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Research article

Complete convergence and complete integration convergence for weighted sums of arrays of rowwise m-END under sub-linear expectations space

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Abstract: In this paper, we study the complete convergence and the complete integration convergence for weighted sums of m-extended negatively dependent (m-END) random variables under sub-linear expectations space with the condition of $\hat{\mathbb{E}}|X|^p \leqslant C_{\mathbb{V}}(|X|^p) < \infty$, $p > 1/\alpha$ and $\alpha > 3/2$. We obtain the results that can be regarded as the extensions of complete convergence and complete moment convergence under classical probability space. In addition, the Marcinkiewicz-Zygmund type strong law of large numbers for weighted sums of m-END random variables under the sub-linear expectations space is proved.

Keywords: sub-linear expectations space; rowwise m-END random variables; complete convergence; complete integration convergence
Mathematics Subject Classification: 60F15

1. Introduction

In the era of information modernization, limit theorems are widely used in real-life economics, information, and risk measurement. Limit theory of classical probability space considers that additive probability and additive expectation, which is suitable for the condition of model certainty. But the problems of financial and economic have different degrees of uncertainty. In order to analyze and calculate the problems under uncertainty, Peng [1, 2] came up with a new conception of the sub-linear expectations, and constructed the basic structure of the sub-linear expectations. Sub-linear expectations relaxes the additivity of probability and expectation of the classical probability. Hence, the theory of sub-linear expectations is more complex and challenging. Under the sub-linear expectations, Peng [3] established the central limit theorem. Enlightened by Peng's main articles, many researchers try to explore the results of sub-linear expectations. Chen and Gan [4] obtained the limiting behavior of weighted sums of independent and identically distributed sequences. Hu and Zhou [5] mainly demonstrated some multi-dimensional central limit theorems and laws of large numbers. Zhang [6–8] gained a series of important inequalities under sub-linear expectations. In addition, Zhang and Lin [9] also studied the Kolmogorov's strong law of large numbers. Lan and Zhang [10] proved the several moment inequalities, including Bernstein's inequalities, Kolmogorov's inequalities and Rademacher's inequalities. Guo and Zhang [11] obtained moderate deviation principle for m-dependent random variables under the sub-linear expectation.

In 1947, the notion of complete convergence was raised by Hsu and Robbins [12] as follows. Let $\{X_n, n \geqslant 1\}$ be a sequence of independent and identically distributed random variables in a probability space $(\Omega, \mathcal{F}, P)$ with $EX_1 = 0$ and $EX_1^2 < \infty$, $S_n = \sum_{k=1}^{n} X_k$,

$$\sum_{n=1}^{\infty} P(|S_n| > n\varepsilon) < \infty, \quad for\ all\ \varepsilon > 0.$$

In 1988, Chow [13] established the complete moment convergence. The complete moment convergence is stronger than the complete convergence. In the classical probability space, the complete convergence and the complete moment convergence for different sequences have been relatively mature. For example, Yu et al. [14] proved the complete convergence for weighted sums of arrays of rowwise m-END random variables. Wu et al. [16, 17] and Wang et al. [18] did a series of studies about extended negatively dependent (END) random variables. Meng et al. [15] and Ding et al. [19] respectively demonstrated the complete convergence and the complete moment convergence for END random variables and widely orthant dependent (WOD) random variables. Based on the basic framework of sub-linear expectations, researchers extended the theories and properties of classical probability space to the sub-linear expectations. For instance, Feng et al. [20] researched the complete convergence and the complete moment convergence for weighted sums of arrays of rowwise negatively dependent (ND) random variables. Zhong and Wu [21], Jia and Wu [22], Lu and Meng [23] , their recent papers had new results about complete convergence and complete integral convergence.

This paper aims to prove the complete convergence and the complete integral convergence for weighted sums of arrays of rowwise m-END under sub-linear expectations space. The rest of the paper is organized as follows. In section 2, we generally recall some basic notations and definitions, related properties under sub-linear expectations and preliminary lemmas that are useful to prove the main theorems. In section 3, the complete convergence, complete integral convergence and Marcinkiewicz-Zygmund type strong law of large numbers under sub-linear expectations space are established. In the last section, the proofs of these theorems are stated.

2. Preliminaries

We use the framework and notions of Peng [1,2]. Let $(\Omega, \mathcal{F})$ be a given measurable space and $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, X_2, \cdots, X_n \in \mathcal{H}$ then $\varphi(X_1, \cdots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}_n)$, where $C_{l,Lip}(\mathbb{R}_n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(x) - \varphi(y)| \leqslant c(1 + |x|^m + |y|^m)|x - y|, \quad \forall x, y \in \mathbb{R}_n,$$

for some $c > 0$, $m \in \mathbb{N}$ depending on φ. $\mathcal{H}$ is considered as a space of random variables. In this case we denote $X \in \mathcal{H}$.

Definition 2.1. A sub-linear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a function $\hat{\mathbb{E}}:\mathcal{H}\to\bar{\mathbb{R}}$ satisfying the following properties: for all $X, Y\in\mathcal{H}$, we have

(a) Monotonicity: if $X\geqslant Y$ then $\hat{\mathbb{E}}[X]\geqslant\hat{\mathbb{E}}[Y]$;

(b) Constant preserving: $\hat{\mathbb{E}}[c]=c$;

(c) Sub-additivity: $\hat{\mathbb{E}}[X+Y]\leqslant\hat{\mathbb{E}}[X]+\hat{\mathbb{E}}[Y]$;

(d) Positive homogeneity: $\hat{\mathbb{E}}[\lambda X]=\lambda\hat{\mathbb{E}}[X]$, $\lambda\geqslant 0$.

Here $\bar{\mathbb{R}}=[-\infty,\infty]$. The triple $(\Omega,\mathcal{H},\hat{\mathbb{E}})$ is called a sub-linear expectation space. Given a sub-linear expectation $\hat{\mathbb{E}}$, let us denote the conjugate expectation $\hat{\varepsilon}$ of $\hat{\mathbb{E}}$ by

$$\hat{\varepsilon}[X]=-\hat{\mathbb{E}}[-X],\quad \forall X\in\mathcal{H}.$$

From the definition, it is easily shown that for all $X, Y\in\mathcal{H}$

$$\hat{\varepsilon}[X]\leqslant\hat{\mathbb{E}}[X],\quad \hat{\mathbb{E}}[X+c]=\hat{\mathbb{E}}[X]+c,$$
$$|\hat{\mathbb{E}}[X-Y]|\leqslant\hat{\mathbb{E}}|X-Y|,\quad \hat{\mathbb{E}}[X-Y]\geqslant\hat{\mathbb{E}}[X]-\hat{\mathbb{E}}[Y].$$

Definition 2.2. Let $\mathcal{G}\subset\mathcal{F}$, a function $V:\mathcal{G}\to[0,1]$ is called a capacity if

(1) $V(\Phi)=0$, $V(\Omega)=1$;

(2) $V(A)\leqslant V(B)$, $\forall A\subset B$, $A, B\in\mathcal{G}$.

It is called to be sub-additive if $A, B\in\mathcal{G}$, $A\cup B\in\mathcal{G}$, $V(A\cup B)\leqslant V(A)+V(B)$.

$$\mathbb{V}(A)=\inf\{\hat{\mathbb{E}}[\xi]: I(A)\leqslant\xi,\xi\in\mathcal{H}\},\quad \mathcal{V}(A)=1-\mathbb{V}(A^c),\forall A\in\mathcal{F},$$

where A^c is the complement set of A. It is obvious that $\mathbb{V}$ is sub-additive and

$$\mathcal{V}(A)\leqslant\mathbb{V}(A),\quad \forall A\in\mathcal{F},$$

$$\mathbb{V}(A):=\hat{\mathbb{E}}[I_A],\quad \mathcal{V}(A):=\hat{\varepsilon}[I_A],\quad \text{if } I_A\in\mathcal{H},$$

$$\hat{\mathbb{E}}[f]\leqslant\mathbb{V}(A)\leqslant\hat{\mathbb{E}}[g],\quad \hat{\varepsilon}[f]\leqslant\mathcal{V}(A)\leqslant\hat{\varepsilon}[g],\quad \text{if } f\leqslant I_A\leqslant g, f, g\in\mathcal{H}.$$

For all $X\in\mathcal{H}$, $p>0$ and $x>0$,

$$I(|X|>x)\leqslant\frac{|X|^p}{x^p}I(|X|>x)\leqslant\frac{|X|^p}{x^p}.$$

Definition 2.3. We define the Choquet integrals $(C_{\mathbb{V}}, C_{\mathcal{V}})$ by

$$C_V[X]=\int_0^\infty V(X\geqslant t)dt+\int_{-\infty}^0[V(X\geqslant t)-1]dt,$$

with V being replaced by $\mathbb{V}$ and $\mathcal{V}$ respectively.

Definition 2.4. [3] (Identical distribution) Let X_1 and X_2 be two n-dimensional random vectors defined respectively in the sub-linear expectations spaces $(\Omega_1,\mathcal{H}_1,\hat{\mathbb{E}}_1)$ and $(\Omega_2,\mathcal{H}_2,\hat{\mathbb{E}}_2)$. They are called identically distributed, denoted by $X_1\overset{d}{=}X_2$ if

$$\hat{\mathbb{E}}_1\left(\varphi(X_1)\right)=\hat{\mathbb{E}}_2(\varphi(X_2)),\quad \forall\varphi\in C_{l,Lip}(\mathbb{R}),$$

whenever the sub-linear expectations are finite. A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to be identically distributed if $X_i \overset{d}{=} X_1$ for each $i \geqslant 1$.

Definition 2.5. [7] (END) In a sub-linear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, random variables $\{X_n, n \geqslant 1\}$ are called to be upper (resp. lower) extended negatively dependent if there is some dominating constant $K \geqslant 1$ such that

$$\hat{\mathbb{E}}\left(\prod_{i=1}^{n} \varphi_i(X_i)\right) \leqslant K \prod_{i=1}^{n} \hat{\mathbb{E}}(\varphi_i(X_i)), \quad n \geqslant 1,$$

whenever the non-negative functions $\varphi_i \in C_{l,Lip}(R), i = 1, 2, \cdots$ are all non-decreasing (resp. all non-increasing). They are called END if they are both upper extended negatively dependent and lower extended negatively dependent.

Definition 2.6. (m-END) Let $m \geqslant 1$ be a fixed positive integer. In a sub-linear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, random variables $\{X_n, n \geqslant 1\}$ is said to be m-END if for any $n \geqslant 2$ and any $i_1, i_2, \cdots, i_n$ such that $|i_k - i_j| \geqslant m$ for all $1 \leqslant k \neq j \leqslant n$, we have that $X_{i_1}, X_{i_2}, \cdots, X_{i_n}$ are END, i.e.

$$\hat{\mathbb{E}}\left(\prod_{k=1}^{n} \varphi_k(X_{i_k})\right) \leqslant K \prod_{k=1}^{n} \hat{\mathbb{E}}(\varphi_k(X_{i_k})), \quad n \geqslant 1,$$
$$|i_k - i_j| \geqslant m, \quad 1 \leqslant k \neq j \leqslant n,$$

where $K \geqslant 1$ is some dominating constant, the non-negative functions $\varphi_i \in C_{l,Lip}(R), i = 1, 2, \cdots$ are all non-decreasing or non-increasing. An array of random variables $\{X_{ni}, n \geqslant 1, i \geqslant 1\}$ is called rowwise m-END random variables if for every $n \geqslant 1$, $\{X_{ni}, i \geqslant 1\}$ is a sequence of m-END random variables, with a dominating sequence $\{K_n \geqslant 1\}$.

It is distinct that if $\{X_n, n \geqslant 1\}$ is a sequence of m-END random variables and $f_1(x), f_2(x), \cdots \in C_{l,Lip}(\mathbb{R})$ are all non-decreasing (or non-increasing), then $\{f_n(X_n), n \geqslant 1\}$ is also a sequence of m-END random variables.

In the following, let $\{X_n, n \geqslant 1\}$ be a sequence of random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. The symbol C is on behalf of a generic positive constant which may differ from one place to another; $I(\cdot)$ denote an indicator function. The following five lemmas are needed in the proofs of our theorems.

Lemma 2.1. [20] (i) Markov inequality: for all $X \in \mathcal{H}$,

$$\mathbb{V}(|X| \geqslant x) \leqslant \hat{\mathbb{E}}(|X|^p)/x^p, \quad \forall x > 0, p > 0.$$

(ii) Hölder inequality: for all $X, Y \in \mathcal{H}$ and $p, q > 1$ satisfying $p^{-1} + q^{-1}=1$,

$$\hat{\mathbb{E}}(|XY|) \leqslant (\hat{\mathbb{E}}(|X|^p))^{1/p}(\hat{\mathbb{E}}(|Y|^q))^{1/q}.$$

(iii) Jensen inequality: for all $X \in \mathcal{H}$ and $0 < r < s$,

$$(\hat{\mathbb{E}}(|X|^r))^{1/r} \leqslant (\hat{\mathbb{E}}(|X|^s))^{1/s}.$$

Lemma 2.2. [21] (i) Suppose $X \in \mathcal{H}, \alpha > 0, p > 0$, for any $c > 0$,

$$C_V(|X|^p) < \infty \Leftrightarrow \sum_{n=1}^{\infty} n^{\alpha p-1} V(|X| > cn^{\alpha}) < \infty. \tag{2.1}$$

(ii) If $C_V(|X|^p) < \infty$, then for any $\theta > 1$ and $c > 0$,

$$\sum_{k=1}^{\infty} \theta^{k\alpha p} V(|X| > c\theta^{k\alpha}) < \infty. \tag{2.2}$$

Lemma 2.3. [7] (Rosenthal's inequalities) Let $\{X_n, n \geqslant 1\}$ be a sequence of END random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ with $\hat{\mathbb{E}}X_k \leqslant 0$. And set $S_n = \sum_{k=1}^{n} X_k$, $B_n = \sum_{k=1}^{n} \hat{\mathbb{E}}X_k^2$, $M_{n,p} = \sum_{k=1}^{n} \hat{\mathbb{E}}|X_k|^p$. For any $p \geqslant 2$ and for all $x > 0$, then

$$\mathbb{V}(S_n \geqslant x) \leqslant (1 + K\mathrm{e})\frac{B_n}{x^2}, \tag{2.3}$$

there K is some dominating constant and exists a constant $C_p \geqslant 1$, such that for all $x > 0$ and $0 < \delta \leqslant 1$,

$$\mathbb{V}(S_n \geqslant x) \leqslant C_p \delta^{-2p} K \frac{M_{n,p}}{x^p} + K\exp\left\{-\frac{x^2}{2B_n(1+\delta)}\right\}. \tag{2.4}$$

With Lemma 2.3 in hand, we can get the following Rosenthal's inequalities for m-END random variables.

Lemma 2.4. (Rosenthal's inequalities) Let $\{X_n, n \geqslant 1\}$ be a sequence of m-END random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ with $\hat{\mathbb{E}}X_k \leqslant 0$. And set $S_n = \sum_{k=1}^{n} X_k$, $B_n = \sum_{k=1}^{n} \hat{\mathbb{E}}X_k^2$, $M_{n,p} = \sum_{k=1}^{n} \hat{\mathbb{E}}|X_k|^p$. For any $p \geqslant 2$ and for all $x > 0$, then

$$\mathbb{V}(S_n \geqslant x) \leqslant m^2(1 + K\mathrm{e})\frac{B_n}{x^2}, \tag{2.5}$$

there K is some dominating constant and exists a constant $C_p \geqslant 1$, such that for all $x > 0$ and $0 < \delta \leqslant 1$,

$$\mathbb{V}(S_n \geqslant x) \leqslant C_p \delta^{-2p} m^p K \frac{M_{n,p}}{x^p} + mK\exp\left\{-\frac{x^2}{2m^2 B_n(1+\delta)}\right\}. \tag{2.6}$$

Proof. Let $r = [\frac{n}{m}]$, define

$$X_i' = \begin{cases} X_i & 1 \leqslant i \leqslant n; \\ 0 & i > n. \end{cases}$$

Note that $S'_{mr+j} = \sum_{i=0}^{r} X'_{mi+j}$, $j = 1, 2, \cdots, m$, then

$$S_n = \sum_{j=1}^{m} \sum_{i=0}^{r} X'_{mi+j} = \sum_{j=1}^{m} S'_{mr+j},$$

for all $x > 0$ and $n \geqslant m$,

$$(S_n \geqslant x) \subset (S'_{mr+1} \geqslant \frac{x}{m}) \cup \cdots \cup (S'_{mr+m} \geqslant \frac{x}{m}) = \bigcup_{j=1}^{m} (S'_{mr+j} \geqslant \frac{x}{m}). \tag{2.7}$$

It follows by the definition of m-END random variables that $X'_j, X'_{m+j}, \cdots, X'_{mr+j}$ are END random variables for each $j = 1, 2, \cdots, m$. Hence, by (2.3) and (2.7) that for all $x > 0$ and $n \geqslant m$, we have

$$\begin{aligned}
\mathbb{V}(S_n \geqslant x) &\leqslant \mathbb{V}\left(\bigcup_{j=1}^{m}\left(S'_{mr+j} \geqslant \frac{x}{m}\right)\right) \\
&\leqslant \sum_{j=1}^{m} \mathbb{V}\left(S'_{mr+j} \geqslant \frac{x}{m}\right) \\
&\leqslant \sum_{j=1}^{m}(1+K\mathrm{e})\frac{\sum_{i=0}^{r}\hat{\mathbb{E}}(X'_{mi+j})^2}{(\frac{x}{m})^2} \\
&= m^2(1+K\mathrm{e})\frac{B_n}{x^2},
\end{aligned}$$

which implies (2.5).

By (2.4) and (2.7) that for all $x > 0, n \geqslant m$ and $p \geqslant 2$, we get

$$\begin{aligned}
\mathbb{V}(S_n \geqslant x) &\leqslant \sum_{j=1}^{m} \mathbb{V}\left(S'_{mr+j} \geqslant \frac{x}{m}\right) \\
&\leqslant \sum_{j=1}^{m}\left(C_p\delta^{-2p}K\frac{\sum_{i=0}^{r}\hat{\mathbb{E}}|X'_{mi+j}|^p}{(\frac{x}{m})^p} + K\exp\left\{-\frac{x^2}{2m^2\sum_{i=0}^{r}\hat{\mathbb{E}}(X'_{mi+j})^2(1+\delta)}\right\}\right) \\
&\leqslant C_p\delta^{-2p}Km^p\frac{M_{n,p}}{x^p} + mK\exp\left\{-\frac{x^2}{2m^2B_n(1+\delta)}\right\},
\end{aligned}$$

which implies (2.6).

This finishes the proof of Lemma 2.4. □

Lemma 2.5. [7] (Borel-Cantelli Lemma) $\{A_n, n \geqslant 1\}$ is a sequence of events in $\mathcal{F}$. Suppose that V is a countably sub-additive capacity. If $\sum_{n=1}^{\infty} V(A_n) < \infty$, then $V(A_n, i.o.) = 0$, where$\{A_n, \text{i.o.}\} = \bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty} A_i$.

3. Main results

Theorem 3.1. Let $\{X, X_{ni}, n \geqslant 1, 1 \leqslant i \leqslant n\}$ be an array of rowwise m-END and identically distributed random variables under sub-linear expectations. $\hat{\mathbb{E}}(X_{ni}) = \hat{\varepsilon}(X_{ni}) = 0$ and $\{a_{ni}, n \geqslant 1, 1 \leqslant i \leqslant n\}$ is an array of real numbers, suppose $\alpha > 3/2$, $p > 1/\alpha$, and $q > \max\{2, p\}$,

$$\sum_{i=1}^{n}|a_{ni}|^q = O(n), \tag{3.1}$$

and

$$\hat{\mathbb{E}}|X|^p \leqslant C_{\mathbb{V}}(|X|^p) < \infty, \tag{3.2}$$

then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} \mathbb{V}\left\{\left|\sum_{i=1}^{n} a_{ni} X_{ni}\right| > \varepsilon n^{\alpha}\right\} < \infty. \tag{3.3}$$

Theorem 3.2. Suppose that the conditions of Theorem 3.1 hold, and $0 < r < p$, then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} C_{\mathbb{V}}\left\{\left|\sum_{i=1}^{n} a_{ni} X_{ni}\right| - \varepsilon n^{\alpha}\right\}_{+}^{r} < \infty. \tag{3.4}$$

Theorem 3.3. Suppose that the conditions of Theorem 3.1 hold, and $\alpha p = 2$, then for any $\varepsilon > 0$,

$$n^{-2/p} \sum_{i=1}^{n} a_{ni} X_{ni} \to 0, a.s.\mathbb{V}, n \to \infty. \tag{3.5}$$

Remark 3.1. Theorems 3.1 and Theorem 3.3 extend the corresponding results of Yu et al. [14] from the classical probability space to sub-linear expectations space.

Remark 3.2. Under sub-linear expectations, the main purpose of our paper is to improve the result of Zhong and Wu [21] from END random variables to arrays of rowwise m-END random variables, and extend the range of p.

Remark 3.3. According to Definition 2.6, we can see that if $m = 1$, then the concept of m-END random variables reduces to END random variables under sub-linear expectations. Hence, the concept of m-END random variables is a natural extension of END random variables, m-END random variables include END random variables and ND random variables. So Theorem 3.1, Theorem 3.2 and Theorem 3.3 also hold for the arrays of END random variables and ND random variables under sub-linear expectations.

4. Proof

Proof of Theorem 3.1. According to

$$\sum_{i=1}^{n} a_{ni} X_{ni} = \sum_{i=1}^{n} a_{ni}^{+} X_{ni} - \sum_{i=1}^{n} a_{ni}^{-} X_{ni},$$

then for any $\varepsilon > 0$,

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{V}\left\{\left|\sum_{i=1}^{n} a_{ni} X_{ni}\right| > \varepsilon n^{\alpha}\right\} \leqslant & \sum_{n=1}^{\infty} \mathbb{V}\left\{\left|\sum_{i=1}^{n} a_{ni}^{+} X_{ni}\right| > \frac{\varepsilon n^{\alpha}}{2}\right\} \\ & + \sum_{n=1}^{\infty} \mathbb{V}\left\{\left|\sum_{i=1}^{n} a_{ni}^{-} X_{ni}\right| > \frac{\varepsilon n^{\alpha}}{2}\right\}. \end{aligned} \tag{4.1}$$

Without loss of generality, we can assume $a_{ni} \geqslant 0$ for all $n \geqslant 1$ and $1 \leqslant i \leqslant n$, which implies that

$$\sum_{n=1}^{\infty} n^{\alpha p-2} \mathbb{V}\left\{\sum_{i=1}^{n} a_{ni} X_{ni} > \varepsilon n^{\alpha}\right\} < \infty, \quad \forall \varepsilon > 0. \tag{4.2}$$

Because of considering $\{-X_{ni}, n \geqslant 1, i \geqslant 1\}$ still satisfies the conditions in Theorem 3.1, we have

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left\{\sum_{i=1}^{n} a_{ni}X_{ni} < -\varepsilon n^{\alpha}\right\} < \infty, \quad \forall \varepsilon > 0. \tag{4.3}$$

Hence, we can imply (3.3) by (4.2) and (4.3).

In the following, we prove (4.2). For all $n \geqslant 1$ and $1 \leqslant i \leqslant n$, denote that

$$\begin{aligned} X_{ni}^{'} =& - n^{\alpha}I(X_{ni} < -n^{\alpha}) + X_{ni}I(|X_{ni}| \leqslant n^{\alpha}) + n^{\alpha}I(X_{ni} > n^{\alpha}), \\ X_{ni}^{''} =& X_{ni} - X_{ni}^{'} = (X_{ni} + n^{\alpha})I(X_{ni} < -n^{\alpha}) + (X_{ni} - n^{\alpha})I(X_{ni} > n^{\alpha}). \end{aligned} \tag{4.4}$$

By Definition 2.6, we know that $\{X_{ni}^{'}, n \geqslant 1, 1 \leqslant i \leqslant n\}$ and $\{a_{ni}X_{ni}^{'}, n \geqslant 1, 1 \leqslant i \leqslant n\}$ are still arrays of rowwise m-END random variables. For any $0 < \beta \leqslant q$, by Hölder inequality and (3.1), we obtain that

$$\left(\sum_{i=1}^{n} a_{ni}^{\beta}\right) \leqslant \left(\sum_{i=1}^{n} a_{ni}^{q}\right)^{\frac{\beta}{q}} \left(\sum_{i=1}^{n} 1\right)^{1-\frac{\beta}{q}} \leqslant Cn. \tag{4.5}$$

For any $\varepsilon > 0$,

$$\left\{\sum_{i=1}^{n} a_{ni}X_{ni} > \varepsilon n^{\alpha}\right\} \subset \left\{\bigcup_{i=1}^{n}(|X_{ni}| > n^{\alpha})\right\} \bigcup \left\{\sum_{i=1}^{n} a_{ni}X_{ni}^{'} > \varepsilon n^{\alpha}\right\},$$

it is easy to see that

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left(\sum_{i=1}^{n} a_{ni}X_{ni} > \varepsilon n^{\alpha}\right) \\ \leqslant&\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left\{\bigcup_{i=1}^{n}(|X_{ni}| > n^{\alpha}) \bigcup \left(\sum_{i=1}^{n} a_{ni}X_{ni}^{'} > \varepsilon n^{\alpha}\right)\right\} \\ \leqslant&\sum_{n=1}^{\infty} n^{\alpha p-2}\sum_{i=1}^{n}\mathbb{V}(|X_{ni}| > n^{\alpha}) + \sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left(\sum_{i=1}^{n} a_{ni}X_{ni}^{'} > \varepsilon n^{\alpha}\right) \\ \doteq& H_1 + H_2. \end{aligned}$$

Hence, we need to prove $H_1 < \infty$ and $H_2 < \infty$.

For $0 < \mu < 1$, let $g(x)$ be a decreasing function when $x \geqslant 0$ and $g(x) \in C_{l,Lip}(\mathbb{R})$, $0 \leqslant g(x) \leqslant 1$ for all $x \in \mathbb{R}$, $g(x) = 1$, if $|x| \leqslant \mu$; $g(x) = 0$ if $|x| > 1$. Then

$$I(|x| \leqslant \mu) \leqslant g(|x|) \leqslant I(|x| \leqslant 1), \quad I(|x| > 1) \leqslant 1 - g(|x|) \leqslant I(|x| > \mu). \tag{4.6}$$

By (4.6) and Lemma 2.2 (2.1),

$$H_1 \leqslant\sum_{n=1}^{\infty} n^{\alpha p-2}\sum_{i=1}^{n}\hat{\mathbb{E}}\left(1 - g\left(\frac{|X_{ni}|}{n^{\alpha}}\right)\right)$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} n^{\alpha p-1} \hat{\mathbb{E}}\left(1 - g\left(\frac{|X|}{n^{\alpha}}\right)\right) \\
&\leqslant \sum_{n=1}^{\infty} n^{\alpha p-1} \mathbb{V}(|X| > \mu n^{\alpha}) < \infty.
\end{aligned}
$$

Next we estimate $H_2 < \infty$. For any $q > 0$, by c_r inequality, (4.4) and (4.6), which implies that

$$
\begin{aligned}
|X_{ni}^{'}|^q \leqslant& |X_{ni}|^q I(|X_{ni}| \leqslant n^{\alpha}) + n^{\alpha q} I(|X_{ni}| > n^{\alpha}) \\
\leqslant& |X_{ni}|^q g\left(\frac{\mu |X_{ni}|}{n^{\alpha}}\right) + n^{\alpha q}\left(1 - g\left(\frac{|X_{ni}|}{n^{\alpha}}\right)\right),
\end{aligned}
$$

furthermore,

$$
\begin{aligned}
\hat{\mathbb{E}}|X_{ni}^{'}|^q \leqslant& \hat{\mathbb{E}}\left(|X|^q g\left(\frac{\mu|X|}{n^{\alpha}}\right)\right) + n^{\alpha q}\hat{\mathbb{E}}\left(1 - g\left(\frac{|X|}{n^{\alpha}}\right)\right) \\
\leqslant& \hat{\mathbb{E}}\left(|X|^q g\left(\frac{\mu|X|}{n^{\alpha}}\right)\right) + n^{\alpha q}\mathbb{V}(|X| > \mu n^{\alpha}). \qquad (4.7)
\end{aligned}
$$

Case A_1: $0 < p < 1$.
By (4.5), (4.7), Markov inequality and $\alpha p > 1$, we get

$$
\begin{aligned}
n^{-\alpha}\left|\sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}X_{ni}^{'}\right| \leqslant& n^{-\alpha}\sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}|X_{ni}^{'}| \\
\leqslant& n^{-\alpha}\sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}\left(|X_{ni}| g\left(\frac{\mu|X_{ni}|}{n^{\alpha}}\right)\right) + \sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}\left(1 - g\left(\frac{|X_{ni}|}{n^{\alpha}}\right)\right) \\
\leqslant& n^{1-\alpha}\hat{\mathbb{E}}|X| I(|X| \leqslant \frac{1}{\mu}n^{\alpha}) + n\mathbb{V}(|X| > \mu n^{\alpha}) \\
\leqslant& C n^{1-\alpha p}\hat{\mathbb{E}}|X|^p \to 0, \quad n \to \infty.
\end{aligned}
$$

Case A_2: $p \geqslant 1$.
By (4.5), $\hat{\mathbb{E}}X_{ni} = 0$ and $\alpha p > 1$, one can get that

$$
\begin{aligned}
n^{-\alpha}\left|\sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}X_{ni}^{'}\right| \leqslant& n^{-\alpha}\sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}|X_{ni} - X_{ni}^{'}| \\
=& n^{-\alpha}\sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}|X_{ni}^{''}| \\
\leqslant& n^{-\alpha}\sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}[(|X_{ni}| - n^{\alpha}) I(|X_{ni}| > n^{\alpha})] \\
\leqslant& n^{-\alpha}\sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}\left[|X_{ni}|\left(1 - g\left(\frac{|X_{ni}|}{n^{\alpha}}\right)\right)\right]
\end{aligned}
$$

$$\leqslant Cn^{1-\alpha}\hat{\mathbb{E}}\left[|X|\left(1-g\left(\frac{|X|}{n^{\alpha}}\right)\right)\right]$$
$$\leqslant Cn^{1-\alpha p}\hat{\mathbb{E}}|X|^{p}\to 0,\quad n\to\infty.$$

It follows that for all n large enough,

$$n^{-\alpha}\left|\sum_{i=1}^{n}a_{ni}\hat{\mathbb{E}}X'_{ni}\right|<\frac{\varepsilon}{2},$$

which implies that

$$H_2\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2}\mathbb{V}\left\{\sum_{i=1}^{n}a_{ni}(X'_{ni}-\hat{\mathbb{E}}X'_{ni})>\frac{\varepsilon n^{\alpha}}{2}\right\}\doteq H_3.$$

By Definition 2.6, we know that $\{a_{ni}(X'_{ni}-\hat{\mathbb{E}}X'_{ni}), n\geqslant 1, 1\leqslant i\leqslant n\}$ are still arrays of rowwise m-END random variables, and $\hat{\mathbb{E}}(a_{ni}(X'_{ni}-\hat{\mathbb{E}}X'_{ni}))=0$. In order to prove $H_2<\infty$, we need to show $H_3<\infty$.

Case B_1: $p<2$.

By c_r inequality, Jensen inequality, and (2.5) in Lemma 2.4, combine with (4.5), (4.9), (4.10) and (4.13), we get

$$\begin{aligned}
H_3\leqslant & C\sum_{n=1}^{\infty}n^{\alpha p-2}(4(1+Ke))m^2\frac{\sum_{i=1}^{n}\hat{\mathbb{E}}(a_{ni}(X'_{ni}-\hat{\mathbb{E}}X'_{ni}))^2}{(\varepsilon n^{\alpha})^2}\\
\leqslant & C\sum_{n=1}^{\infty}n^{\alpha p-2-2\alpha}\sum_{i=1}^{n}\hat{\mathbb{E}}(a_{ni}(X'_{ni}-\hat{\mathbb{E}}X'_{ni}))^2\\
\leqslant & C\sum_{n=1}^{\infty}n^{\alpha p-2-2\alpha}\sum_{i=1}^{n}a_{ni}^2\hat{\mathbb{E}}(X'_{ni})^2\\
\leqslant & C\sum_{n=1}^{\infty}n^{\alpha p-1-2\alpha}\left[\hat{\mathbb{E}}\left(|X|^2 g\left(\frac{\mu|X|}{n^{\alpha}}\right)\right)+n^{2\alpha}\mathbb{V}(|X|>\mu n^{\alpha})\right]\\
\leqslant & C\sum_{n=1}^{\infty}n^{\alpha p-1-2\alpha}\hat{\mathbb{E}}\left(|X|^2 g\left(\frac{\mu|X|}{n^{\alpha}}\right)\right)+C\sum_{n=1}^{\infty}n^{\alpha p-1}\mathbb{V}(|X|>\mu n^{\alpha})\\
\doteq & H_{31}+H_{32}.
\end{aligned}$$

By (2.1), which implies that $H_{32}<\infty$. Next we prove $H_{31}<\infty$.

For $0<\mu<1$, let $g_k(x)\in C_{l,Lip}(\mathbb{R})$, $k\geqslant 1$ such that $0\leqslant g_k(x)\leqslant 1$ for all $x\in\mathbb{R}$, and $g_k(\frac{x}{2^{k\alpha}})=1$ if $2^{(k-1)\alpha}<|X|\leqslant 2^{k\alpha}$; $g_k(\frac{x}{2^{k\alpha}})=0$ if $|x|\leqslant\mu 2^{(k-1)\alpha}$ or $|x|>(1+\mu)2^{k\alpha}$. Then

$$g_k\left(\frac{|X|}{2^{k\alpha}}\right)\leqslant I(\mu 2^{(k-1)\alpha}<|X|\leqslant(1+\mu)2^{k\alpha}),$$
$$|X|^l g\left(\frac{|X|}{2^{j\alpha}}\right)\leqslant 1+\sum_{k=1}^{j}|X|^l g_k\left(\frac{|X|}{2^{k\alpha}}\right),\forall l>0. \tag{4.8}$$

By (4.8) and $g(x)$ is a decreasing function if $x \geqslant 0$,

$$
\begin{aligned}
H_{31} \leqslant & C \sum_{j=1}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} n^{\alpha p-2\alpha-1} \hat{\mathbb{E}}\left(X^2 g\left(\frac{\mu|X|}{n^\alpha}\right)\right) \\
\leqslant & C \sum_{j=1}^{\infty} 2^{(\alpha p-2\alpha-1)j} 2^j \hat{\mathbb{E}}\left(X^2 g\left(\frac{\mu|X|}{2^{\alpha(j+1)}}\right)\right) \\
\leqslant & C \sum_{j=1}^{\infty} 2^{\alpha(p-2)j} \hat{\mathbb{E}}\left(1 + \sum_{k=1}^{j} X^2 g_k\left(\frac{\mu|X|}{2^{(k+1)\alpha}}\right)\right) \\
\leqslant & C \sum_{j=1}^{\infty} 2^{\alpha(p-2)j} + C \sum_{j=1}^{\infty} 2^{\alpha(p-2)j} \sum_{k=1}^{j} \hat{\mathbb{E}}\left(X^2 g_k\left(\frac{\mu|X|}{2^{\alpha(k+1)}}\right)\right) \\
\doteq & H_{311} + H_{312}.
\end{aligned}
\tag{4.9}
$$

By $p < 2$, we obtain that $H_{311} < \infty$. For H_{312}, by (4.8) and (2.2) in Lemma 2.2, we get

$$
\begin{aligned}
H_{312} \leqslant & C \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} 2^{\alpha(p-2)j} \hat{\mathbb{E}}\left(X^2 g_k\left(\frac{\mu|X|}{2^{\alpha(k+1)}}\right)\right) \\
\leqslant & C \sum_{k=1}^{\infty} 2^{\alpha pk} \mathbb{V}(|X| > 2^{\alpha k}) < \infty.
\end{aligned}
\tag{4.10}
$$

Case B_2: $p \geqslant 2$.

By $q > p \geqslant 2$ and $n \geqslant m$, $\delta = 1$ and (2.6) in Lemma 2.4, we have

$$
\begin{aligned}
H_3 \leqslant & \sum_{n=1}^{\infty} n^{\alpha p-2} C_p \delta^{-2p} m^p K \frac{\sum_{i=1}^{n} \hat{\mathbb{E}}|a_{ni}(X'_{ni} - \hat{\mathbb{E}}X'_{ni})|^q}{(\varepsilon n^\alpha)^q} \\
& + \sum_{n=1}^{\infty} n^{\alpha p-2} m K \exp\left\{-\frac{(\varepsilon n^\alpha)^2}{8m^2 \sum_{i=1}^{n} \hat{\mathbb{E}}(a_{ni}(X'_{ni} - \hat{\mathbb{E}}X'_{ni}))^2 (1+\delta)}\right\} \\
\leqslant & C \sum_{n=1}^{\infty} n^{\alpha p-2} \frac{\sum_{i=1}^{n} a_{ni}^q \hat{\mathbb{E}}|(X'_{ni} - \hat{\mathbb{E}}X'_{ni})|^q}{(\varepsilon n^\alpha)^q} \\
& + C \sum_{n=1}^{\infty} n^{\alpha p-2} \exp\left\{-\frac{(\varepsilon n^\alpha)^2}{16m^2 \sum_{i=1}^{n} a_{ni}^2 \hat{\mathbb{E}}(X'_{ni} - \hat{\mathbb{E}}X'_{ni})^2}\right\} \\
\leqslant & C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q} \sum_{i=1}^{n} a_{ni}^q \hat{\mathbb{E}}|X'_{ni} - \hat{\mathbb{E}}X'_{ni}|^q
\end{aligned}
$$

$$+ C\sum_{n=1}^{\infty} n^{\alpha p-2}\exp\left\{-\frac{(\varepsilon n^{\alpha})^2}{16n^2\sum\limits_{i=1}^{n} a_{ni}^2\hat{\mathbb{E}}(X_{ni}'-\hat{\mathbb{E}}X_{ni}')^2}\right\}$$
$$\doteq I_1 + I_2.$$

Next we establish that $I_1 < \infty$ and $I_2 < \infty$. For I_1, by $\hat{\mathbb{E}}|X|^p < \infty$, c_r inequality, Jensen inequality and (4.7), we have that

$$\begin{aligned}
I_1 \leqslant& C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q}\sum_{i=1}^{n} a_{ni}^q\hat{\mathbb{E}}|X_{ni}'|^q \\
\leqslant& C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q}\sum_{i=1}^{n} a_{ni}^q\left(\hat{\mathbb{E}}|X|^q g\left(\frac{|X|}{n^{\alpha}}\right)+n^{\alpha q}\mathbb{V}(|X|>\mu n^{\alpha})\right) \\
\leqslant& C\sum_{i=1}^{\infty}\sum_{2^{i-1}\leqslant n<2^i} n^{\alpha p-\alpha q-1}\hat{\mathbb{E}}\left(|X|^q g\left(\frac{\mu|X|}{n^{\alpha}}\right)\right)+C\sum_{n=1}^{\infty} n^{\alpha p-1}\mathbb{V}(|X|>\mu n^{\alpha}) \\
\doteq& I_{11}+I_{12}.
\end{aligned} \tag{4.11}$$

By (2.1), it is obvious that that $I_{12} < \infty$. We only need to prove $I_{11} < \infty$. By (2.1) and (4.8), it is easy to prove that

$$\begin{aligned}
I_{11} \leqslant& C\sum_{i=1}^{\infty} 2^{i(\alpha p-\alpha q)}\hat{\mathbb{E}}\left(|X|^q g\left(\frac{\mu|X|}{2^{i\alpha}}\right)\right) \\
\leqslant& C\sum_{i=1}^{\infty} 2^{i(\alpha p-\alpha q)} + C\sum_{i=1}^{\infty} 2^{i(\alpha p-\alpha q)}\sum_{k=1}^{i}\hat{\mathbb{E}}\left(|X|^q g_k\left(\frac{\mu|X|}{2^{k\alpha}}\right)\right) \\
\leqslant& C\sum_{k=1}^{\infty}\sum_{i=k}^{\infty} 2^{i(\alpha p-\alpha q)}\hat{\mathbb{E}}\left(|X|^q g_k\left(\frac{\mu|X|}{2^{k\alpha}}\right)\right) \\
\leqslant& C\sum_{k=1}^{\infty} 2^{k\alpha p}\mathbb{V}(|X|>c2^{k\alpha}) < \infty.
\end{aligned} \tag{4.12}$$

For $\alpha > 3/2$, $2\alpha - 3 > 0$, which implies that for all n large enough,

$$\frac{\varepsilon^2}{16}n^{2\alpha-3} \geqslant \alpha p\ln n.$$

By (3.2), we can imply that

$$\begin{aligned}
I_2 \leqslant& C\sum_{n=1}^{\infty} n^{\alpha p-2}\exp\left\{\frac{\varepsilon^2}{16}n^{2\alpha-3}\right\} \\
\leqslant& C\sum_{n=1}^{\infty} n^{\alpha p-2}\exp\{\ln n^{-\alpha p}\} \\
\leqslant& C\sum_{n=1}^{\infty} n^{-2} < \infty.
\end{aligned}$$

Hence $H_2 < \infty$. This finishes the proof of Theorem 3.1.

Proof of Theorem 3.2. Without loss of generality, assume $a_{ni} \geqslant 0$ for all $n \geqslant 1$ and $1 \leqslant i \leqslant n$. For any $\varepsilon > 0$, by Theorem 3.1 we have that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} C_{\mathbb{V}} \left\{ \left| \sum_{i=1}^{n} a_{ni} X_{ni} \right| - \varepsilon n^{\alpha} \right\}_{+}^{r} \\
=& \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} \int_0^{\infty} \mathbb{V}\left(\left| \sum_{i=1}^{n} a_{ni} X_{ni} \right| - \varepsilon n^{\alpha} > x^{1/r} \right) dx \\
=& \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} \int_0^{n^{\alpha r}} \mathbb{V}\left(\left| \sum_{i=1}^{n} a_{ni} X_{ni} \right| - \varepsilon n^{\alpha} > x^{1/r} \right) dx \\
&+ \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} \int_{n^{\alpha r}}^{\infty} \mathbb{V}\left(\left| \sum_{i=1}^{n} a_{ni} X_{ni} \right| - \varepsilon n^{\alpha} > x^{1/r} \right) dx \\
\leqslant& \sum_{n=1}^{\infty} n^{\alpha p-2} \mathbb{V}\left(\left| \sum_{i=1}^{n} a_{ni} X_{ni} \right| > \varepsilon n^{\alpha} \right) \\
&+ \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} \int_{n^{\alpha r}}^{\infty} \mathbb{V}\left(\left| \sum_{i=1}^{n} a_{ni} X_{ni} \right| > x^{1/r} \right) dx \\
\leqslant& \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} \int_{n^{\alpha r}}^{\infty} \mathbb{V}\left(\sum_{i=1}^{n} a_{ni} X_{ni} > x^{1/r} \right) dx \doteq J.
\end{aligned}$$

Hence, it suffices to show that $J < \infty$.

For all $n \geqslant 1$ and $1 \leqslant i \leqslant n$, denote that

$$\begin{aligned}
Y_{ni}' &= -x^{1/r} I(X_{ni} < -x^{1/r}) + X_{ni} I(|X_{ni}| \leqslant x^{1/r}) + x^{1/r} I(X_{ni} > x^{1/r}), \\
Y_{ni}'' &= (X_{ni} + x^{1/r}) I(X_{ni} < -x^{1/r}) + (X_{ni} - x^{1/r}) I(X_{ni} > x^{1/r}),
\end{aligned}$$

then

$$\begin{aligned}
J \leqslant& \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} \int_{n^{\alpha r}}^{\infty} \sum_{i=1}^{n} \mathbb{V}(|X_{ni}| > x^{1/r}) dx + \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} \int_{n^{\alpha r}}^{\infty} \mathbb{V}\left(\sum_{i=1}^{n} a_{ni} Y_{ni}' > x^{1/r} \right) dx \\
\leqslant& \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} \int_{n^{\alpha r}}^{\infty} \sum_{i=1}^{n} \mathbb{V}(|X_{ni}| > x^{1/r}) dx \\
&+ \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} \int_{n^{\alpha r}}^{\infty} \mathbb{V}\left(\sum_{i=1}^{n} a_{ni} (Y_{ni}' - \hat{\mathbb{E}}(Y_{ni}')) > x^{1/r} - \left| \sum_{i=1}^{n} a_{ni} \hat{\mathbb{E}}(Y_{ni}') \right| \right) dx \\
\doteq& J_1 + J_2.
\end{aligned}$$

In order to estimate $J < \infty$, we only to show that $J_1 < \infty$ and $J_2 < \infty$. Thus by (4.5), (2.1) in Lemma 2.2 and $g(x)$ is a decreasing function when $x \geqslant 0$, we get

$$J_1 \leqslant \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} \int_{n^{\alpha r}}^{\infty} \sum_{i=1}^{n} \hat{\mathbb{E}}\left(1 - g\left(\frac{|X_{ni}|}{x^{1/r}} \right) \right) dx$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-1} \int_{n^{\alpha r}}^{\infty} \hat{\mathbb{E}}\left(1-g\left(\frac{|X|}{x^{1/r}}\right)\right)dx \\
&= \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-1} \sum_{m=n}^{\infty} \int_{m^{\alpha r}}^{(m+1)^{\alpha r}} \hat{\mathbb{E}}\left(1-g\left(\frac{|X|}{x^{1/r}}\right)\right)dx \\
&\leqslant \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-1} \sum_{m=n}^{\infty} [(m+1)^{\alpha r}-m^{\alpha r}]\hat{\mathbb{E}}\left(1-g\left(\frac{|X|}{m^{\alpha}}\right)\right) \\
&\leqslant \sum_{m=1}^{\infty} m^{\alpha r-1}\mathbb{V}(|X|>\mu m^{\alpha}) \sum_{n=1}^{m} n^{\alpha p-\alpha r-1} \\
&\leqslant \sum_{m=1}^{\infty} m^{\alpha p-1}\mathbb{V}(|X|>\mu m^{\alpha}) < \infty.
\end{aligned}
$$

Next we prove $J_2 < \infty$. By (4.5) and c_r inequality, for all $\gamma > 0$

$$
\begin{aligned}
\hat{\mathbb{E}}|Y'_{ni}|^{\gamma} &\leqslant \hat{\mathbb{E}}|X|^{\gamma} g\left(\frac{\mu|X|}{x^{1/r}}\right) + x^{\gamma/r}\hat{\mathbb{E}}\left(1-g\left(\frac{|X|}{x^{1/r}}\right)\right) \\
&\leqslant \hat{\mathbb{E}}\left(|X|^{\gamma} g\left(\frac{\mu|X|}{x^{1/r}}\right)\right) + x^{\gamma/r}\mathbb{V}(|X|>\mu x^{1/r}). \qquad (4.13)
\end{aligned}
$$

Case C_1: $p \geqslant 1$.
By (4.5), $\hat{\mathbb{E}}X_{ni} = 0$ and $\alpha p > 1$, it is sufficient to see that

$$
\begin{aligned}
\sup_{x\geqslant n^{\alpha r}} x^{-1/r}\left|\sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}Y'_{ni}\right| &\leqslant \sup_{x\geqslant n^{\alpha r}} x^{-1/r} \sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}|X_{ni}-Y'_{ni}| \\
&\leqslant \sup_{x\geqslant n^{\alpha r}} x^{-1/r} \sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}|Y''_{ni}| \\
&= \sup_{x\geqslant n^{\alpha r}} x^{-1/r} \sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}[(|X_{ni}|-x^{-1/r})I(|X_{ni}|>x^{-1/r})] \\
&\leqslant n^{-\alpha} \sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}[|X_{ni}|I(|X_{ni}|>n^{\alpha})] \\
&\leqslant n^{-\alpha} \sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}\left[|X_{ni}|\left(1-g\left(\frac{|X_{ni}|}{n^{\alpha}}\right)\right)\right] \\
&\leqslant Cn^{1-\alpha}\hat{\mathbb{E}}\left[|X|\left(1-g\left(\frac{|X|}{n^{\alpha}}\right)\right)\right] \\
&\leqslant Cn^{1-\alpha p}\hat{\mathbb{E}}|X|^{p} \to 0, \quad n\to\infty.
\end{aligned}
$$

Case C_2: $0 < p < 1$.
By (4.5), (4.13), Markov inequality and $\alpha p > 1$, we show that

$$
\sup_{x\geqslant n^{\alpha r}} x^{-1/r}\left|\sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}Y'_{ni}\right| \leqslant \sup_{x\geqslant n^{\alpha r}} x^{-1/r} \sum_{i=1}^{n} a_{ni}\hat{\mathbb{E}}|Y'_{ni}|
$$

$$\begin{aligned}
&\leqslant \sup_{x \geqslant n^{\alpha r}} x^{-1/r} \sum_{i=1}^{n} a_{ni} \hat{\mathbb{E}}\left(|X_{ni}| g\left(\frac{\mu |X_{ni}|}{x^{1/r}} \right) \right) \\
&\quad + \sup_{x \geqslant n^{\alpha r}} x^{-1/r} \sum_{i=1}^{n} a_{ni} x^{-1/r} \hat{\mathbb{E}}\left(1 - g\left(\frac{|X_{ni}|}{x^{1/r}} \right) \right) \\
&\leqslant \sup_{x \geqslant n^{\alpha r}} x^{-1/r} n \hat{\mathbb{E}} |X| I(|X| \leqslant \frac{1}{\mu} x^{1/r}) + \sup_{x \geqslant n^{\alpha r}} n \mathbb{V}(|X| > \mu x^{1/r}) \\
&\leqslant C n^{1-\alpha p} \hat{\mathbb{E}} |X|^p + n \mathbb{V}(|X| > \mu n^{\alpha}) \\
&\leqslant C n^{1-\alpha p} \hat{\mathbb{E}} |X|^p \to 0, \quad n \to \infty.
\end{aligned}$$

Hence, it follows that for all n large enough,

$$\sup_{x \geqslant n^{\alpha r}} x^{-1/r} \left| \sum_{i=1}^{n} a_{ni} \hat{\mathbb{E}} Y_{ni}^{'} \right| < \frac{1}{2},$$

which implies that

$$J_2 \leqslant \sum_{n=1}^{\infty} n^{\alpha p - \alpha r - 2} \int_{n^{\alpha r}}^{\infty} \mathbb{V}\left(\sum_{i=1}^{n} a_{ni} (Y_{ni}^{'} - \hat{\mathbb{E}}(Y_{ni}^{'})) > \frac{x^{1/r}}{2} \right) dx \doteq J_3.$$

By Definition 2.6, we know that $\{a_{ni}(Y_{ni}^{'} - \hat{\mathbb{E}} Y_{ni}^{'}), n \geqslant 1, 1 \leqslant i \leqslant n\}$ are still arrays of rowwise m-END random variables, and $\hat{\mathbb{E}}(a_{ni}(Y_{ni}^{'} - \hat{\mathbb{E}} Y_{ni}^{'})) = 0$. In order to prove $J_2 < \infty$, we have to show $J_3 < \infty$.

Case D_1: $p < 2$.

By c_r inequality, Jensen inequality, and (2.5) in Lemma 2.4, combine with (4.5), (4.9), (4.10) and (4.13) that

$$\begin{aligned}
J_3 &\leqslant \sum_{n=1}^{\infty} n^{\alpha p - \alpha r - 2} (4(1 + Ke)) m^2 \int_{n^{\alpha r}}^{\infty} \frac{\sum_{i=1}^{n} \hat{\mathbb{E}}(a_{ni}(Y_{ni}^{'} - \hat{\mathbb{E}} Y_{ni}^{'}))^2}{x^{-2/r}} dx \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p - \alpha r - 2} \int_{n^{\alpha r}}^{\infty} x^{-2/r} \sum_{i=1}^{n} a_{ni}^2 \hat{\mathbb{E}}(Y_{ni}^{'})^2 dx \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p - \alpha r - 1} \int_{n^{\alpha r}}^{\infty} x^{-2/r} \hat{\mathbb{E}}\left(|X|^2 g\left(\frac{\mu |X|}{x^{1/r}} \right) \right) dx \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p - \alpha r - 1} \int_{n^{\alpha r}}^{\infty} \hat{\mathbb{E}}\left(1 - g\left(\frac{\mu |X|}{x^{1/r}} \right) \right) dx \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p - \alpha r - 1} \sum_{k=n}^{\infty} \int_{k^{\alpha r}}^{(k+1)^{\alpha r}} x^{-2/r} \hat{\mathbb{E}}\left(|X|^2 g\left(\frac{\mu |X|}{x^{1/r}} \right) \right) dx \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p - \alpha r - 1} \sum_{k=n}^{\infty} k^{\alpha r - 1 - 2\alpha} \hat{\mathbb{E}}\left(|X|^2 g\left(\frac{\mu |X|}{k^{\alpha}} \right) \right) \\
&\leqslant C \sum_{k=1}^{\infty} k^{\alpha r - 1 - 2\alpha} \hat{\mathbb{E}}\left(|X|^2 g\left(\frac{\mu |X|}{k^{\alpha}} \right) \right) \sum_{n=1}^{k} n^{\alpha p - \alpha r - 1}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\sum_{k=1}^{\infty}k^{\alpha r-1-2\alpha}\hat{\mathbb{E}}\left(|X|^2 g\left(\frac{\mu|X|}{k^\alpha}\right)\right)k^{\alpha p-\alpha r}\\
&\leqslant C\sum_{k=1}^{\infty}k^{\alpha r-1-2\alpha}\hat{\mathbb{E}}\left(|X|^2 g\left(\frac{\mu|X|}{k^\alpha}\right)\right)<\infty.
\end{aligned}
$$

Case D_2: $p \geqslant 2$.

For $q > p \geqslant 2$ and $n \geqslant m$, by (2.6) in Lemma 2.4, c_r inequality and Jensen inequality, let $\delta = 1$, we have

$$
\begin{aligned}
J_3 \leqslant& C\sum_{n=1}^{\infty}n^{\alpha p-\alpha r-2}\int_{n^{\alpha r}}^{\infty}\frac{\sum_{i=1}^{n}\hat{\mathbb{E}}|a_{ni}(Y'_{ni}-\hat{\mathbb{E}}Y'_{ni})|^q}{x^{q/r}}dx\\
&+C\sum_{n=1}^{\infty}n^{\alpha p-\alpha r-2}\int_{n^{\alpha r}}^{\infty}\exp\left\{-\frac{x^{2/r}}{8m^2\sum_{i=1}^{n}\hat{\mathbb{E}}(a_{ni}(Y'_{ni}-\hat{\mathbb{E}}Y'_{ni}))^2(1+\delta)}\right\}dx\\
\leqslant& C\sum_{n=1}^{\infty}n^{\alpha p-\alpha r-2}\int_{n^{\alpha r}}^{\infty}x^{-q/r}\sum_{i=1}^{n}a_{ni}^q\hat{\mathbb{E}}|Y'_{ni}-\hat{\mathbb{E}}Y'_{ni}|^q dx\\
&+C\sum_{n=1}^{\infty}n^{\alpha p-\alpha r-2}\int_{n^{\alpha r}}^{\infty}\exp\left\{-\frac{x^{2/r}}{16m^2\sum_{i=1}^{n}a_{ni}^2\hat{\mathbb{E}}(Y'_{ni}-\hat{\mathbb{E}}Y'_{ni})^2}\right\}dx\\
\leqslant& C\sum_{n=1}^{\infty}n^{\alpha p-\alpha r-2}\int_{n^{\alpha r}}^{\infty}x^{-q/r}\sum_{i=1}^{n}a_{ni}^q\hat{\mathbb{E}}|Y'_{ni}|^q dx\\
&+C\sum_{n=1}^{\infty}n^{\alpha p-\alpha r-2}\int_{n^{\alpha r}}^{\infty}\exp\left\{-\frac{x^{2/r}}{16m^2\sum_{i=1}^{n}a_{ni}^2\hat{\mathbb{E}}(Y'_{ni}-\hat{\mathbb{E}}Y'_{ni})^2}\right\}dx\\
\doteq& J_{31}+J_{32}.
\end{aligned}
$$

Next we prove $J_{31} < \infty$ and $J_{32} < \infty$. By c_r inequality, Jensen inequality, and (2.5), combine with (4.5), (4.11), (4.12) and (4.13), then

$$
\begin{aligned}
J_{31} \leqslant& C\sum_{n=1}^{\infty}n^{\alpha p-\alpha r-1}\int_{n^{\alpha r}}^{\infty}x^{-q/r}\left(\hat{\mathbb{E}}\left(|X|^q g\left(\frac{\mu|X|}{x^{1/r}}\right)\right)+x^{q/r}\hat{\mathbb{E}}\left(1-g\left(\frac{\mu|X|}{x^{1/r}}\right)\right)\right)dx\\
\leqslant& C\sum_{n=1}^{\infty}n^{\alpha p-\alpha r-1}\int_{n^{\alpha r}}^{\infty}x^{-q/r}\hat{\mathbb{E}}\left(|X|^q g\left(\frac{\mu|X|}{x^{1/r}}\right)\right)dx\\
&+C\sum_{n=1}^{\infty}n^{\alpha p-\alpha r-1}\int_{n^{\alpha r}}^{\infty}\hat{\mathbb{E}}\left(1-g\left(\frac{\mu|X|}{x^{1/r}}\right)\right)dx\\
\leqslant& C\sum_{n=1}^{\infty}n^{\alpha p-\alpha r-1}\sum_{k=n}^{\infty}\int_{k^{\alpha r}}^{(k+1)^{\alpha r}}x^{-q/r}\hat{\mathbb{E}}\left(|X|^q g\left(\frac{\mu|X|}{x^{1/r}}\right)\right)dx
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-1}\sum_{k=n}^{\infty} k^{\alpha r-1-\alpha q}\hat{\mathbb{E}}\left(|X|^q g\left(\frac{\mu|X|}{k^\alpha}\right)\right)\\
&\leqslant C\sum_{k=1}^{\infty} k^{\alpha r-1-\alpha q}\hat{\mathbb{E}}\left(|X|^q g\left(\frac{\mu|X|}{k^\alpha}\right)\right)\sum_{n=1}^{k} n^{\alpha p-\alpha r-1}\\
&\leqslant C\sum_{k=1}^{\infty} k^{\alpha r-1-\alpha q}\hat{\mathbb{E}}\left(|X|^q g\left(\frac{\mu|X|}{k^\alpha}\right)\right)k^{\alpha p-\alpha r}\\
&\leqslant C\sum_{k=1}^{\infty} k^{\alpha r-1-2\alpha}\hat{\mathbb{E}}\left(|X|^q g\left(\frac{\mu|X|}{k^\alpha}\right)\right)<\infty.
\end{aligned}
$$

Let $\beta > \max\{\frac{\alpha p-1}{2\alpha-3}, \frac{r}{2}\}$, and $\frac{2\beta}{r} > 1$, $2-\alpha p+(2\alpha-3)\beta > 1$, it follows that all s large enough, $\mathrm{e}^s > s^\beta$, take $x = n^{\alpha r}t$, noting that by (3.2),

$$
\begin{aligned}
J_{32} &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2}\int_{n^{\alpha r}}^{\infty}\exp\left\{-\frac{x^{2/r}}{16n^3\hat{\mathbb{E}}(Y'_{ni}-\hat{\mathbb{E}}Y'_{ni})^2}\right\}dx\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2}\int_{n^{\alpha r}}^{\infty}\exp\left\{-\frac{x^{2/r}}{n^3}\right\}dx\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2}n^{\alpha r}\int_{1}^{\infty}\exp\left\{\frac{n^{2\alpha}t^{2/r}}{n^3}\right\}dt\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2}n^{\alpha r}\int_{1}^{\infty}\left(\frac{n^{2\alpha}t^{2/r}}{n^3}\right)^{-\beta}dt\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-(2\alpha-3)\beta}\int_{1}^{\infty}\frac{1}{t^{2\beta/r}}dt\\
&\leqslant C\sum_{n=1}^{\infty}\frac{1}{n^{2-\alpha p+(2\alpha-3)\beta}}<\infty.
\end{aligned}
$$

By $\hat{\mathbb{E}}(X_{ni}) = \hat{\varepsilon}(X_{ni}) = 0$, $\{-X_{ni}, n \geqslant 1, i \geqslant 1\}$ also satisfies the conditions of Theorem 3.2, we obtain

$$
\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2}\int_{n^{\alpha r}}^{\infty}\mathbb{V}\left(\sum_{i=1}^{n} a_{ni}X_{ni} < -x^{1/r}\right)dx < \infty.
$$

Hence, the proof of Theorem 3.2 is finished.

Proof of Theorem 3.3. Take $\alpha p = 2$ in Theorem 3.1, we get

$$
\sum_{n=1}^{\infty}\mathbb{V}\left\{\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| > \varepsilon n^\alpha\right\} < \infty.
$$

By Lemma 2.5, then

$$
\mathbb{V}\left\{\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| > \varepsilon n^\alpha, i.o.\right\} = 0,
$$

and

$$\mathcal{V}\left\{\bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty}\left(\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| \leqslant \varepsilon n^{\alpha}\right)\right\} = 1,$$

furthermore,

$$\mathbb{V}\left\{\bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty}\left(\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| \leqslant \varepsilon n^{\alpha}\right)\right\} = 1.$$

Then

$$\left(\bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty}\left(\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| \leqslant \varepsilon n^{\alpha}\right)\right) \subset \left(n^{-\alpha}\sum_{i=1}^{n} a_{ni}X_{ni} \longrightarrow 0\right).$$

When $\alpha = 2/p$, we have

$$\mathbb{V}\left\{\left(n^{-2/p}\sum_{i=1}^{n} a_{ni}X_{ni}\right) \longrightarrow 0\right\} = 1.$$

Above all, the proof of Theorem 3.3 is completed.

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Conflict of interest

All authors declare no conflict of interest in this paper.

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