open sets forms a topology. That is, $\sigma_{\zeta_\Gamma^*0}=\{M\subseteq U : M\subseteq i(c(\Psi_{\zeta_\Gamma^*}(M)))\}$ is a topology on U.*

Proof. It is obvious that $\emptyset, U\in\sigma_{\zeta_\Gamma^*0}$. Let $M,K\in\sigma_{\zeta_\Gamma^*0}$. So, $M\subseteq i(c(\Psi_{\zeta_\Gamma^*}(M)))$ and $K\subseteq i(c(\Psi_{\zeta_\Gamma^*}(K)))$. From Theorem 4.2-a), c) and Lemma 4.8, we have $M\cap K\subseteq i(c(\Psi_{\zeta_\Gamma^*}(M)))\cap i(c(\Psi_{\zeta_\Gamma^*}(K)))=i(c(\Psi_{\zeta_\Gamma^*}(M)\cap\Psi_{\zeta_\Gamma^*}(K)))=i(c(\Psi_{\zeta_\Gamma^*}(M\cap K)))$. Therefore, $M\cap K\in\sigma_{\zeta_\Gamma^*0}$. Let $\{M_\alpha\}_{\alpha\in I}$ be a family of subsets of $\sigma_{\zeta_\Gamma^*0}$ for any index set I. Since $M_\alpha\subseteq i(c(\Psi_{\zeta_\Gamma^*}(M_\alpha)))$ for every $\alpha\in I$, $M_\alpha\subseteq i(c(\Psi_{\zeta_\Gamma^*}(M_\alpha)))\subseteq i(c(\Psi_{\zeta_\Gamma^*}(\cup_{\alpha\in I}M_\alpha)))$. Then $\cup_{\alpha\in I}M_\alpha\subseteq i(c(\Psi_{\zeta_\Gamma^*}(\cup_{\alpha\in I}M_\alpha)))$. Therefore, $\cup_{\alpha\in I}M_\alpha\in\sigma_{\zeta_\Gamma^*}$. Consequently, $\sigma_{\zeta_\Gamma^*0}$ is a topology on U. □

Theorem 4.10. *In any $\mathcal{I}$-space (U,τ), every σ_0-open subset is $\sigma_{\zeta_\Gamma^*0}$-open.*

Proof. From Lemma 4.5, $\Psi_\Gamma(M)\subseteq\Psi_{\zeta_\Gamma^*}(M)$ and so $i(c(\Psi_\Gamma(M)))\subseteq i(c(\Psi_{\zeta_\Gamma^*}(M)))$. □

Theorem 4.11. *In any $\mathcal{I}$-space (U,τ), every $\sigma_{\zeta_\Gamma^*}$-open set is a $\sigma_{\zeta_\Gamma^*0}$-open set.*

Proof. Let M be $\sigma_{\zeta_\Gamma^*}$-open. Therefore, $M\subseteq\Psi_{\zeta_\Gamma^*}(M)$. Since Theorem 4.2-a), $M\subseteq\Psi_{\zeta_\Gamma^*}(M)\subseteq i(c(\Psi_{\zeta_\Gamma^*}(M)))$. Consequently, M is $\sigma_{\zeta_\Gamma^*0}$-open. □

From Theorems 4.6, 4.10, 4.11 and Diagram I, we obtain the following:

$$\begin{array}{ccc} \theta\text{-open} & \rightarrow & \text{open} \\ \downarrow & & \\ \sigma\text{-open} & \rightarrow & \sigma_0\text{-open} \\ \downarrow & & \downarrow \\ \sigma_{\zeta_\Gamma^*}\text{-open} & \rightarrow & \sigma_{\zeta_\Gamma^* 0}\text{-open} \end{array}$$

Diagram II.

The necessary examples for Diagram II are given below.

Example 4.12. *Let $\tau = \{U, \emptyset, \{a\}, \{c\}, \{a,c\}, \{a,c,d\}, \{a,b,c\}\}$ be a topology on $U = \{a,b,c,d\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{c\}, \{a,c\}\}$. For the subset $M = \{a,b\}$, $\Psi_\Gamma(M) = i(c(\Psi_\Gamma(M))) = \emptyset$, $\Psi_{\zeta_\Gamma^*}(M) = \{a,b,c\}$ and $i(c(\Psi_{\zeta_\Gamma^*}(M))) = U$. Although M is both $\sigma_{\zeta_\Gamma^*}$-open and $\sigma_{\zeta_\Gamma^* 0}$-open but it is neither σ-open set nor σ_0-open.*

Example 4.13. *Let's consider the topological space in the Example 4.12 with the ideal $\mathcal{I} = \{\emptyset, \{c\}\}$. The subset $M = \{a\}$ is open subset in this space. Since $\Psi_{\zeta_\Gamma^*}(M) = i(c(\Psi_{\zeta_\Gamma^*}(M))) = \{c\}$, $M = \{a\}$ is neither $\sigma_{\zeta_\Gamma^*}$-open nor $\sigma_{\zeta_\Gamma^* 0}$-open. For the subset $K = \{a,b,d\}$, $\Psi_{\zeta_\Gamma^*}(K) = i(c(\Psi_{\zeta_\Gamma^*}(K))) = U$. Therefore the subset K is both $\sigma_{\zeta_\Gamma^*}$-open and $\sigma_{\zeta_\Gamma^* 0}$-open. But K is not an open set.*

Example 4.14. *Let us consider the usual topology on the set of real numbers $\mathbb{R}$ with the ideal $\mathcal{I}_{cd}$ on $\mathbb{R}$. For the subset $M = \mathbb{R} \setminus \{\frac{1}{n} : n \in \mathbb{N}\}$,*

$$\Psi_{\zeta_\Gamma^*}(M) = \mathbb{R} \setminus \zeta_\Gamma^*(\mathbb{R} \setminus M) = \mathbb{R} \setminus \zeta_\Gamma^*(\{\frac{1}{n} : n \in \mathbb{N}\}) = \mathbb{R} \setminus \{0\}.$$

So, the subset M is not $\sigma_{\zeta_\Gamma^*}$-open. Since $i(c(\Psi_{\zeta_\Gamma^*}(M))) = \mathbb{R}$, the subset M is $\sigma_{\zeta_\Gamma^* 0}$-open. Similarly, $\Psi_\Gamma(M) = \mathbb{R} \setminus \{0\}$ and $i(c(\Psi_\Gamma(M))) = \mathbb{R}$. Therefore, the subset M is not σ-open and it is σ_0-open.

"Is there an example which shows that $\sigma \subsetneq \sigma_0$?" was asked in [19]. This question was answered by Pavlović in [20]. Pavlović gave an example that there exists a σ_0-open set but it is not a σ-open set. In Example 4.14, since the subset M is not σ-open and it is σ_0-open, this example is a new alternative answer to the question in [19]. In addition, the following example is an answer to the mentioned question. In this way, we have given two different answers to the open question given in [19].

Example 4.15. *Let us consider the topology $\tau_1 = \{M \subseteq \mathbb{R} : 1 \notin M\} \cup \{\mathbb{R}\}$ on $\mathbb{R}$ with the ideal $\mathcal{I}_{fin}$ on $\mathbb{R}$. For the subset $M = \{1\}$,*

$$\Psi_\Gamma(M) = \mathbb{R} \setminus \Gamma(\mathbb{R} \setminus \{1\}) = \mathbb{R} \setminus \{1\}.$$

Therefore $M = \{1\}$ is not a σ-open set. But, it is a σ_0-open set since $M \subseteq i(c(\Psi_\Gamma(M))) = \mathbb{R}$.

Theorem 4.16. *The subset M is closed in $(U, \sigma_{\zeta_\Gamma^*})$ if and only if $\zeta_\Gamma^*(M) \subseteq M$.*

Proof.

$$\begin{aligned} M \text{ is closed in } (U, \sigma_{\zeta_\Gamma^*}) &\Leftrightarrow U \setminus M \text{ is open in } (U, \sigma_{\zeta_\Gamma^*}) \\ &\Leftrightarrow U \setminus M \subseteq \Psi_{\zeta_\Gamma^*}(U \setminus M) \\ &\Leftrightarrow U \setminus M \subseteq U \setminus \zeta_\Gamma^*(M) \\ &\Leftrightarrow \zeta_\Gamma^*(M) \subseteq M. \end{aligned}$$

□

Theorem 4.17. *The subset M is closed in $(U, \sigma_{\zeta_\Gamma^* 0})$ if and only if $c(i(\zeta_\Gamma^*(M))) \subseteq M$.*

Proof.

$$\begin{aligned} M \text{ is closed in } (U, \sigma_{\zeta_\Gamma^* 0}) &\Leftrightarrow U \setminus M \text{ is open in } (U, \sigma_{\zeta_\Gamma^* 0}) \\ &\Leftrightarrow U \setminus M \subseteq i(c(\Psi_{\zeta_\Gamma^*}(U \setminus M)) \\ &\Leftrightarrow U \setminus M \subseteq i(c(U \setminus \zeta_\Gamma^*(M))) \\ &\Leftrightarrow U \setminus M \subseteq U \setminus (c(i(\zeta_\Gamma^*(M)))) \\ &\Leftrightarrow c(i(\zeta_\Gamma^*(M))) \subseteq M. \end{aligned}$$

□

ζ_Γ^*-local function does not always have to be idempotent. That is, it doesn't always have to $\zeta_\Gamma^*(\zeta_\Gamma^*(M)) \subseteq \zeta_\Gamma^*(M)$. This situation is illustrated in the example below.

Example 4.18. *Consider the topological space in Example 3.3 with minimal $\mathcal{I} = \{\emptyset\}$. For the subset $M = \{d\}$, $\zeta_\Gamma^*(M) = \{b, d\}$ and $\zeta_\Gamma^*(\zeta_\Gamma^*(M)) = U$. That is, $\zeta_\Gamma^*(\zeta_\Gamma^*(M)) \nsubseteq \zeta_\Gamma^*(M)$.*

Theorem 4.19. *Let M be a subset in any $\mathcal{I}$-space (U, τ). Then,*

$$\zeta_\Gamma^*(\zeta_\Gamma^*(M)) \subseteq \zeta_\Gamma^*(M) \Leftrightarrow \Psi_{\zeta_\Gamma^*}(U \setminus M) \subseteq \Psi_{\zeta_\Gamma^*}(\Psi_{\zeta_\Gamma^*}(U \setminus M)).$$

Proof.

$$\begin{aligned} \zeta_\Gamma^*(\zeta_\Gamma^*(M)) \subseteq \zeta_\Gamma^*(M) &\Leftrightarrow U \setminus \zeta_\Gamma^*(M) \subseteq U \setminus \zeta_\Gamma^*(\zeta_\Gamma^*(M)) \\ &\Leftrightarrow U \setminus \zeta_\Gamma^*(U \setminus (U \setminus M)) \subseteq U \setminus \zeta_\Gamma^*(U \setminus (U \setminus \zeta_\Gamma^*(U \setminus (U \setminus M)))) \\ &\Leftrightarrow \Psi_{\zeta_\Gamma^*}(U \setminus M) \subseteq \Psi_{\zeta_\Gamma^*}(\Psi_{\zeta_\Gamma^*}(U \setminus M)). \end{aligned}$$

□

Corollary 4.20. *The following conditions are equivalent in any $\mathcal{I}$-space:*

a) $\zeta_\Gamma^*(\zeta_\Gamma^*(M)) \subseteq \zeta_\Gamma^*(M)$ *for every subset M.*

b) $\Psi_{\zeta_\Gamma^*}(M) \subseteq \Psi_{\zeta_\Gamma^*}(\Psi_{\zeta_\Gamma^*}(M))$ *for every subset M.*

Theorem 4.21. *Let M be a subset in any $\mathcal{I}$-space (U, τ). Then,*

$$c(i(\zeta_\Gamma^*(c(i(\zeta_\Gamma^*(M)))))) \subseteq c(i(\zeta_\Gamma^*(M))) \Leftrightarrow i(c(\Psi_{\zeta_\Gamma^*}(U \setminus M))) \subseteq i(c(\Psi_{\zeta_\Gamma^*}(i(c(\Psi_{\zeta_\Gamma^*}(U \setminus M)))))).$$

Proof. If we pursue the following, we obtain the desired result:

$$c(i(\zeta_\Gamma^*(c(i(\zeta_\Gamma^*(M)))))) \subseteq c(i(\zeta_\Gamma^*(M)))$$
$$\Leftrightarrow$$
$$U \setminus c(i(\zeta_\Gamma^*(M))) \subseteq U \setminus c(i(\zeta_\Gamma^*(c(i(\zeta_\Gamma^*(M))))))$$
$$\Leftrightarrow$$
$$i(c(U \setminus \zeta_\Gamma^*(U \setminus U \setminus M))) \subseteq i(c(U \setminus \zeta_\Gamma^*(c(i(\zeta_\Gamma^*(M))))))$$
$$\Leftrightarrow$$

$$i(c(\Psi_{\zeta_\Gamma^*}(U \setminus M))) \subseteq i(c(U \setminus \zeta_\Gamma^*(c(i(U \setminus U \setminus \zeta_\Gamma^*(U \setminus U \setminus M))))))$$
$$\Leftrightarrow$$
$$i(c(\Psi_{\zeta_\Gamma^*}(U \setminus M))) \subseteq i(c(U \setminus \zeta_\Gamma^*(c(i(U \setminus \Psi_{\zeta_\Gamma^*}(U \setminus M))))))$$
$$\Leftrightarrow$$
$$i(c(\Psi_{\zeta_\Gamma^*}(U \setminus M))) \subseteq i(c(U \setminus \zeta_\Gamma^*(U \setminus i(c(\Psi_{\zeta_\Gamma^*}(U \setminus M))))))$$
$$\Leftrightarrow$$
$$i(c(\Psi_{\zeta_\Gamma^*}(U \setminus M))) \subseteq i(c(\Psi_{\zeta_\Gamma^*}(i(c(\Psi_{\zeta_\Gamma^*}(U \setminus M)))))).$$

□

Corollary 4.22. *The following conditions are equivalent in any $\mathcal{I}$-space:*

a) $c(i(\zeta_\Gamma^*(c(i(\zeta_\Gamma^*(M)))))) \subseteq c(i(\zeta_\Gamma^*(M)))$ *for every subset M.*

b) $i(c(\Psi_{\zeta_\Gamma^*}(M))) \subseteq i(c(\Psi_{\zeta_\Gamma^*}(i(c(\Psi_{\zeta_\Gamma^*}(M))))))$ *for every subset M.*

The closure of the set M with respect to the topology $(U, \sigma_{\zeta_\Gamma^*})$, we denote by $c_{\sigma_{\zeta_\Gamma^*}}(M)$.

Theorem 4.23. *Let (U, τ) be an $\mathcal{I}$-space and $\zeta_\Gamma^*(\zeta_\Gamma^*(M)) \subseteq \zeta_\Gamma^*(M)$ for every $M \subseteq U$. Then, the subset $M \cup \zeta_\Gamma^*(M)$ is the smallest closed set in $(U, \sigma_{\zeta_\Gamma^*})$ containing the subset M . That is, $c_{\sigma_{\zeta_\Gamma^*}}(M) = M \cup \zeta_\Gamma^*(M)$.*

Proof. Since Theorem 4.16 and

$$\begin{aligned}\zeta_\Gamma^*(M \cup \zeta_\Gamma^*(M)) &= \zeta_\Gamma^*(M) \cup \zeta_\Gamma^*(\zeta_\Gamma^*(M))\\ &\subseteq \zeta_\Gamma^*(M)\\ &\subseteq M \cup \zeta_\Gamma^*(M).\end{aligned}$$

$(M \cup \zeta_\Gamma^*(M))$ is closed in $(U, \sigma_{\zeta_\Gamma^*})$. Let us show that it is the smallest closed set in $(U, \sigma_{\zeta_\Gamma^*})$ containing M. Let $x \in M \cup \zeta_\Gamma^*(M)$. If $x \in M$, then $x \in c_{\sigma_{\zeta_\Gamma^*}}(M)$. Let $x \in \zeta_\Gamma^*(M)$. Then $O^* \cap M \notin \mathcal{I}$ for every $O \in \tau(x)$. From the definition of ideal, $(O^* \cap M) \subseteq (O^* \cap c_{\sigma_{\zeta_\Gamma^*}}(M)) \notin \mathcal{I}$, Therefore, $x \in \zeta_\Gamma^*(c_{\sigma_{\zeta_\Gamma^*}}(M))$. Since Theorem 4.16 and $c_{\sigma_{\zeta_\Gamma^*}}(M)$ is closed set in $(U, \sigma_{\zeta_\Gamma^*})$, $x \in \zeta_\Gamma^*(c_{\sigma_{\zeta_\Gamma^*}}(M)) \subseteq c_{\sigma_{\zeta_\Gamma^*}}(M)$. As a result $M \cup \zeta_\Gamma^*(M) \subseteq c_{\sigma_{\zeta_\Gamma^*}}(M)$. Since $c_{\sigma_{\zeta_\Gamma^*}}(M)$ is the smallest closed set in $(U, \sigma_{\zeta_\Gamma^*})$ containing M, $M \cup \zeta_\Gamma^*(M) = c_{\sigma_{\zeta_\Gamma^*}}(M)$. □

The closure of the set M with respect to the topology $(U, \sigma_{\zeta_\Gamma^* 0})$, we denote by $c_{\sigma_{\zeta_\Gamma^* 0}}(M)$.

Theorem 4.24. *Let (U, τ) be an $\mathcal{I}$-space and $\zeta_\Gamma^*(c(i(\zeta_\Gamma^*(M)))) \subseteq c(i(\zeta_\Gamma^*(M)))$ for every $M \subseteq U$. Then, the subset $M \cup c(i(\zeta_\Gamma^*(M)))$ is the smallest closed set in $(U, \sigma_{\zeta_\Gamma^* 0})$ containing the subset M . That is, $c_{\sigma_{\zeta_\Gamma^* 0}}(M) = M \cup c(i(\zeta_\Gamma^*(M)))$.*

Proof. Since Theorem 4.17 and $\zeta_\Gamma^*(M)$ is closed set,

$$\begin{aligned}c[i[\zeta_\Gamma^*(M \cup c(i(\zeta_\Gamma^*(M))))]] &= c[i[\zeta_\Gamma^*(M) \cup \zeta_\Gamma^*(c(i(\zeta_\Gamma^*(M))))]]\\ &\subseteq c[i[\zeta_\Gamma^*(M) \cup c(i(\zeta_\Gamma^*(M))]]\\ &= c(i(\zeta_\Gamma^*(M)))\\ &\subseteq M \cup c(i(\zeta_\Gamma^*(M))).\end{aligned}$$

From Theorem 4.17, $M \cup c(i(\zeta_\Gamma^*(M)))$ is closed in $(U, \sigma_{\zeta_\Gamma^* 0})$. Let us show that it is the smallest closed set in $(U, \sigma_{\zeta_\Gamma^* 0})$ containing M . Let $x \in M \cup c(i(\zeta_\Gamma^*(M)))$. If $x \in M$, then $x \in c_{\sigma_{\zeta_\Gamma^* 0}}(M)$. Let $x \in c(i(\zeta_\Gamma^*(M)))$. Since $M \subseteq c_{\sigma_{\zeta_\Gamma^* 0}}(M)$, $x \in c(i(\zeta_\Gamma^*(M))) \subseteq c(i(\zeta_\Gamma^*(c_{\sigma_{\zeta_\Gamma^* 0}}(M))))$. Since $c_{\sigma_{\zeta_\Gamma^* 0}}(M)$ is closed in $(U, \sigma_{\zeta_\Gamma^* 0})$ and Theorem 4.17, $x \in c_{\sigma_{\zeta_\Gamma^* 0}}(M)$. As a result $M \cup c(i(\zeta_\Gamma^*(M))) \subseteq c_{\sigma_{\zeta_\Gamma^* 0}}(M)$. Since $c_{\sigma_{\zeta_\Gamma^* 0}}(M)$ is the smallest closed set in $(U, \sigma_{\zeta_\Gamma^* 0})$ containing M, $M \cup c(i(\zeta_\Gamma^*(M))) = c_{\sigma_{\zeta_\Gamma^* 0}}(M)$. □

Theorem 4.25. *Let ζ_Γ^* be idempotent in $\mathcal{I}$-space (U, τ). Then $(\Psi_{\zeta_\Gamma^*}(M) \setminus K) \in \sigma_{\zeta_\Gamma^*}$ for the subset $M \subseteq U$ and $K \in \mathcal{I}$.*

Proof. Using Corollary 4.20 and Theorem 4.2-g),

$$\begin{aligned}(\Psi_{\zeta_\Gamma^*}(M) \setminus K) &\subseteq \Psi_{\zeta_\Gamma^*}(M)\\ &\subseteq \Psi_{\zeta_\Gamma^*}(\Psi_{\zeta_\Gamma^*}(M))\\ &= \Psi_{\zeta_\Gamma^*}(\Psi_{\zeta_\Gamma^*}(M) \setminus K).\end{aligned}$$

Therefore $\Psi_{\zeta_\Gamma^*}(M) \setminus K$ is $\sigma_{\zeta_\Gamma^*}$-open. □

The following result is obtained by Theorems 4.11 and 4.25.

Corollary 4.26. *Let ζ_Γ^* be idempotent in $\mathcal{I}$-space (U, τ). Then $(\Psi_{\zeta_\Gamma^*}(M) \setminus K) \in \sigma_{\zeta_\Gamma^* 0}$ for the subset $M \subseteq U$ and $K \in \mathcal{I}$.*

Theorem 4.27. *Let $c(i(\zeta_\Gamma^*(c(i(\zeta_\Gamma^*(M)))))) \subseteq c(i(\zeta_\Gamma^*(M)))$ for every subset M in $\mathcal{I}$-space (U, τ). Then $(i(c(\Psi_{\zeta_\Gamma^*}(M))) \setminus K) \in \sigma_{\zeta_\Gamma^* 0}$ for the subset $M \subseteq U$ and $K \in \mathcal{I}$.*

Proof. Using Corollary 4.22 and Theorem 4.2-g),

$$\begin{aligned}(i(c(\Psi_{\zeta_\Gamma^*}(M))) \setminus K) &\subseteq i(c(\Psi_{\zeta_\Gamma^*}(M)))\\ &\subseteq i(c(\Psi_{\zeta_\Gamma^*}(i(c(\Psi_{\zeta_\Gamma^*}(M))))))\\ &= i(c(\Psi_{\zeta_\Gamma^*}((i(c(\Psi_{\zeta_\Gamma^*}(M))) \setminus K)))).\end{aligned}$$

Therefore $(i(c(\Psi_{\zeta_\Gamma^*}(M))) \setminus K)$ is $\sigma_{\zeta_\Gamma^* 0}$-open. □

5. ζ_Γ^*-compatibility

Definition 5.1. *Let (U, τ) be an $\mathcal{I}$-space. If for every $M \subseteq U$ the condition*

$$\forall\ x \in M\ \exists O \in \tau(x)\ (O \cap M) \in \mathcal{I}$$

implies $M \in \mathcal{I}$, τ is compatible [5] with $\mathcal{I}$, denoted by $\tau \sim \mathcal{I}$.

If for every $M \subseteq U$ the condition

$$\forall\ x \in M\ \exists O \in \tau(x)\ (c(O) \cap M) \in \mathcal{I}$$

implies $M \in \mathcal{I}$, τ is closure compatible [19] with $\mathcal{I}$, denoted by $\tau \sim_\Gamma \mathcal{I}$.

Definition 5.2. *Let (U, τ) be an $\mathcal{I}$-space. If for every $M \subseteq U$ the condition*

$$\forall\ x \in M\ \exists O \in \tau(x)\ (O^*(\mathcal{I}, \tau) \cap M) \in \mathcal{I}$$

implies $M \in \mathcal{I}$, then we say τ is ζ^*_Γ-compatible with $\mathcal{I}$, denoted by $\tau \sim_{\zeta^*_\Gamma} \mathcal{I}$.

Theorem 5.3. *Let (U, τ) be an $\mathcal{I}$-space. If $\tau \sim_{\zeta^*_\Gamma} \mathcal{I}$, then $\tau \sim_\Gamma \mathcal{I}$.*

Proof. Let $\tau \sim_{\zeta^*_\Gamma} \mathcal{I}$ and $M \subseteq U$. Suppose that for every $x \in M$ there exists an $O \in \tau(x)$ such that $c(O) \cap M \in \mathcal{I}$. Since $O^* \subseteq c(O)$, $O^* \cap M \in \mathcal{I}$. Therefore, $M \in \mathcal{I}$. Consequently $\tau \sim_\Gamma \mathcal{I}$. □

In the following example [21], we show that the converse of this theorem is not true.

Example 5.4. *Let κ be an infinite cardinal number and $\kappa+1$ be the ordinal succeeding κ. Let the family $\mathcal{B}_\kappa = \{\{\lambda, \kappa\} : \lambda < \kappa\} \cup \{\{\kappa\}\}$ be the base of the topology τ_κ on $\kappa + 1$. If this space is considered with the ideal of the set of cardinality less than κ, $\mathcal{I}_\kappa = \{M \subseteq \kappa + 1 : |M| < \kappa\}$, then $\tau_\kappa \sim_\Gamma \mathcal{I}_\kappa$ [21]. For the same reason as in [21], τ_κ is not ζ^*_Γ-compatible with the ideal $\mathcal{I}_\kappa$.*

Question 2: Are there any examples that the concepts of compatibility and ζ^*_Γ-compatibility are different from each other?

Theorem 5.5. *In any $\mathcal{I}$-space (U, τ), the following are equivalent:*

a) $\tau \sim_{\zeta^*_\Gamma} \mathcal{I}$.

b) If M has a cover of open sets each of whose local function intersection with M is in $\mathcal{I}$, then $M \in \mathcal{I}$.

c) For every $M \subseteq U$, $M \cap \zeta^*_\Gamma(M) = \emptyset$ implies $M \in \mathcal{I}$.

d) For every $M \subseteq U$, $M \setminus \zeta^*_\Gamma(M) \in \mathcal{I}$.

e) For every $M \subseteq U$, M contains no nonempty the subset K such that $K \subseteq \zeta^*_\Gamma(K)$, then $M \in \mathcal{I}$.

Proof. a) ⇒ b) It is obvious.
b) ⇒ c) Suppose that $M \cap \zeta^*_\Gamma(M) = \emptyset$. So, $x \notin \zeta^*_\Gamma(M)$ if $x \in M$. Then, for every $x \in M$, there exists an $O_x \in \tau(x)$ such that $O^*_x \cap M \in \mathcal{I}$. Since $M \subseteq \bigcup\{O_x \in \tau : x \in M\}$ and b), $M \in \mathcal{I}$.
c) ⇒ d) For every $M \subseteq U$, since $M \setminus \zeta^*_\Gamma(M) \subseteq M$, $(M \setminus \zeta^*_\Gamma(M)) \cap \zeta^*_\Gamma(M \setminus \zeta^*_\Gamma(M)) \subseteq (M \setminus \zeta^*_\Gamma(M)) \cap \zeta^*_\Gamma(M) = \emptyset$. By c), $M \setminus \zeta^*_\Gamma(M) \in \mathcal{I}$.
d) ⇒ e) For every $M \subseteq U$, by d), $M \setminus \zeta^*_\Gamma(M) \in \mathcal{I}$. Since $M = (M \setminus \zeta^*_\Gamma(M)) \cup (M \cap \zeta^*_\Gamma(M))$,

$$\begin{aligned}\zeta^*_\Gamma(M) &= \zeta^*_\Gamma((M \setminus \zeta^*_\Gamma(M)) \cup (M \cap \zeta^*_\Gamma(M)) \\ &= \zeta^*_\Gamma((M \setminus \zeta^*_\Gamma(M)) \cup \zeta^*_\Gamma((M \cap \zeta^*_\Gamma(M)) \\ &= \zeta^*_\Gamma(M \cap \zeta^*_\Gamma(M)).\end{aligned}$$

Moreover $M \cap \zeta^*_\Gamma(M) = M \cap \zeta^*_\Gamma(M \cap \zeta^*_\Gamma(M)) \subseteq \zeta^*_\Gamma(M \cap \zeta^*_\Gamma(M))$ and $M \cap \zeta^*_\Gamma(M) \subseteq M$. By assumption $M \cap \zeta^*_\Gamma(M) = \emptyset$, $M = M \setminus \zeta^*_\Gamma(M) \in \mathcal{I}$.
e) ⇒ a) Let $M \subseteq U$. Suppose that for every $x \in M$ there exists an $O \in \tau(x)$ such that $O^* \cap M \in \mathcal{I}$. Therefore, $M \cap \zeta^*_\Gamma(M) = \emptyset$. If M contains the subset K such that $K \subseteq \zeta^*_\Gamma(K)$, then $K = K \cap \zeta^*_\Gamma(K) \subseteq M \cap \zeta^*_\Gamma(M) = \emptyset$. So, M contains no nonempty subset K such that $K \subseteq \zeta^*_\Gamma(K)$. By e), $M \in \mathcal{I}$. □

Theorem 5.6. *If $\tau \sim_{\zeta_\Gamma^*} \mathcal{I}$ in any $\mathcal{I}$-space (U,τ), then the following statements are equivalent for every $M \subseteq U$:*

a) $M \cap \zeta_\Gamma^*(M) = \emptyset$ implies $\zeta_\Gamma^*(M) = \emptyset$

b) $\zeta_\Gamma^*(M \setminus \zeta_\Gamma^*(M)) = \emptyset$

c) $\zeta_\Gamma^*(M \cap \zeta_\Gamma^*(M)) = \zeta_\Gamma^*(M)$.

Proof. Let $\tau \sim_{\zeta_\Gamma^*} \mathcal{I}$ and $M \cap \zeta_\Gamma^*(M) = \emptyset$. Then, from Theorem 5.5-c), $M \in \mathcal{I}$. Therefore, $\zeta_\Gamma^*(M) = \emptyset$. That is, a) is satisfied if $\tau \sim_{\zeta_\Gamma^*} \mathcal{I}$.
a) $\Rightarrow$ b) Suppose that a) is satisfied. Then,

$$\begin{aligned}(M \setminus \zeta_\Gamma^*(M)) \cap \zeta_\Gamma^*(M \setminus \zeta_\Gamma^*(M)) &= (M \cap (U \setminus \zeta_\Gamma^*(M))) \cap \zeta_\Gamma^*(M \cap (U \setminus \zeta_\Gamma^*(M))) \\ &\subseteq (M \cap (U \setminus \zeta_\Gamma^*(M))) \cap (\zeta_\Gamma^*(M) \cap \zeta_\Gamma^*(U \setminus \zeta_\Gamma^*(M))) \\ &= \emptyset.\end{aligned}$$

From a), $\zeta_\Gamma^*(M \setminus \zeta_\Gamma^*(M)) = \emptyset$.

b) $\Rightarrow$ c) Since $M = (M \setminus \zeta_\Gamma^*(M)) \cup (M \cap \zeta_\Gamma^*(M))$ and b),

$$\begin{aligned}\zeta_\Gamma^*(M) &= \zeta_\Gamma^*((M \setminus \zeta_\Gamma^*(M)) \cup (M \cap \zeta_\Gamma^*(M))) \\ &= \zeta_\Gamma^*((M \setminus \zeta_\Gamma^*(M)) \cup \zeta_\Gamma^*((M \cap \zeta_\Gamma^*(M)) \\ &= \zeta_\Gamma^*(M \cap \zeta_\Gamma^*(M)).\end{aligned}$$

c) $\Rightarrow$ a) Let $M \cap \zeta_\Gamma^*(M) = \emptyset$ for every $M \subseteq U$. From c), $\zeta_\Gamma^*(M \cap \zeta_\Gamma^*(M)) = \zeta_\Gamma^*(\emptyset) = \emptyset = \zeta_\Gamma^*(M)$.

□

Theorem 5.7. *In any $\mathcal{I}$-space (U,τ), $\tau \sim_{\zeta_\Gamma^*} \mathcal{I}$ if and only if $\Psi_{\zeta_\Gamma^*}(M) \setminus M \in \mathcal{I}$ for every $M \subseteq U$.*

Proof. We have $M \setminus K = (U \setminus K) \setminus (U \setminus M)$ for every $M, K \subseteq U$. Moreover, using Theorem 5.5- d),

$$\begin{aligned}\tau \sim_{\zeta_\Gamma^*} \mathcal{I} &\Leftrightarrow \forall\ M \subseteq U,\ M \setminus \zeta_\Gamma^*(M) \in \mathcal{I} \\ &\Leftrightarrow \forall\ M \subseteq U, (U \setminus M) \setminus \zeta_\Gamma^*(U \setminus M) \in \mathcal{I} \\ &\Leftrightarrow \forall\ M \subseteq U, (U \setminus M) \setminus [U \setminus (U \setminus \zeta_\Gamma^*(U \setminus M))] \in \mathcal{I} \\ &\Leftrightarrow \forall\ M \subseteq U, (U \setminus M) \setminus (U \setminus \Psi_{\zeta_\Gamma^*}(M)) \in \mathcal{I} \\ &\Leftrightarrow \forall\ M \subseteq U, \Psi_{\zeta_\Gamma^*}(M) \setminus M \in \mathcal{I}.\end{aligned}$$

□

Theorem 5.8. *Let ζ_Γ^* be idempotent in $\mathcal{I}$-space (U,τ) and $\tau \sim_{\zeta_\Gamma^*} \mathcal{I}$. Then*

$$\sigma_{\zeta_\Gamma^*} = \{\Psi_{\zeta_\Gamma^*}(M) \setminus K : M \subseteq U \text{ and } K \in \mathcal{I}\}.$$

Proof. From Theorem 4.25, $\{\Psi_{\zeta^*_\Gamma}(M) \setminus K : M \subseteq U \text{ and } K \in \mathcal{I}\} \subseteq \sigma_{\zeta^*_\Gamma}$.
Conversely, let $M \in \sigma_{\zeta^*_\Gamma}$. Therefore $M \subseteq \Psi_{\zeta^*_\Gamma}(M)$. From Theorem 5.7, we have $K = (\Psi_{\zeta^*_\Gamma}(M) \setminus M) \in \mathcal{I}$. Then

$$\begin{aligned} M &= \Psi_{\zeta^*_\Gamma}(M) \cap ((U \setminus \Psi_{\zeta^*_\Gamma}(M)) \cup M) \\ &= \Psi_{\zeta^*_\Gamma}(M) \cap (U \setminus (\Psi_{\zeta^*_\Gamma}(M) \cap (U \setminus M)) \\ &= \Psi_{\zeta^*_\Gamma}(M) \setminus (\Psi_{\zeta^*_\Gamma}(M) \setminus M) \\ &= \Psi_{\zeta^*_\Gamma}(M) \setminus K. \end{aligned}$$

Therefore $M \in \{\Psi_{\zeta^*_\Gamma}(M) \setminus K : M \subseteq U \text{ and } K \in \mathcal{I}\}$. Consequently, we obtain $\sigma_{\zeta^*_\Gamma} = \{\Psi_{\zeta^*_\Gamma}(M) \setminus K : M \subseteq U \text{ and } K \in \mathcal{I}\}$.

□

Theorem 5.9. *In any $\mathcal{I}$-space (U, τ),*

a) Let $M^* = \zeta^*_\Gamma(M)$ for every $M \subseteq U$. Then, $\tau \sim_{\zeta^*_\Gamma} \mathcal{I}$ if and only if $\tau \sim \mathcal{I}$.

b) Let $\Gamma(M) = \zeta^*_\Gamma(M)$ for every $M \subseteq U$. Then, $\tau \sim_{\zeta^*_\Gamma} \mathcal{I}$ if and only if $\tau \sim_\Gamma \mathcal{I}$.

Proof. a) It is obtained by Theorem 4.5-c) in [5] and Theorem 5.5-c).

b) It is obtained by Theorem 3.4-3 in [19] and Theorem 5.5-c).

□

Corollary 5.10. *Let (U, τ) be an $\mathcal{I}_{nw}$-space. Then,*

$$\tau \sim_{\zeta^*_\Gamma} \mathcal{I}_{nw} \Leftrightarrow \tau \sim \mathcal{I}_{nw} \Leftrightarrow \tau \sim_\Gamma \mathcal{I}_{nw}$$

Proof. It is obtained by Theorems 5.9 and 3.5. □

6. $*$-Nearly discrete and τ^*-nearly discrete spaces

A topological space is nearly discrete (or sometimes said to be locally finite) if each point has a finite neigborhood. In [5], Jankovic and Hamlet showed that in $\mathcal{I}_{fin}$-space (U, τ), "(U, τ) is nearly discrete if and only if $U^*(\mathcal{I}_{fin}) = \emptyset$". In [20], Pavlović showed that "if $\Gamma(U)(\mathcal{I}_{fin}) = \emptyset$, (U, τ) is nearly discrete". Pavlović also gave an example in which the reverse of this theorem is not true. Now, we give the definitions of two new spaces.

Definition 6.1. *Let (U, τ) be an $\mathcal{I}$-space. (U, τ) is a $*$-nearly discrete space if for every $x \in U$, there exists $M \in \tau(x)$ such that M^* is finite.*

Definition 6.2. *Let (U, τ) be an $\mathcal{I}$-space. (U, τ) is a τ^*-nearly discrete space if every $x \in U$ has a finite τ^*-open neighborhood.*

Theorem 6.3. *Every nearly discrete space is a τ^*-nearly discrete space.*

Proof. Since $\tau \subseteq \tau^*$, it is obvious. □

The converse of this theorem is not true in general. Moreover, concepts of $*$-nearly discrete space and τ^*-nearly discrete space, nearly discrete space are independent of each other.

Example 6.4. *Consider the usual topology τ_u on the set of real numbers $\mathbb{R}$ with the ideal $\mathcal{I} = \mathcal{P}(\mathbb{R})$. Since the topology $\tau_u^* = \mathcal{P}(\mathbb{R})$, $(\mathbb{R}, \tau_u)$ is τ_u^*-nearly discrete space. But $(\mathbb{R}, \tau_u)$ is not nearly discrete.*

Example 6.5. *Let us consider the topology $\tau_1 = \{M \subseteq \mathbb{R} : 1 \notin M\} \cup \{\mathbb{R}\}$ on $\mathbb{R}$ with the ideal $\mathcal{I}_{fin}$ on $\mathbb{R}$. For every $M \in \tau_1$, $M^* = \emptyset$ or $M^* = \{1\}$. Therefore this space is a $*$-nearly discrete space. The point $x = 1$ has not finite neighborhood in both τ_1 and τ_1^*. That is, this space is neither nearly discrete nor τ^*-nearly discrete.*

Example 6.6. *Let us consider the topology $\tau_1' = \{M \subseteq \mathbb{R} : 1 \in M\} \cup \{\emptyset\}$ on $\mathbb{R}$ with the ideal $\mathcal{I} = \{\emptyset\}$. This space is a nearly discrete and so $\tau_1'^*$-nearly discrete. Since $M^* = \mathbb{R}$ for every $M \in \tau_1' \setminus \{\emptyset\}$, this is not a $*$-nearly discrete space.*

Theorem 6.7. *Let (U, τ) be an $\mathcal{I}_{fin}$-space. $\zeta_\Gamma^*(U)(\mathcal{I}_{fin}) = \emptyset$ if and only if (U, τ) is a $*$-nearly discrete space.*

Proof. Let $\zeta_\Gamma^*(U)(\mathcal{I}_{fin}) = \emptyset$. For every $x \in U$ there exists an $O \in \tau(x)$ such that $O^* \cap U = O^* \in \mathcal{I}_{fin}$. Therefore this space is a $*$-nearly discrete space.
Conversely, let it be a $*$-nearly discrete space. So, for every $x \in U$ there exists an $O \in \tau(x)$ such that $O^* \in \mathcal{I}_{fin}$. Consequently, $\zeta_\Gamma^*(U)(\mathcal{I}_{fin}) = \emptyset$. □

Theorem 6.8. *Let (U, τ) be an $\mathcal{I}_{fin}$-space. If (U, τ) is nearly discrete, then it is a $*$-nearly discrete space.*

Proof. Let (U, τ) be an $\mathcal{I}_{fin}$-space. Then $U^*(\mathcal{I}_{fin}) = \emptyset$. Therefore for every $O \in \tau$, $O^*(\mathcal{I}_{fin}) = \emptyset$ and $O^*(\mathcal{I}_{fin}) \cap U = \emptyset \in \mathcal{I}_{fin}$. That is $\zeta_\Gamma^*(U)(\mathcal{I}_{fin}) = \emptyset$. From Theorem 6.7, (U, τ) is a $*$-nearly discrete space. □

In Example 6.5, it is shown that converse of this theorem is not true.

Corollary 6.9. *Let (U, τ) be an $\mathcal{I}_{fin}$-space. If $U^*(\mathcal{I}_{fin}) = \emptyset$, then (U, τ) is a $*$-nearly discrete space.*

Theorem 6.10. *Let (U, τ) be an $\mathcal{I}$-space and $\mathcal{I} \cap \tau = \{\emptyset\}$. If (U, τ) is a $*$-nearly discrete space, then it is a nearly discrete.*

Proof. Let (U, τ) be a $*$-nearly discrete space. Then for every $x \in U$ there exists an $O \in \tau_x$ such that O^* is finite. From Theorem 2.4, $O \subseteq O^*$ Therefore O is finite. □

In Example 6.6, it is shown that converse of this theorem is not true.

Corollary 6.11. *Let (U, τ) be an $\mathcal{I}$-space and $\mathcal{I} \cap \tau = \{\emptyset\}$. If $\zeta_\Gamma^*(U) = \emptyset$, then (U, τ) is a nearly discrete space.*

Corollary 6.12. *Let (U, τ) be an $\mathcal{I}_{fin}$-space and $\mathcal{I}_{fin} \cap \tau = \{\emptyset\}$. (U, τ) is a nearly discrete space if and only if it is a $*$-nearly discrete space.*

7. Conclusions

We defined the concepts of ζ_Γ^*-local function, the operator $\Psi_{\zeta_\Gamma^*}$, ζ_Γ^*-compatibility, $*$-nearly discrete space and τ^*-nearly discrete space. We obtained two new topologies with help the operator $\Psi_{\zeta_\Gamma^*}$. We answered the open question in [19] with two different examples apart from the example given in [20]. In addition, we have given two open questions in this text.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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