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Research article

On the oscillation of first-order differential equations with deviating arguments and oscillatory coefficients

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Abstract: In this work, we study the oscillation problem of first-order differential equations with deviating arguments and oscillatory coefficients. We generalize and improve the work of Kwong [30] such that the delay (advanced) and the coefficient functions do not need to be monotone and nonnegative, respectively. This method essentially improves many known oscillation conditions. The significance and the substantial improvement of our results are shown by two illustrative examples.

Keywords: oscillation; differential equations; nonmonotone delays; oscillatory coefficients
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1. Introduction

Functional differential equations arise widely in many fields such as mathematical biology, economy, physics, or biology, see [16, 19, 28, 40]. This explains the great interest in the qualitative properties of these kinds of equations. Oscillation phenomena appear in various models from real world applications; see, e.g., the papers [12, 35, 38] for models from mathematical biology where oscillation and/or delay actions may be formulated by means of cross-diffusion terms. As a part of this approach, the oscillation theory of this type of equation has been extensively developed, see [1–45]. In particular, the oscillation criteria of first-order differential equations with deviating arguments have numerous applications in the study of higher-order functional differential equations (e.g., one can study the oscillatory behavior of higher-order functional differential equations by relating oscillation of these equations to that of associated first-order functional differential equations); see, e.g., the papers [13, 36, 37].

Recently, there has been great interest in studying the oscillation of all solutions of the first-order delay differential equation

$$x'(t) + b(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{1.1}$$

and its dual advanced equation

$$x'(t) - c(t)x(\sigma(t)) = 0, \quad t \geq t_0, \tag{1.2}$$

where $b, c, \tau, \sigma \in C([t_0, \infty), [0, \infty))$ such that $\tau(t) \leq t$, $\lim_{t\to\infty} \tau(t) = \infty$, and $\sigma(t) \geq t$. In most of these works, the delay (advanced) function is assumed to be nondecreasing, see [14, 16, 29, 31–34, 42, 45] and the references therein. As shown in [8], the oscillation character of Eq (1.1) with nonmontone delay, is not an easy extension to the oscillation problem for the nondecreasing delay case. Many authors [1, 3, 5–11, 15, 20, 25, 27, 39, 43] have developed and generalized the methods used to study the oscillation of equations (1.1) and (1.2) with monotone delays and to study this property for the nonmonotone case. Only a few works, however, dealt with the oscillation of equations (1.1) and (1.2) with oscillatory coefficients. For example, [16, 44] studied the oscillation of Eq (1.1) where the delay function $\tau(t)$ is assumed to be nondecreasing and constant (i.e., $\tau(t) = t - \alpha,\ \alpha > 0$), respectively. Also, Kulenovic and Grammatikopoulos [29] studied the oscillation of a first-order nonlinear functional differential equation that contains both equations (1.1) and (1.2). The authors obtained liminf and limsup oscillation criteria for the case when the coefficient function does not need to be nonnegative. However, the delay (advanced) and the coefficient functions are assumed to be nondecreasing (for limsup conditions) and nonnegative on a sequence of intervals $\{(r_n,\ s_n)\}_{n\geq 0}$ such that $\lim_{n\to\infty}\ (s_n - r_n) = \infty$ (for liminf conditions), respectively.

Our aim in this work is to obtain oscillation criteria for equations (1.1) and (1.2) where $b(t)$ and $c(t)$ are continuous functions on $[t_0, \infty)$. We relax the nonnegative restriction on the coefficient functions $b(t)$ and $c(t)$. To accomplish this goal, using the ideas of [27], we develop and enhance the work of Kwong [30]. This procedure leads to new sufficient oscillation criteria that improve and generalize those mentioned in [16, 29, 44].

2. Main results

From now on, we assume that $b(t)$ and $c(t)$ are only continuous functions on $[t_0, \infty)$.

2.1. Delay differential equations

Let $\Lambda(t)$ and $\Lambda_i(t)$, $t \geq t_0$, $i \in \mathbb{N}$ be defined as follows (see [27]):

$$\begin{aligned} \Lambda(t) &= \max\{u \geq t : \ \tau(u) \leq t\}, \\ \Lambda_1(t) &= \Lambda(t), \qquad \Lambda_i(t) = \Lambda(t) \circ \Lambda_{i-1}(t),\ i = 2, 3, \dots . \end{aligned} \tag{2.1}$$

Also, we define the function $g(t)$ and the sequence $\{Q_n(v,u)\}_{n=0}^{\infty}$, $\tau(v) \leq u \leq v$, as follows:

$$g(t) = \sup_{u\leq t} \tau(u), \quad t \geq t_0 \tag{2.2}$$

and

$$Q_0(v, u) = 1,$$

$$Q_n(v,u) = \exp\left(\int_u^v b(\zeta)Q_{n-1}(\zeta,\tau(\zeta))d\zeta\right), \quad n \in \mathbb{N}.$$

The proofs of our main results are essentially based on the following lemma.

Lemma 2.1. *Let $n \in \mathbb{N}_0$, $T^* > t_0$, $T \geq T^*$ and $x(t)$ be a solution of Eq (1.1) such that $x(t) > 0$ for all $t \geq T^*$. If $b(t) \geq 0$ on $[T, T_1]$, $T_1 \geq \Lambda_{n+2}(T)$, then*

$$\frac{x(u)}{x(v)} \geq Q_n(v,u), \qquad \tau(v) \leq u \leq v, \qquad \textit{for } v \in [\Lambda_{n+2}(T),\ T_1]. \tag{2.3}$$

Proof. It follows from Eq (1.1) that $x'(t) \leq 0$ on $[\Lambda_1(T),\ T_1]$. Therefore,

$$\frac{x(u)}{x(v)} \geq 1 = Q_0(v,u), \qquad \tau(v) \leq u \leq v, \qquad \text{for } v \in [\Lambda_2(T),\ T_1].$$

Dividing Eq (1.1) by $x(t)$ and integrating from u to v, $\tau(v) \leq u \leq v$, we obtain

$$\frac{x(u)}{x(v)} = \exp\left(\int_u^v b(\zeta)\frac{x(\tau(\zeta))}{x(\zeta)}d\zeta\right). \tag{2.4}$$

Since $x'(t) \leq 0$ on $[\Lambda_1(T),\ T_1]$, we get

$$\frac{x(u)}{x(v)} \geq \exp\left(\int_u^v b(\zeta)d\zeta\right) = \exp\left(\int_u^v b(\zeta)Q_0(\zeta,\tau(\zeta))d\zeta\right) = Q_1(v,u), \quad \tau(v) \leq u \leq v$$

for $v \in [\Lambda_3(T),\ T_1]$ and consequently, for $u \leq \zeta \leq v$, we have

$$\frac{x(\tau(\zeta))}{x(\zeta)} \geq Q_1(\zeta,\tau(\zeta)), \quad \tau(v) \leq u \leq v \quad \text{for } v \in [\Lambda_4(T),\ T_1].$$

Substituting in (2.4), we get

$$\frac{x(u)}{x(v)} \geq \exp\left(\int_u^v b(\zeta)Q_1(\zeta,\tau(\zeta))d\zeta\right) = Q_2(v,u) \qquad \text{for } v \in [\Lambda_4(T),\ T_1].$$

Repeating this argument n times, we obtain

$$\frac{x(u)}{x(v)} \geq \exp\left(\int_u^v b(\zeta)Q_{n-1}(\zeta,\tau(\zeta))d\zeta\right) = Q_n(v,u) \qquad \text{for } t \in [\Lambda_{n+2}(T),\ T_1].$$

The proof of the lemma is complete. □

Let $\{T_k\}_{k\geq 0}$ be a sequence of real numbers such that $\lim_{k\to\infty} T_k = \infty$ and

$$b(t) \geq 0 \qquad \text{for } t \in [T_k,\ \Lambda_{n+4}(T_k)], \qquad \text{for all } k \in \mathbb{N} \qquad \text{for some } n \in \mathbb{N}_0. \tag{2.5}$$

Also, we define the sequence $\{\beta_n\}_{n\geq 1}, \beta_n > 1$ for all $n \in \mathbb{N}$ as follows:

$$Q_n(t,\ g(t)) > \beta_n, \qquad t \in \left[g(\Lambda_{n+3}(T_k)), \Lambda_{n+3}(T_k)\right] \quad \text{for all } k \in \mathbb{N}_0 \quad \text{for some } n \in \mathbb{N}. \tag{2.6}$$

Theorem 2.1. *Let $n \in \mathbb{N}$ such that (2.5) and (2.6) are satisfied. If*

$$\int_{g(\Lambda_{n+4}(T_k))}^{\Lambda_{n+4}(T_k)} b(\zeta) Q_{n+1}(g(\zeta), \tau(\zeta))\, d\zeta \geq \frac{\ln(\beta_{n+1}) + 1}{\beta_{n+1}} \qquad \text{for all } k \in \mathbb{N}_0,$$

then every solution of Eq (1.1) is oscillatory.

Proof. Assume, for the sake of contradiction, that $x(t)$ is an eventually positive solution of Eq (1.1). Then there exists a sufficiently large $T^* > t_0$ such that $x(t) > 0$ for $t > T^*$. Suppose that $T_{k_1} \in \{T_k\}_{k \geq 0}$ such that $T_{k_1} > T^*$. In view of (2.3), (2.5) and (2.6), it follows that

$$\frac{x\left(g(\Lambda_{n+4}(T_{k_1}))\right)}{x(\Lambda_{n+4}(T_{k_1}))} \geq Q_{n+1}(\Lambda_{n+4}(T_{k_1}), g(\Lambda_{n+4}(T_{k_1}))) > \beta_{n+1} > 1.$$

Then there exists $t_* \in (g(\Lambda_{n+4}(T_{k_1})), \Lambda_{n+4}(T_{k_1}))$ such that

$$\frac{x\left(g(\Lambda_{n+4}(T_{k_1}))\right)}{x(t_*)} = \beta_{n+1}. \tag{2.7}$$

Integrating Eq (1.1) from t_* to t, we get

$$x(\Lambda_{n+4}(T_{k_1})) - x(t_*) + \int_{t_*}^{\Lambda_{n+4}(T_{k_1})} b(\zeta) x(\tau(\zeta)) d\zeta = 0. \tag{2.8}$$

It is easy to see that

$$x(\tau(\zeta)) = x(g(\zeta)) \exp\left(\int_{\tau(\zeta)}^{g(\zeta)} b(\zeta_1) \frac{x(\tau(\zeta_1))}{x(\zeta_1)} d\zeta_1\right). \tag{2.9}$$

Substituting in (2.8), we have

$$x(\Lambda_{n+4}(T_{k_1})) - x(t_*) + \int_{t_*}^{\Lambda_{n+4}(T_{k_1})} b(\zeta) \exp\left(\int_{\tau(\zeta)}^{g(\zeta)} b(\zeta_1) \frac{x(\tau(\zeta_1))}{x(\zeta_1)} d\zeta_1\right) x(g(\zeta)) d\zeta = 0.$$

Since $x'(t) \leq 0$ on $[\Lambda_1(T_{k_1}),\ \Lambda_{n+4}(T_{k_1})]$, it follows that

$$x(\Lambda_{n+4}(T_{k_1})) - x(t_*) + x(g(\Lambda_{n+4}(T_{k_1}))) \int_{t_*}^{\Lambda_{n+4}(T_{k_1})} b(\zeta) \exp\left(\int_{\tau(\zeta)}^{g(\zeta)} b(\zeta_1) \frac{x(\tau(\zeta_1))}{x(\zeta_1)} d\zeta_1\right) d\zeta \leq 0.$$

By (2.3) and $\zeta_1 \in [\Lambda_{n+2}(T_{k_1}), \Lambda_{n+3}(T_{k_1})]$ for $\tau(\zeta) < \zeta_1 < g(\zeta)$, $g(\Lambda_{n+4}(T_{k_1})) < \zeta < \Lambda_{n+4}(T_{k_1})$, we get

$$\int_{t_*}^{\Lambda_{n+4}(T_{k_1})} b(\zeta) \exp\left(\int_{\tau(\zeta)}^{g(\zeta)} b(\zeta_1) Q_n(\zeta_1, \tau(\zeta_1)) d\zeta_1\right) d\zeta \leq \frac{x(t_*)}{x(g(\Lambda_{n+4}(T_{k_1})))} - \frac{x(\Lambda_{n+4}(T_{k_1}))}{x(g(\Lambda_{n+4}(T_{k_1})))} < \frac{1}{\beta_{n+1}}. \tag{2.10}$$

Dividing Eq (1.1) by $x(t)$ and integrating from $g(\Lambda_{n+4}(T_{k_1}))$ to t_*, we obtain

$$-\int_{g(\Lambda_{n+4}(T_{k_1}))}^{t_*} \frac{x'(\zeta)}{x(\zeta)} d\zeta = \int_{g(\Lambda_{n+4}(T_{k_1}))}^{t_*} b(\zeta) \frac{x(\tau(\zeta))}{x(\zeta)} d\zeta.$$

Using (2.9), we get

$$\ln\left(\frac{x(g(\Lambda_{n+4}(T_{k_1})))}{x(t_*)}\right)=\int_{g(\Lambda_{n+4}(T_{k_1}))}^{t_*}b(\zeta)\,\frac{x(g(\zeta))}{x(\zeta)}\exp\left(\int_{\tau(\zeta)}^{g(\zeta)}b(\zeta_1)\frac{x(\tau(\zeta_1))}{x(\zeta_1)}d\zeta_1\right)d\zeta.$$

From this, (2.3) and (2.6), we get

$$\int_{g(\Lambda_{n+4}(T_{k_1}))}^{t_*}b(\zeta)Q_{n+1}(\zeta,g(\zeta))\,\exp\left(\int_{\tau(\zeta)}^{g(\zeta)}b(\zeta_1)Q_n(\zeta_1,\tau(\zeta_1))d\zeta_1\right)d\zeta\le\ln\left(\frac{x(g(\Lambda_{n+4}(T_{k_1})))}{x(t_*)}\right).$$

It follows from (2.6) and (2.7) that

$$\int_{g(\Lambda_{n+4}(T_{k_1}))}^{t_*}b(\zeta)\,\exp\left(\int_{\tau(\zeta)}^{g(\zeta)}b(\zeta_1)Q_n(\zeta_1,\tau(\zeta_1))d\zeta_1\right)d\zeta\le\frac{\ln(\beta_{n+1})}{\beta_{n+1}}.$$

Combining this and (2.10) we get

$$\int_{g(\Lambda_{n+4}(T_{k_1}))}^{\Lambda_{n+4}(T_{k_1})}b(\zeta)\,\exp\left(\int_{\tau(\zeta)}^{g(\zeta)}b(\zeta_1)Q_n(\zeta_1,\tau(\zeta_1))d\zeta_1\right)d\zeta<\frac{\ln(\beta_{n+1})+1}{\beta_{n+1}},$$

that is

$$\int_{g(\Lambda_{n+4}(T_k))}^{\Lambda_{n+4}(T_k)}b(\zeta)Q_{n+1}(g(\zeta),\tau(\zeta))\,d\zeta<\frac{\ln(\beta_{n+1})+1}{\beta_{n+1}}.$$

The proof of the theorem is complete. □

Theorem 2.2. *Let $n\in\mathbb{N}_0$ such that (2.5) is satisfied. If*

$$\int_{g(\Lambda_{n+4}(T_k))}^{\Lambda_{n+4}(T_k)}b(\zeta)Q_{n+1}(g(\Lambda_{n+4}(T_k)),\tau(\zeta))d\zeta\ge 1\qquad\textit{for all } k\in\mathbb{N}_0,\tag{2.11}$$

then every solution of Eq (1.1) is oscillatory.

Proof. Let $x(t)$ be an eventually positive solution of Eq (1.1). Then the exists $T^*>t_0$ such that $x(t)>0$ for all $t\ge T^*$. It is not difficult to prove that

$$\begin{aligned}&x(\Lambda_{n+4}(T_{k_1}))-x(g(\Lambda_{n+4}(T_{k_1})))\\&+x(g(\Lambda_{n+4}(T_{k_1})))\int_{g(\Lambda_{n+4}(T_{k_1}))}^{\Lambda_{n+4}(T_{k_1})}b(\zeta)\exp\left(\int_{\tau(\zeta)}^{g(\Lambda_{n+4}(T_{k_1}))}b(\zeta_1)\frac{x(\tau(\zeta_1))}{x(\zeta_1)}d\zeta_1\right)d\zeta=0,\end{aligned}$$

where $T_{k_1}\in\{T_k\}$ such that $T_{k_1}>T^*$. Using (2.3), we get

$$x(\Lambda_{n+4}(T_{k_1}))+x(g(\Lambda_{n+4}(T_{k_1})))\left(\int_{g(\Lambda_{n+4}(T_{k_1}))}^{\Lambda_{n+4}(T_{k_1})}b(\zeta)\exp\left(\int_{\tau(\zeta)}^{g(\Lambda_{n+4}(T_{k_1}))}b(\zeta_1)Q_n(\zeta_1,\tau(\zeta_1))d\zeta_1\right)d\zeta-1\right)\le 0.$$

Using the positivity of $x(\Lambda_{n+4}(T_{k_1}))$ we have

$$\int_{g(\Lambda_{n+4}(T_{k_1}))}^{\Lambda_{n+4}(T_{k_1})}b(\zeta)\exp\left(\int_{\tau(\zeta)}^{g(\Lambda_{n+4}(T_{k_1}))}b(\zeta_1)Q_n(\zeta_1,\tau(\zeta_1))d\zeta_1\right)d\zeta<1.$$

Then

$$\int_{g(\Lambda_{n+4}(T_k))}^{\Lambda_{n+4}(T_k)}b(\zeta)Q_{n+1}(g(\Lambda_{n+4}(T_k)),\tau(\zeta))d\zeta<1,$$

which contradicts (2.11). The proof of the theorem is complete. □

Remark 2.1.

(1) It should be noted that the monotonicity of the delay function $\tau(t)$ is required in many previous works to study the oscillation of Eq (1.1) with oscillating coefficients; see, for example, [16, 29, 44]. In this work, the sequence $\{\Lambda_i(t)\}_{i\geq 0}$ plays a central role in the derivation of our results. In fact the delay function $\tau(t)$ does not need to be monotone. Therefore, our results substantially improve and generalize [29, Theroems 6, 7] for Eq (1.1). Furthermore, using our approach, many previous oscillation studies for Eq (1.1) with monotone delays can be used to study the oscillation of Eq (1.1) with general delays (the delay does not need to be monotone) and oscillating coefficients.

(2) There are numerous lower bounds for the quotient $\frac{x(\tau(t))}{x(t)}$, where $x(t)$ is a positive solution of Eq (1.1) with a nonnegative continuous function $b(t)$, see [6, 16, 22, 24, 42]. For example, [22, Lemma 1] and Lemma 2.1 can be used instead of Lemma 2.1 to improve our results in the case where $b(t)$ is a nonnegative continuous function. In this case, the adjusted version of the results improves Theorems 2.1 and 2.2. Even in the case where $\tau(t)$ is nondecreasing, the improvement is substantial.

2.2. Advanced differential equations

Similar results for the (dual) advanced differential equation (1.2) can be obtained easily. The details of the proofs are omitted since they are quite similar to Eq (1.1).

We will use the following notation:

$$h(t) = \inf_{u\geq t} \sigma(u), \quad t \geq t_0 \tag{2.12}$$

$$\Omega_1(t) = \Omega(t), \qquad \Omega_i(t) = \Omega(t) \circ \Omega_{i-1}(t),\ t \geq t_0,\ i = 2, 3, \ldots, \tag{2.13}$$

where

$$\Omega(t) = \min\{t_0 \leq u \leq t : \ \sigma(u) \geq t\}.$$

Also, we define the sequence $\{R_n(u, v)\}_{n=0}^{\infty}$, $v \leq u \leq \sigma(v)$ as follows:

$$\begin{aligned} R_0(u, v) &= 1, \\ R_n(u, v) &= \exp\left(\int_v^u b(\zeta)R_{n-1}(\sigma(\zeta), \zeta)d\zeta\right), \quad n \in \mathbb{N}. \end{aligned}$$

In order to obtain the oscillation criteria for Eq (1.2) we need the following conditions:
Let the sequence $\{T_k\}_{k\geq 0}$ be a sequence of real numbers such that $\lim_{k\to\infty} T_k = \infty$ and

$$c(t) \geq 0 \qquad \text{for } t \in [\Omega_{n+4}(T_k),\ T_k] \qquad \text{for all } k \in \mathbb{N}_0 \qquad \text{for some } n \in \mathbb{N}. \tag{2.14}$$

Also, we define the sequence $\{\gamma_n\}_{n\geq 1}$, $\gamma_n > 1$ for all $n \in \mathbb{N}$ as follows:

$$R_n(h(t),\ t) > \gamma_n, \qquad t \in [\Omega_{n+3}(T_k), h(\Omega_{n+3}(T_k))] \quad \text{for all } k \in \mathbb{N}_0 \quad \text{for some } n \in \mathbb{N}. \tag{2.15}$$

Theorem 2.3. *Let $n \in \mathbb{N}$ such that (2.14) and (2.15) are satisfied. If*

$$\int_{\Omega_{n+4}(T_k)}^{h(\Omega_{n+4}(T_k))} c(\zeta)R_{n+1}(\sigma(\zeta), h(\zeta))d\zeta \geq \frac{\ln(\gamma_{n+1}) + 1}{\gamma_{n+1}} \qquad \text{for all } k \in \mathbb{N}_0,$$

then every solution of Eq (1.2) is oscillatory.

Theorem 2.4. *Let $n \in \mathbb{N}_0$ such that (2.14) is satisfied. If*

$$\int_{\Omega_{n+4}(T_k)}^{h(\Omega_{n+4}(T_k))} c(\zeta)R_{n+1}(\sigma(\zeta), h(\Omega_{n+4}(T_k)))d\zeta \geq 1 \qquad \text{for all } k \in \mathbb{N}_0,$$

then every solution of Eq (1.2) is oscillatory.

3. Numerical examples

Example 3.1. Consider the delay differential equation

$$x'(t) + b(t)x(\tau(t)) = 0, \quad t \geq 1, \tag{3.1}$$

where $b \in C([1, \infty), \mathbb{R})$ such that

$$b(t) = \eta > 0 \qquad \text{for } t \in \left[3r_k,\ 3r_k + \frac{645}{121}\right] \qquad \text{for all } k \in \mathbb{N}_0,$$

$\{r_k\}_{k\geq 0}$ is a sequence of positive integers such that $r_{k+1} > r_k + \frac{215}{121}$ and $\lim_{k\to\infty}\ r_k = \infty$, and

$$\tau(t) = \begin{cases} t-1 & \text{if } t \in [3l,\ 3l+2], \\ -t+6l+3 & \text{if } t \in [3l+2,\ 3l+2.1],\ l \in \mathbb{N}_0. \\ \frac{11}{9}t - \frac{2}{3}l - \frac{5}{3} & \text{if } t \in [3l+2.1,\ 3l+3], \end{cases}$$

In view of (2.1) and (2.2) , it is easy to see that

$$g(t) = \begin{cases} t-1 & \text{if } t \in [3l,\ 3l+2], \\ 3l+1 & \text{if } t \in [3l+2,\ 3l+\frac{24}{11}], \quad l \in \mathbb{N}_0 \\ \frac{11}{9}t - \frac{2}{3}l - \frac{5}{3} & \text{if } t \in [3l++\frac{24}{11},\ 3l+3], \end{cases}$$

and

$$\Lambda(t) = \begin{cases} t+1 & \text{if } t \in [3l,\ 3l+0.9], \\ \frac{9}{11}t + \frac{6}{11}l + \frac{15}{11} & \text{if } t \in [3l+0.9,\ 3l+2],\ l \in \mathbb{N}_0, \\ t+1 & \text{if } t \in [3l+2,\ 3l+3], \end{cases}$$

respectively.
Letting $T_k = 3r_k$, $k \in \mathbb{N}_0$, so $\Lambda_5(T_k) = 3r_k + \frac{645}{121}$, and hence

$$b(t) = \eta \qquad \text{for } t \in [T_k,\ \Lambda_5(T_k)] \qquad \text{for all } k \in \mathbb{N}_0. \tag{3.2}$$

It is obvious that $g(\Lambda_5(T_k)) = 3r_k + \frac{46}{11}$ and

$$t - 1.2 \leq \tau(t) \leq g(t) \leq t-1 \qquad \text{for all } t \geq 1.$$

Therefore,

$$Q_2(t, g(t)) = \exp\left(\int_{g(t)}^{t} b(\zeta)\exp\left(\int_{\tau(\zeta)}^{\zeta} b(\zeta_1)d\zeta_1\right)d\zeta\right)$$

$$\geq \exp\left(\int_{t-1}^{t} b(\zeta)\exp\left(\int_{\zeta-1}^{\zeta} b(\zeta_1)d\zeta_1\right)d\zeta\right) \geq \exp\left(\eta\exp\left(\eta\right)\right)$$

for $t \in \left[3r_k + \frac{46}{11},\, 3r_k + \frac{645}{121}\right]$. Denote $\beta_2 = \exp\left(\eta\exp\left(\eta\right)\right) > 1$. Then

$$Q_2(t,\, g(t)) > \beta_2 \qquad \text{for } t \in \left[g(\Lambda_5(T_k)), \Lambda_5(T_k)\right] \quad \text{for all } k \in \mathbb{N}_0. \tag{3.3}$$

Also,

$$\int_{g(\Lambda_5(T_k))}^{\Lambda_5(T_k)} b(\zeta)Q_2(g(\zeta),\tau(\zeta))d\zeta = \int_{3r_k+\frac{46}{11}}^{3r_k+\frac{645}{121}} b(\zeta)\exp\left(\int_{\tau(\zeta)}^{g(\zeta)} Q_1(\zeta_1,\tau(\zeta_1))b(\zeta_1)\right)d\zeta$$
$$= \frac{117}{121}\eta + \frac{20\left(\exp\left(\frac{1}{10}\eta\exp\left(\eta\right)\right) - 1\right)\exp\left(-\eta\right)}{11} > 0.707$$

for all $\eta \geq 0.61$ and $k \in \mathbb{N}_0$ and

$$\left(\frac{1+\ln\left(\beta_2\right)}{\beta_2}\right) < 0.691 \quad \text{for all } \eta \geq 0.61 \text{ and } k \in \mathbb{N}_0.$$

It is obvious that

$$\int_{g(\Lambda_5(T_k))}^{\Lambda_5(T_k)} b(\zeta)Q_2(g(\zeta),\tau(\zeta))d\zeta > \left(\frac{1+\ln\left(\beta_2\right)}{\beta_2}\right) \qquad \text{for all } \eta \geq 0.61 \text{ and } k \in \mathbb{N}_0.$$

In view of this, (3.2) and (3.3), all conditions of Theorem 2.1 with $n = 1$ are satisfied for all $\eta \geq 0.61$. Therefore all solutions of Eq (3.1) are oscillatory for $\eta \geq 0.61$.

However, if we assume that $a_k = 3r_k$ and $b_k = 3r_k + \frac{645}{121}$, then $b_k - a_k = \frac{645}{121} < \infty$. It follows that [29, Theorem 3] cannot be applied to Eq (3.1). Note also that since τ is not monotone, [29, Theorem 6] cannot be applied to this example.

Example 3.2. Consider the advanced differential equation

$$x'(t) - c(t)x(\sigma(t)) = 0, \quad t \geq 0, \tag{3.4}$$

where $c \in C([0,\infty),\mathbb{R})$ such that

$$c(t) = \begin{cases} \delta\left(1+\sin\left(9\pi t\right)\right) & \text{for } t \in \left[4r_k - \frac{68}{9},\, 4r_k - \frac{41}{9}\right], \\ \left(\alpha-\delta\right)\left(9t - 36r_k + 41\right) + \delta & \text{for } t \in \left[4r_k - \frac{41}{9},\, 4r_k - \frac{40}{9}\right], \\ \alpha & \text{for } t \in \left[4r_k - \frac{40}{9},\, 4r_k + \frac{10}{3}\right], \end{cases} \quad k \in \mathbb{N}_0, \tag{3.5}$$

where $\alpha, \delta \geq 0$ and $\{r_k\}_{k\geq 0}$ is a sequence of positive integers such that $r_{k+1} > r_k + \frac{49}{18}$ and $\lim_{k\to\infty} r_k = \infty$, and

$$\sigma(t) = \begin{cases} t+3 & \text{if } t \in [4l,\, 4l+2], \\ -t+8l+7 & \text{if } t \in [4l+2,\, 4l+3], \\ 3t-8l-5 & \text{if } t \in [4l+3,\, 4l+4], \end{cases} \quad l \in \mathbb{N}_0.$$

In view of (2.12) and (2.13), it follows that

$$h(t) = \begin{cases} t+3 & \text{if } t \in [4l, 4l+2], \\ 4l+5 & \text{if } t \in [4l+2, 4l+\frac{10}{3}], \ l \in \mathbb{N}_0 \\ 3t-8l-5 & \text{if } t \in [4l+\frac{10}{3}, 4l+4], \end{cases}$$

and

$$\Omega(t) = \begin{cases} t-3 & \text{if } t \in [4l, 4l+1], \\ \frac{1}{3}t+\frac{8}{3}l-1 & \text{if } t \in [4l+1, 4l+3], \ l \in \mathbb{N}_0, \\ t-3 & \text{if } t \in [4l+3, 4l+4], \end{cases}$$

respectively.
Clearly,

$$t+1 \leq h(t) \leq \sigma(t) \leq t+3, \qquad \text{for all } t \geq 1.$$

If we assume that $T_k = 4r_k + \frac{10}{3}$, $k \in \mathbb{N}_0$, then $\Omega_4(T_k) = 4r_k - \frac{68}{9}$. It follows from (3.5) that

$$c(t) \geq 0 \qquad \text{for } t \in [\Omega_4(T_k), T_k] \qquad \text{for all } k \in \mathbb{N}_0.$$

Thus

$$\begin{aligned} \int_{\Omega_4(T_k)}^{h(\Omega_4(T_k))} c(\zeta) R_1(h(\Omega_4(T_k)), \tau(\zeta)) d\zeta &\geq \int_{4r_k-\frac{68}{9}}^{4r_k-\frac{41}{9}} c(\zeta) d\zeta \\ &= \int_{4r_k-\frac{68}{9}}^{4r_k-\frac{41}{9}} \delta\,(1+\sin(9\pi\zeta))\, d\zeta \\ &= \frac{\delta\ (2+27\,\pi)}{9\pi} \geq 1, \quad \text{for all } \delta \geq \tfrac{9\pi}{2+27\,\pi} \text{ and } k \in \mathbb{N}_0. \end{aligned}$$

Therefore all conditions of Theorem 2.4 with $n = 0$ are satisfied for all $\delta \geq \frac{9\pi}{2+27\,\pi}$, and hence Eq (3.4) is oscillatory for $\delta \geq \frac{9\pi}{2+27\,\pi}$.

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Conflicts of interest

The authors declare that they have no competing of interests regarding the publication of this paper.

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