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Research article

On Modified Erdős-Ginzburg-Ziv constants of finite abelian groups

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Abstract: Let G be a finite abelian group with exponent $\exp(G)$ and S be a sequence with elements of G. We say S is a zero-sum sequence if the sum of the elements in S is the zero element of G. For a positive integer t, let $\mathsf{s}_{t\exp(G)}(G)$ (respectively, $\mathsf{s}'_{t\exp(G)}(G)$) denote the smallest integer ℓ such that every sequence (respectively, zero-sum sequence) S over G with $|S| \ge \ell$ contains a zero-sum subsequence of length $t\exp(G)$. The invariant $\mathsf{s}_{t\exp(G)}(G)$ (respectively, $\mathsf{s}'_{t\exp(G)}(G)$) is called the Generalized Erdős-Ginzburg-Ziv constant (respectively, Modified Erdős-Ginzburg-Ziv constant) of G. In this paper, we discuss the relationship between Generalized Erdős-Ginzburg-Ziv constant and Modified Erdős-Ginzburg-Ziv constant, and determine $\mathsf{s}'_{t\exp(G)}(G)$ for some finite abelian groups.

Keywords: zero-sum sequence; Erdős-Ginzburg-Ziv Theorem; Generalized Erdős-Ginzburg-Ziv constant; Modified Erdős-Ginzburg-Ziv constant
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1. Introduction and main results

Let $\mathbb{Z}_n$ denote the cyclic group of n elements. Let G be a finite abelian group. Up to isomorphism, we can write G as $\mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_r}$ with $1 < n_1 \mid \ldots \mid n_r$, where $n_r = \exp(G)$ is the *exponent* of G. Whenever $n_1 = n_2 = \cdots = n_r = n$, we denote G by $\mathbb{Z}_n^r$. For convenience, we write $g \in \mathbb{Z}_n^r$ as $g = (a_1, a_2, \ldots, a_r)$, where integers $a_i \in [0, n-1]$ for every $i \in [1, r]$.

Let G be an abelian group and $G_0 \subseteq G$ be a subset. We consider sequences over G_0 as elements in the free abelian monoid with basis G_0. So a sequence S over G_0 can be written in the form

$$S = g_1 \cdot \ldots \cdot g_\ell = \prod_{g \in G_0} g^{\mathsf{v}_g(S)},$$

where $\mathsf{v}_g(S) \in \mathbb{N} \cup \{0\}$ denotes the *multiplicity* of g in S. We denote $|S| = \ell \in \mathbb{N} \cup \{0\}$ the *length* of S, $\sigma(S) = \sum_{i=1}^{\ell} g_i = \sum_{g \in G} \mathsf{v}_g(S) g \in G$ the *sum* of S. We say the sequence S over G_0 is a *zero-sum* sequence if $\sigma(S) = 0 \in G$. For an element $g \in G$, let $g + S$ denote the sequence $(g + g_1) \cdot \ldots \cdot (g + g_\ell)$

over $g + G_0$. Let H be a subgroup of G and $\varphi : G \to G/H$ be the natural homomorphism. Then $\varphi(S)$ denotes the sequence $\varphi(g_1) \cdot \ldots \cdot \varphi(g_\ell)$ over G_0/H. A sequence T is called a *subsequence* of S if $\mathsf{v}_g(T) \le \mathsf{v}_g(S)$ for all $g \in G_0$. Whenever T is a subsequence of S, let ST^{-1} denote the subsequence with T deleted from S. If S_1 and S_2 are two sequences over G, let S_1S_2 denote the sequence satisfying that $\mathsf{v}_g(S_1S_2) = \mathsf{v}_g(S_1) + \mathsf{v}_g(S_2)$ for all $g \in G_0$. If $S_1 = S_2$, we denote S_1S_2 by S_1^2.

For a finite abelian group G and a positive integer t, let $\mathsf{s}_{t\exp(G)}(G)$ denote the smallest integer ℓ such that every sequence S over G with $|S| \ge \ell$ contains a zero-sum subsequence of length $t\exp(G)$. For $t = 1$, we abbreviate $\mathsf{s}_{\exp(G)}(G)$ to $\mathsf{s}(G)$, which is called the *Erdős-Ginzburg-Ziv* constant of G. For $t \ge 2$, the invariant $\mathsf{s}_{t\exp(G)}(G)$ is called the *Generalized Erdős-Ginzburg-Ziv* constant of G. The classical Erdős-Ginzburg-Ziv Theorem [7], proved in 1961, states that

$$\mathsf{s}(\mathbb{Z}_n) = 2n - 1.$$

This theorem was viewed as one of the starting points for many problems involving zero-sum sequences over finite abelian groups. These problems occur naturally in various branches of combinatorics and number theory (see [1, 3, 16] for some classical papers). Since 1960s, the invariant $\mathsf{s}_{t\exp(G)}(G)$ has been studied by many authors (for recent progress see [6, 10, 12–15, 17–20]).

Our main motivation of this paper is the following *Modified Erdős-Ginzburg-Ziv* constant, which was introduced by Augspurger et al. [2] in 2017.

Definition 1.1. Let G be an abelian group and $G_0 \subseteq G$ be a subset. For a positive integer w, let $\mathsf{s}'_w(G_0)$ denote the smallest integer ℓ such that every zero-sum sequence S over G_0 with $|S| \ge \ell$ contains a zero-sum subsequence of length w. If no such ℓ exists, then let $\mathsf{s}'_w(G_0) = \infty$. For $w = \exp(G)$, we abbreviate $\mathsf{s}'_{\exp(G)}(G_0)$ to $\mathsf{s}'(G_0)$. The invariant $\mathsf{s}'_w(G_0)$ is called the *Modified Erdős-Ginzburg-Ziv* constant of G_0.

There has been lots of results on Generalized Erdős-Ginzburg-Ziv constant, but little is known about Modified Erdős-Ginzburg-Ziv constant.

For the infinite abelian group $\mathbb{Z}$, let $[-k, k] = \{i \mid -k \le i \le k\} \subset \mathbb{Z}$, where k is a positive integer. Augspurger et al. [2] gave some conditions for which $\mathsf{s}'_w([-k, k])$ is finite. They also gave the first results on $\mathsf{s}'_w([-k, k])$ for some positive integers w, k. In 2019, Berger [4] completely determined $\mathsf{s}'_w([-k, k])$ for all positive integers w, k.

Let G be a finite abelian group and g be an element of G with $\mathrm{ord}(g) = \exp(G)$. Let w be a positive integer with $\exp(G) \nmid w$. Then for every positive integer ℓ, both $S_1 = g^\ell \cdot (-\ell g)$ and $S_2 = g^{\ell+1} \cdot (-(\ell+1)g)$ are zero-sum sequences over G. But either S_1 or S_2 contains no zero-sum subsequence of length w. It follows that $\mathsf{s}'_w(G)$ is infinite. So we always consider the case when $w = t\exp(G)$, where t is a positive integer. In 2009, Berger and Wang [5] determined the invariants $\mathsf{s}'_{tn}(\mathbb{Z}_n)$ for every positive integer t and $\mathsf{s}'(\mathbb{Z}_n^2)$.

By the definition of $\mathsf{s}_{t\exp(G)}(G)$ and $\mathsf{s}'_{t\exp(G)}(G)$, we clearly have

$$\mathsf{s}'_{t\exp(G)}(G) \le \mathsf{s}_{t\exp(G)}(G) \tag{1.1}$$

for every finite abelian group G and every positive integer t.

Let G be a finite abelian group. In 2017, Zhong (see [2, Lemma 1]) proved that if $\gcd(\mathsf{s}(G) - 1, \exp(G)) = 1$, then $\mathsf{s}'(G) = \mathsf{s}(G)$. In this paper, we can slightly improve this result as

Theorem 1.2. *Let G be a finite abelian group. If $\gcd(\mathsf{s}_{t\exp(G)}(G) - i, \exp(G)) = 1$ for some positive integer i, then $\mathsf{s}'_{t\exp(G)}(G) > \mathsf{s}_{t\exp(G)}(G) - i$. Moreover, if $\gcd(\mathsf{s}_{t\exp(G)}(G) - 1, \exp(G)) = 1$, then $\mathsf{s}'_{t\exp(G)}(G) = \mathsf{s}_{t\exp(G)}(G)$.*

By Theorem 1.2, we are able to determine $\mathsf{s}'_{t\exp(G)}(G)$ for the following cases.

Theorem 1.3. *(1) Let $G = \mathbb{Z}_n^2$. If $t \geq 2$ and $\gcd(n, 3) = 1$, then $\mathsf{s}'_{tn}(G) = \mathsf{s}_{tn}(G) = tn + 2n - 2$.*

(2) Let $G = \mathbb{Z}_m \oplus \mathbb{Z}_n$ with $1 \leq m \mid n$. If $\gcd(2m - i, n) = 1$ for some integer $i \geq 4$, then $\mathsf{s}'(G) \geq 2m + 2n - (i - 1)$. Moveover, if $i = 4$, then $\mathsf{s}'(G) = \mathsf{s}(G) = 2m + 2n - 3$.

(3) Let $G = \mathbb{Z}_2^d$, where d is even. If $2t > d$, then $\mathsf{s}'_{2t}(G) = \mathsf{s}_{2t}(G) = 2t + d$.

We also find some cases such that $\mathsf{s}'_{t\exp(G)}(G) \neq \mathsf{s}_{t\exp(G)}(G)$. We need the following definition associated with $\mathsf{s}(G)$.

Definition 1.4. [8, Definition 1.1] A pair (n, d) of positive integers is said to have *Property D* if $(n - 1) \mid (\mathsf{s}(\mathbb{Z}_n^d) - 1)$ and every sequence S over $\mathbb{Z}_n^d$ of length $\mathsf{s}(\mathbb{Z}_n^d) - 1$ having no zero-sum subsequence of length n is of the form T^{n-1}, where T is a sequence over $\mathbb{Z}_n^d$ with $|T| = (\mathsf{s}(\mathbb{Z}_n^d) - 1)/(n - 1)$.

Let $G = \mathbb{Z}_n^r$ with $n \geq 3$ and $r \geq 2$. Suppose $\mathsf{s}(G) = c(n - 1) + 1$ for some positive integer c and (n, r) has Property D. Zhong (see [2, Lemma 1]) proved that if $\gcd(\mathsf{s}(G) - 1, n) = c$, then $\mathsf{s}'(G) < \mathsf{s}(G)$. In this paper, we obtain that

Theorem 1.5. *(1) Let n, d, k_1 be positive integers and $n > 1$. Suppose $\mathsf{s}(\mathbb{Z}_{n^k}^d) = c(n^k - 1) + 1$ for every positive integer k, where c is a constant depending only on n and d. If $\mathsf{s}'(\mathbb{Z}_{n^{k_1}}^d) \leq \mathsf{s}(\mathbb{Z}_{n^{k_1}}^d) - i$ for some positive integer i, then $\mathsf{s}'(\mathbb{Z}_{n^k}^d) \leq \mathsf{s}(\mathbb{Z}_{n^k}^d) - i$ for every integer $k \geq k_1$.*

(2) Let $G = \mathbb{Z}_m \oplus \mathbb{Z}_n$ with $1 \leq m \mid n$ and $H = \mathbb{Z}_d \oplus \mathbb{Z}_d$ with $d \mid m$. If $\mathsf{s}'(H) \leq \mathsf{s}(H) - i$ for some positive integer i, then $\mathsf{s}'(G) \leq \mathsf{s}(G) - i$.

By using Theorem 1.2 and Theorem 1.5, we obtain the following results.

Theorem 1.6. *Let k, d be two positive integers.*

(1) If $k < d$, then $\mathsf{s}'(\mathbb{Z}_{2^k}^d) = \mathsf{s}(\mathbb{Z}_{2^k}^d) = 2^d(2^k - 1) + 1$.
(2) If $k \geq d$, then $\mathsf{s}'(\mathbb{Z}_{2^k}^d) = \mathsf{s}(\mathbb{Z}_{2^k}^d) - 1 = 2^d(2^k - 1)$.
(3) If $k \geq 2$, then $\mathsf{s}'(\mathbb{Z}_{3^k}^3) = \mathsf{s}(\mathbb{Z}_{3^k}^3) - 1 = 9(3^k - 1)$.

We remark that the first two results in Theorem 1.6 confirm a conjecture of Berger and Wang (see [5, Conjecture 4.2]).

This paper is organized as follows. In Section 2, we deal with Theorems 1.2 and 1.3. In Section 3, we prove Theorems 1.5 and 1.6. In the final section, some additional results are given.

2. Proofs of Theorems 1.2 and 1.3

In this section, we deal with some cases when $\mathsf{s}'_{t\exp(G)}(G) = \mathsf{s}_{t\exp(G)}(G)$.

Proof of Theorem 1.2. Suppose $\gcd(\mathsf{s}_{t\exp(G)}(G) - i, \exp(G)) = 1$ for some positive integer i. We need to find a zero-sum sequence of length $\mathsf{s}_{t\exp(G)}(G) - i$, which contains no zero-sum subsequence of length $t\exp(G)$. By the definition of $\mathsf{s}_{t\exp(G)}(G)$, there exists a sequence S with $|S| = \mathsf{s}_{t\exp(G)}(G) - i$ and S has no zero-sum subsequence of length $t\exp(G)$. Assume that $\sigma(S) = h \in G$ and let $x \in \mathbb{N}$ be an integer such that

$$(\mathsf{s}_{t\exp(G)}(G) - i)x \equiv 1 \pmod{\exp(G)}.$$

Such integer x exists because $\gcd(\mathsf{s}_{t\exp(G)}(G) - i, \exp(G)) = 1$. Let

$$S' = -(xh) + S.$$

Since $\sigma(S') = \sigma(S) - (\mathsf{s}_{t\exp(G)}(G) - i)xh = 0$, we obtain that S' is a zero-sum sequence of length $\mathsf{s}_{t\exp(G)}(G) - i$. Next we prove that S' has no zero-sum subsequence of length $t\exp(G)$. Assume to the contrary that T' is a zero-sum subsequence of S' with $|T'| = t\exp(G)$. Then $\sigma(T') + t\exp(G)(xh) = 0$. It follows that $T = (xh) + T'$ is a zero-sum subsequence of S with $|T| = t\exp(G)$, yielding a contradiction to the definition of S. So S' has no zero-sum subsequence of length $t\exp(G)$. It follows that $\mathsf{s}'_{t\exp(G)}(G) > \mathsf{s}_{t\exp(G)}(G) - i$, and we are done.

Moreover, if $\gcd(\mathsf{s}_{t\exp(G)}(G) - 1, \exp(G)) = 1$, then $\mathsf{s}'_{t\exp(G)}(G) > \mathsf{s}_{t\exp(G)}(G) - 1$. By (1.1), we obtain that $\mathsf{s}'_{t\exp(G)}(G) \le \mathsf{s}_{t\exp(G)}(G)$. Therefore, $\mathsf{s}'_{t\exp(G)}(G) = \mathsf{s}_{t\exp(G)}(G)$. □

We need the following lemmas to prove Theorem 1.3.

Lemma 2.1. *[9, Proposition 4.1] Suppose $t \ge 2$, then $\mathsf{s}_{tn}(\mathbb{Z}_n^2) = tn + 2n - 2$.*

Lemma 2.2. *[11, Theorem 5.8.3] Let $G = \mathbb{Z}_m \oplus \mathbb{Z}_n$ with $1 \le m \mid n$. Then $\mathsf{s}(G) = 2m + 2n - 3$.*

Lemma 2.3. *[20, Theorem 5.9] $\mathsf{s}_{2t}(\mathbb{Z}_2^d) = 2t + d$ for $d < 2t$.*

Proof of Theorem 1.3. (1). Let $G = \mathbb{Z}_n^2$. Suppose $t \ge 2$. By Lemma 2.1, we have that $\mathsf{s}_{tn}(G) = tn + 2n - 2$. Since $\gcd(n, 3) = 1$, we infer that $\gcd(\mathsf{s}_{tn}(G) - 1, n) = 1$. It follows from Theorem 1.2 that $\mathsf{s}'_{tn}(G) = \mathsf{s}_{tn}(G) = tn + 2n - 2$.

(2). Let $G = \mathbb{Z}_m \oplus \mathbb{Z}_n$. By Lemma 2.2, we have that $\mathsf{s}(G) = 2m + 2n - 3$. Since $\gcd(2m - i, n) = 1$ for some integer $i \ge 4$, we infer that $\gcd(\mathsf{s}(G) - (i - 3), \exp(G)) = 1$. By Theorem 1.2, we obtain that $\mathsf{s}'(G) \ge \mathsf{s}(G) - (i - 3) + 1 = 2m + 2n - (i - 1)$.

If $i = 4$, we infer that $\gcd(\mathsf{s}(G) - 1, \exp(G)) = 1$. By Theorem 1.2, we obtain that $\mathsf{s}'(G) = \mathsf{s}(G) = 2m + 2n - 3$.

(3). Let $G = \mathbb{Z}_2^d$. Since $d < 2t$, by Lemma 2.3, we have that $\mathsf{s}_{2t}(G) = 2t + d$. Since d is even, we obtain that $\gcd(2t + d - 1, 2) = 1$, i.e. $\gcd(\mathsf{s}_{2t}(G) - 1, \exp(G)) = 1$. By Theorem 1.2, we obtain that $\mathsf{s}'_{2t}(G) = \mathsf{s}_{2t}(G) = 2t + d$. □

3. Proof of Theorems 1.5 and 1.6

In this section, we study some cases when $\mathsf{s}'_{t\exp(G)}(G) \ne \mathsf{s}_{t\exp(G)}(G)$.

Proof of Theorem 1.5. (1). We proceed by induction on k. If $k = k_1$, then by the hypothesis of the theorem we have $\mathsf{s}'(\mathbb{Z}_{n^{k_1}}^d) \le \mathsf{s}(\mathbb{Z}_{n^{k_1}}^d) - i$.

Suppose that the theorem is true for $k - 1 \ge k_1$. We will show that $\mathsf{s}'(\mathbb{Z}_{n^k}^d) \le \mathsf{s}(\mathbb{Z}_{n^k}^d) - i$. Let S be a zero-sum sequence over $\mathbb{Z}_{n^k}^d$ with $|S| = \mathsf{s}(\mathbb{Z}_{n^k}^d) - i = c(n^k - 1) + 1 - i$. It suffices to show that S contains a zero-sum subsequence of length n^k. Let

$$\phi : \mathbb{Z}_{n^k}^d \to \mathbb{Z}_{n^{k-1}}^d$$

be the natural homomorphism with $\ker\phi \cong \mathbb{Z}_n^d$. Then $\phi(S)$ is a zero-sum sequence over $\mathbb{Z}_{n^{k-1}}^d$ with $|\phi(S)| = c(n^k-1)+1-i = c(n-1)\times n^{k-1} + c(n^{k-1}-1)+1-i > c(n^{k-1}-1)+1-i$. By the assumption, we have $\mathsf{s}'(\mathbb{Z}_{n^{k-1}}^d) \le c(n^{k-1}-1)+1-i$. So we can find a subsequence S_1 of S such that $\phi(S_1)$ is a zero-sum sequence over $\mathbb{Z}_{n^{k-1}}^d$ of length n^{k-1}. It follows that $\phi(SS_1^{-1})$ is a zero-sum sequence over $\mathbb{Z}_{n^{k-1}}^d$ with $|\phi(SS_1^{-1})| = c(n^k-1)+1-i-n^{k-1} = (c(n-1)-1)\times n^{k-1} + c(n^{k-1}-1)+1-i > c(n^{k-1}-1)+1-i$ ($n > 1$). Then we can find a subsequence S_2 of SS_1^{-1} such that $\phi(S_2)$ is a zero-sum sequence over $\mathbb{Z}_{n^{k-1}}^d$ of length n^{k-1}. Repeating the above procedure, one can find $c(n-1)+1$ disjoint zero-sum subsequences $\phi(S_1),\dots,\phi(S_{c(n-1)+1})$ of $\phi(S)$ such that $|\phi(S_1)| = \cdots = |\phi(S_{c(n-1)+1})| = n^{k-1}$. Since $\phi(\sigma(S_i)) = \sigma(\phi(S_i)) = \overline{0} \in \mathbb{Z}_{n^{k-1}}^d$, we infer that $\sigma(S_i) \in \ker\phi \cong \mathbb{Z}_n^d$ for $i = 1,\dots,c(n-1)+1$. Let

$$T = \sigma(S_1)\cdot\ldots\cdot\sigma(S_{c(n-1)+1}).$$

Then T is a sequence over $\ker\phi \cong \mathbb{Z}_n^d$. By the hypothesis of the theorem, we have $\mathsf{s}(\mathbb{Z}_n^d) = c(n-1)+1$. So T has a zero-sum subsequence of length n. Without loss of generality we may assume that $\sigma(S_1)\sigma(S_2)\cdot\ldots\cdot\sigma(S_n)$ is a zero-sum subsequence of T over $\ker\phi \cong \mathbb{Z}_n^d$. Since $\sigma(S_1)+\sigma(S_2)+\cdots+\sigma(S_n) = 0 \in \ker\phi \subseteq \mathbb{Z}_{n^k}^d$ and $|S_1S_2\cdot\ldots\cdot S_n| = |\phi(S_1)\phi(S_2)\cdot\ldots\cdot\phi(S_n)| = n^k$, we obtain that $S_1S_2\cdot\ldots\cdot S_n$ is a zero-sum subsequence of S of length n^k. It follows that $\mathsf{s}'(\mathbb{Z}_{n^k}^d) \le \mathsf{s}(\mathbb{Z}_{n^k}^d) - i$, and we are done.

(2). Let $G = \mathbb{Z}_m \oplus \mathbb{Z}_n$ with $1 \le m \mid n$ and $H = \mathbb{Z}_d \oplus \mathbb{Z}_d$ with $d \mid m$. By Lemma 2.2, we have $\mathsf{s}(G) = 2m+2n-3$. Let S be a zero-sum sequence over G with $|S| = \mathsf{s}(G) - i = 2m+2n-3-i$. It suffices to show that S contains a zero-sum subsequence of length n. Let

$$\phi : G \to H$$

be the natural homomorphism with $\ker\phi \cong G/H = \mathbb{Z}_{\frac{m}{d}} \oplus \mathbb{Z}_{\frac{n}{d}}$. By the assumption, we have $\mathsf{s}'(H) \le 4d-3-i$. Since $\phi(S)$ is a zero-sum sequence over H with $|\phi(S)| = 2m+2n-3-i = (2\frac{m}{d}+2\frac{n}{d}-4)d+4d-3-i$, similar with (1), we can find $2\frac{m}{d}+2\frac{n}{d}-3$ disjoint zero-sum subsequences $\phi(S_1),\dots,\phi(S_{2\frac{m}{d}+2\frac{n}{d}-3})$ such that $|\phi(S_1)| = \cdots = |\phi(S_{2\frac{m}{d}+2\frac{n}{d}-3})| = d$. By Lemma 2.2, we have $\mathsf{s}(G/H) = 2\frac{m}{d}+2\frac{n}{d}-3$. So we can prove that S has a zero-sum subsequence of length n. It follows that $\mathsf{s}'(G) \le \mathsf{s}(G) - i$, and we are done. □

In 2003, Gao and Thangadurai proved the following result.

Lemma 3.1. *[8, Corollary 1.1] The pairs $(2^k, d)$, $(3^k, 3)$ have Property D for any positive integers k and d.*

We need the following results.

Lemma 3.2. *(1)* $\mathsf{s}(\mathbb{Z}_{2^k}^d) = 2^d(2^k-1)+1$ *for every $k \ge 1$ and $d \ge 1$. (Harborth [16])*
(2) $\mathsf{s}(\mathbb{Z}_{3^k}^3) = 9(3^k-1)+1$ *for every $k \ge 1$. (Gao and Thangadurai [8])*

Now we are in a position to prove Theorem 1.6.

Proof of Theorem 1.6. (1). We will prove $\mathsf{s}'(\mathbb{Z}_{2^k}^d) = 2^d(2^k-1)+1$ for $k \le d-1$. By (1.1) and Lemma 3.2.(1), we have $\mathsf{s}'(\mathbb{Z}_{2^k}^d) \le \mathsf{s}(\mathbb{Z}_{2^k}^d) = 2^d(2^k-1)+1$. Let

$$W = \prod_{a_1,a_2,\dots,a_d \in \{0,1\}} (a_1,a_2,\dots,a_d)^{2^k-1}.$$

Since $k \le d-1$, it is easy to check that W is a zero-sum sequence of length $2^d(2^k-1)$, and W contains no zero-sum subsequence of length 2^k. It follows that $\mathsf{s}'(\mathbb{Z}_{2^k}^d) \ge 2^d(2^k-1)+1$. Therefore, $\mathsf{s}'(\mathbb{Z}_{2^k}^d) = 2^d(2^k-1)+1$ for $k \le d-1$.

(2). Assume that $k \geq d$. By (1.1) and Lemma 3.2.(1), we have $\mathsf{s}'(\mathbb{Z}_{2^k}^d) \leq \mathsf{s}(\mathbb{Z}_{2^k}^d) = 2^d(2^k-1)+1$.

We first prove that $\mathsf{s}'(\mathbb{Z}_{2^k}^k) \leq 2^k(2^k-1)$. Let S be a zero-sum sequence over $\mathbb{Z}_{2^k}^k$ with $|S| = 2^k(2^k-1)$. We need to prove that S contains a zero-sum subsequence of length 2^k. Assume to the contrary that S contains no zero-sum subsequence of length 2^k. Note that $|S| = \mathsf{s}(\mathbb{Z}_{2^k}^k) - 1$. By Lemma 3.1, we obtain that the pair $(2^k, k)$ has Property D. Therefore, S is of form T^{2^k-1} for some sequence T with $|T| = \frac{s(\mathbb{Z}_{2^k}^k)-1}{2^k-1} = 2^k$. Since $0 = \sigma(S) = (2^k-1)\sigma(T)$ and $\gcd(2^k-1, 2^k) = 1$, we infer that $\sigma(T) = 0$. Therefore, T is a zero-sum subsequence of S with $|T| = 2^k$, yielding a contradiction. So S contains a zero-sum subsequence of length 2^k, and this proves that

$$\mathsf{s}'(\mathbb{Z}_{2^k}^k) \leq 2^k(2^k-1) = \mathsf{s}(\mathbb{Z}_{2^k}^k) - 1.$$

By Theorem 1.5, we have that $\mathsf{s}'(\mathbb{Z}_{2^k}^d) \leq \mathsf{s}(\mathbb{Z}_{2^k}^d) - 1 = 2^d(2^k-1)$ for every $k \geq d$. Since $\gcd(2^d(2^k - 1) - 1, 2^k) = \gcd(\mathsf{s}(\mathbb{Z}_{2^k}^d) - 2, \exp(\mathbb{Z}_{2^k}^d)) = 1$, by Theorem 1.2, we obtain that

$$\mathsf{s}'(\mathbb{Z}_{2^k}^d) > \mathsf{s}(\mathbb{Z}_{2^k}^d) - 2 = 2^d(2^k-1) - 1.$$

Therefore, $\mathsf{s}'(\mathbb{Z}_{2^k}^d) = 2^d(2^k-1)$ for every $k \geq d$.

(3). By Lemma 3.2.(2), we have $\mathsf{s}(\mathbb{Z}_{3^k}^3) = 9(3^k-1)+1$. Since $(3^k, 3)$ has Property D, by using a similar argument as in (2), we can prove that $\mathsf{s}'(\mathbb{Z}_{3^k}^3) = 9(3^k-1)$ for every integer $k \geq 2$. □

4. Concluding remarks

In this section, we give some additional results. By (1.1), we have $\mathsf{s}'_{t\exp(G)}(G) \leq \mathsf{s}_{t\exp(G)}(G)$ for every finite abelian group G and every positive integer t. By Theorem 1.6, the equality $\mathsf{s}'_{t\exp(G)}(G) = \mathsf{s}_{t\exp(G)}(G)$ does not always hold. A natural problem asks that

Problem 4.1. *When do we have* $\mathsf{s}'_{t\exp(G)}(G) = \mathsf{s}_{t\exp(G)}(G)$*?*

By Theorem 1.2, if $\gcd(\mathsf{s}_{t\exp(G)}(G) - 1, \exp(G)) = 1$, then $\mathsf{s}'_{t\exp(G)}(G) = \mathsf{s}_{t\exp(G)}(G)$. However, $\gcd(\mathsf{s}_{t\exp(G)}(G) - 1, \exp(G)) = 1$ is not necessary for $\mathsf{s}'_{t\exp(G)}(G) = \mathsf{s}_{t\exp(G)}(G)$. In this paper, we are able to prove the following result.

Theorem 4.2. *Let* $G = \mathbb{Z}_2^{2t}$. *Then* $\mathsf{s}'_{2t}(G) = \mathsf{s}_{2t}(G) = 4t+1$.

In order to prove Theorem 4.2, we need the following result due to Sidorenko [20].

Lemma 4.3. *[20, Theorem 5.10]* $\mathsf{s}_{2t}(\mathbb{Z}_2^{2t}) = 4t+1$.

Proof of Theorem 4.2. By (1.1) and Lemma 4.3, we have

$$\mathsf{s}'_{2t}(\mathbb{Z}_2^{2t}) \leq \mathsf{s}_{2t}(\mathbb{Z}_2^{2t}) = 4t+1.$$

Next we need to prove $\mathsf{s}'_{2t}(\mathbb{Z}_2^{2t}) \geq 4t+1$. We denote $\mathbf{e}_j = (a_1, a_2, \ldots, a_{2t})$ for every $j = 1, 2, \ldots, 2t$, where $a_j = 1$ and $a_i = 0$ for every $i \neq j$. Let

$$T = \mathbf{0}^{2t-1}\mathbf{e}_1 \cdot \mathbf{e}_2 \cdot \ldots \cdot \mathbf{e}_{2t} \cdot (\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_{2t}).$$

Then T is a zero-sum sequence with $|T| = 4t$ and T contains no zero-sum subsequence of length $2t$. Hence, $\mathsf{s}'_{2t}(\mathbb{Z}_2^{2t}) \geq 4t+1$. Therefore, $\mathsf{s}'_{2t}(\mathbb{Z}_2^{2t}) = 4t+1$. □

Among other results, we can also determine some cases when $\mathsf{s}'_{t\exp(G)}(G) < \mathsf{s}_{t\exp(G)}(G)$ holds.

Theorem 4.4. *(1) If $G = \mathbb{Z}_n$ with $\gcd(n, 2) \neq 1$, then $\mathsf{s}'_{tn}(G) < \mathsf{s}_{tn}(G)$ for every integer $t \geq 1$.*
(2) If $G = \mathbb{Z}_n^2$ with $4 \mid n$, then $\mathsf{s}'(G) < \mathsf{s}(G)$.
(3) If $G = \mathbb{Z}_m \oplus \mathbb{Z}_n$ with $4 \mid m \mid n$, then $\mathsf{s}'(G) < \mathsf{s}(G)$.

In order to prove Theorem 4.4, we need the following results.

Lemma 4.5. *[5, Theorem 1.3] $\mathsf{s}'_{tn}(\mathbb{Z}_n) = (t+1)n - \ell + 1$ for every integer $t \geq 1$, where ℓ is the smallest integer such that $\ell \nmid n$.*

Lemma 4.6. *[5, Theorem 1.4] $\mathsf{s}'(\mathbb{Z}_n^2) = 4n - \ell + 1$, where ℓ is the smallest integer such that $\ell \geq 4$ and $\ell \nmid n$.*

The following result is a consequence of the Erdős-Ginzburg-Ziv Theorem.

Lemma 4.7. *Let t, n be two positive integers. Then $\mathsf{s}_{tn}(\mathbb{Z}_n) = (t+1)n - 1$.*

Proof of Theorem 4.4. (1). By Lemma 4.5, we infer that $\mathsf{s}'_{tn}(\mathbb{Z}_n) = (t+1)n - \ell + 1$ for every integer $t \geq 1$, where ℓ is the smallest integer such that $\ell \nmid n$. Since $\gcd(n, 2) \neq 1$, we infer that $\ell \geq 3$. It follows that $\mathsf{s}'_{tn}(\mathbb{Z}_n) \leq (t+1)n - 2$. By Lemma 4.7, we have that $\mathsf{s}'_{tn}(\mathbb{Z}_n) < \mathsf{s}_{tn}(\mathbb{Z}_n)$.

(2). By Lemma 4.6, we infer that $\mathsf{s}'(\mathbb{Z}_n^2) = 4n - \ell + 1$, where ℓ is the smallest integer such that $\ell \geq 4$ and $\ell \nmid n$. Since $4 \mid n$, we infer that $\ell \geq 5$. It follows that $\mathsf{s}'(\mathbb{Z}_n^2) \leq 4n - 4$. By Lemma 2.2, we get that $\mathsf{s}'(\mathbb{Z}_n^2) < \mathsf{s}(\mathbb{Z}_n^2)$.

(3). Let $H = \mathbb{Z}_4^2$. By (2), we obtain that $\mathsf{s}'(H) \leq \mathsf{s}(H) - 1$. By Theorem 1.5, we infer that $\mathsf{s}'(\mathbb{Z}_m \oplus \mathbb{Z}_n) < \mathsf{s}(\mathbb{Z}_m \oplus \mathbb{Z}_n)$. □

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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