Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(3): 6749–6762.
DOI:10.3934/math.2023343
Received: 26 October 2022
Revised: 20 December 2022
Accepted: 23 December 2022
Published: 09 January 2023

Research article

Some results for a variation-inequality problem with fourth order p(x)-Kirchhoff operator arising from options on fresh agricultural products

Tao Wu[1,2,*]

[1] Guizhou Institute for Rural Vitalization, Guizhou Education University, Guiyang 550018, China
[2] Guizhou Institute of Minority Education, Guizhou Education University, Guiyang 550018, China

* **Correspondence:** Email: wolftao1982@163.com; wutao@gznc.edu.cn.

Abstract: In this paper, we study variation-inequality initial-boundary value problems with fouth order $p(x)$-Kirchhoff operators. First, an operator is constructed based on the Leray Schauder principle, and the existence of solutions is obtained. Secondly, the stability and uniqueness of the solution are analyzed after the conditions are appropriately relaxed on the Kirchhoff operators.

Keywords: parabolic variation-inequality problems; weak solution; fouth order $p(x)$-Kirchhoff operator; existence; uniqueness; stability
Mathematics Subject Classification: 35K99, 97M30

1. Introduction

Let $\Omega \subset \mathbb{R}_N (N \geq 2)$ be a bounded simply connected domain, $0 < T < \infty$ and $Q_T = \Omega \times [0, T]$. We study the following variation-inequality initial-boundary value problems

$$\begin{cases} \min\{L\phi, \phi - \phi_0\} = 0, & (x,t) \in Q_T, \\ \phi(0,x) = \phi_0(x), & x \in \Omega, \\ \phi(t,x) = 0, & (x,t) \in \partial\Omega \times (0,T), \end{cases} \tag{1}$$

with fourth-order p-Laplacian Kirchhoff operators,

$$L\phi = \partial_t \phi - \Delta\left((1 + \lambda \|\Delta\phi\|^{p(x)}_{L^{p(x)}(\Omega)})|\Delta\phi|^{p(x)-2}\Delta\phi\right) + \gamma\phi,$$

$\gamma \geq 0$ and $p(x) > 2$. Here $L^{p(x)}(\Omega)$ stands for

$$L^{p(x)}(\Omega) = \{u | u \,\text{is mesurable real} - \text{valued function}, \int_\Omega |u|^{p(x)} \mathrm{d}x < \infty\}.$$

Our consideration of this paper is motivated by the model proposed by Chen [1] and Yi [1,2], who studied the problem (1) with parabolic operator

$$Lu = \frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} + r\frac{\partial u}{\partial x} - ru.$$

Many interesting results have been established for the variation-inequality initial-boundary value problem, for example, [3–8] and references therein. Some papers have studied the existence of solutions to variational inequalities [3–5]. Some scholars have studied the approximate solutions of variational inequalitie [6,7]. Reference [8] attempted to obtain numerical solutions from the perspective of numerical difference.

At present, there are many literatures about the initial boundary value problem of Kirchhoff operator[9–12]. Long and Deng in [9] make use of minimax methods and invariant sets of descending flow to study the existence and non uniqueness of solutions. Later, Chen and Zhou by the Leray-Schauder principle study the existence of solutions for the following p-Laplacian Kirchhoff equation[10]. Huang and Deng also prove the existence of a positive ground state solution for Kirchhoff type problem[11,12].

The authors of this paper investigate a class of variation-inequality initial-boundary value problems with fourth-order $p(x)$-Kirchhoff operators, and study the existence, uniqueness and stability of solutions. One of the innovations of this paper is to construct a mapping based on the Leray Schauder principle and introduce a penalty function to prove the existence of the solution. Another innovation of this paper is to obtain the uniqueness and stability of solutions from weak solutions by using inequality amplification techniques.

The structure of this paper is as follows: The second section gives the main results of this paper and the application background in fresh agricultural products Section 3. The 4th section analyzes the uniqueness and stability of the solution.

2. The main results of weak solutions and application background

Before giving the main conclusions of this paper, here we first consider an application case of the variational inequality problem. Here we consider the ordering strategy for a fresh agricultural product retailer. As the production lead time of fresh agricultural products is long and the sales period is short, retailers have no chance to replenish, so they need to replenish before the sales season comes. Retailers can consider the supplier's call option contract: They have the right to purchase a certain amount of fresh rural products with the agreed price of c at time 0. Then the value of the option is

$$\exp\{-rT\}E[\max\{p(T) + g - c, 0\}]$$

where p is the retail price of fresh agricultural products per unit. g represents the unit penalty cost incurred due to the retailer's failure to meet the market demand. Assume that the retail price of fresh products meets the following B-S equation

$$\mathrm{d}p(t) = rp(t)\mathrm{d}t + \sigma p(t)\mathrm{d}B(t), \;\; p(0) = p_0,$$

where μ and σ represent the expected return and volatility respectively. $\{B(t), t \geq 0\}$ stands for standard Brownian movement, which contains the noise of fresh agricultural product market. And r is the risk-free interest rate.

In [1], the value of the options provided by the supplier meets the following variational inequality problem

$$\begin{cases} \min\{Lv, v - \max\{p + g - c, 0\}\} = 0, & p \times t \in \mathrm{R}_+ \times [0, T], \\ v(T, p) = \max\{p + g - c, 0\}, & p \in \mathrm{R}_+, \\ v(t, 0) = 0, & t \in [0, T]. \end{cases}$$

In this paper, we consider more complex variation-inequality problems. Combining the ideas of [1–3], we will use the following maximal monotone operators to prove the conclusions of this paper

$$G(x) = \begin{cases} 0, & x > 0, \\ M_0, & x = 0. \end{cases} \tag{2}$$

Here, M_0 is positive constant which will be chosen later. The purpose of this paper is to obtain the existence, uniqueness and stablity of weak solutions of (1), and the weak solution is defined as

Definition 2.1. Function (ϕ, ξ) is called a generalized solution of the systems (1.1), if

$$\phi \in L^\infty(0, T; W^{2,p(x)}(\Omega)), \partial_t u \in L^2(\Omega_T), \xi \in L^\infty(0, T; L^\infty(\Omega)),$$

and satisfies

(a) $u(x, t) \geq u_0(x)$, (b) $u(x, 0) = u_0(x)$, (c) $\xi \in G(u - u_0)$,

(d) for each test-function $\varphi \in C^1(\bar{Q}_T)$, there holds

$$\begin{aligned} &\int\int_{\Omega_T} (\partial_t u \cdot \varphi + (1 + \lambda \|\Delta\phi\|^{p(x)}_{L^{p(x)}(\Omega)})|\Delta\phi|^{p(x)-2}\Delta\phi\Delta\varphi + \gamma\phi\varphi)\mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_\Omega \xi \cdot \varphi \,\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3}$$

Let us summarize the main results as follows.

Theorem 2.1. Assume that $\phi_0 \in L^\infty(\Omega_T)$, then (1) has a solution ϕ in the sense of Definition 2.1.

Theorem 2.2. Assume that (ϕ_i, ξ_i) is a generalized of (1) with different initial values conditions $\phi(0, x) = \phi_{0,i}(x), \quad x \in \Omega, i = 1, 2$. If $\lambda = 0$, then there exists a positive constant C such that

$$\|\phi_1 - \phi_2\|_{L^2(\Omega_T)} \leq \|\phi_{0,1}(\cdot) - \phi_{0,2}(\cdot)\|_{L^2(\Omega)}, \tag{4}$$

$$\|\phi_1 - \phi_2\|_{L^2(0,T;W^{2,p(x)}\Omega_T)} \leq C\|\phi_{0,1}(\cdot) - \phi_{0,2}(\cdot)\|_{L^2(\Omega)}. \tag{5}$$

Furthermore, the solution of (1) is unique.

3. Some estimates

Now, we try to decompose the existence of solution of problem (1). Unfortunately, we cannot deal with the operator $L\phi$ like the classical parabolic initial boundary value problem, because $|\Delta\phi|^{p(x)-2}\Delta\phi$ is coupled with $(1 + \lambda\|\Delta\phi\|^{p(x)}_{L^{p(x)}(\Omega)})$ in $L\phi$. We also need to introduce penalty function to deal with the inequality restriction in problem (1), so as to approximate it. To this end, we introduce the following operator

$$M : L^\infty(0, T; W_0^{2,p(x)}(\Omega)) \times [0, 1] \to L^\infty(0, T; W_0^{2,p(x)}(\Omega)) \tag{6}$$

in such a way, that for every function $\omega \in L^\infty(0, T; W_0^{2,p(x)}(\Omega))$ and $\theta \in [0, 1]$, $u = M(\omega, \theta)$ is a solution of the equation

$$\begin{cases} L_\varepsilon^{\theta,\omega}\phi_\varepsilon = -\theta\beta_\varepsilon(\phi_\varepsilon - \phi_0), & (x, t) \in \Omega_T, \\ \phi_\varepsilon(x, 0) = \phi_{0\varepsilon}(x) = \phi_0 + \varepsilon, & x \in \Omega, \\ \phi_\varepsilon(x, t) = \varepsilon, & (x, t) \in \partial\Omega_T, \end{cases} \tag{7}$$

with a operator

$$L_\varepsilon^{\theta,\omega}\phi_\varepsilon = \partial_t u_\varepsilon - \Delta\left((1+\theta\lambda\|\Delta\omega_\varepsilon\|^{p(x)}_{L^{p(x)}(\Omega)})|\Delta\phi_\varepsilon|^{p(x)-2}\Delta\phi_\varepsilon\right) + \gamma\phi_\varepsilon, \tag{8}$$

and a penalty function $\beta_\varepsilon(\cdot)$ satisfies

$$\begin{aligned} &\varepsilon \in (0,1), \beta_\varepsilon(\cdot) \in C^2(\mathrm{R}), \beta_\varepsilon(x) \le 0, \beta'_\varepsilon(x) \ge 0, \beta''_\varepsilon(x) \le 0, \\ &\beta_\varepsilon(x) = \begin{cases} 0 & x \ge \varepsilon, \\ -M_0 & x = 0, \end{cases} \quad \lim_{\varepsilon\to 0+}\beta(x) = \begin{cases} 0, & x > 0, \\ -M_0, & x = 0. \end{cases} \end{aligned} \tag{9}$$

Thus, we can prove the existence of the following problem

$$\begin{cases} L\phi_\varepsilon = -\beta_\varepsilon(\phi_\varepsilon - u_0), & (x,t) \in Q_T, \\ \phi_\varepsilon(x,0) = \phi_{0,\varepsilon}(x), & x \in \Omega, \\ \phi_\varepsilon(x,t) = \varepsilon, & (x,t) \in \partial Q_T, \end{cases} \tag{10}$$

by showing the existence of the fixed point of operator $M(\cdot,1)$ in $L^\infty(0,T;W_0^{2,p(x)}(\Omega))$. Then one can use the penalty function $\beta_\varepsilon(\cdot)$ to make (10) approach the solution of (1) by $\varepsilon \to 0$.

Choosing $t = 0$ in (7), we get the following estimates

$$L_\varepsilon^{\theta,\omega}\phi_{0,\varepsilon} = -\theta\beta_\varepsilon(\phi_{0,\varepsilon} - \phi_0) = 0, L_\varepsilon^{\theta,\omega}\phi_\varepsilon = -\theta\beta_\varepsilon(\phi_\varepsilon - \phi_0) \le 0,$$

such that for any . In view of comparison principle [1,11],

$$|\phi_0|_\infty + \varepsilon \ge \phi_\varepsilon \ge \phi_{0,\varepsilon} \text{ for any } (\mathrm{x},\mathrm{t}) \in \mathrm{Q_T}. \tag{11}$$

Here, we show some estimates of problem (1), which will be used later.

Lemma 3.1. For any $(x,t) \in Q_T$, the solution of problem (7) satisfies

$$\phi_\varepsilon \in L^\infty(0,T;L^2(\Omega)) \cap L^\infty(0,T;W^{2,p(x)}(\Omega)). \tag{12}$$

Proof. For any $t \in (0,T]$, multiply the first line of (7) by ϕ_ε and integrate it over Ω , such that

$$\begin{aligned} &\textstyle\int_0^t\int_\Omega \partial_t\phi_\varepsilon \cdot \phi_\varepsilon + (1+\theta\lambda\|\Delta\omega\|^{p(x)}_{L^{p(x)}(\Omega)})|\Delta\phi_\varepsilon|^{p(x)} + \gamma|\phi_\varepsilon|^2 \mathrm{d}x\mathrm{d}\tau \\ &= -\textstyle\int_0^t\int_\Omega \beta_\varepsilon(\phi_\varepsilon - \phi_0)\phi_\varepsilon \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{13}$$

It follows by differential transformation technique that

$$\int_0^t\int_\Omega \partial_\tau\phi_\varepsilon \cdot \phi_\varepsilon \mathrm{d}x\mathrm{d}\tau = \frac{1}{2}\int_0^t\int_\Omega \partial_\tau(\phi_\varepsilon)^2 \mathrm{d}x\mathrm{d}\tau = \frac{1}{2}\int_\Omega \phi_\varepsilon^2(\cdot,t) - \phi_\varepsilon^2(\cdot,0)\mathrm{d}x. \tag{14}$$

From (9), (11) and $\theta \in [0,1]$, we use Holder and Young inequalities to infer that

$$\theta\left|\int_0^t\int_Q \beta_\varepsilon(\phi_\varepsilon - \phi_0)\cdot\phi_\varepsilon \mathrm{d}x\mathrm{d}t\right| \le \frac{1}{2}M_0^2\theta^2|\Omega|T + \frac{1}{2}\int_0^t \|\phi_\varepsilon(\cdot,\tau)\|^2_{L^2(\Omega)}\mathrm{d}\tau. \tag{15}$$

Combining (13)–(15), and dropping the nonnegative term $(1+\theta\lambda\|\nabla\omega\|^{p(x)}_{L^{p(x)}(\Omega)})\cdot|\nabla\phi_\varepsilon|^{p(x)}$ in (13),

$$\frac{1}{2}\|\phi_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)} + \left(\gamma - \frac{1}{2}\right)\int_0^t \|\phi_\varepsilon(\cdot,\tau)\|^2_{L^2(\Omega)}\mathrm{d}\tau \le \frac{1}{2}M_0^2\theta^2|\Omega|T + \frac{1}{2}\|\phi_\varepsilon(\cdot,0)\|^2_{L^2(\Omega)}. \tag{16}$$

Next we will discuss (16) in two case: $\gamma-\frac{1}{2}\leq 0$ and $\gamma-\frac{1}{2}>0$. In case of $\gamma-\frac{1}{2}\leq 0$, $\phi_\varepsilon\in L^\infty(0,T;L^2(\Omega))$ is an immediate result of (16). If $\gamma-\frac{1}{2}>0$, using Gronwall inequality,

$$\|\phi_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)}\leq\left(M_0^2\theta^2\left|\Omega\right|T+\|\phi_\varepsilon(\cdot,0)\|^2_{L^2(\Omega)}\right)\exp\{(2\gamma-1)T\}. \tag{17}$$

Thus $\phi_\varepsilon\in L^\infty(0,T;L^2(\Omega))$ is still valid.

Combining (13)–(15), and dropping the nonnegative term $\frac{1}{2}\left\|\phi_\varepsilon(\cdot,t)\right\|^2_{L^2(\Omega)}$, we get the following estimate

$$\begin{aligned}&\int_0^t(1+\theta\lambda\|\Delta\omega\|^{p(x)}_{L^{p(x)}(\Omega)})\int_\Omega|\Delta\phi_\varepsilon|^{p(x)}\mathrm{d}x\mathrm{d}\tau+\left(\gamma-\tfrac{1}{2}\right)\int_0^t\|\phi_\varepsilon(\cdot,\tau)\|^2_{L^2(\Omega)}\mathrm{d}\tau\\&\leq\tfrac{1}{2}M_0^2\theta^2\left|\Omega\right|T+\tfrac{1}{2}\left\|\phi_\varepsilon(\cdot,0)\right\|^2_{L^2(\Omega)}.\end{aligned} \tag{18}$$

Now we also analyze (18) in two case: $\gamma-\frac{1}{2}\leq 0$ and $\gamma-\frac{1}{2}>0$. In case of $\gamma-\frac{1}{2}>0$, we drop $\left(\gamma-\frac{1}{2}\right)\int_0^t\|\phi_\varepsilon(\cdot,\tau)\|^2_{L^2(\Omega)}\mathrm{d}\tau$ to arrive at

$$\int_0^t(1+\theta\lambda\|\Delta\omega\|^{p(x)}_{L^{p(x)}(\Omega)})\int_\Omega|\Delta\phi_\varepsilon|^{p(x)}\mathrm{d}x\mathrm{d}\tau\leq\frac{1}{2}M_0^2\theta^2\left|\Omega\right|T+\frac{1}{2}\left\|\phi_\varepsilon(\cdot,0)\right\|^2_{L^2(\Omega)}. \tag{19}$$

When $\gamma-\frac{1}{2}\leq 0$, it follows from (17) that

$$\begin{aligned}&\int_0^t(1+\theta\lambda\|\Delta\omega\|^{p(x)}_{L^{p(x)}(\Omega)})\int_\Omega|\Delta\phi_\varepsilon|^{p(x)}\mathrm{d}x\mathrm{d}\tau\\&\leq\tfrac{1}{2}M_0^2\theta^2\left|\Omega\right|T+\tfrac{1}{2}\left\|\phi_\varepsilon(\cdot,0)\right\|^2_{L^2(\Omega)}\\&\quad+\gamma\left(M_0^2\theta^2\left|\Omega\right|T+\|\phi_\varepsilon(\cdot,0)\|^2_{L^2(\Omega)}\right)\exp\{(2\gamma-1)T\}.\end{aligned} \tag{20}$$

This and (19), combined with $\omega\in L^\infty(0,T;W^{2,p(x)}(\Omega))$, imply

$$\int_0^t\int_\Omega|\Delta\phi_\varepsilon|^{p(x)}\mathrm{d}x\mathrm{d}\tau\leq C(T,\left|\Omega\right|,\|\phi_\varepsilon(\cdot,0)\|^2_{L^2(\Omega)}).$$

Thus, using Sobolev embedding, that is,

$$\int_\Omega\phi_\varepsilon^2\mathrm{d}x\leq C\int_\Omega|\nabla\phi_\varepsilon|^{p(x)}\mathrm{d}x\leq C\int_\Omega|\Delta\phi_\varepsilon|^{p(x)}\mathrm{d}x,$$

$\phi_\varepsilon\in L^\infty(0,T;W^{2,p(x)}(\Omega))$ follows. □

Lemma 3.2. Assume ϕ_ε is the solution of problem (7), such that

$$\partial_t\phi_\varepsilon\in L^\infty(0,T;L^2(\Omega)). \tag{21}$$

Proof. For any $t\in(0,T]$, multiplying the first line of (7) by $\partial_t\phi_\varepsilon$ and integrating the resulting relation over Ω_t , we have that

$$\int_0^t\|\partial_\tau\phi_\varepsilon(\cdot,\tau)\|^2_{L^2(\Omega)}\mathrm{d}\tau=-A_1+A_2+A_3, \tag{22}$$

where

$$A_1=\int_0^t\int_\Omega(1+\theta\lambda\|\nabla\omega\|^{p(x)}_{L^{p(x)}(\Omega)})|\Delta\phi_\varepsilon|^{p(x)}\Delta\phi_\varepsilon\Delta\partial_t\phi_\varepsilon\mathrm{d}x\mathrm{d}\tau,$$

$$A_2=\gamma\int_0^t\int_\Omega\phi_\varepsilon\cdot\partial_\tau\phi_\varepsilon\mathrm{d}x\mathrm{d}\tau, A_3=\theta\int_0^t\int_\Omega\beta_\varepsilon(\phi_\varepsilon-\phi_0)\phi_\varepsilon\mathrm{d}x\mathrm{d}\tau.$$

Using some differential transforms obtains

$$A_1 = -\tfrac{1}{2}\int\int_{Q_T}\rho^{\alpha}\Big(|\Delta\phi_\varepsilon|^2+\varepsilon\Big)^{\frac{p(x)-2}{2}}\partial_t\left(|\Delta\phi_\varepsilon|^2+\varepsilon\right)\mathrm{d}x\mathrm{d}t$$
$$= -\tfrac{1}{p}\int\int_{Q_T}\partial_t\Big(|\Delta\phi_\varepsilon|^2+\varepsilon\Big)^{\frac{p(x)}{2}}\mathrm{d}x\mathrm{d}t.$$

Since $u_{0\varepsilon}(x) = u_0 + \varepsilon$, then

$$\begin{aligned}A_1 &= \textstyle\int_0^t\int_\Omega (1+\theta\lambda\|\Delta\omega\|^{p(x)}_{L^{p(x)}(\Omega)})\partial_t|\Delta\phi_\varepsilon|^{p(x)}\mathrm{d}x\mathrm{d}\tau\\ &\le \sup_{x\in[0,T]}(1+\lambda\|\Delta\omega\|^{p(x)}_{L^{p(x)}(\Omega)})\left(\|\Delta\phi_\varepsilon(\cdot,t)\|^{p(x)}_{L^{p(x)}(\Omega)}+\|\Delta\phi_\varepsilon(\cdot,0)\|^{p(x)}_{L^{p(x)}(\Omega)}\right).\end{aligned} \tag{23}$$

Applying Holder and Young inequalities, we have that

$$|A_2| \le 2\gamma^2\int_0^t\|\phi_\varepsilon(\cdot,\tau)\|^2_{L^2(\Omega)}\mathrm{d}\tau + \frac{1}{8}\int_0^t\|\partial_\tau\phi_\varepsilon(\cdot,\tau)\|^2_{L^2(\Omega)}\mathrm{d}\tau, \tag{24}$$

$$A_3 \le 2M_0^2\theta^2|\Omega|T + \frac{1}{8}\int_0^t\|\partial_\tau\phi_\varepsilon(\cdot,\tau)\|^2_{L^2(\Omega)}\mathrm{d}\tau. \tag{25}$$

Then, submitting (23)–(25) into (22), we infer that

$$\begin{aligned}&\tfrac{3}{4}\textstyle\int_0^t\|\partial_\tau\phi_\varepsilon(\cdot,\tau)\|^2_{L^2(\Omega)}\,\mathrm{d}\tau\\ &\le \sup_{x\in[0,T]}(1+\lambda\|\Delta\omega\|^{p(x)}_{L^{p(x)}(\Omega)})\left(\|\Delta\phi_\varepsilon(\cdot,t)\|^{p(x)}_{L^{p(x)}(\Omega)}+\|\Delta\phi_\varepsilon(\cdot,0)\|^{p(x)}_{L^{p(x)}(\Omega)}\right)\\ &\quad+2\gamma^2\textstyle\int_0^t\|\phi_\varepsilon(\cdot,\tau)\|^2_{L^2(\Omega)}\mathrm{d}\tau+2M_0^2\theta^2|\Omega|T,\end{aligned} \tag{26}$$

which, combining (12) and $\omega\in L^\infty(0,T;W^{2,p(x)}(\Omega))$ implies that (21) follows. □

Lemma 3.3. For any $t\in(0,T]$, it holds

$$\Delta(|\Delta\phi_\varepsilon|^{p(x)-2}\Delta\phi_\varepsilon)\in L^2(\Omega). \tag{27}$$

Proof. Replace $\partial_t\phi_\varepsilon$ with $-\Delta\left(|\Delta\phi_\varepsilon|^{p(x)-2}\Delta\phi_\varepsilon\right)$ in (22), and (22) becomes

$$\int_\Omega(1+\lambda\theta\|\Delta\omega\|^{p(x)}_{L^{p(x)}(\Omega)})|\Delta(|\Delta\phi_\varepsilon|^{p(x)-2}\Delta\phi_\varepsilon)|^2\mathrm{d}x = A_4+A_5+A_6, \tag{28}$$

where

$$A_4 = \int_\Omega\partial_t\phi_\varepsilon\Delta(|\Delta\phi_\varepsilon|^{p(x)-2}\Delta\phi_\varepsilon)\mathrm{d}x, A_5 = \gamma\int_\Omega\phi_\varepsilon\Delta(|\Delta\phi_\varepsilon|^{p(x)-2}\Delta\phi_\varepsilon)\mathrm{d}x,$$

$$A_6 = \theta\int_\Omega\beta_\varepsilon(\phi_\varepsilon-\phi_0)\Delta(|\Delta\phi_\varepsilon|^{p(x)-2}\Delta\phi_\varepsilon)\mathrm{d}x.$$

Using Holder and Young inequalities, we infer that

$$|A_4| \le 2\int_\Omega|\partial_t\phi_\varepsilon|^2\mathrm{d}x + \frac{1}{8}\int_\Omega\left|\Delta(|\Delta\phi_\varepsilon|^{p(x)-2}\Delta\phi_\varepsilon)\right|^2\mathrm{d}x, \tag{29}$$

$$|A_5| \le 2\gamma^2\int_\Omega|\phi_\varepsilon|^2\mathrm{d}x + \frac{1}{8}\int_\Omega\left|\Delta(|\Delta\phi_\varepsilon|^{p(x)-2}\Delta\phi_\varepsilon)\right|^2\mathrm{d}x, \tag{30}$$

$$\begin{aligned}|A_6| &\leq \int_\Omega M_0\theta\left|\Delta(|\Delta\phi_\varepsilon|^{p(x)-2}\Delta\phi_\varepsilon)\right|\mathrm{d}x\\ &\leq 2M_0^2\theta^2|\Omega|+\tfrac{1}{8}\int_\Omega\left|\Delta(|\Delta\phi_\varepsilon|^{p(x)-2}\Delta\phi_\varepsilon)\right|^2\mathrm{d}x.\end{aligned}\tag{31}$$

Here we use $-\frac{1}{2}\leq\beta_\varepsilon(\phi_\varepsilon-\phi_0)\leq 0$ in (31) which is obtained by (11). Thus combining (28)–(31), one can drop the nonnegative term $\|\Delta\omega\|^{p(x)}_{L^{p(x)}(\Omega)}$ to arrive at

$$\begin{aligned}&\tfrac{5}{8}\int_\Omega|\Delta(|\Delta\phi_\varepsilon|^{p(x)-2}\Delta\phi_\varepsilon)|^2\mathrm{d}x\\ &\leq 2\gamma^2\int_\Omega|\phi_\varepsilon|^2\mathrm{d}x+2M_0^2\theta^2|\Omega|+2\int_\Omega|\partial_t\phi_\varepsilon|^2\mathrm{d}x,\end{aligned}\tag{32}$$

which, from (12) and (21),implies that (27) follows.□

Now, we analyze the continuity of $M(\cdot,\theta)$ for fixed θ. Define

$$\phi_{1,\varepsilon}:=M(\omega_1,\theta)\mathrm{and}\phi_{2,\varepsilon}:=M(\omega_2,\theta)$$

are the solution of (7) with different parameter ω_1 and ω_2, that is,

$$\partial_t\phi_{1,\varepsilon}-\nabla\left((1+\theta\lambda\|\Delta\omega_1\|^{p(x)}_{L^{p(x)}(\Omega)})\left|\Delta\phi_{1,\varepsilon}\right|^{p(x)-2}\Delta\phi_{1,\varepsilon}\right)-\gamma\phi_{1,\varepsilon}=-\theta\beta_\varepsilon(\phi_{1,\varepsilon}-u_0),$$

$$\partial_t\phi_{2,\varepsilon}-\Delta\left((1+\theta\lambda\|\Delta\omega_2\|^{p(x)}_{L^{p(x)}(\Omega)})\left|\Delta\phi_{2,\varepsilon}\right|^{p(x)-2}\Delta\phi_{2,\varepsilon}\right)-\gamma\phi_{2,\varepsilon}=-\theta\beta_\varepsilon(\phi_{2,\varepsilon}-\phi_0).$$

From (11), $\beta_\varepsilon(\phi_{1,\varepsilon}-\phi_0)=0$ and $\beta_\varepsilon(\phi_{2,\varepsilon}-\phi_0)=0$, then defining $\phi_\varepsilon:=\phi_{1,\varepsilon}-\phi_{2,\varepsilon}$ implies

$$\begin{aligned}&\partial_t\phi_\varepsilon-(1+\theta\lambda\|\Delta\omega_1\|^{p(x)}_{L^p(\Omega)})\nabla\left(|\Delta\phi_\varepsilon|^{p(x)-2}\Delta\phi_\varepsilon\right)+\gamma\phi_\varepsilon\\ &\leq\theta\lambda\left|\Delta\phi_{2,\varepsilon}\right|^{p(x)-2}\Delta\phi_{2,\varepsilon}\cdot\|\omega_1-\omega_2\|_{L^\infty(0,T;W_0^{2,p(x)}(\Omega))},\end{aligned}\tag{33}$$

with $\phi_\varepsilon(x,0)=0$ in Ω .

Lemma 3.4. Letting $\|\omega_1-\omega_2\|_{L^\infty(0,T;W_0^{2,p(x)}(\Omega))}\to 0$, there is

$$\|\phi_{1,\varepsilon}-\phi_{2,\varepsilon}\|_{L^\infty(0,T;W_0^{2,p(x)}(\Omega))}\to 0.\tag{34}$$

Proof. First, multiplying (33) by ϕ_ε and integrating over Ω ,

$$\begin{aligned}&\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\phi_\varepsilon|^2\mathrm{d}x+(1+\theta\lambda\|\Delta\omega_1\|^{p(x)}_{L^{p(x)}(\Omega)})\int_\Omega|\Delta\phi_\varepsilon|^{p(x)}\mathrm{d}x+\gamma\int_\Omega\phi_\varepsilon^2\mathrm{d}x\\ &\leq\|\omega_1-\omega_2\|_{L^\infty(0,T;W_0^{2,p(x)}(\Omega))}\int_\Omega\Delta(\left|\Delta\phi_{2,\varepsilon}\right|^{p(x)-2}\Delta\phi_{2,\varepsilon})\phi_\varepsilon\mathrm{d}x.\end{aligned}\tag{35}$$

Using Holder and young inequalities with parameter $(\frac{1}{2},\frac{1}{2})$, we have the following estimate

$$\begin{aligned}&\int_\Omega\Delta(\left|\Delta\phi_{2,\varepsilon}\right|^{p(x)-2}\Delta\phi_{2,\varepsilon})\phi_\varepsilon\mathrm{d}x\\ &\leq\tfrac{1}{2}\left\|\Delta(\left|\Delta\phi_{2,\varepsilon}\right|^{p(x)-2}\Delta\phi_{2,\varepsilon})\right\|^2_{L^2(\Omega)}+\tfrac{1}{2}\int_\Omega|\phi_\varepsilon|^2\mathrm{d}x.\end{aligned}\tag{36}$$

Using Sobolev embedding and Holder inequalities

$$\int_\Omega\phi_\varepsilon^2\mathrm{d}x\leq C\int_\Omega|\Delta\phi_\varepsilon|^2\mathrm{d}x\leq C\int_\Omega|\Delta\phi_\varepsilon|^{p(x)}\mathrm{d}x.\tag{37}$$

For any fixed $t \in (0, T]$, integrating (35) over $[0, t]$, and substituting (36) and (37) into (35),

$$\begin{aligned}&\tfrac{1}{2}\int_\Omega |\phi_\varepsilon|^2 \mathrm{d}x + A_7 \int_0^t \int_\Omega \phi_\varepsilon^2 \mathrm{d}x \\ &\leq \tfrac{1}{2} T \|\omega_1 - \omega_2\|_{L^\infty(0,T;W_0^{2,p(x)}(\Omega))} \left\| \Delta(\left|\Delta\phi_{2,\varepsilon}\right|^{p(x)-2} \Delta\phi_{2,\varepsilon}) \right\|_{L^2(\Omega)}^2 ,\end{aligned} \tag{38}$$

where

$$A_7 = 1 + \gamma + \theta\lambda \|\Delta\omega_1\|_{L^{p(x)}(\Omega)}^{p(x)} - \frac{1}{2}\|\omega_1 - \omega_2\|_{L^\infty(0,T;W_0^{2,p(x)}(\Omega))},$$

$$A_8 = \gamma - \frac{1}{2}\|\omega_1 - \omega_2\|_{L^\infty(0,T;W_0^{2,p(x)}(\Omega))}.$$

By Gronwall inequality and $\phi(0, x) = 0$, we have

$$\begin{aligned}\int_\Omega |\phi_\varepsilon|^2 \mathrm{d}x \leq\ & T \|\omega_1 - \omega_2\|_{L^\infty(0,T;W_0^{2,p(x)}(\Omega))} \\ & \times \sup_{t\in[0,T]} \left\| \Delta(\left|\Delta\phi_{2,\varepsilon}\right|^{p(x)-2} \nabla\phi_{2,\varepsilon}) \right\|_{L^2(\Omega)}^2 \exp\{A_8 T\}.\end{aligned} \tag{39}$$

Second, replacing ϕ_ε with $\partial_t\phi_\varepsilon$ and (35) becomes

$$\begin{aligned}&\int_\Omega |\partial_t\phi_\varepsilon|^2 \mathrm{d}x + (1 + \theta\lambda \|\Delta\omega_1\|_{L^{p(x)}(\Omega)}^{p(x)}) \int_\Omega \Delta(|\Delta\phi_\varepsilon|^{p(x)-2}\Delta\phi_\varepsilon)\partial_t\Delta\phi_\varepsilon \mathrm{d}x \\ &\leq \lambda \int_0^T \|\omega_1 - \omega_2\|_{L^\infty(0,T;W_0^{2,p(x)}(\Omega))} \int_\Omega \Delta(\left|\Delta\phi_{2,\varepsilon}\right|^{p(x)-2} \Delta\phi_{2,\varepsilon})\partial_t\phi_\varepsilon \mathrm{d}x \\ &\quad - \gamma \int_\Omega \phi_\varepsilon \partial_t\phi_\varepsilon \mathrm{d}x.\end{aligned} \tag{40}$$

Using Holder and Young inequalities,

$$\gamma \int_\Omega \phi_\varepsilon \partial_t\phi_\varepsilon \mathrm{d}x \leq 2\gamma^2 \int_\Omega \phi_\varepsilon^2 \mathrm{d}x + \frac{1}{8}\int_\Omega |\partial_t\phi_\varepsilon|^2 \mathrm{d}x, \tag{41}$$

$$\begin{aligned}&\int_\Omega \Delta(\left|\Delta\phi_{2,\varepsilon}\right|^{p(x)-2} \Delta\phi_{2,\varepsilon})\partial_t\phi_\varepsilon \mathrm{d}x \\ &\leq 2 \left\| \Delta(\left|\Delta u_{2,\varepsilon}\right|^{p(x)-2} \Delta u_{2,\varepsilon}) \right\|_{L^2(\Omega)}^2 + \tfrac{1}{8}\int_\Omega |\partial_t\phi_\varepsilon|^2 \mathrm{d}x.\end{aligned} \tag{42}$$

Combining (40) and (41) to (42) , we infer that

$$\begin{aligned}&A_9 \int_\Omega |\partial_t\phi_\varepsilon|^2 \mathrm{d}x + (1 + \theta\lambda \|\Delta\omega_1\|_{L^{p(x)}(\Omega)}^{p(x)}) \tfrac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\Delta\phi_\varepsilon|^{p(x)} \mathrm{d}x \\ &\leq A_{10}(t) + 2\gamma^2 \int_\Omega |\phi_\varepsilon|^2 \mathrm{d}x,\end{aligned} \tag{43}$$

where

$$A_9 = \frac{7}{8} - \frac{1}{8}\lambda \|\omega_1 - \omega_2\|_{L^\infty(0,T;W_0^{2,p(x)}(\Omega))},$$

$$A_{10} = 2\lambda \left\| \Delta(\left|\Delta\phi_{2,\varepsilon}\right|^{p(x)-2} \Delta\phi_{2,\varepsilon}) \right\|_{L^2(\Omega)}^2 \|\omega_1 - \omega_2\|_{L^\infty(0,T;W_0^{2,p(x)}(\Omega))}.$$

It is noteworthy that $A_9 \int_\Omega |\partial_t\phi_\varepsilon|^2 \mathrm{d}x > 0$ if $\|\omega_1 - \omega_2\|_{L^\infty(0,T;W_0^{2,p(x)}(\Omega))}$ is small enough. Thus we drop the nonnegative term $A_9 \int_\Omega |\partial_t\phi_\varepsilon|^2 \mathrm{d}x$ and $\|\Delta\omega_1\|_{L^{p(x)}(\Omega)}^{p(x)}$ to have

$$\int_\Omega |\nabla\phi_\varepsilon|^{p(x)} \mathrm{d}x \leq A_{10} T + 2\gamma^2 T \int_\Omega |\phi_\varepsilon|^2 \mathrm{d}x. \tag{44}$$

From (39) and Lemma 3.3, $A_{10} \to 0$, $\int_\Omega |\phi_\varepsilon|^2 \mathrm{d}x \to 0$ ($\|\omega_1 - \omega_2\|_{L^\infty(0,T;W_0^{2,p(x)}(\Omega))} \to 0$), such that using Poincars inequality gives

$$\int_\Omega |\phi_\varepsilon|^{p(x)} \mathrm{d}x \le C \int_\Omega |\Delta\phi_\varepsilon|^{p(x)} \mathrm{d}x \to 0 \text{ as } \|\omega_1 - \omega_2\|_{L^\infty(0,T;W_0^{2,p(x)}(\Omega))} \to 0.$$

Hence, the proof is completed. □

We can also consider the continuity of $M(\omega, \cdot)$ for fixed ω . For any fixed $\omega \in L^\infty(0, T; W^{2,p(x)}(\Omega))$, let

$$\phi_{1,\varepsilon} := M(\omega, \theta_1), \phi_{2,\varepsilon} := M(\omega, \theta_2)$$

be the solution of (7) with different θ_1 and θ_2 , and define $\phi_\varepsilon := \phi_{1,\varepsilon} - \phi_{2,\varepsilon}$. Following a similar way of (33), we have that

$$\begin{aligned} &\partial_t\phi_\varepsilon - (1 + \theta_2\lambda\|\Delta\omega\|^{p(x)}_{L^{p(x)}(\Omega)})\Delta\left(|\Delta\phi_\varepsilon|^{p(x)-2}\Delta\phi_\varepsilon\right) + \gamma\phi_\varepsilon \\ &\le |\theta_1 - \theta_2| \cdot \left|\Delta\phi_{1,\varepsilon}\right|^{p(x)-2}\Delta\phi_{1,\varepsilon} \cdot \|\Delta\omega\|_{L^{p(x)}(\Omega)}. \end{aligned} \tag{45}$$

Thus, the continuity of $M(\omega, \cdot)$ with respect to θ , that is,

$$\int_\Omega |\phi_\varepsilon|^{p(x)} \mathrm{d}x \le C \int_\Omega |\Delta\phi_\varepsilon|^{p(x)} \mathrm{d}x \to 0 \text{ as } |\theta_1 - \theta_2| \to 0$$

will be verified in a similar way as in Lemma 3.4.

Lemma 3.5. For any fixed θ, the operator $M(\cdot, \theta)$ is compact.

Proof. Assume B is a bounded domain in $L^\infty(0, T; W_0^{2,p(x)}(\Omega))$. From Lemma 3.1, we have that $M(B, \theta)$ is bounded in $L^\infty(0, T; W_0^{2,p(x)}(\Omega))$. Lemma 3.2 implies that $\partial_t\phi_\varepsilon$ is bounded in $L^\infty(0, T; L^2(\Omega))$. Lemmas 3.1 and 3.4 show that $M(B, \theta)$ is compact in $L^\infty(0, T; W_0^{2,p(x)}(\Omega))$. Hence, the operator $M(\cdot, \theta)$, by Aubin-Lions lemma [14], is compact.□

By the same argument as in [6,12], we can get that equation

$$\partial_t\phi_\varepsilon - \Delta(|\Delta\phi_\varepsilon|^{p(x)-2}\Delta\phi_\varepsilon) + \gamma\phi_\varepsilon + x = 0 \tag{46}$$

with Dirichlet boundary condition has a unique solution in $L^\infty(0, T; W_0^{2,p(x)}(\Omega))$ for any $x \in L^\infty(0, T; W_0^{2,p(x)}(\Omega))$. Thus, combining Lemma 3.4, Lemma 3.5, (46), according to Leray-Schauder principle [11], problem (10) admits a solution

$$\phi_\varepsilon \in L^\infty(0, T; W_0^{2,p(x)}(\Omega)),\ \partial_t\phi_\varepsilon \in L^2(\Omega_T). \tag{47}$$

Using integral by part, the solution of problem (10) satisfies

$$\begin{aligned} &\int\int_{\Omega_T} (\partial_t\phi_\varepsilon \cdot \varphi + (1 + \lambda\|\Delta\phi_\varepsilon\|^{p(x)}_{L^{p(x)}(\Omega)})|\Delta\phi_\varepsilon|^{p(x)-2}\Delta\phi_\varepsilon\Delta\varphi + \gamma\phi_\varepsilon\varphi)\mathrm{d}x\mathrm{d}t \\ &= -\int\int_\Omega \beta_\varepsilon(\phi_\varepsilon - \phi_0)\varphi \,\mathrm{d}x\mathrm{d}t \end{aligned} \tag{48}$$

with $\varphi \in C^1(\bar{\Omega}_T)$.

4. Proof of the Theorem 2.1

This section plan to prove Theorem 1.1. By (12), (21) and (27), there exist a ubsequence of $\{\phi_\varepsilon\}$ (still use the same notation) and a function

$$\phi \in L^\infty(0,T;W_0^{2,p(x)}(\Omega))$$

such that as $\varepsilon \to 0$,

$$\phi_\varepsilon \overset{weak}{\to} \phi \ \text{ in } L^\infty(0,T;W_0^{2,p(x)}(\Omega)), \tag{49}$$

$$\partial_t\phi_\varepsilon \overset{weak}{\to} \partial_t\phi \text{ in } L^2(\Omega_T), \tag{50}$$

$$(1+\lambda\|\Delta\phi_\varepsilon\|^{p(x)}_{L^{p(x)}(\Omega)})|\Delta\phi_\varepsilon|^{p(x)-2}\phi_\varepsilon \overset{weak}{\to}(1+\lambda\|\Delta\phi\|^{p(x)}_{L^{p(x)}(\Omega)})|\Delta\phi|^{p(x)}\phi \text{ in } L^1(Q_T), \tag{51}$$

where $\overset{weak}{\to}$ stands for weak convergence. To have (51), we need some proofs similar to those in [3].
Lemma 4.1. Assume that ϕ_ε is a solution of (10). One has the following limit formula

$$-\beta_\varepsilon(\phi_\varepsilon-\phi_0)\to\xi\in G(\phi-\phi_0) \text{ as } \varepsilon\to 0. \tag{52}$$

Proof. Using (11) and the definition of β_ε, give

$$-\beta_\varepsilon(\phi_\varepsilon-\phi_0)\to\xi \text{ as } \varepsilon\to 0.$$

Now, we prove $\xi\in G(u-u_0)$ and only need to prove that

$$\xi(x_0,t_0)=0 \text{ if } \phi(x_0,t_0)>\phi_0(x_0).$$

In deed, when $\phi(x_0,t_0)>\phi_0(x_0)$, there exist a constant $h>0$ and a δ-neighborhood $B_\delta(x_0,t_0)$ such that

$$\phi_\varepsilon(x,t)\geq\phi_0+\varepsilon+h, \forall(x,t)\in B_\delta(x_0,t_0).$$

If ε is small enough, one has

$$0\geq\beta_\varepsilon(\phi_\varepsilon-\phi_0)\geq\beta_\varepsilon(h)=0, \forall(x,t)\in B_\delta(x_0,t_0).$$

Let $\varepsilon\to 0$, then the following limit results will be obtained

$$\xi(x,t)=0, \forall(x,t)\in Q_T.$$

Hence, (52) follows. □

Combining (10), (11), and Lemma 4.1,

$$\phi(x,0)=\phi_0(x) \text{ in } \Omega,\ \phi(x,t)\geq\phi_0(x) \text{ in } \Omega_T, \xi\in G(\phi-\phi_0),$$

such that (a)–(c) of Definition 2.1 follows. Therefore, the classic theory of parabolic problems (for details, see [3]) ensures the remainder arguments of Theorem 2.1.

5. Proof of the Theorem 2.2

In this section, we first prove the stability of solution in Theorem 2.2 by using two generalized solution (u_1, ξ_1) and (u_2, ξ_2) of (1) with two different initial conditions

$$\phi(0, x) = \phi_{0,i}(x), \quad x \in \Omega, i = 1, 2. \tag{53}$$

Recall that $\lambda = 0$ in this section such that for any $\varphi \in L^\infty(0, T; W^{2,p(x)}(\Omega))$,

$$\int_0^t \int_\Omega (\partial_t\phi_1 \cdot \varphi + |\Delta\phi_1|^{p(x)-2}\Delta\phi_1\Delta\varphi + \gamma\phi_1\varphi)\mathrm{d}x = \int_0^t \int_\Omega \xi_1 \cdot \varphi \,\mathrm{d}x\mathrm{d}t, \tag{54}$$

$$\int_0^t \int_{\Omega_T} (\partial_t\phi_2 \cdot \varphi + |\Delta\phi_2|^{p(x)-2}\Delta\phi_2\Delta\varphi + \gamma\phi_2\varphi)\mathrm{d}x\mathrm{d}t = \int_0^t \int_\Omega \xi_2 \cdot \varphi \,\mathrm{d}x\mathrm{d}t. \tag{55}$$

Defining $\varphi = \phi_1 - \phi_2$, so one gets

$$\begin{aligned}&\int_0^t \int_{\Omega_T} \partial_t\varphi \cdot \varphi + \left(|\Delta\phi_1|^{p(x)-2}\Delta\phi_1 - |\Delta\phi_2|^{p(x)-2}\Delta\phi_2\right)\Delta\varphi + \gamma\varphi^2\mathrm{d}x\mathrm{d}\tau\\ &= \int_0^t \int_\Omega (\xi_1 - \xi_2) \cdot \varphi \,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{56}$$

From [11], one gets the following inequalities

$$\left(|\Delta\phi_1|^{p(x)-2}\Delta\phi_1 - |\Delta\phi_2|^{p(x)-2}\Delta\phi_2\right)\Delta\varphi \geq 2^{-p^+}|\Delta\phi_1 - \Delta\phi_2|^{p(x)} \geq 0. \tag{57}$$

Now, we give the following result before proving,

$$\int_0^t \int_\Omega (\xi_1 - \xi_2) \cdot \varphi \,\mathrm{d}x\mathrm{d}t \leq 0, \forall t \in [0, T]. \tag{58}$$

In deed, we may analyze it in two cases: $\phi_1(x, t) < \phi_2(x, t)$ and $\phi_1(x, t) > \phi_2(x, t)$. In case of $\phi_1(x, t) > \phi_2(x, t)$, one gets $\phi_1(x, t) > \phi_{1,0}(x)$. It follows from (52) that

$$\xi_1 = 0 \leq \xi_2, \ (\xi_1 - \xi_2) \cdot \varphi = (\xi_1 - \xi_2) \cdot (\phi_1 - \phi_2) \leq 0. \tag{59}$$

When $\phi_1(x, t) < \phi_2(x, t)$, $\xi_i \geq 0 = \zeta_i$, such that (58) still holds.

On the one hand, dropping the nonnegative term (57) and $\int_0^t \int_\Omega \varphi^2\mathrm{d}x$ and removing non positive term (58) in (56),

$$\int_0^t \int_{\Omega_T} \partial_t\varphi \cdot \varphi\mathrm{d}x\mathrm{d}t \leq 0. \tag{60}$$

This finishes the proof of (4).

On the other hand, if (57) is not deleted, (58) becomes

$$\int_0^T \int_\Omega \partial_\tau\varphi \cdot \varphi + \left(|\Delta\phi_1|^{p(x)-2}\Delta\phi_1 - |\Delta\phi_2|^{p(x)-2}\Delta\phi_2\right)\Delta\varphi\mathrm{d}x\mathrm{d}t \leq 0. \tag{61}$$

It is easy to verify that

$$\int_0^T \int_\Omega \partial_t\varphi \cdot \varphi\mathrm{d}x\mathrm{d}t = \frac{1}{2}\int_0^T \int_{\Omega_T} \partial_t\varphi^2\mathrm{d}x\mathrm{d}t = \int_\Omega \varphi(x, T)^2\mathrm{d}x - \int_\Omega \varphi(x, 0)^2\mathrm{d}x,$$

which, combined with (61) and $\int_\Omega \varphi^2 \mathrm{d}x \ge 0$, implies

$$\begin{aligned}&\int\int_{\Omega_T} \left(|\Delta\phi_1|^{p(x)-2}\Delta\phi_1 - |\Delta\phi_2|^{p(x)-2}\Delta\phi_2\right)\Delta\varphi \mathrm{d}x\mathrm{d}t\\ &\le \frac{1}{2}\int_\Omega |\phi_{0,1}(x) - \phi_{0,2}(x)|^2 \mathrm{d}x.\end{aligned} \tag{62}$$

Thus, from (57), we have

$$\int\int_{\Omega_T} |\Delta\phi_1 - \Delta\phi_2|^{p(x)} \mathrm{d}x\mathrm{d}t \le C\int_\Omega |\phi_{0,1}(x) - \phi_{0,2}(x)|^2 \mathrm{d}x. \tag{63}$$

It follows by Poincarés inequality that

$$\begin{aligned}&\int\int_{\Omega_T} |\phi_1 - \phi_2|^{p(x)} \mathrm{d}x\mathrm{d}t\\ &\le C\int\int_{\Omega_T} |\nabla\phi_1 - \nabla\phi_2|^{p(x)} \mathrm{d}x\mathrm{d}t \le C\int\int_{\Omega_T} |\Delta\phi_1 - \Delta\phi_2|^{p(x)} \mathrm{d}x\mathrm{d}t.\end{aligned} \tag{64}$$

It gives (5) in Theorem2.2 by combining (63) and (64).

Final, we consider the uniqueness of solution in Theorem 2.2. Assume that (ϕ_1, ξ_1) and (ϕ_2, ξ_2) are weak solutions of (1). Because they have the same initial value condition, that is

$$\phi_1(0, x) = \phi_2(0, x) = \phi_0(x), \quad x \in \Omega,$$

uniqueness of solution can be obtained by choosing $\phi_{0,1}(x) = \phi_{0,2}(x)$ in (4) or (5).

6. Conclusions

This paper study the following variation-inequality initial-boundary value problems

$$\begin{cases} \min\{L\phi, \phi - \phi_0\} = 0, & (x, t) \in Q_T,\\ \phi(0, x) = \phi_0(x), & x \in \Omega,\\ \phi(t, x) = 0, & (x, t) \in \partial\Omega \times (0, T),\end{cases}$$

with fourth-order p-Laplacian Kirchhoff operators,

$$L\phi = \partial_t\phi - \Delta\left((1 + \lambda\|\Delta\phi\|^{p(x)}_{L^{p(x)}(\Omega)})|\Delta\phi|^{p(x)-2}\Delta\phi\right) + \gamma\phi,$$

$\gamma \ge 0$ and $p(x) > 2$. Here we construct a mapping based on the Leray Schauder principle and introduce a penalty function to prove the existence of the solution. Further, this paper is to obtain the uniqueness and stability of solutions from weak solutions by using inequality amplification techniques.

At present, this paper has the following shortcomings: Lemma 3.1 is not valid when $1 < p(x) < 2$, because Sobolev embedding theorem cannot be used at this time. In Lemma 3.4, we use the condition $\gamma > 0$ to prove (40) Lemma 3.4 is valid only if γ is not greater than 0 or at least γ is greater than -1. In view of these problems, we will continue to study.

Compared with [8–12], this paper extends equality to inequality and studies the initial boundary value problem in the case of variational inequality. The advantage of this paper compared with [1–3] is to analyze the variation-inequality problem under the more general fourth order Kirchhoff operator. Reference [4] only studies the existence of solutions, while this paper continues to study the stability and uniqueness of solutions.

Acknowledgments

The author sincerely thanks the editors and anonymous reviewers for their insightful comments and constructive suggestions, which greatly improved the quality of the paper. This work is supported by the National Social Science Fund of China in 2020(No.20BMZ043) and the Doctoral Project of Guizhou Education University (No.2021BS037).

Conflict of interest

The author declares no conflict of interest.

References

1. W. Chen, T. Zhou, Existence of solutions for p-Laplacian parabolic Kirchhoff equation, *Appl. Math. Lett.*, **122** (2021),107527. http://dx.doi.org/10.1016/j.aml.2021.107527
2. I. Lasiecka, J. H. Rodrigues, Weak and strong semigroups in structural acoustic Kirchhoff-Boussinesq interactions with boundary feedback, *J. Differ. Equations*, **298** (2021), 387–429. http://dx.doi.org/10.1016/j.jde.2021.07.009
3. C. Vetro, Variable exponent $p(x)$-Kirchhoff type problem with convection, *J. Math. Anal. Appl.*, **506** (2022), 125721. http://dx.doi.org/10.1016/j.jmaa.2021.125721
4. N. D. Phuong, N. H. Tuan, Z. Hammouch, R. Sakthivel, On a pseudo-parabolic equations with a non-local term of the Kirchhoff type with random Gaussian white noise, *Chaos Soliton. Fract.*, **145** (2021), 110771. http://dx.doi.org/10.1016/j.chaos.2021.110771
5. M. Xiang, D. Hu, D. Yang, Least energy solutions for fractional Kirchhoff problems with logarithmic nonlinearity, *Nonlinear Anal.*, **198** (2020), 111899. http://dx.doi.org/10.1016/j.na.2020.111899
6. M. Xiang, D. Yang, Nonlocal Kirchhoff problems: Extinction and non-extinction of solutions, *J. Math. Anal. Appl.*, **477** (2019), 133–152. http://dx.doi.org/10.1016/j.jmaa.2019.04.020
7. Y. Han, Q. Li, Threshold results for the existence of global and blow-up solutions to Kirchhoff equations with arbitrary initial energy, *Comput. Math. Appl.*, **75** (2018), 3283–3297. http://dx.doi.org/10.48550/arXiv.1703.09094
8. B. Guo, H. Zhou, Output feedback stabilization for multi-dimensional Kirchhoff plate with general corrupted boundary observation, *Eur. J. Control*, **28** (2016), 38–48. http://dx.doi.org/10.1016/j.ejcon.2015.12.004
9. I. Lasiecka, M. Pokojovy, X. Wand, Global existence and exponential stability for a nonlinear thermoelastic Kirchhoff-Love plate, *Nonlinear Anal.-Real*, **38** (2017), 184–221. http://dx.doi.org/10.1016/J.NONRWA.2017.04.001
10. M. Ghisi, M. Gobbino, Optimal decay-error estimates for the hyperbolic-parabolic singular perturbation of a degenerate nonlinear equation, *J. Differ. Equations*, **254** (2013), 911–932. http://dx.doi.org/10.1016/j.jde.2012.10.005

11. Z. Yang, Longtime behavior of the Kirchhoff type equation with strong damping on RN, *J. Differ. Equations*, **242** (2007), 269–286. http://dx.doi.org/10.1016/j.jde.2007.08.004
12. I. Lasiecka, M. Pokojovy, X. Wan, Long-time behavior of quasilinear thermoelastic Kirchhoff-Love plates with second sound, *Nonlinear Anal.*, **186** (2019), 219–258. http://dx.doi.org/10.48550/arXiv.1811.01138
13. T. Boudjeriou, M. K. Hamdani, M. Bayrami-Aminloue, Global existence, blow-up and asymptotic behavior of solutions for a class of $p(x)$-Choquard diffusion equations in RN, *J. Math. Anal. Appl.*, **506** (2022), 125720. http://dx.doi.org/10.1016/j.jmaa.2021.125720
14. Y. Sun, H. Wang, Study of weak solutions for a class of degenerate parabolic variational inequalities with variable exponent, *Symmetry*, **14** (2022), 1255. http://dx.doi.org/10.3390/sym14061255
15. D. Adak, G. Manzini, S. Natarajan, Virtual element approximation of two-dimensional parabolic variational inequalities, *Comput. Math. Appl.*, **116** (2022), 48–70. http://dx.doi.org/10.1016/j.camwa.2021.09.007
16. J. Dabaghi, V. Martin, M. Vohralík, A posteriori estimates distinguishing the error components and adaptive stopping criteria for numerical approximations of parabolic variational inequalities, *Comput. Method. Appl. M.*, **367** (2020), 113105. http://dx.doi.org/10.1016/j.cma.2020.113105
17. J. Li, C. Bi, Study of weak solutions of variational inequality systems with degenerate parabolic operators and quasilinear terms arising Americian option pricing problems, *AIMS Math.*, **7** (2022), 19758–19769. http://dx.doi.org/10.3934/math.20221083
18. X. Chen, F. Yi, Parabolic variational inequality with parameter and gradient constraints, *J. Math. Anal. Appl.*, **385** (2012), 928–946. http://dx.doi.org/10.1016/j.jmaa.2011.07.025
19. X. Chen, F. Yi, L. Wang, American lookback option with fixed strike price 2-D parabolic variational inequality, *J. Differ. Equations*, **251** (2011), 3063–3089. http://dx.doi.org/10.1016/j.jde.2011.07.027

AIMS Press