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Research article

Some new characterizations and results for fuzzy contractions in fuzzy b-metric spaces and applications

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Abstract: In this work, we initiate the notion of a fuzzy cyclic (α,β)-admissibility to establish some fixed point results for contraction mappings involving a generalized simulation function in the class of fuzzy b-metric spaces. We give some illustrative examples to validate the new concepts and obtained results. At the end, we present an application on a Fredholm integral equation.

Keywords: fuzzy b-metric space; simulation function; fixed point; integral equation
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1. Introduction and preliminaries

Because of its simplicity, usefulness and applications, the Banach fixed point theorem [1] has become a very popular tool in solving the existence problems in many branches of mathematical analysis. Recently, many authors have improved, extended, and generalized this result in several ways. The metric theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear phenomena. Particularly, fixed point methods have been applied in diverse fields such as: biology, chemistry, economics, engineering, game theory, computer science, physics, geometry, astronomy, fluid and elastic mechanics, physics, control theory, image processing, and economics. For related works, we cite [2–5]. Coming back to the limitations of metrics, some real problems entail uncertainty in itself when the measurement between objects has to be carried out. Fuzzy theory seems to be a proper framework to tackle such situations. In this framework, we can find the notion of the fuzzy metric introduced by Kramosil and Michalek in [6]. This metric constitutes a fuzzy version of the concept of a metric, which provides a degree of nearness between two objects with respect to a parameter.

Mathematically, a fuzzy set is a set with elements that are with a membership degree. It is considered as one of the uncertainty approaches to construct mathematical models compatible with real world problems in engineering, life science, economics, medicine, language theory, and so on. This concept is established by Zadeh [7]. It was particularly outlined for appearing instability in mathematics and for managing with unclearness in numerous real-life problems, it is reasonable for approximating thinking mathematical models that are difficult to infer or giving a choice with fragmented information. After that, Heilpern [8] performs the concept of the fuzzy contraction and ensured the existence of a fixed point for fuzzy mappings on metric spaces. The fixed point result of Heilpern [8] is parallel to the Banach fixed point theorem in the frame of fuzzy sets. Thereafter, several authors have studied and applied fuzzy fixed point results in different settings, see, for example the works of Došenović et al. [9], Georgi and Veeramani [10, 11], Gregori and Sapana [12], Jebril et al. [13], Shahzad et al. [14], Sedghi et al. [15], Uddin el. [16], Wairojjana et al. [17] and Miheţ [18].

On the other hand, Bakhtin [19] and Czerwik [20] stated the framework of a b-metric space, where the triangular inequality was observed as a weaker condition. Many authors have studied and proved several generalizations of this setting.

Definition 1.1. *[19, 20] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d_{bms} : X \times X \to [0, \infty)$ is called a b-metric on X if the following hold for all $x, y, \kappa \in X$:*

$(B1)$ $d_{bms}(x, y) = 0$ *if and only if* $x = y$;
$(B2)$ $d_{bms}(x, y) = d_{bms}(y, x)$;
$(B3)$ $d_{bms}(x, \kappa) \leq s[d_{bms}(x, y) + d_{bms}(y, \kappa)]$.

Note that the class of b-metric spaces is greater than the class of metric spaces. When $s = 1$, a b-metric is a metric.

Example 1.1. *Take $X = [0, 1]$. Define $d_{bms} : X \times X \to [0, \infty)$ by $d_{bms}(x, y) = |x - y|^2$ for all $x, y \in X$. Here, $(X, d_{bms}, 2)$ is a b-metric space, but not a metric space.*

Definition 1.2. *[21] A triangular norm (shortly, t-norm) is a binary operation on the unit interval $[0, 1]$, i.e., a function $\mathbb{T} : [0, 1] \times [0, 1] \to [0, 1]$ such that for all $a, b, c \in [0, 1]$, the following four axioms are satisfied:*
$(Tn1)$ $\mathbb{T}(a, \mathbb{T}(b, c)) = \mathbb{T}(\mathbb{T}(a, b), c)$ *(associativity);*
$(Tn2)$ $\mathbb{T}(a, b) = \mathbb{T}(b, a)$ *(commutativity);*
$(Tn3)$ $\mathbb{T}(a, 1) = a$ *(boundary condition);*
$(Tn4)$ $\mathbb{T}(a, b) \leq \mathbb{T}(a, c)$*, whenever $b \leq c$ (monotonicity).*

Example 1.2. *The basic t-norms are given as*

$(t1)$ $\mathbb{T}_P(a, b) = a \cdot b$,
$(t2)$ $\mathbb{T}_{min}(a, b) = \min\{a, b\}$,

(t3) $\mathbb{T}_L(a, b) = \max\{a + b - 1, 0\}$,

for all $a, b \in [0, 1]$.

Definition 1.3. *[22] Let* $\mathbb{T}$ *be a t-norm and let* $T_n : [0, 1] \to [0, 1]$ *(*$n \in \mathbb{N}$*) be defined in the following way:*

$$T_1(a) = \mathbb{T}(a, a),\ \mathbb{T}_{n+1}(a) = \mathbb{T}(T_n(a), a),\ n \in \mathbb{N},\ a \in [0, 1].$$

If the family $\{T_n(a)\}_{n\in\mathbb{N}}$ *is equicontinuous at* $a = 1$, *we say that the t-norm* $\mathbb{T}$ *is of H-type.*

Example 1.3. *[23] The following are some examples of t-norms of H-type:*

(i) $\mathbb{T}_{min}(a_1, \ldots, a_n) = \min(a_1, \ldots, a_n)$;
(ii) $\mathbb{T}_L(a_1, \ldots, a_n) = \max(\sum_{i=1}^{n} a_i - (n - 1),\ 0)$;
(iii) $\mathbb{T}_P(a_1, \ldots, a_n) = \prod_{i=1}^{n} a_i$.

Let $\mathbb{T}$ be a t-norm. $\mathbb{T}$ can indeed be generalized to a counted operation (see [21]) taking for any sequence $\{x_n\}_{n\in\mathbb{N}}$ from [0, 1] the value

$$\mathbb{T}_{i=1}^{\infty} x_i = \lim_{n\to\infty} \mathbb{T}_{i=1}^{n} x_i.$$

As a result, the sequence $\{\mathbb{T}_{i=1}^{n} x_i\}_{n\in\mathbb{N}}$ is nonincreasing and bounded from below, and so the limit $\mathbb{T}_{i=1}^{\infty} x_i$ exists.

In fixed point theory (see [23]), it is of interest to investigate the classes of t-norms $\mathbb{T}$ and sequences $\{x_n\}_{n\in\mathbb{N}}$ from the interval [0, 1] so that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent to 1 and

$$\lim_{n\to\infty} \mathbb{T}_{i=n}^{\infty} x_i = \lim_{n\to\infty} \mathbb{T}_{i=1}^{n} x_{n+i} = 1.$$

In this regard, we state the following proposition.

Proposition 1.1. *[23] Let* $\{x_n\}$ *be a sequence in* [0, 1] *so that* $\lim_{n\to\infty} x_n = 1$, *and let* $\mathbb{T}$ *be a t-norm of H-type. Then,* $\lim_{n\to\infty} \mathbb{T}_{i=n}^{\infty} x_i = \lim_{n\to\infty} \mathbb{T}_{i=1}^{\infty} x_{n+i} = 1$.

Definition 1.4. *[10, 11] The 3-tuple* $(X, \mu, \mathbb{T})$ *is a fuzzy metric space if* X *is an arbitrary set,* $\mathbb{T}$ *is a t-norm and* $\mu : X^2 \times (0, \infty) \to [0, 1]$ *is a fuzzy set on* $X^2 \times (0, \infty)$ *satisfying the following conditions for all* $x, y, \kappa \in X$ *and* $r, t > 0$*:*
(f1) $\mu(x, y, t) > 0$;
(f2) $\mu(x, y, t) = 1 \Leftrightarrow x = y$;
(f3) $\mu(x, y, t) = \mu(y, x, t)$;
(f4) $\mathbb{T}(\mu(x, y, t), \mu(y, \kappa, r)) \leq \mu(x, \kappa, t + r)$;
(f5) $\mu(x, y, .) : (0, \infty) \to [0, 1]$ *is continuous.*

A fuzzy b-metric space (denoted by a FbMS) is introduced as follows.

Definition 1.5. *[24] Let* $b \geq 1$ *be a given real number. The 3-tuple* $(X, \mu, \mathbb{T})$ *is a fuzzy b-metric space if* X *is an arbitrary set,* $\mathbb{T}$ *is a t-norm and* μ *is a fuzzy set on* $X^2 \times (0, \infty)$ *satisfying the following conditions for all* $x, y, \kappa \in X$ *and* $r, t > 0$*:*

$(bf1)$ $\mu(x,y,t) > 0$;
$(bf2)$ $\mu(x,y,t) = 1 \Leftrightarrow x = y$;
$(bf3)$ $\mu(x,y,t) = \mu(y,x,t)$;
$(bf4)$ $\mathbb{T}(\mu(x,y,\frac{t}{b}),\mu(y,\kappa,\frac{r}{b})) \leq \mu(x,\kappa,t+r)$;
$(bf5)$ $\mu(x,y,.) : (0,\infty) \to [0,1]$ *is continuous.*

Clearly, the class of FbMSs is larger than the class of fuzzy metric spaces. A FbMS turns to be a fuzzy metric space when $b = 1$.

Example 1.4. *[25] Let $X = \mathbb{R}$. For $x, y \in X$ and $t > 0$, choose $\mu(x,y,t) = e^{\frac{-|x-y|^p}{t}}$, with $p > 1$. Take $\mathbb{T}_P(a,b) = a \cdot b$ for $a, b \in [0,1]$. Then μ is a FbM on X with $b = 2^{p-1}$. While, for $p = 2$, $(X,\mu,\mathbb{T})$ is not really a fuzzy metric space.*

Example 1.5. *[25] Let $\mu(x,y,t) = \frac{t}{t+d(x,y)}$, where d is a b-metric on X and $\mathbb{T}(a,c) = \min(a,c)$ for all $a, c \in [0,1]$. Then, $(X,\mu,\mathbb{T})$ is a FbMS.*

Definition 1.6. *[25] A function $f : \mathbb{R} \to \mathbb{R}$ is called b-nondecreasing, if $x > by$ implies $f(x) \geq f(y)$, for all $x, y \in \mathbb{R}$.*

Lemma 1.1. *[24] Let $(X,\mu,\mathbb{T})$ be a FbMS. Then, $\mu(x,y,t)$ is b-nondecreasing with respect to $t > 0$ for all $x, y \in X$.*

Definition 1.7. *[24] Let $(X,\mu,\mathbb{T})$ be a FbMS and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. Then,*

(a1) $\{x_n\}_{n\in\mathbb{N}}$ converges to $x \in X$ if and only if $\mu(x_n,x,t) \to 1$ as $n \to \infty$ (for each $t > 0$). In this context, we can write $\lim_{n\to\infty} x_n = x$;
(a2) $\{x_n\}_{n\in\mathbb{N}}$ is called as a Cauchy sequence if for all $\varepsilon \in (0,1)$ and $t > 0$, there is $n_0 \in \mathbb{N}$ such that $\mu(x_n,x_m,t) > 1-\varepsilon$ for all $n, m \geq n_0$;
(a3) a FbMS is said to be complete if each Cauchy sequence is convergent.

Proposition 1.2. *[26] Let $(X,\mu,\mathbb{T})$ be a FbMS and $\{x_n\}_{n\in\mathbb{N}}$ be a convergent sequence to x in X. Then,*

$$\mu(x,y,\frac{t}{b}) \leq \limsup_{n\to\infty} \mu(x_n,y,t) \leq \mu(x,y,bt),$$

and

$$\mu(x,y,\frac{t}{b}) \leq \liminf_{n\to\infty} \mu(x_n,y,t) \leq \mu(x,y,bt).$$

In 2015, Khojasteh et al. [27] initiated the notion of the standard simulation functions in the following way.

Definition 1.8. *[27] The function $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is a called as a simulation function if the following conditions hold:*

(b1) $\zeta(0,0) = 0$;
(b2) $\zeta(t,s) < s - t$ for all if $t, s > 0$;
(b3) if $\{t_n\}_{n\in\mathbb{N}}$ and $\{s_n\}_{n\in\mathbb{N}}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then,

$$\limsup_{n\to\infty} \zeta(t_n,s_n) < 0.$$

Using Definition 1.8, Khojasteh et al. [27] provided a novel approach to prove the existence of fixed points by exploring the concept of simulation functions, which exhibit a significant unifying power. Accordingly, several researchers extended and enriched this notion in various distinct metric spaces (see [28–30]). Later in 2019, Perveen et al. [31] introduced a specific control function, named as a generalized simulation function. It is given as follows.

Definition 1.9. *[31] The function $\zeta : (0,1]\times(0,1] \to \mathbb{R}$ is said to be a generalized simulation function if the following conditions hold:*

(c1) $\zeta(t,s) < \frac{1}{t} - \frac{1}{s}$ *for all* $t, s \in (0,1)$*;*
(c2) If $\{t_n\}_{n\in\mathbb{N}}$ *and* $\{s_n\}_{n\in\mathbb{N}}$ *are sequences in* $(0,1]$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = y \in (0,1)$ *and* $t_n < s_n$*, then,*

$$\lim_{n\to\infty} \sup \zeta(t_n, s_n) < 0.$$

Example 1.6. *[31] The following functions $\zeta : (0,1]\times(0,1] \to \mathbb{R}$ are generalized simulation functions (for all* $t, s \in (0,1]$*):*

(d1) $\zeta(t,s) = \phi(\frac{1}{t} - 1) - (\frac{1}{s} - 1)$, *where* $\phi : [0,\infty) \to (0,\infty)$ *is a right continuous function such that* $\phi(u) < u$ *for all* $u > 0$*;*
(d1) $\zeta(t,s) = (\frac{1}{t} - 1) - \phi(\frac{1}{t} - 1) - (\frac{1}{s} - 1)$, *where* $\phi : [0,\infty) \to [0,\infty)$ *is a function such that* $\phi(u) > 0$ *for every* $u > 0$ *and* $\phi(0) = 0$.

Denote by $\boldsymbol{Z}$ the class of generalized simulation functions. Alizadeh et al. [32] introduced the notion of cyclic (α,β)-admissible mappings in order to generalize, unify and modify some recent theorems in the literature (for instance, see [33]).

Definition 1.10. *[32] Let f be a self-mapping on a nonempty set X and $\alpha,\beta : X \to [0,\infty)$ be two functions. f is termed as a cyclic (α,β)-admissible mapping if*

$$x \in X,\ \alpha(x) \geq 1 \Rightarrow \beta(fx) \geq 1$$

and

$$x \in X,\ \beta(x) \geq 1 \Rightarrow \alpha(fx) \geq 1.$$

The above notion of cyclic admissibility is developed in this work via the fuzzy setting. By involving a generalized simulation function, we aim to give some fixed point results. Note that fuzzy differential equations and fuzzy integral equations play significant roles in modeling dynamic systems in which uncertainties or vague notions flourish. These concepts have been established in different theoretical directions, and a large number of applications in practical problems have been studied (see, for example, [34–36]). Going in same direction and using a fixed point technique, we ensure the existence of (fuzzy) solution of a Fredholm integral equation.

2. Main results

An extension of Definition 1.10 within the idea of the fuzzy setting is given as follows.

Definition 2.1. *Let X be a nonempty set and f be a self-mapping on X. Let $\alpha,\beta : X\times X\times(0,\infty) \to [0,\infty)$ be a pair of functions. Such f is a fuzzy cyclic (α,β)-admissible mapping if for all $t > 0$,*

$$x, y \in X,\ \alpha(x, y, t) \geq 1 \Rightarrow \beta(fx, fy, t) \geq 1$$

and

$$x, y \in X,\ \beta(x, y, t) \geq 1 \Rightarrow \alpha(fx, fy, t) \geq 1.$$

An illustrated example to Definition 2.1 is as follows.

Example 2.1. Take $X = \mathbb{R}$. Let $f : X \to X$ and $\alpha,\beta : X \times X \times (0,\infty) \to [0,\infty)$ be defined by

$$fx = \begin{cases} -x^3, & x \in [0,\infty), \\ -\frac{x}{2}, & \text{otherwise,} \end{cases}$$

$$\alpha(x, y, t) = \begin{cases} e^{x+y}, & x, y \in [0, 1], \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\beta(x, y, t) = \begin{cases} e^{-(x+y)}, & x, y \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Here, f is an (α,β)-cyclic fuzzy admissible mapping. Indeed, if $\alpha(x, y, t) \geq 1$, then $e^{x+y} > 1$, and so x and y are in $[0, 1]$. That is, fx and fy are nonpositive values. Hence, $\beta(fx, fy, t) = e^{-(fx+fy)} \geq 1$. While, if $\beta(x, y, t) \geq 1$, then $x = y = 0$, and so $\alpha(fx, fy, t) = 1$.

From now on, for a FbMS $(X, \mu, \mathbb{T})$, we make an additional condition in the context of Definition 1.5, that is,

$$\lim_{t\to\infty} \mu(x, y, t) = 1 \tag{2.1}$$

for all $x, y \in X$.

Lemma 2.1. *Let X be a nonempty set and $f : X \to X$ be a fuzzy cyclic (α,β)-admissible mapping. Assume that there is $x_0 \in X$ such that for each $t > 0$,*

$$\alpha(x_0, fx_0, t) \geq 1 \quad \text{and} \quad \beta(x_0, fx_0, t) \geq 1.$$

Take $x_n = f^n x_0 = fx_{n-1}$ for any integer $n \geq 1$. Then,

$$\alpha(x_n, x_{n+1}, t) \geq 1 \quad \text{and} \quad \beta(x_n, x_{n+1}, t) \geq 1,$$

for all $n \geq 0$ and $t > 0$.

Proof. Since $f : X \to X$ is a fuzzy cyclic (α,β)-admissible mapping and $\alpha(x_0, fx_0, t) = \alpha(x_0, x_1, t) \geq 1$, we have

$$\beta(x_1, x_2, t) = \beta(fx_0, fx_1, t) \geq 1,$$

which implies

$$\alpha(x_2, x_3, t) = \alpha(fx_1, fx_2, t) \geq 1.$$

By maintaining the process described above, we get

$$\alpha(x_{2n}, x_{2n+1}, t) \geq 1 \quad \text{and} \quad \beta(x_{2n-1}, x_{2n}, t) \geq 1 \tag{2.2}$$

for each $t > 0$ and $n \geq 1$. Again, $f : X \to X$ is a fuzzy cyclic (α,β)-admissible mapping and $\beta(x_0, fx_0, t) = \beta(x_0, x_1, t) \geq 1$, we have

$$\alpha(x_1, x_2, t) = \alpha(fx_0, fx_1, t) \geq 1$$

which implies

$$\beta(x_2, x_3, t) = \beta(fx_1, fx_2, t) \geq 1.$$

Again, proceeding similarly, one writes

$$\alpha(x_{2n-1}, x_{2n}, t) \geq 1 \quad \text{and} \quad \beta(x_{2n}, x_{2n+1}, t) \geq 1 \tag{2.3}$$

for each $t > 0$ and $n \geq 1$. Consequently, in view of (2.2) and (2.3), we obtain for each integer n,

$$\alpha(x_n, x_{n+1}, t) \geq 1 \quad \text{and} \quad \beta(x_n, x_{n+1}, t) \geq 1.$$

□

Lemma 2.2. *Let* $(X, \mu, \mathbb{T})$ *be a FbMS. If for all* $x, y \in X$ *and* $t > 0$*,*

$$\mu(x, y, t) \geq \mu(x, y, \frac{t}{\lambda}), \tag{2.4}$$

where $\lambda \in (0, 1)$*, then* $x = y$*.*

Proof. Condition (2.4) implies that

$$\mu(x, y, t) \geq \mu(x, y, \frac{t}{\lambda^n}), \ n \in \mathbb{N}, \ t > 0.$$

Recall that the FbMS is assumed to verify the hypothesis (2.1), so, for each $t > 0$,

$$\lim_{n\to\infty} \mu(x, y, \frac{t}{\lambda^n}) = 1.$$

Since $\lambda \in (0, 1)$, we deduce at the limit as $n \to \infty$,

$$1 \geq \mu(x, y, t) \geq \lim_{n\to\infty} \mu(x, y, \frac{t}{\lambda^n}) = 1 \quad \text{for each } t > 0.$$

That is, $\mu(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$, and due to condition ($f2$), one gets $x = y$. □

Definition 2.2. *Let* $(X, \mu, \mathbb{T})$ *be a FbMS (*$b \geq 1$ *is a given real number). Given two functions* $\alpha, \beta : X \times X \times (0, \infty) \to [0, \infty)$*. A self-mapping* f *on* X *is called a* **Z**-*contraction mapping if there are a generalized simulation function* $\zeta \in \mathbf{Z}$ *and* $\lambda \in (0, \frac{1}{b})$ *such that for all* $t > 0$*,*

$$x, y \in X, \alpha(x, y, t)\beta(x, y, t) \geq 1 \Rightarrow \zeta(\mu(x, y, \frac{t}{\lambda}), \mu(fx, fy, t)) \geq 0. \tag{2.5}$$

Now, we are ready to prove our first theorem.

Theorem 2.1. *Let $(X,\mu,\mathbb{T})$ be a FbMS such that $\mathbb{T}$ is a continuous t-norm of H-type and $b \geq 1$ is a given real. Given two functions $\alpha,\beta : X \times X \times (0,\infty) \to [0,\infty)$. Assume that f is a* **Z***-contraction mapping such that:*

(1) f is a fuzzy cyclic (α,β)-admissible mapping;
(2) There are $x_0 \in X$ and $\nu \in (0,1)$ such that $\alpha(x_0, fx_0, t) \geq 1$, $\beta(x_0, fx_0, t) \geq 1$, and

$$\lim_{n\to\infty} \mathbb{T}_{k=n}^{\infty}\mu(x_0, fx_0, \frac{t}{\nu^k}) = 1,\ t > 0. \tag{2.6}$$

Then, f has a fixed point. Moreover, if $\alpha(x, y, t) \geq 1$ and $\beta(x, y, t) \geq 1$ for all $x, y \in Fix(f)$ ($Fix(f)$ is the ste of fixed points of f), then, f has a unique fixed point.

Proof. We supposed that the element $x_0 \in X$ verifies $\alpha(x_0, fx_0, t) \geq 1$ and $\beta(x_0, fx_0, t) \geq 1$. Choose $x_n = f^n x_0$ for each $n \geq 0$. Due to loss of generality, assume that $x_n \neq x_{n+1}$ for each $n \neq 0$. Now, using Lemma 2.1, one writes

$$\alpha(x_n, x_{n+1}, t) \geq 1 \quad \text{and} \quad \beta(x_n, x_{n+1}, t) \geq 1. \tag{2.7}$$

We also have

$$\alpha(x_{n-1}, x_n, t)\beta(x_{n-1}, x_n, t) \geq 1,$$

for each $n \geq 1$. Then, using implication (2.5), we have for each $t > 0$,

$$\zeta(\mu(x_{n-1}, x_n, \frac{t}{\lambda}), \mu(fx_{n-1}, fx_n, t)) \geq 0.$$

That is,

$$\zeta(\mu(x_{n-1}, x_n, \frac{t}{\lambda}), \mu(x_n, x_{n+1}, t)) \geq 0.$$

In view of $(b2)$, we have

$$0 \leq \zeta(\mu(x_{n-1}, x_n, \frac{t}{\lambda}), \mu(x_n, x_{n+1}, t)) < \frac{1}{\mu(x_{n-1}, x_n, \frac{t}{\lambda})} - \frac{1}{\mu(x_n, x_{n+1}, t)},$$

for all $n \geq 1$ and $t > 0$. Thus,

$$\frac{1}{\mu(x_n, x_{n+1}, t)} < \frac{1}{\mu(x_{n-1}, x_n, \frac{t}{\lambda})}.$$

Consequently, for all $n \geq 1$ and $t > 0$,

$$\mu(x_n, x_{n+1}, t) > \mu(x_{n-1}, x_n, \frac{t}{\lambda}).$$

Now, we demonstrate that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Choose $\sigma \in (\lambda b, 1)$, then, there exists $n_0 \in \mathbb{N}$ such that $\sum_{i=n}^{\infty} \sigma^i < 1$, for every $n > n_0$. Let $n > m > n_0$. Since μ is b- nondecreasing with respect to the third variable $t > 0$, by $(bf4)$, for every $T > 0$, one writes

$$
\begin{aligned}
\mu(x_n, x_{n+m}, t) &\geq \mu(x_n, x_{n+m}, \frac{t\sum_{i=n}^{n+m-1}\sigma^i}{b}) \\
&\geq \mathbb{T}(\mu(x_n, x_{n+1}, \frac{t\sigma^n}{b^2}), \mu(x_{n+1}, x_{n+m}, \frac{t\sum_{i=n+1}^{n+m-1}\sigma^i}{b^2})) \\
&\geq \mathbb{T}(\mu(x_n, x_{n+1}, \frac{t\sigma^n}{b^2}), \mathbb{T}(\mu(x_{n+1}, x_{n+2}, \frac{t\sigma^{n+1}}{b^3}), \ldots, \mu(x_{n+m-1}, x_{n+m}, \frac{t\sigma^{n+m-1}}{b^m}) \ldots)) \\
&\geq \mathbb{T}(\mu(x_0, x_1, \frac{t\sigma^n}{b^2\lambda^n}), \mathbb{T}(\mu(x_0, x_1, \frac{t\sigma^{n+1}}{b^3\lambda^{n+1}}), \ldots, \mu(x_0, x_1, \frac{t\sigma^{n+m-1}}{b^{m+1}\lambda^{n+m-1}}) \ldots)) \\
&\geq \mathbb{T}_{i=n}^{n+m-1}\mu(x_0, x_1, \frac{t\sigma^i}{b^{i-n+2}\lambda^i}) \\
&\geq \mathbb{T}_{i=n}^{n+m-1}\mu(x_0, x_1, \frac{t\sigma^i}{b^i\lambda^i}) \\
&\geq \mathbb{T}_{i=n}^{\infty}\mu(x_0, x_1, \frac{t}{\nu^i}),
\end{aligned}
$$

where $\nu = \frac{b\lambda}{\sigma}$. Since $\nu \in (0, 1)$, by (2.6), we obtain that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy. Since $(X, \mu, \mathbb{T})$ is complete, there is $x \in X$ such that $\lim_{n\to\infty} x_n = x$, that is for all $t > 0$,

$$
\lim_{n\to\infty} \mu(x, x_n, t) = 1. \tag{2.8}
$$

We claim that x is a fixed point of f. We have

$$
\begin{aligned}
\mu(fx, x, t) &\geq \mathbb{T}(\mu(fx, x_n, \frac{t}{2b}), \mu(x_n, x, \frac{t}{2b})) \\
&\geq \mathbb{T}(\mu(x, x_{n-1}, \frac{t}{2b\lambda}), \mu(x_n, x, \frac{t}{2b})),
\end{aligned}
$$

for all $t > 0$. By using (2.8) and the continuity of the t-norm $\mathbb{T}$, we get as $n \to \infty$,

$$
\mu(fx, x, t) \geq \mathbb{T}(1, 1) = 1.
$$

Hence, $\mu(fx, x, t) = 1$ for every $t > 0$. Due to $(bf2)$, we find that $fx = x$, that is, x is a fixed point of f. Assume that f has another fixed point, say $y \in X$. If $\alpha(x, y, t) \geq 1$ and $\beta(x, y, t) \geq 1$, in view of (2.5), one writes

$$
\zeta(\mu(x, y, \frac{t}{\lambda}), \mu(x, y, t)) = \zeta(\mu(x, y, \frac{t}{\lambda}), \mu(fx, fy, t)) \geq 0.
$$

Thus,

$$
0 < \frac{1}{\mu(x, y, \frac{t}{\lambda})} - \frac{1}{\mu(x, y, t)}.
$$

That is, for each $t > 0$,

$$
\mu(x, y, t) > \mu(x, y, \frac{t}{\lambda}).
$$

Since $\lambda \in (0, \frac{1}{b}) \subset (0, 1)$, using Lemma 2.2, one gets $x = y$. That is, the fixed point of f is unique. □

Now, we give the following definition, which is a particular case of Definition 2.1.

Definition 2.3. *Let X be a nonempty set and f be a self-mapping on X. Let $\alpha : X\times X\times(0,\infty) \to [0,\infty)$ be a function. f is called as a fuzzy cyclic α-admissible mapping if for all $t > 0$,*

$$x, y \in X,\ \alpha(x, y, t) \geq 1 \Rightarrow \alpha(fx, fy, t) \geq 1.$$

Corollary 2.1. *Let $(X, \mu, \mathbb{T})$ be a FbMS such that $\mathbb{T}$ is a continuous t-norm of H-type and $b \geq 1$ is a given real number. Given a function $\alpha : X \times X \times (0,\infty) \to [0,\infty)$. Assume that:*

(1) There are $\zeta \in \mathbf{Z}$ and $\lambda \in (0, \frac{1}{b})$ such that if $x, y \in X$ with $\alpha(x, y, t) \geq 1$, then,

$$\zeta(\mu(x, y, \frac{t}{\lambda}), \mu(fx, fy, t)) \geq 0; \tag{2.9}$$

(2) f is a fuzzy cyclic α-admissible mapping;
(3) There are $x_0 \in X$ and $\nu \in (0, 1)$ such that $\alpha(x_0, fx_0, t) \geq 1$, and

$$\lim_{n\to\infty} \mathbb{T}_{k=n}^{\infty}\mu(x_0, fx_0, \frac{t}{\nu^k}) = 1,\ \ t > 0. \tag{2.10}$$

Then, f has a fixed point. Moreover, if $\alpha(x, y, t) \geq 1$ for all $x, y \in Fix(f)$, then, f has a unique fixed point.

Proof. Take $\beta = \alpha$ in Theorem 2.1. □

Corollary 2.2. *Let $(X, \mu, \mathbb{T})$ be a FbMS such that $\mathbb{T}$ is a continuous t-norm of H-type and $b \geq 1$ is a given real number. Given a function $\alpha : X \times X \times (0,\infty) \to [0,\infty)$. Assume that:*

(1) There are $k \in [0, 1)$ and $\lambda \in (0, \frac{1}{b})$ such that if $x, y \in X$ with $\alpha(x, y, t) \geq 1$, then,

$$(\frac{1}{\mu(fx, fy, t)} - 1) \leq k(\frac{1}{\mu(x, y, \frac{t}{\lambda})} - 1); \tag{2.11}$$

(2) f is a fuzzy cyclic α-admissible mapping;
(3) There are $x_0 \in X$ and $\nu \in (0, 1)$ such that $\alpha(x_0, fx_0, t) \geq 1$, and

$$\lim_{n\to\infty} \mathbb{T}_{k=n}^{\infty}\mu(x_0, fx_0, \frac{t}{\nu^k}) = 1,\ \ t > 0. \tag{2.12}$$

Then, f has a fixed point. Moreover, if $\alpha(x, y, t) \geq 1$ for all $x, y \in Fix(f)$, then, f has a unique fixed point.

Proof. It suffices to take $\zeta(r, s) = k(\frac{1}{r} - 1) - (\frac{1}{s} - 1)$ in Corollary 2.1. □

Corollary 2.3. *Let $(X, \mu, \mathbb{T})$ be a FbMS such that $\mathbb{T}$ is a continuous t-norm of H-type and $b \geq 1$ is a given real number. Given a function $\alpha : X \times X \times (0,\infty) \to [0,\infty)$. Assume that:*

(1) There are $\lambda \in (0, \frac{1}{b})$ and $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) > 0$ for all $t > 0$ and $\varphi(0) = 0$ in order that if $x, y \in X$ with $\alpha(x, y, t) \geq 1$, then,

$$(\frac{1}{\mu(fx, fy, t))} - 1) \leq \varphi(\frac{1}{\mu(x, y, \frac{t}{\lambda})} - 1); \tag{2.13}$$

(2) f is a fuzzy cyclic α-admissible mapping;
(3) There are $x_0 \in X$ and $\nu \in (0,1)$ such that $\alpha(x_0, fx_0, t) \geq 1$, and

$$\lim_{n\to\infty} \mathbb{T}_{k=n}^{\infty} \mu(x_0, fx_0, \frac{t}{\nu^k}) = 1, \ t > 0. \tag{2.14}$$

Then, f has a fixed point. Moreover, if $\alpha(x,y,t) \geq 1$ for all $x, y \in Fix(f)$, then, f has a unique fixed point.

Proof. It suffices to take $\zeta(r,s) = \phi(\frac{1}{r} - 1) - (\frac{1}{s} - 1)$ in Corollary 2.1. □

Example 2.2. *Going back to Example 1.5, let $X = \mathbb{R}$ and $d(x,y) = (x-y)^2$. Here, (X,d) is a b-metric space ($b = 2$). Take $\mu(x,y,t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$, $t > 0$ and $\mathbb{T}(a,c) = \min(a,c)$ for all $a, c \in [0,1]$. Clearly, the FbMS $(X, \mu, \mathbb{T})$ is complete. Here, the condition (2.1) is verified. Choose $fx = kx$, where $k \in (0, \frac{1}{2})$. The inequality (2.11) is equivalent to*

$$(\frac{t + d(fx, fy)}{t} - 1) \leq k(\frac{\frac{t}{\lambda} + d(x,y)}{\frac{t}{\lambda}} - 1).$$

That is,

$$d(fx, fy) \leq k\lambda d(x,y).$$

It means that

$$k^2(x-y)^2 \leq k\lambda(x-y)^2,$$

which holds by choosing $k \leq \lambda < \frac{1}{2}$. All conditions of Corollary 2.2 hold (with $\alpha(x,y,t) = 1$). Here, f has a unique fixed point.

3. An application

Soliton theory is an important branch of applied mathematics and mathematical physics. An active and productive field of research, it has important applications in fluid mechanics, nonlinear optics, classical and quantum fields theories, etc. It has also many applications in pure and applied mathematics, particularly in areas such as differential equations, Lie groups, Lie algebras, differential and algebraic geometry. The existence of soliton solutions to a nonlinear PDE can also lead to certain exact solutions. Among the used methods to ensure the existence of such solutions, we may consider fixed point techniques. For further techniques, we refer the readers to the works [37–42]. Integral equations have applications in a variety of scientific fields, such as biology, chemistry, physics, or engineering. Furthermore, fuzzy integral equations constitute one of the important branches of fuzzy analysis theory and play a major role in numerical analysis. One of the approaches followed for the study of integral equations is the application of fixed point theory directly to the mapping defined by the right-hand side of the equation, or by the development of homotopy methods, which are largely considered in fixed point theory (see [43,44]). In this section, we use a fixed point theorem to ensure the existence of a unique solution of a Fredholm integral equation. For this, consider the integral equation

$$x(t) = \int_a^b F(t,s)h(x(s))ds + g(t), \tag{3.1}$$

for all $t \in I = [a,b]$, where $F \in C(I \times I, \mathbb{R})$ and $g, h \in C(I, \mathbb{R})$. Our next result ensures the existence of a unique solution to Eq (3.1).

Theorem 3.1. *Assume the following hold:*
$(p1)$ *There exists* $k \in (0, \frac{1}{\sqrt{2}})$ *such that for all* $x, y \in C(I, \mathbb{R})$*,*

$$| h(x) - h(y) | \leq \sqrt{k} \, | x - y |;$$

$(p2)$ $\sup\limits_{t \in I} \int_a^b F^2(t, s)\, ds = \lambda < \frac{1}{\sqrt{2}}$.
Then, Eq (3.1) *has a unique solution.*

Proof. Let $X = C(I, \mathbb{R})$. For all $x, y \in X$, take

$$d(x, y) = \sup_{t \in I} |x(t) - y(t)|^2.$$

Choose $\mathbb{T}(a, c) = ac$ for all $a, c \in [0, 1]$ and $\mu(x, y, r) = \frac{r}{r + d(x,y)}$ for $x, y \in X$ and $r > 0$. Obviously, $(X, \mu, \mathbb{T})$ is a complete FbMS ($b = 2$). Define $f : X \to X$ by

$$fx(t) = \int_a^b F(t, s) h(x(s)) ds + g(t).$$

For all $x, y \in X$, we have

$$\begin{aligned}
fx(t) - fy(t) &= \int_a^b F(t, s)(h(x(s)) - h(y(s))) ds \\
&\leq \sqrt{k} \int_a^b F(t, s) \mid x(s) - y(s) \mid ds \\
&\leq \sqrt{k} [\int_a^b F^2(t, s) ds]^{\frac{1}{2}} [\int_a^b (\mid x(s) - y(s) \mid)^2 ds]^{\frac{1}{2}} \\
&\leq \sqrt{k} \sqrt{\lambda} (d(x, y))^{\frac{1}{2}}.
\end{aligned}$$

We deduce for all $x, y \in X$,

$$d(fx, fy) \leq k\lambda d(x, y).$$

This inequality is equivalent to

$$(\frac{1}{\mu(fx, fy, t)} - 1) \leq k(\frac{1}{\mu(x, y, \frac{t}{\lambda})} - 1).$$

Hence, all hypotheses of Corollary 2.2 are fulfilled (with $\alpha(x, y, t) = 1$), and then the solution to Eq (3.1) is unique. □

4. Conclusions

With a generalized simulation function, we have demonstrated the existence and uniqueness of a fixed point for self-mappings in FbMSs via non-linear contractions involving fuzzy cyclic (α,β)-admissibility. We also demonstrate how our findings may be applied to resolve a (fuzzy) Fredholm integral equation. In addition, as perspectives, we suggest the following question: What about the extension of our obtained results on further fuzzy generalized metric spaces, like fuzzy partial b-metric spaces, fuzzy (double) controlled metric spaces, etc.?

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Conflict of interest

We declare that we have no conflicts of interest.

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