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Research article

Invariant properties of modules under smash products from finite dimensional algebras

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Abstract: We give the relationship between indecomposable modules over the finite dimensional k-algebra A and the smash product $A\sharp G$ respectively, where G is a finite abelian group satisfying $G \subseteq Aut(A)$, and k is an algebraically closed field with the characteristic not dividing the order of G. More precisely, we construct all indecomposable $A\sharp G$-modules from indecomposable A-modules and prove that an $A\sharp G$-module is indecomposable if and only if it is an indecomposable G-stable module over A. Besides, we give the relationship between simple, projective and injective modules in $modA$ and those in $modA\sharp G$.

Keywords: finite dimensional algebra; smash product; indecomposable module; G-stable module; stable category; abelian
Mathematics Subject Classification: 16G10, 16G20

1. Introduction

Let A be a finite dimensional algebra over an algebraically closed field k with characteristic p, and G be an arbitrary finite group each element of which acts as an algebra automorphism on A. Then we have the skew group algebra of A by G, denoted by $A\sharp G$. The representation theory of skew group algebras has been widely studied [2, 3, 11, 12]. It is well known that the smash product $A\sharp G$ retains many properties from A when the order of G is invertible in k. For example, A is of finite representation type (1-Gorenstein, selfinjective, of finite global dimension) if and only if $A\sharp G$ is.

Since the algebras A and $A\sharp G$ have a lot of properties in common, it is of interest to study the relationship between modules over A and $A\sharp G$ and consider whether A-modules can induce $A\sharp G$-modules and if an A-module can induce $A\sharp G$-modules, how to describe all non-isomorphic classes of such induced $A\sharp G$-modules.

In [13], the authors give the relationship under the assumption $A = kQ$ is a path algebra and G is a cyclic group. In [7], the authors discuss the relationship for algebra $A = kQ/I$ and G is a finite abelian subgroup of automorphism group of bound quiver (Q, I). In this paper, we investigate the relationship between indecomposable modules over the finite dimensional k-algebra A and the smash product $A\sharp G$ respectively, where G is a finite abelian group satisfying $G \subseteq Aut(A)$, and k is an algebraically closed field with the characteristic not dividing the order of G. We prove that an $A\sharp G$-module is indecomposable if and only if it is indecomposable G-stable module and describe all the $A\sharp G$-module structures from the same G-stable module.

It should be noted that if algebra A is not basic, it is no longer isomorphic to a quotient of the path algebra. In general, the representations of a finite dimensional (non-basic) algebra is characterized via its corresponding basic algebra. In this paper, we show that the representations of skew group algebra of a finite dimensional non-basic algebra can be induced directly not through the representations of its corresponding basic algebra.

The article is organized as follows. In the next section, we introduce the basic notations in the context of smash products and G-stable modules. We devote Section 3 to induced $A\sharp G$-modules from G-stable modules. Section 4 focuses on the construction of all indecomposable $A\sharp H_M$-modules from an indecomposable A-module M with maximal stable subgroup H_M of G and describes the number of non-isomorphic indecomposable induced $A\sharp H_M$-modules from an indecomposable A-module M. Section 5 states the main theorem which constructs all indecomposable $A\sharp G$-modules from an indecomposable A-module and gives the number of non-isomorphic indecomposable induced $A\sharp G$-modules from an indecomposable A-module M. Then the relationship between simple, projective and injective modules in *modA* and those in *modA*$\sharp G$ is discussed. In the last section, we give the relation between the stable category of a path algebra and the corresponding smash products to be abelian.

2. Notations and preliminaries

We fix an algebraically closed field k. Let $Q = (Q_0, Q_1)$ be a finite quiver given by the vertex set Q_0 and the arrow set Q_1. For an arrow a in a quiver, we write $s(a)$ and $t(a)$ for its source and target respectively. Arrows in quivers are composed as functions, that is if ab is a path then $s(a) = t(b)$. The path algebra kQ is the algebra generated by all paths (including those of length zero) of Q, with multiplication induced by composition of paths. In this paper, algebras are assumed to be associative unital finite dimensional k-algebras and all modules are always unital and finitely generated.

We introduce the definition of skew group algebras as smash products, which was well-known in the theory of Hopf algebras.

Definition 2.1. *Let G be a finite group acting on an algebra A over a field by automorphisms. The smash product A♯G of A by G is the algebra defined by:*

(i) its underlying vector space is $A \otimes_k kG$;

(ii) multiplication is given by

$$(a \otimes g)(b \otimes h) = ag(b) \otimes gh$$

for $a, b \in A$ and $g, h \in G$, extended by linearity and distributivity.

Usually, we also call A♯G the skew group algebra of A by G.

From now on, algebra A is finite dimensional and $G \subseteq Aut(A)$ is a finite group with order n. Let k be an algebraically closed field whose characteristic does not divide the order of G. In this paper, We will deal with the smash product $A\sharp G$.

For $M \in modA$ and $g \in G$, we define an A-module ${}^gM$ by taking the same underlying vector space as M with the new module action

$$a \cdot m = g^{-1}(a)m.$$

Let $\phi : M \to N$ be a module homomorphism, and set ${}^g\phi = \phi$ as a linear map. Then ${}^g\phi : {}^gM \to {}^gN$ can be viewed as a homomorphism of modules under the new module action. Indeed,

$$\phi(a \cdot m) = \phi(g^{-1}(a)m) = g^{-1}(a)\phi(m) = a \cdot \phi(m).$$

Definition 2.2. *An A-module M is G-stable module if* ${}^gM \cong M$ *for any* $g \in G$.

We say that an A-module M is an indecomposable G-stable module if it is not isomorphic to the proper direct sum of two G-stable modules. Let $H_M = \{g \in G | {}^gM \cong M\}$. Then H_M is a subgroup of G. We call H_M maximal stable subgroup of G for M. Denote $R_M = \{g_1, g_2, \cdots, g_s\}$ a complete set of left coset representatives of H_M in G. By the Krull-Remak-Schmidt theorem for modules over finite dimensional k-algebras, we have the following lemma

Lemma 2.1. *[7] With the above notations, any indecomposable G-stable module M is precisely the representation of the form*

$$M \cong \bigoplus_{g \in R_N} {}^gN = {}^{g_1}N \oplus {}^{g_2}N \oplus \cdots \oplus {}^{g_s}N,$$

where N is an indecomposable A-module. Moreover, the Krull-Remak-Schmidt theorem holds for G-stable modules.

Proof. First, define $f : {}^h({}^gM) \to {}^{hg}M$ such that $f(m) = m$ for all $m \in M$, $g, h \in G$. Since

$$f(a \cdot m) = f(h^{-1}(a) \cdot m) = f(g^{-1}h^{-1}(a) \cdot m) = f((hg)^{-1}(a)m) = a \cdot f(m),$$

we have f is an A-module automorphism. Therefore, ${}^h({}^gM) \cong {}^{hg}M$ and $\bigoplus_{g \in R_N} {}^gN$ is an indecomposable G-stable module.

Let M be an indecomposable G-stable module. Then ${}^gM \cong M$ for any $g \in G$. If N is a summand of M as A-module, we have the isomorphism classes of $\{{}^gN | g \in G\}$ are summands of M as A-module. Therefore $M \cong \bigoplus_{g \in R_N} {}^gN$.

Let X be a G-stable module. Then ${}^gX \cong X$ as A-module for any $g \in G$. So we can write

$$X \cong M_1 \oplus M_2 \oplus \cdots \oplus M_t,$$

where each M_i is of the form

$$M_i \cong \bigoplus_{g \in R_{N_i}} {}^gN_i = {}^{g_{i1}}N_i \oplus {}^{g_{i2}}N_i \oplus \cdots \oplus {}^{g_{is}}N_i$$

with N_i an indecomposable A-module, $R_{N_i} = \{g_{i1}, g_{i2}, \cdots, g_{is}\}$ a complete set of left coset representatives of H_{N_i} in G. The lemma follows from the Krull-Remak-Schmidt theorem for A-modules. □

3. Induced $A\sharp G$-modules from G-stable modules

In this section, the conclusions can be found in [5–7, 13] when A is a finite dimensional path algebra or a path algebra with relation. Here we give their proofs by the similar method when A is a finite dimensional algebra.

Lemma 3.1. *Every module M of the smash product $A\sharp G$ is a G-stable A-module.*

Proof. Define $f : {}^gM \to M$ such that $f(m) = g(m)$ for all $m \in M$, $g \in G$. It is well-defined since M is an $A\sharp G$ module. Then we have that for any $a \in A$,

$$f(a \cdot m) = g(a \cdot m) = g(g^{-1}(a)m) = (a\sharp g)m = a(g(m)) = af(m).$$

Then f is an A-module homomorphism. It is easy to check that f is an isomorphism if we define $f^{-1} : M \to {}^gM$ such that $f^{-1}(m) = g^{-1}(m)$ for all $m \in M$. Therefore, $M \cong {}^gM$ as A-modules which means M is a G-stable A-module. □

We will show that any G-stable A module induces an $A\sharp G$-module.

Proposition 3.1. *[4] Let M be a G-stable A-module. Then for any $g \in G$ there exists an isomorphism $\phi_g : {}^gM \to M$ such that $\phi_g^n = \phi_g {}^g\phi_g \cdots {}^{g^{n-1}}\phi_g$ is the identity.*

Proof. Let M be a G-stable A-module and $u : {}^gM \to M$ be an isomorphism.

If $w = u^gu \cdots {}^{g^{n-1}}u$ is identity, then we get the required isomorphism $\phi_g = u$.

If $w = u^gu \cdots {}^{g^{n-1}}u$ is not identity, we can find some isomorphism $y : M \to M$ such that $\phi_g = yu$ satisfying that $\phi_g^n = \phi_g {}^g\phi_g \cdots {}^{g^{n-1}}\phi_g$ is the identity.

For this purpose, we note that ${}^gw = u^{-1}wu$, hence that ${}^g(w^m) = u^{-1}w^mu$ and ${}^g\psi = u^{-1}\psi u$ for all $m \in \mathbb{N}$ and all $\psi \in k[w] \subset End(M)$.

Suppose $y \in k[w]$ and set $\phi_g = yu$. Then by induction

$$\begin{aligned}
\phi_g^n = \phi_g {}^g\phi_g \cdots {}^{g^{n-1}}\phi_g &= yu^gy^gu \cdots {}^{g^{n-1}}y^{g^{n-1}}u \\
&= yuu^{-1}yu^gu \cdots {}^{g^{n-1}}y^{g^{n-1}}u \\
&= y^2u^gu^gu^{-1}u^{-1}yu^gu^{g^2}u \cdots {}^{g^{n-1}}y^{g^{n-1}}u \\
&= y^3u^gu^{g^2}u \cdots {}^{g^{n-1}}y^{g^{n-1}}u \\
&= \cdots = y^nu^gu \cdots {}^{g^{n-1}}u \\
&= y^nw.
\end{aligned}$$

Since n does not divide the characteristic of k, the equation $y^nw = 1$ has a solution $y = \sqrt[n]{w^{-1}}$ in $k[w]$. Therefore $\phi_g = yu$ is the required isomorphism. □

Theorem 3.1. *Let M be a G-stable A-module. Then M has an induced $A\sharp G$-module structure.*

Proof. We define $g(m) = \phi_g(m)$ for any $g \in G$, $m \in M$. First, by Proposition 3.1, $g^i(m) = \phi_g^i(m)$ and $\phi_{g_1}\phi_{g_2} = \phi_{g_1g_2}$. Hence, g is well-defined. Since for $a\sharp g_1, b\sharp g_2 \in A\sharp G$, $m \in M$,

$$(a\sharp g_1)((b\sharp g_2)(m)) = (a\sharp g_1)(b\phi_{g_2}(m)) = a\phi_{g_1}(b\phi_{g_2}(m))$$

$$= a\phi_{g_1}(g_1(b)\cdot\phi_{g_2}(m))) = ag_1(b)(\phi_{g_1}\phi_{g_2}(m))$$
$$= ag_1(b)(\phi_{g_1g_2}(m)) = (ag_1(b)\sharp g_1g_2)(m)$$
$$= ((a\sharp g_1)(b\sharp g_2))(m),$$

we have that M has an induced $A\sharp G$-module structure. □

4. Construction of indecomposable $A\sharp H_M$-modules for a maximal stable subgroup H_M

The main purpose of this section is to construct all indecomposable $A\sharp H_M$-modules from an indecomposable A-module M with maximal stable subgroup H_M of G and give the number of non-isomorphic indecomposable induced $A\sharp H_M$-modules from an indecomposable A-module M.

Here are two lemmas which will be used.

Lemma 4.1. *[12] Let M, N be indecomposable A-modules, and $G \subseteq Aut(A)$ be a finite subgroup of the k-automorphism group of A with order n. Then:*

(i) $(A\sharp G)\otimes_A M \cong \bigoplus_{g\in G} {}^gM$ as A-modules;

(ii) $(A\sharp G)\otimes_A M \cong (A\sharp G)\otimes_A N$ as $A\sharp G$-modules if and only if $N \cong {}^gM$ for some $g \in G$;

(iii) The number of summands in the decomposition of $(A\sharp G)\otimes_A M$ into a direct sum of indecomposables is at most the order of H_M, where $H_M = \{g \in G, {}^gM \cong M\}$.

Lemma 4.2. *[8, 9] Let H be a finite dimensional semisimple Hopf algebra and A be a finite dimensional H-module algebra. Then, for any $A\sharp H$ -module M, it holds that M is a direct summand of $A\sharp H \otimes_A M$ as an $A\sharp H$ -module.*

For an indecomposable A-module M, denote $H_M = \{g \in G, {}^gM \cong M\}$ and $R_M = \{g_1, g_2, \cdots, g_s\}$ a complete set of left coset representatives of H_M in G.

In the following discussion, we assume H_M is abelian. In particular, we can assume G is an abelian group. Since kH_M is semisimple, we have

$$kH_M \cong \bigoplus_{i=1}^{r=n/s} H_i \tag{4.1}$$

as kH_M -modules, where H_i is one dimensional irreducible H_M-representation for $i = 1, 2, \cdots, r$.

By Theorem 3.1, the H_M-stable A-module M induces an $A\sharp H_M$-module structure. Then $H_i \otimes_k M$ has an $A\sharp H_M$-module structure if we define

$$(a\sharp g)(1\otimes m) = g(1)\otimes a\sharp g(m)$$

for any $i \in \{1, 2, \cdots r\}$, $a\sharp g \in A\sharp H_M$, $1\otimes m \in H_i \otimes_k M$.

Lemma 4.3. *With the above notations and assumpution, we have*

(i) $H_i \otimes_k M \cong M$ as A-modules for $i = 1, 2, \cdots, r$;

(ii) $H_i \otimes_k M$ is an indecomposable $A\sharp H_M$-module for any $i = 1, 2, \cdots, r$;

(iii) $H_i \otimes_k M \ncong H_j \otimes_k M$ as $A\sharp H_M$-modules if $i \neq j$.

Proof. (i) Define $f : M \to H_i \otimes_k M$ such that $f(m) = 1 \otimes m$ for all $m \in M$. Then f is bijective. Since $f(a(m)) = 1 \otimes a(m) = a(1 \otimes m) = af(m)$ for all $a \in A, m \in M$, then f is an A-module isomorphism.

(ii) Since for any $a\sharp g_1, b\sharp g_2 \in A\sharp G$, $m \in M$,

$$(1\sharp 1)(1 \otimes m) = 1(1) \otimes 1\sharp 1(m) = 1 \otimes m$$

$$\begin{aligned}((a\sharp g_1)(b\sharp g_2))(1 \otimes m) &= (ag_1(b)\sharp g_1g_2)(1 \otimes m)\\ &= g_1g_2(1) \otimes (ag_1(b)\sharp g_1g_2)(m)\\ &= g_1g_2(1) \otimes ((a\sharp g_1)(b\sharp g_2))(m)\\ &= (a\sharp g_1)(g_2(1) \otimes (b\sharp g_2)(m)\\ &= (a\sharp g_1)((b\sharp g_2)(1 \otimes m)),\end{aligned}$$

we have $H_i \otimes_k M$ is an $A\sharp H_M$-module. By (i), it is an indecomposable A-module. Then $H_i \otimes_k M$ is an indecomposable $A\sharp H_M$-module for any $i = 1, 2, \cdots, r$.

(iii) Before proving (iii), we claim that $Hom_A(M, H_i \otimes_k M) \cong H_i \otimes_k End_A(M)$ as $A\sharp H_M$-modules for any $i = 1, 2, \cdots, r$.

First, $Hom_A(M, H_i \otimes_k M)$ has an $A\sharp H_M$-module structure with

$$(a\sharp l)(f)(m) = (a\sharp l)f(m), \quad for\ any\ f \in Hom_A(M, H_i \otimes_k M), a\sharp l \in A\sharp H_M, m \in M.$$

$End_A(M)$ has an $A\sharp H_M$-module structure with

$$(a\sharp l)(f)(m) = (a\sharp l)f(m), \quad for\ any\ f \in End_A(M), a\sharp l \in A\sharp H_M, m \in M.$$

$H_i \otimes_k End_A(M)$ has an $A\sharp H_M$-module structure with

$$(a\sharp l)(1 \otimes f) = l(1) \otimes (a\sharp l)f, \quad for\ any\ f \in End_A(M), a\sharp l \in A\sharp H_M.$$

Now we define $\Phi : Hom_A(M, H_i \otimes_k M) \to H_i \otimes_k End_A(M)$ by $\Phi(f) = 1 \otimes \overline{f}$ for any $f \in Hom_A(M, H_i \otimes_k M)$, where $\overline{f}$ is defined by $\overline{f}(m) = h_f m_f$ if $f(m) = h_f \otimes m_f$ for any $m \in M$.

Since for any $a \in A, m \in M$,

$$f(am) = af(m) = a(h_f \otimes m_f) = h_f \otimes a(m_f) = 1 \otimes a(h_f m_f),$$

we have

$$\overline{f}(am) = h_f a(m_f) = a(h_f m_f).$$

Thus $\overline{f} \in End_A(M)$ and Φ is well-defined.

Since for any $a \in A, f \in Hom_A(M, H_i \otimes_k M)$,

$$(af)(m) = a(f(m)) = a(h_f \otimes m_f) = h_f \otimes a(m_f),$$

we have

$$\Phi(af) = 1 \otimes \overline{af} = 1 \otimes a\overline{f} = a(1 \otimes \overline{f}) = a\Phi(f).$$

Thus Φ is an A-module homomorphism.

Since

$$(gf)(m) = g(f(m)) = g(h_f \otimes m_f),$$

we have

$$\Phi(gf) = 1 \otimes \overline{gf} = g(1 \otimes \overline{f}) = g\Phi(f).$$

It means that Φ is an $A\sharp H_M$-module homomorphism.

Since Φ is injective and

$$dim_k Hom_A(M, H_i \otimes_k M) = dim_k H_i \otimes_k End_A(M),$$

we have the homomorphism Φ is an $A\sharp H_M$-module isomorphism.

Now we prove (iii). Assume $H_i \otimes_k M \cong H_j \otimes_k M$ as $A\sharp H_M$-modules for some $i \neq j$. Then $H_i \otimes_k End_A(M) \cong H_j \otimes_k End_A(M)$ by the claim . Since $End_A(M)$ is local, we have $End_A(M)/radEnd_A(M) \cong k$ as algebras. Then

$$H_i \otimes_k End_A(M)/radEnd_A(M) \cong H_j \otimes_k End_A(M)/radEnd_A(M),$$

which means $H_i \cong H_j$ as kH_M-modules. We get a contradiction to $i \neq j$. □

Theorem 4.1. *Let M be an indecomposable A-module M with H_M =$\{g \in G, {}^gM \cong M\}$. Suppose H_M is abelian and $R_M = \{g_1, g_2, \cdots, g_s\}$ is a complete set of left coset representatives of H_M in G. H_i is defined in (4.1). Then we have*

(i) $A\sharp H_M \otimes_A M \cong \bigoplus_{i=1}^{r} H_i \otimes_k M$ as $A\sharp H_M$-modules.

(ii) For any $A\sharp H_M$-module N, if $M \cong N$ as A-modules, then there exists a unique $i \in \{1, 2 \cdots, r\}$ such that $N \cong H_i \otimes_k M$ as $A\sharp H_M$-modules.

Proof. (i) By Lemma 4.2, $H_i \otimes_k M$ is a direct summand of $A\sharp H_M \otimes_A (H_i \otimes_k M)$ for any $i \in \{1, 2 \cdots, r\}$. By Lemma 4.3(i), $A\sharp H_M \otimes_A (H_i \otimes_k M) \cong A\sharp H_M \otimes_A M$. Then we have $H_i \otimes_k M$ is a direct summand of $A\sharp H_M \otimes_A M$. By Lemma 4.3(iii), if $i \neq j$, $H_i \otimes_k M \ncong H_j \otimes_k M$, then by Krull-Remak-Schmidt theorem, $\bigoplus_{i=1}^{r} H_i \otimes_k M$ is a direct summand of $A\sharp H_M \otimes_A M$. And by Lemma 4.1(iii), $A\sharp H_M \otimes_A M$ has at most r summands. It shows that

$$A\sharp H_M \otimes_A M \cong \bigoplus_{i=1}^{r} H_i \otimes_k M$$

as $A\sharp H_M$-modules.

(ii) For an $A\sharp H_M$-module N, if $N \cong M$ as A-modules, then N is an indecomposable $A\sharp H_M$-module. By Lemma 4.2, N is a direct summand of $A\sharp H_M \otimes_A N \cong A\sharp H_M \otimes_A M$. By (i), Lemma 4.3 and the Krull-Remak-Schmidt theorem, it is easy to see that there exists a unique $i \in \{1, 2 \cdots, r\}$ such that $N \cong H_i \otimes_k M$. □

5. Construction of indecomposable $A\sharp G$-modules from an indecomposable A-module

In this section, we give the main results which construct all induced indecomposable $A\sharp G$-modules from an indecomposable A-module and give the number of non-isomorphic indecomposable induced $A\sharp G$-modules from an indecomposable A-module M.

Lemma 5.1. *Let G be a finite group with order n and M be an indecomposable A-module with $H_M = \{g \in G, {}^gM \cong M\}$. Suppose H_M is abelian and $R_M = \{g_1, g_2, \cdots, g_s\}$ is a complete set of left coset representatives of H_M in G. H_i is defined in (4.1). Then*

(i) $A\sharp G \otimes_{A\sharp H_M} (H_i \otimes_k M) \cong \bigoplus_{g\in R_M} {}^gM$ as A-modules;

(ii) $A\sharp G \otimes_{A\sharp H_M} (H_i \otimes_k M)$ is an indecomposable $A\sharp G$ -module;

(iii) $A\sharp G \otimes_{A\sharp H_M} (H_i \otimes_k M) \ncong A\sharp G \otimes_{A\sharp H_M} (H_j \otimes_k M)$ as $A\sharp G$ -modules, if $i \neq j$;

(iv) $A\sharp G \otimes_A M \cong \bigoplus_{i=1}^{r} A\sharp G \otimes_{A\sharp H_M} (H_i \otimes_k M)$.

Proof. (i) Define $f : {}^gM \to g \otimes M$ such that $f(m) = g \otimes m$ for any $g \in G, m \in M$. Then f is bijection. Since for any $a \in A, m \in A$,

$$f(a \cdot m) = f(g^{-1}(a)m) = g \otimes g^{-1}(a)m = g(g^{-1}(a)) \otimes m = a(g \otimes m) = af(m),$$

f is an A-module isomorphism. Therefore, ${}^gM \cong g \otimes M$ as A-modules.

Since $A\sharp G \otimes_{A\sharp H_M} (H_i \otimes_k M) \cong \bigoplus_{g\in R_M} g \otimes H_i \otimes_k M$ as $A\sharp H_M$-modules and $H_i \otimes_k M \cong M$ as A-modules, we have $A\sharp G \otimes_{A\sharp H_M} (H_i \otimes_k M) \cong \bigoplus_{g\in R_M} g \otimes M \cong \bigoplus_{g\in R_M} {}^gM$ as A-modules.

(ii) By (i) and Lemma 2.1, $A\sharp G \otimes_{A\sharp H_M} (H_i \otimes_k M)$ is an indecomposable G-stable A-module. By Lemma 4.1, $A\sharp G \otimes_{A\sharp H_M} (H_i \otimes_k M)$ is an indecomposable $A\sharp G$-module.

(iii) If $i \neq j$, $A\sharp G \otimes_{A\sharp H_M} (H_i \otimes_k M) \cong A\sharp G \otimes_{A\sharp H_M} (H_j \otimes_k M)$ as $A\sharp G$ -modules, by

$$A\sharp G \otimes_{A\sharp H_M} (H_i \otimes_k M) \cong \bigoplus_{g\in R_M} g \otimes H_i \otimes_k M$$

as $A\sharp H_M$-modules, we have $h \otimes H_i \otimes_k M$ is a direct summand of $A\sharp G \otimes_{A\sharp H_M} (H_j \otimes_k M) \cong \bigoplus_{g\in R_M} g \otimes H_j \otimes_k M$, where $h \in R_M$.

If $h \otimes H_i \otimes_k M \cong h \otimes H_j \otimes_k M$, then $H_i \otimes_k M \cong H_j \otimes_k M$ as $A\sharp H_M$-modules. By Lemma 4.3(iii), It is a contradiction.

If $h \otimes H_i \otimes_k M \cong g \otimes H_j \otimes_k M$ for some $h \neq g \in R_M$, by $g \otimes H_j \otimes M \cong {}^gM$, we have ${}^hM \cong {}^gM$. It is also a contradiction.

(iv) By Lemma 4.2, $A\sharp G \otimes_{A\sharp H_M} (H_i \otimes_k M)$ is a direct summand of

$$A\sharp G \otimes_A A\sharp G \otimes_{A\sharp H_M} (H_i \otimes_k M).$$

By (i),

$$A\sharp G \otimes_{A\sharp H_M} (H_i \otimes_k M) \cong \bigoplus_{g\in R_M} {}^gM$$

is a direct summand of $A\sharp G \otimes_A (\bigoplus_{g\in R_M} {}^gM)$. By (ii) and Lemma 4.1(ii), $A\sharp G \otimes_{A\sharp H_M} (H_i \otimes_k M)$ is a direct summand of $A\sharp G \otimes_A M$. By (iii) and the Krull-Schmidt theorem, we have

$$\bigoplus_{i=1}^{r} A\sharp G \otimes_{A\sharp H_M} (H_i \otimes_k M)$$

is a direct summand of $A\sharp G \otimes_A M$. By Lemma 4.1(iii), $A\sharp G \otimes_A M$ has at most r indecomposable summands. Therefore

$$A\sharp G \otimes_A M \cong \bigoplus_{i=1}^{r} A\sharp G \otimes_{A\sharp H_M} (H_i \otimes_k M).$$

□

Next, we construct all induced indecomposable $A\sharp G$-modules from an indecomposable A-module and give the number of non-isomorphic indecomposable $A\sharp G$ -modules induced from the corresponding G-stable A-module.

Theorem 5.1. *Let G be a finite group with order n and M be an indecomposable A-module with maximal stable subgroup H_M =$\{g \in G, {}^gM \cong M\}$. Suppose H_M is abelian and $R_M = \{g_1, g_2, \cdots, g_s\}$ is a complete set of left coset representatives of H_M in G. H_i is defined in (4.1). Then for any $A\sharp G$-module N, if $N \cong \bigoplus_{g\in R_M} {}^gM$ as A-modules, there exists a unique $i \in \{1, 2 \cdots, r\}$ such that $N \cong A\sharp G \otimes_{A\sharp H_M} (H_i \otimes_k M)$ as $A\sharp G$ -modules. That is, there are r non-isomorphic indecomposable $A\sharp G$ -modules induced from the same indecomposable G-stable A -module.*

Proof. For any $A\sharp G$-module N, if $N \cong \bigoplus_{g\in R_M} {}^gM$ as A-modules, then by Lemma 4.2, N is a direct summand of

$$A\sharp G \otimes_A N \cong A\sharp G \otimes_A \bigoplus_{g\in R_M} {}^gM.$$

By Lemma 4.1(ii), Lemma 5.1(iv) and the Krull-Schmidt theorem, there exists a unique $i \in \{1, 2 \cdots, r\}$ such that $N \cong A\sharp G \otimes_{A\sharp H_M} (H_i \otimes_k M)$. That is, there are r non-isomorphic indecomposable $A\sharp G$ -modules induced from the same indecomposable G-stable A -module. □

Theorem 5.2. *Suppose $G \subseteq Aut(A)$ is an abelian group. Any indecomposable $A\sharp G$-module is an indecomposable G-stable A-module. Conversely, for any indecomposable G-stable A-module, the corresponding canonical induced $A\sharp G$-module is indecomposable.*

Proof. First, for any indecomposable $A\sharp G$-module M, by Lemma 2.1,

$$M \cong \bigoplus_{j=1}^{t} M_j \quad M_j \cong \bigoplus_{g\in R_{N_j}} {}^gN_j$$

with N_j an indecomposable A-module, $R_{N_j} = \{g_{j1}, g_{j2}, \cdots, g_{js}\}$ a complete set of left coset representatives of H_{N_j} in G. By Lemma 4.2, we have M is direct summand of

$$A\sharp G \otimes_A M \cong \bigoplus_{j=1}^{t} \bigoplus_{g\in R_{N_j}} A\sharp G \otimes_A {}^gN_j.$$

Then by Lemma 4.1(ii) and the Krull-Schmidt theorem, there exists j such that M is a direct summand of $A\sharp G \otimes_A N_j$. Therefore, by Theorem 5.1, we have

$$M \cong \bigoplus_{g\in R_{N_j}} {}^gN_j$$

as A-modules. By Lemma 2.1, M is an indecomposable G-stable A-module.

Conversely, by Lemma 4.1 it is obvious that for any indecomposable G-stable A-module, the corresponding canonical induced $A\sharp G$-module is indecomposable. □

According to Theorems 5.1 and 5.2, for a skew group algebra $A\sharp G$ where A is a finite dimensional algebra and $G \subseteq Aut(A)$ is abelian, all finite dimensional $A\sharp G$-modules can be obtained from G-stable modules. The number of non-isomorphic indecomposable $A\sharp G$ -modules induced from the same G-stable A-module can be given. In this case, for any indecomposable A-module M, the G-stable A-module $\bigoplus_{g\in R_M} {}^gM$ has r non-isomorphic $A\sharp G$-module structures, where $r = |H_M| = n/s$.

We give the relation between simple, projective and injective modules in $modA$ and those in $modA\sharp G$.

Theorem 5.3. *Suppose $G \subseteq Aut(A)$ is an abelian group. Let M be an $A\sharp G$-module. Then*

(i) M is simple if and only if there exists a simple A-module S such that $M \cong \bigoplus\limits_{g\in R_S} {}^gS$.

(ii) M is projective if and only if there exists an indecomposable projective A-module P such that $M \cong \bigoplus\limits_{g\in R_P} {}^gP$.

(iii) M is injective if and only if there exists an indecomposable injective A-module I such that $M \cong \bigoplus\limits_{g\in R_I} {}^gI$.

Proof. According to Theorems 5.1 and 5.2, we need only to prove that M is a simple (projective, injective) $A\sharp G$-module if and only if M is a simple (projective, injective) A-module.

(i) Assume $M \cong \bigoplus\limits_{g\in R_S} {}^gS$ for some simple A-module S. If it is not simple, we have for its proper submodule N, $N \cong \bigoplus\limits_{g\in E} {}^gS$, where E is a proper subset of R_S. By Lemma 4.1, N is G-stable. It is a contradiction.

(ii) By [8], M is an indecomposable projective $A\sharp G$ -module if and only if M is an indecomposable projective A -module.

By duality, we get (iii). □

6. The stable category of a skew group algebra under a cyclic group

Let Q be a connected finite quiver without oriented cycles and $\sigma \in Aut(Q)$ with order n. I. Reiten and Chrisine Riedtmann in [12] constructed the dual quiver with automorphism $(\widetilde{Q}, \widetilde{\sigma})$, where $\widetilde{Q}$ is the Ext-quiver of $KQ\sharp\langle\sigma\rangle$ and $\widetilde{\sigma}$ is the automorphism of $k\widetilde{Q}$ induced from an admissible automorphism. Fix a primitive n-th root of unity ζ, we give the definition of dual quiver.

Definition 6.1. *[12] Let $G = \langle\sigma\rangle$ and $\mathcal{E}$ be a set of representatives of the G-orbits of vertices of Q. The dual quiver $\widetilde{Q}$ is described as follows: Each vertex of Q in $\mathcal{E}$ whose G-orbit has size s gives rise to n/s vertices of $\widetilde{Q}$. Let ε and ε' lie in $\mathcal{E}$ and have orbits of size s and s', and let $\alpha : \varepsilon \to \sigma^{-t}(\varepsilon')$ be an arrow of Q. If $\eta_0, \eta_1, \cdots, \eta_{m-1}$ and $\eta'_0, \eta'_1, \cdots, \eta'_{m'-1}$ are the vertices of $\widetilde{Q}$ arising from $G\varepsilon$ and $G\varepsilon'$, there is an arrow from η_μ to $\eta'_{\mu'}$ in $\widetilde{Q}$ if and only if $\mu \equiv \mu' + a$ module(m, m'), where a is defined by*

$$\sigma^{[s,s']}(\alpha) = \zeta^{[s,s']a}\alpha.$$

Then we have the following proposition.

Proposition 6.1. *[12] the skew-group algebra $KQ\sharp\langle\sigma\rangle$ is Morita equivalent to the path algebra $k\widetilde{Q}$ of dual quiver $\widetilde{Q}$.*

Let A be a k-algebra. The corresponding stable module category has modules as objects, while its morphisms are equivalence classes of modulo homomorphisms factoring through projectives. The category of left A-modules is denoted by $modA$, and the corresponding stable category is denoted by $\underline{mod}A$ with the set of morphisms denoted by $\underline{Hom}(M, N)$ for any A-modules M and N. The stable category is additive.

Definition 6.2. *[1] An A-module monomorphism $u : M \to N$ in mod A is minimal if every nonzero submodule X of N has a nonzero intersection with Imu. A monomorphism $u : M \to E$ in mod A is called an injective envolope of M if E is an injective module and u is a minimal monomorphism.*

In this section we give the relationship between the stable category of a path algebra and corresponding smash product to be abelian.

Theorem 6.1. *[10] Let $A = KQ$ be a hereditary algebra. The stable category $\underline{mod}A$ of left A-modules is abelian if and only if the injective envelope of $_AA$ is projective.*

Theorem 6.2. *[12] The hereditary algebra KQ is 1-Gorenstein if and only $KQ\sharp\langle\sigma\rangle$ is.*

Then we have the following theorem

Theorem 6.3. *The stable category of a path algebra KQ is abelian if and only if the stable category of skew group algebra $KQ\sharp\langle\sigma\rangle$ is.*

Proof. Let $I(A)$ denote the injective envelope of algebra A.

"if": Since the smash product $kQ\sharp\langle\sigma\rangle$ is Morita to $k\widetilde{Q}$, by the Theorem 6.1, $I(kQ\sharp\langle\sigma\rangle)$ is projective, which means that $kQ\sharp\langle\sigma\rangle$ is 1-Gorenstein. By the Theorem 6.2, the hereditary algebra KQ is 1-Gorenstein, which means that $I(kQ)$ is projective,. Hence the stable category of a path algebra KQ is abelian by Theorem 6.1.

"only if": By the Theorem 6.1, the stable category $\underline{mod}kQ$ of left KQ-modules is abelian if and only if the injective envelope of $I(kQ)$ is projective, which means that kQ is 1-Gorenstein. By the Theorem 6.2, $kQ\sharp\langle\sigma\rangle$ is 1-Gorenstein, which means that $I(kQ\sharp\langle\sigma\rangle)$ is projective. Hence the stable category of a path algebra $kQ\sharp\langle\sigma\rangle$ is abelian by Theorem 6.1. □

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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