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Research article

Spherical q-linear Diophantine fuzzy aggregation information: Application in decision support systems

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Abstract: The main goal of this article is to reveal a new generalized version of the q-linear Diophantine fuzzy set (q-LDFS) named spherical q-linear Diophantine fuzzy set (Sq-LDFS). The existing concepts of intuitionistic fuzzy set (IFS), q-rung orthopair fuzzy set (q-OFS), linear Diophantine fuzzy set (LDFS), and spherical fuzzy set have a wide range of applications in decision-making problems, but they all have strict limitations in terms of membership degree, non-membership degree, and uncertainty degree. We moot the article of the spherical q-linear Diophantine fuzzy set (Sq-LDFS) with control factors to alleviate these limitations. A Spherical q-linear Diophantine fuzzy number structure is independent of the selection of the membership grades because of its control parameters in three membership grades. An Sq-LDFS with a parameter estimation process can be extremely useful for modeling uncertainty in decision-making (DM). By using control factors, Sq-LDFS may classify a physical system. We highlight some of the downsides of q-LDFSs. By using algebraic norms, we offer some novel operational laws for Sq-LDFSs. We also introduced the weighted average and weighted geometric aggregation operators and their fundamental laws and properties. Furthermore, we proposed the algorithms for a multicriteria decision-making approach with graphical representation. Moreover, a numerical illustration of using the proposed methodology for Sq-LDF data for emergency decision-making is presented. Finally, a comparative analysis is presented to examine the efficacy of our proposed approach.

Keywords: spherical q-linear Diophantine fuzzy set; operational laws; aggregation operators; decision making

1. Introduction

Along with intricacies and ambiguity inherent in such situations, classical mathematics is not always effective when solving real-world problems. Zadeh [52] established the fuzzy set notion by assigning grades to possibilities with limit $[0, 1]$. Since Zadeh's approach to the fuzzy collection, fuzzy logic has been used to characterize imprecision, ambiguity and obscureness in a variety of fields [29, 44, 46]. Uncertainty-related problems arise frequently in DM, however they are difficult to predict and control on account of extensive modelling and regulating situation of these uncertainties.

The additional terms of non-membership degrees (NMD) to membership degrees (MD) with the premise that by adding MD and NMD restricted by unity, atanassov [2] gave notion about the thought of intuitionistic fuzzy set (IFS) as an extend form of FS. IF elements were used in a geometric description by attanassov [1]. Xu [47] represent weighted geometric notations for IFNs. Garg [20] utilized Einstein's t-norm operating rules for IFS. Apart from that, the researchers, create a new useful technique for collecting the OWA weights. By giving these reasons a low weight, the technique can reduce the impact of unjust arguments on the outcome of the decision. Xue [48] give their views about Measure-based granular uncertainty decision-making with intuitionistic fuzzy sets by using Choquet's integral, measure and representative payoffs. It is capable of handling problems in intuitionistic fuzzy environments. Khalil [31] look into two new distances: the absolute normalized Euclidean distance and the square hamming distance, both of which are used in decision-making as intuitionistic fuzzy sets.

The Pythagorean fuzzy set (PyFS) was introduced by Yager [49], which is an expansion of the IFS conception that satisfies the criteria that the summation of square of its MD and NMD is not exceeding by one. Farhadinia [17] proposed Pythagorean fuzzy decision technique to make decisions using similarity. Yager [50] added multiple aggregate operators (AOs) to the PyFS framework and Garg [23] updated PyFSs to include more integrated operational rules and associated aggregation operatives. Some Pythagorean fuzzy Dombi aggregation operators were constructed and explained in [33]. Garg in [21, 22] utilized Einstein t-norm operational standards towards PyFNs. [36] created a couple of symmetric PyF AOs. Zeng [54] reported the information on probabilistic and ordered weighted averaging (OWA). Under the PyF framework, Garg [24] put forward various strategic DM ways to tackle MCDM problems with imminent probability. Deqing [35] suggest several distance measures that account for the four Pythagorean fuzzy set parameters for Pythagorean fuzzy sets and Pythagorean fuzzy numbers. Utilizing triangle conorms, Firozja [19] presents a novel similarity measure for Pythagorean fuzzy sets (S-norm). For more details, we refer to [13–15]

A wide range of applications in various real-world sectors are concept based on IFSs and PyFSs. FSs, IFSs along with PyFSs ideas have a broad range of implementations in various physical-world sectors, but allow them hold their individual set of constraints that are appropriate to MD and NMD. To control these restrictions, Riaz [42] put forward the innovative invention of Linear Diophantine Fuzzy Set (LDFS) by adding control factors (CFs). The LDFS concept is most effectual and multifaceted than other models due to the inclusion of CFs. This collection expanded the space for MD and NMD by filling the cracks left by current structures and by adding CPs. LDFSs also gave two grades about information, the sum of which is not exceed by one and total of its factors such as the total of product of control factors with MD and NMD can not be surpass 1. Many researcher contribute to the LDFSs like: Iampan [16] introduced linear Diophantine fuzzy Einstein aggregation operators for multi-criteria decision-making problems for the established post-acute care (PAC) model network for the health restoration of patients with cerebrovascular disorders (CVDs). Decision making with linear Diophantine fuzzy relations and their algebraic properties was first described by Ayub [12]. Mahmood [37] presented linear Diophantine uncertain linguistic setting-based generalized hamacher aggregation operators and their applications in decision-making problems. Riaz [41] expressed his thoughts on a novel method for choosing third-party reverse logistics providers using linear Diophantine fuzzy prioritized aggregation operators. Farid [18] proposed using Einstein prioritized for linear Diophantine fuzzy aggregation operators to pick suppliers of sustainable thermal power equipment.

As a generalization of FS, IFS and PyFS, the concepts of spherical fuzzy sets (SFS) was introduced by Ashraf et al. [11]. The human thoughts are not ristricted to MD and NMD like [27, 28, 45]. So, Ashraf [9] presented the concept of spherical fuzzy set and its aggregation operators. The sphercal fuzzy set is an extention of picture fuzzy set containing three grades namely MD, ND (neutral degree) and NMD with limitation $0 \leq \eth(\wp) + \delta(\wp) + F(\wp) \leq 1$ but ashraf et al. presented the generalization set picture fuzzy set as spherical fuzzy set with constraint $0 \leq \eth^2(\wp) + \delta^2(\wp) + F^2(\wp) \leq 1$. Also Ashraf et al. developed the spherical fuzzy Dombi aggregation operatives in [10]. The decision-making system based on cosine similarity was put forward by Rafiq et al. [40]. Barukab et al. [5] used Spherical fuzzy distance measurements to identify environmental influences on child development. In [6] Spherical fuzzy sets were utilized by Ashraf et al. to express the spherical fuzzy t-norms and t-conorms. Ashraf et al. [53] used application in the multi-attribute group decision making problem using symmetric sum based aggregation operators for spherical fuzzy information. In [25] Gündodu and Kahraman presented a novel technique for dealing with uncertainty in renewable energy utilising an analytic hierarchy process. The DM methodologies connected to child development effect environmental elements under SFSs were reported by Ashraf et al. [5]. Emergency decision model to deal with COVID19 under spherical fuzzy information in different ways is study in [8] and [7]. Jin et al. [30] used the logarithmic function to aggregate the uncertainty in decision-making issues and established the spherical fuzzy aggregation information. Gündoğdu et al. [26] developed the QFD technique for spherical fuzzy information and explored how it may be used to linear delta robot technology. So, in some physical-world situations the summation of the membership grades such as alternative meets to fulfil attribute given by DM is sometimes beat one, showing that SFS did not carry off his goal concerning to control factors, like $0 \leq \eth^2(\wp) + \delta^2(\wp) + F^2(\wp) \nleq 1$. For more detail decision making methods, we suggest fuzzy decision models [32–34, 51].

Almagrabi [3] introduced the q-linear Diophantine fuzzy set (q-LDFS), an unique generalisation of the Pythagorean fuzzy set, q-rung orthopair fuzzy set, and linear Diophantine fuzzy set, and also highlighted its significant aspects. Furthermore, aggregation operators contribute significantly to the efficient aggregation of uncertainty in decision-making situations. The q-linear Diophantine fuzzy set cover the MD ($\eth$) and NMD (F) with control factors (λ, μ) having restrictions $0 \leq \lambda^q \eth + \mu^q F \leq 1, \forall \wp \in U$ and $0 \leq \lambda^q + \mu^q \leq 1$. It gives us an open choice to select the MD and NMD values. The q-linear Diophantine fuzzy set Qiyas [39] used the q-rung linear Diophantine fuzzy to suggest some new distance and similarity measurements (q-ROLDF). The q-linear Diophantine fuzzy set only deals with MD and NMD but still there is a gap of ND (neutral degree). We need to add new fuzzy numbers in fuzzy set theory in order to close this gap.

This research is driven by the desire to offer a novel hybrid structure of spherical q-linear Diophantine fuzzy set (Sq-LDFS) that includes both SFS and q-LDFS in order to evaluate the best option based on the record of attributes. This collection gives you a bird's-eye view of a variety of SFS generalized shapes. In addition, the research discusses certain aggregation operations for integrating spherical q-linear Diophantine fuzzy information in uncertain emergency condition. These operators are unique in that they can synthesis spherical q-linear Diophantine fuzzy information, further developing and enriching the idea of spherical q-linear Diophantine fuzzy aggregation operators. Furthermore, because the suggested aggregation operators are useful DM tools, they assist the development of multiple-criteria decision-making in the spherical fuzzy setting. This paper's contributions are mostly evident in the following areas:

(1). The novel Sq-LDFS strategy with $q > 1$ is our initial target in filling this knowledge gap. We can solve the IF, PyF, q-OF, LDF and SFS structure (e.g. for $0.9 + 0.7 + 0.4 > 1$), also $\left(0.9^2 + 0.7^2 + 0.4^2 > 1\right)$ entrance of Sq-LDFS such that $(0.9)(0.5) + (0.7)(0.4) + (0.4)(0.3) < 1$, where the triplet $(0.6, 0.7, 0.4)$ can be utilized for MD, neutral grade (NG) and NMD respectively. Because this proposed model is similar to familiar Diophantine equation ($ax + by + cz = c$) of number theory and the insertion of the qth degree of control factors it appears that Sq-LDFS is the best name for the established framework.

(2). The second objective is to implement the qth degree of control factors (CFs) capabilities in Sq-LDFS because qth factors cannot be handled by IFSs, PyFSs, q-ROFSs, LDFSs and SFSs. The designed system improves on current approaches and DM has complete freedom in selecting grades. By changing physical sense of connection this model also characterizes the problem. When setting $q = 1$, the respective assemblage is converted to SLDS. Furthermore, if we qth value increases, the Diophantine space extends giving boundary bounds a larger search space to transmit a broader range of fuzzy data. As a result, we may use Sq- LDFSs to describe a broader range of fuzzy data. By taking it in another way, we can keep adjusting the value of the factor q to decide the information expressive range, making Sq-LDFSs more idea and adaptable for unpredictable environments.

(3). Our third objective is to provide a clear connection between the present research and MADM problems. We derived decision support techniques to deal with multi-attribute difficulties in a parametric manner. Surprisingly both algorithms produce the same outcome.

This research contributions are follows as:

(i). By merging the features of SFSs with q-LDFSs, we may offer some more sophisticated operational laws under spherical q-LDFSs based on algebraic t-norm and t-conorm.

(ii). Under spherical q-linear Diophantine fuzzy numbers, offer a collection of innovative aggregation operators using the defined algebraic t-norm and t-conorm. The significance of some fundamental features between the proposed aggregation operatives is also demonstrated.

(iii). To provide an unique MADM technique for solving decision making issues based on the proposed aggregation operatives.

(iv). A numerical demonstration as well as their complete evaluations, demonstrates the consistency and usefulness of the suggested method.

This work is organized as follows: Section 2 introduces the fundamental ideas of FS, IFS, PyFS, q-OFS, LDFS and SLDFSs. In Section 3, we explain the unique notion of Sq-LDFS and provide illustrations to demonstrate some Sq-LDFS procedures. The concept of Śq-LDFSs is introduced in Section 4 for the Śq-LDFWA, Śq-LDFOWA operatives also provides the concept of Śq-LDFSs for defining Śq-LDFWG, Śq-LDFOWG operatives, as well as distinct score for evaluating Śq-LDFNs of different orders. The notion of MADM mathematical modelling is presented in Section 5 with the help of the Śq-LDFWA and Śq-LDFWG aggregation operators. In Section 6, we compare the proposed method to current methods in detail and examine the aggregated findings as the influence of score functions on the final selection. Section 7 outlines the conclusion of this project.

2. Primal concepts with some premises

In this section, we recall some significant and fundamental concepts of FSs, PyFSs, SFSs, LDFSs, q-LDFSs and SLDFSs. We also introduce some fundamental properties of the mentioned notations used in the study and briefly discuss the ideas and results employed in the rest of the work.

Definition 1. *[52] A fuzzy set F under the action of universal set U, mathematically represented as*

$$F = \{\mho(\wp) | \wp \in U\},$$

where $\mho(\wp) \in [0,1]$ *is membership degree* (MD) *of F in U.*

Definition 2. *[2] An Intuitionistic fuzzy set (IFS)* $\bar{I}$ *under the action of universal set U, mathematically represented as*

$$\bar{I} = \{(\mho(\wp), F(\wp)) | \wp \in U\},$$

where $\mho(\wp) \in [0,1]$ *is membership degree and* $F \in [0,1]$ *is non-membership degree of* $\bar{I}$ *in U with necessary condition* $0 \leq \mho(\wp) + F(\wp) \leq 1$.

Definition 3. *[49] A Pythagorean fuzzy set P under the action of universal set U, mathematically represented as*

$$P = \{(\mho(\wp), F(\wp)) | \wp \in U\},$$

where $\mho(\wp) \in [0,1]$ *is membership degree and* $F \in [0,1]$ *is non-membership degree of P in U with necessary condition* $0 \leq (\mho(\wp))^2 + (F(\wp))^2 \leq 1$.

Definition 4. *[42] A LDFS L under the action of universal set U, mathematically represented as*

$$L = \{(\mho(\wp), F(\wp)), (\lambda, \mu) | \wp \in U\},$$

where $\mho(\wp), F(\wp) \in [0,1]$ *are the MD and NMD and* $\lambda, \mu \in [0,1]$ *are the control factors* (CFs) *with necessary condition* $0 \leq \lambda + \mu \leq 1$. *The degrees met with the criteria* $0 \leq \lambda\mho(\wp) + \mu F(\wp) \leq 1$.
For simplicity, $L = \{(\mho(\wp), F(\wp)), (\lambda, \mu)\}$ *is termed as Linear Diophantine fuzzy number* $(LDFN)$ *with* $0 \leq \lambda\mho(\wp) + \mu F(\wp) \leq 1$ *and* $0 \leq \lambda + \mu \leq 1$.

Definition 5. *[11] The spherical fuzzy set over the non-empty fixed set U reflect the mathematical form as under:*

$$S = \{\wp, (\mho(\wp), \delta(\wp), F(\wp)) | \wp \in U\},$$

where $\mho(\wp), F(\wp), \delta(\wp) \in [0,1]$, *are MD, neutral degree (ND) and* NMD *sequentially with the constraint* $0 \leq \mho^2(\wp) + \delta^2(\wp) + F^2(\wp) \leq 1$. *The hasitation part of the* (SFS) *in U can be taken in the form as below:* $\sqrt{1 - (\mho^2(\wp) + \delta^2(\wp) + F^2(\wp))}$. *A triplet* $(\mho(\wp), \delta(\wp), F(\wp))$ *is taken into the account of Spherical number* SFN.

Definition 6. *[43] A spherical linear Diophantine fuzzy set* $\left(\acute{S}LDFS\right)$ $\Im$ *in set* U, *mathematically represented as*

$$\Im = \{\wp, (\eth(\wp), \delta(\wp), F(\wp)), (\lambda, \mu, \omega) | \wp \in U\},$$

where $\eth(\wp), \delta(\wp), F(\wp) \in [0,1]$ *are MD, neutral degree (ND) and NMD, also* $\lambda, \mu, \omega \in [0,1]$ *are* CF. *The mentioned degrees surely met the constraint* $0 \leq \lambda\eth(\wp) + \mu\delta(\wp) + \omega F(\wp) \leq 1$, $\forall \wp \in U$ *with* $0 \leq \lambda + \mu + \omega \leq 1$.

These comparative parameters may aid in the description and identification of system. We can arrange the framework by changing the manner in given factors.

Definition 7. $\rho\chi(\wp) = 1 - (\lambda\eth(\wp) + \mu\delta(\wp) + \omega F(\wp))$, *where* $\rho\chi$ *serves as rejection portion of* $\left(\acute{S}LDFS\right)$, $((\eth(\wp), \delta(\wp), F(\wp)), (\lambda, \mu, \omega))$ *are stands for* $\left(\acute{S}LDFN\right)$ *with limitation* $0 \leq \lambda\eth(\wp) + \mu\delta(\wp) + \omega F(\wp) \leq 1$ *and* $0 \leq \lambda + \mu + \omega \leq 1$.

Definition 8. *[3] q-Linear Diophantine fuzzy set* $(q - LDFS)$ ϖq *over a fixed set U depicted in the mathematical type as given:*

$$\varpi q = \left\{\wp, (\eth_q(\wp), F_q(\wp)), (\lambda, \mu) | \wp \in U\right\},$$

$\eth_q(\wp), F_q(\wp), \lambda, \mu \in [0,1]$, *are MD*, NMD *and control factors* (CFs) *sequentially. These grades met with essential constraint* $0 \leq \lambda^q\eth_q(\wp) + \mu^q F_q(\wp) \leq 1, \forall \wp \in U$ *and* $0 \leq \lambda^q + \mu^q \leq 1$.

3. Spherical q-linear Diophantine fuzzy set $\left(\acute{S}q - LDFS\right)$

In this section, we initiate a novel notion of a spherical q-linear Diophantine fuzzy set (Sq-LDFS). In pure mathematics, there is a well-known linear Diophantine equation for three independent variables $ax+by+cz = d$, and the framework provided fits it. It's a little more challenging since the participation, abstention, and dissatisfaction categories in the picture fuzzy set, spherical fuzzy set, and SLDFS are confined in certain ways. To address these restrictions, we proposed the concept of Sq-LDFSs based on reference or control parameters. One significant component of this approach is that the decision-maker (DM) is not bound by grade membership (positive, neutral, or negative). This framework is frequently used to categories the problem by selecting several sorts of reference or control criteria. The Sq-LDFSs structure, its visual representation, and the use of diagrams to explain specific principles are all discussed.

Definition 9. *A spherical q-linear Diophantine fuzzy set* $\left(\acute{S}q - LDFS\right)$ *over the non-empty fixed set U reflect the mathematical form as under:*

$$\Xi = \{\wp, (\eth_s(\wp), \delta_s(\wp), F_s(\wp)), (\lambda, \mu, \omega) | \wp \in U\}, \tag{3.1}$$

where $\eth_s(\wp), \delta_s(\wp), F_s(\wp) \in [0,1]$ *are MD, ND and NMD, also* $\lambda, \mu, \omega \in [0,1]$ *are* CF. *The mentioned degrees surely met the constraint*

1). $0 \leq \lambda^q\eth_s(\wp) + \mu^q\delta_s(\wp) + \omega^q F_s(\wp) \leq 1$, $\forall \wp \in U$,

2). $0 \leq \lambda^q + \mu^q + \omega^q \leq 1$.

These comparative factors may aid in the description or identification of pattern. We can arrange the pattern by changing the way in given factors. $\varrho\sigma(\wp) = 1 - \lambda\eth_s(\wp) + \mu\delta_s(\wp) + \omega F_s(\wp)$, *where* $\varrho\sigma$ *serves as refusal part of* $\left(\acute{S}q - LDFS\right)$. $((\eth_s(\wp), \delta_s(\wp), F_s(\wp)), (\lambda, \mu, \omega))$ *are stands for* $\left(\acute{S}q - LDFN\right)$ *with*

$$0 \leq \lambda^q \eth_s(\wp) + \mu^q \delta_s(\wp) + \omega^q F_s(\wp) \leq 1$$

and

$$0 \leq \lambda^q + \mu^q + \omega^q \leq 1.$$

In this part we have present the Spherical q-linear Diophantine fuzzy set which is the extension of q-linear Diophantine fuzzy set by extended the reference parameters and classify as: the summation of qth power of reference factors (RFs) by scalar multiplication with (MD), (ND) and (NMD). Our futuristic model of spherical q-linear Diophantine fuzzy set $\left(\acute{S}q - LDFS\right)$ is most flexible and having more efficacy than (q-LDFS) due to extended control factors (CFs) in spherical form. Our model strong correlation with (MADM) issues.

Table 1. Comparison of Sq-LDFSs with existing approach.

$Set\ theories$	$\eth$	δ	F	$Denail\ part$	$Limits$
FSs	✓	×	×	×	×
$IFSs$	✓	×	✓	×	×
$q - LDFSS$	✓	×	✓	✓	✓
$SFSs$	✓	✓	✓	✓	×
$\acute{S}q - LDFSs$	✓	✓	✓	✓	✓

Now we present the definitions related to (Śq-LDFS) which are absolute spherical q-linear Diophantine fuzzy set and null spherical q-linear Diophantine fuzzy set.

Definition 10. *The absolute spherical q-linear Diophantine fuzzy set is structured as:*

$${}^1\upsilon_s = \{\alpha, (1, 0, 0), (1, 0, 0) : \alpha \in U\}.$$

Definition 11. *The null spherical q-linear Diophantine fuzzy set is the compliment of absolute spherical q-linear Diophantine fuzzy set is structured as:*

$${}^1\upsilon_s^c = \{\alpha, (0, 0, 1), (0, 0, 1) : \alpha \in U\},$$

we are familiar with this, if
(1). *we set* $q = 1$ *in Definition 9, Śq-LDFS becomes ŚLDFS,*
(2). *we set* $q = 2$ *in Definition 9, Śq-LDFS becomes Śpherical quadratic DFS.*
(3). *we set* $q = 3$ *in Definition 9, Śq-LDFS becomes Śpherical cubic DFS,*
(4). *we set* $q = 4$ *in Definition 9, Śq-LDFS becomes Śpherical bi-quadratic DFS.*

These are the major advantages of Sq-LDFS for distinct q values. It should be observed that as we increase the q values, the spherical Diophantine space stretches, giving the boundary parameters a larger search space to generate a larger spectrum of fuzzy data. Setting $q = 1$ gives Riaz's (ŚLDFS), while setting $q = 2$ makes Ashraf's (ŚFS) as displayed in Figure 1:

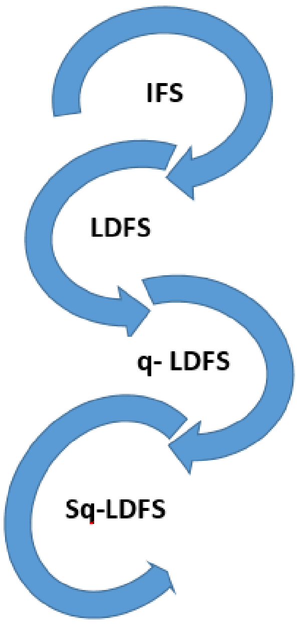

Figure 1. Flow chart about proposed Sq-LDFS.

Any IFS becomes LDFS, each LDFS becomes q-LDFS, also q-LDFS becomes Sq-LDFS by some additional terms as shown in Figure 2.

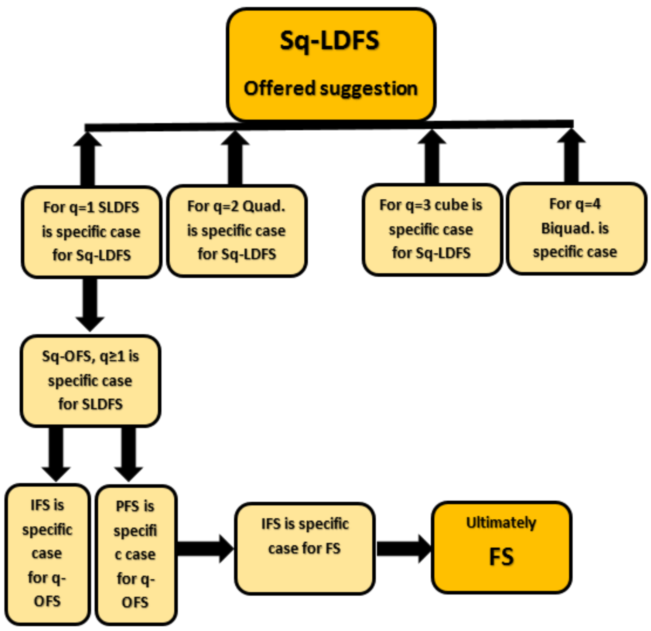

Figure 2. Flow chart expansion of Sq-LDFS.

Definition 12. *Let* $\Gamma_1 = \{(\eth_{s_1}, \delta_{s_1}, F_{s_1}), (\lambda_1, \mu_1, \omega_1)\}, \Gamma_2 = \{(\eth_{s_2}, \delta_{s_2}, F_{s_2}), (\lambda_2, \mu_2, \omega_2)\}$ *be two* $\left(\acute{S}q - LDFS\right)$ *over* ξ *and* $\Omega > 0$*, then*

1). $\Gamma_1^c = \{(F_{s_1}, \delta_{s_1}, \eth_{s_1}), (\omega_1, \mu_1, \lambda_1)\}$,
2). $\Gamma_1 = \Gamma_2 \Leftrightarrow \eth_{s_1} = \eth_{s_2}, \delta_{s_1} = \delta_{s_2}, F_{s_1} = F_{s_2}, \lambda_1 = \lambda_2, \mu_1 = \mu_2, \omega_1 = \omega_2$,
3). $\Gamma_1 \subseteq \Gamma_2 \Leftrightarrow \eth_{s_1} \leq \eth_{s_2}, \delta_{s_1} \geq \delta_{s_2}, F_{s_2} \geq F_{s_2}, \lambda_1 \leq \lambda_2, \mu_1 \geq \mu_2, \omega_1 \geq \omega_2$,
4). $\Gamma_2 \cup \Gamma_2 = \begin{pmatrix} ((\sup(\eth_{s_1}, \eth_{s_2}), \inf(\delta_{s_1}, \delta_{s_2}), \inf(F_{s_1}, F_{s_2})), \\ (\sup(\lambda_1, \lambda_2), \inf(\mu_1, \mu_2), \inf(\omega_1, \omega_2)) \end{pmatrix}$,
5). $\Gamma_1 \cap \Gamma_2 = \begin{pmatrix} ((\inf(\eth_{s_1}, \eth_{s_2}), \inf(\delta_{s_1}, \delta_{s_2}), \sup(F_{s_1}, F_{s_2})), \\ (\inf(\lambda_1, \lambda_2), \inf(\mu_1, \mu_2), \sup(\omega_1, \omega_2)) \end{pmatrix}$,
6). $\Gamma_1 \oplus \Gamma_2 = \begin{pmatrix} \left(\sqrt[q]{(\eth_{s_1})^q + (\eth_{s_2})^q - (\eth_{s_1})^q(\eth_{s_2})^q}, \delta_{s_1}\delta_{s_2}, F_{s_1}F_{s_2}\right), \\ \left(\sqrt[q]{(\lambda_1)^q + (\lambda_2)^q - (\lambda_1)^q(\lambda_2)^q}, \mu_1\mu_2, \omega_1\omega_2\right) \end{pmatrix}$,
7). $\Gamma_1 \otimes \Gamma_2 = \begin{pmatrix} \left(\eth_{s_1}\eth_{s_2}, \delta_{s_1}\delta_{s_2}, \sqrt[q]{(F_{s_1})^q + (F_{s_2})^q - (F_{s_1})^q(F_{s_2})^q}\right), \\ \left(\lambda_1\lambda_2, \mu_1\mu_2, \sqrt[q]{(\omega_1)^q + (\omega_2)^q - (\omega_1)^q(\omega_2)^q}\right) \end{pmatrix}; q \geq 1$,
8). $\Omega\Gamma_1 = \begin{pmatrix} \left(\sqrt[q]{1 - (1 - \eth_{s_1}^q)^\Omega}, (\delta_{s_1})^\Omega, (F_{s_1})^\Omega\right), \\ \left(\sqrt[q]{1 - (1 - \lambda_1^q)^\Omega}, (\mu_1)^\Omega, (\omega_1)^\Omega\right) \end{pmatrix}$,
9). $\Gamma_1^\Omega = \left(\left(\eth_s^\Omega, \delta_s^\Omega, \sqrt[q]{1 - (1 - F_s^q)^\Omega}\right), \left(\lambda_1^\Omega, \mu_1^\Omega, \sqrt[q]{1 - (1 - \omega_1^q)^\Omega}\right)\right)$.

The algebraic rules for spherical q-linear Diophantine fuzzy numbers are therefore limited to the algebraic rules for q-linear Diophantine numbers if $\delta_{s_1} = \delta_{s_2} = 0 = \mu_1 = \mu_2$.

Example 1. *Let* $\Gamma_1 = (0.48, 0.25, 0.34), (0.41, 0.24, 0.23)$ *and* $\Gamma_2 = (0.53, 0.24, 0.13), (0.48, 0.25, 0.26)$ *are two Śq-LDFNs, then*
1). $\Gamma_1^c = ((0.34, 0.25, 0.48), (0.23, 0.24, 0.41))$.
2). *Obvious by definition* $\Gamma_1 \subseteq \Gamma_2$.
3). $\Gamma_1 \cup \Gamma_2 = ((0.53, 0.24, 0.13), (0.48, 0.24, 0.23))$.
4). $\Gamma_1 \cap \Gamma_2 = ((0.48, 0.41, 0.34), (0.41, 0.25, 0.26))$.
5). $\Gamma_1 \oplus \Gamma_2 = ((0.6240, 0.0600, 0.0442), (0.5560, 0.0600, 0.0598))$.
6). $\Gamma_1 \otimes \Gamma_2 = ((0.2544, 0.0600, 0.3460), (0.1968, 0.0600, 0.3091))$.
7). $\Omega\Gamma_1 = ((0.6668, 0.0600, 0.0442), (0.5777, 0.0138, 0.0122))$.
8). $\Gamma_1^\Omega = ((0.1106, 0.0156, 0.4839), (0.0689, 0.014, 0.3304))$.

Theorem 1. *Let* Γ_1 *and* Γ_2 *are two Śq-LDFNs with* $\Omega, \Omega_1, \Omega_2 > 0$,*then these properties must hold:*
1). $\Gamma_1 \oplus \Gamma_2 = \Gamma_2 \oplus \Gamma_1$,
2). $\Gamma_1 \otimes \Gamma_2 = \Gamma_2 \otimes \Gamma_1$,
3). $\Omega(\Gamma_1 \oplus \Gamma_2) = \Omega\Gamma_1 \oplus \Omega\Gamma_2$,
4). $(\Gamma_1 \otimes \Gamma_2)^\Omega = \Gamma_1^\Omega \otimes \Gamma_2^\Omega$,
5). $\Omega_1\Gamma_1 \oplus \Omega_2\Gamma_1 = (\Omega_1 \oplus \Omega_2)\Gamma_1$,
6). $\Gamma_1^{\Omega_1} \otimes \Gamma_1^{\Omega_2} = \Gamma_1^{(\Omega_1 + \Omega_2)}$,
7). $(\Gamma_1^{\Omega_1})^{\Omega_2} = \Gamma_1^{\Omega_1\Omega_2}$.

Proof. We just provide conclusive proof for the 1–3, 5 and 7 equality. According to Definition 12,
1).$\Gamma_1 \oplus \Gamma_2 = \Gamma_2 \oplus \Gamma_1$

$$\Gamma_1 \oplus \Gamma_2 = \begin{pmatrix} \left(\sqrt[q]{(\eth_{s_1})^q + (\eth_{s_2})^q - (\eth_{s_1})^q(\eth_{s_2})^q}, \delta_{s_1}\delta_{s_2}, F_{s_1}F_{s_2}\right), \\ \left(\sqrt[q]{(\lambda_1)^q + (\lambda_2)^q - (\lambda_1)^q(\lambda_2)^q}, \mu_1\mu_2, \omega_1\omega_2\right) \end{pmatrix}; q \geq 1$$

$$= \begin{pmatrix} \left(\sqrt[q]{(\eth_{s_2})^q + (\eth_{s_1})^q - (\eth_{s_2})^q(\eth_{s_1})^q}, \delta_{s_2}\delta_{s_1}, F_{s_2}F_{s_1} \right), \\ \left(\sqrt[q]{(\lambda_2)^q + (\lambda_1)^q - (\lambda_2)^q(\lambda_1)^q}, \mu_2\mu_1, \omega_2\omega_1 \right) \end{pmatrix}; q \geq 1$$

$$= \Gamma_2 \oplus \Gamma_1.$$

Hence it proved.

For equality (2) , we have

2). $\Gamma_1 \otimes \Gamma_2 = \Gamma_2 \otimes \Gamma_1$

$$\Gamma_1 \otimes \Gamma_2 = \begin{pmatrix} \left(\eth_{s_1}\eth_{s_2}, \delta_{s_1}\delta_{s_2}, \sqrt[q]{(F_{s_1})^q + (F_{s_2})^q - (F_{s_1})^q(F_{s_2})^q} \right), \\ \left(\lambda_1\lambda_2, \mu_1\mu_2, \sqrt[q]{(\omega_1)^q + (\omega_2)^q - (\omega_1)^q(\omega_2)^q} \right) \end{pmatrix}; q \geq 1$$

$$= \begin{pmatrix} \left(\eth_{s_2}\eth_{s_1}, \delta_{s_2}\delta_{s_1}, \sqrt[q]{(F_{s_2})^q + (F_{s_1})^q - (F_{s_2})^q(F_{s_1})^q} \right), \\ \left(\lambda_2\lambda_1, \mu_2\mu_1, \sqrt[q]{(\omega_2)^q + (\omega_1)^q - (\omega_2)^q(\omega_1)^q} \right) \end{pmatrix}; q \geq 1$$

$$= \Gamma_2 \otimes \Gamma_1.$$

Also for equality (3) , we have

3). $\Omega(\Gamma_1 \oplus \Gamma_2) = \Omega\Gamma_1 \oplus \Omega\Gamma_2$.

By combining 6 and 8 point of Definition 12, we gain

$$\Omega(\Gamma_1 \oplus \Gamma_2) = \Omega\begin{pmatrix} \left(\sqrt[q]{(\eth_{s_1})^q + (\eth_{s_2})^q - (\eth_{s_1})^q(\eth_{s_2})^q}, \delta_{s_1}\delta_{s_2}, F_{s_1}F_{s_2} \right), \\ \left(\sqrt[q]{(\lambda_1)^q + (\lambda_2)^q - (\lambda_1)^q(\lambda_2)^q}, \mu_1\mu_2, \omega_1\omega_2 \right) \end{pmatrix}; q \geq 1$$

$$= \begin{pmatrix} \left(\sqrt[q]{1 - (1 - (\eth_{s_1})^q + (\eth_{s_2})^q - (\eth_{s_1})^q(\eth_{s_2})^q)^{\Omega}}, (\delta_{s_1}\delta_{s_2})^{\Omega}, (F_{s_1}F_{s_2})^{\Omega} \right), \\ \left(\sqrt[q]{1 - (1 - (\lambda_1)^q + (\lambda_2)^q - (\lambda_1)^q(\lambda_2)^q)^{\Omega}}, (\mu_1\mu_2)^{\Omega}, (\omega_1\omega_2)^{\Omega} \right) \end{pmatrix}; q \geq 1$$

$$= \begin{pmatrix} \left(\sqrt[q]{1 - (1 - (\eth_{s_1})^q)^{\Omega}(1 - (\eth_{s_2})^q)^{\Omega}}, (\delta_{s_1}\delta_{s_2})^{\Omega}, (F_{s_1}F_{s_2})^{\Omega} \right), \\ \left(\sqrt[q]{1 - (1 - (\lambda_1)^q)^{\Omega}(1 - (\eth\lambda_2)^q)^{\Omega}}, (\mu_1\mu_2)^{\Omega}, (\omega_1\omega_2)^{\Omega} \right) \end{pmatrix}; q \geq 1.$$

We can retrieve it by using the right side of the equation.

$$\Omega\Gamma_1 = \begin{pmatrix} \left(\sqrt[q]{1 - (1 - \eth_{s_1}^q)^{\Omega}}, (\delta_{s_1})^{\Omega}, (F_{s_1})^{\Omega} \right), \\ \left(\sqrt[q]{1 - (1 - \lambda_1^q)^{\Omega}}, (\mu_1)^{\Omega}, (\omega_1)^{\Omega} \right) \end{pmatrix},$$

$$\Omega\Gamma_2 = \begin{pmatrix} \left(\sqrt[q]{1 - (1 - \eth_{s_2}^q)^{\Omega}}, (\delta_{s_2})^{\Omega}, (F_{s_2})^{\Omega} \right), \\ \left(\sqrt[q]{1 - (1 - \lambda_2^q)^{\Omega}}, (\mu_2)^{\Omega}, (\omega_2)^{\Omega} \right) \end{pmatrix}.$$

Futhermore, apart from this

$$= \begin{pmatrix} \begin{pmatrix} \sqrt[q]{1 - (1 - \eth_{s_1}^q)^{\Omega} + 1 - (1 - \eth_{s_2}^q)^{\Omega} - (1 - (1 - \eth_{s_1}^q)^{\Omega})(1 - (1 - \eth_{s_2}^q)^{\Omega})}, \\ (\delta_{s_1})^{\Omega}(\delta_{s_2})^{\Omega}, (F_{s_1})^{\Omega}(F_{s_2})^{\Omega} \end{pmatrix}, \\ \begin{pmatrix} \sqrt[q]{1 - (1 - \lambda_1^q)^{\Omega} + 1 - (1 - \lambda_2^q)^{\Omega} - (1 - (1 - \lambda_1^q)^{\Omega})(1 - (1 - \lambda_2^q)^{\Omega})}, \\ (\mu_1)^{\Omega}(\mu_2)^{\Omega}, (\omega_1)^{\Omega}(\omega_2)^{\Omega} \end{pmatrix} \end{pmatrix}; q \geq 1$$

$$= \begin{pmatrix} \left(\sqrt[q]{(1-(1-\eth_{s_1}^{q})^{\Omega}(1-\eth_{s_2}^{q})^{\Omega})}, (\delta_{s_1})^{\Omega}(\delta_{s_2})^{\Omega}, (F_{s_1})^{\Omega}(F_{s_2})^{\Omega}\right), \\ \left(\sqrt[q]{1-(1-\lambda_1^{q})^{\Omega}(1-\lambda_2^{q})^{\Omega}}, (\mu_1)^{\Omega}(\mu_2)^{\Omega}, (\omega_1)^{\Omega}(\omega_2)^{\Omega}\right) \end{pmatrix}; q \geq 1$$

$$= \Omega\Gamma_1 \oplus \Omega\Gamma_2.$$

Thus shown.

In favour of (5) , the proof is given as

5).$\Omega_1\Gamma_1 \oplus \Omega_2\Gamma_1 = (\Omega_1 \oplus \Omega_2)\Gamma_1$

$$\Omega_1\Gamma_1 = \begin{pmatrix} \left(\sqrt[q]{1-(1-\eth_{s_1}^{q})^{\Omega_1}}, (\delta_{s_1})^{\Omega_1}, (F_{s_1})^{\Omega_1}\right), \\ \left(\sqrt[q]{1-(1-\lambda_1^{q})^{\Omega_1}}, (\mu_1)^{\Omega_1}, (\omega_1)^{\Omega_1}\right) \end{pmatrix},$$

$$\Omega_2\Gamma_1 = \begin{pmatrix} \left(\sqrt[q]{1-(1-\eth_{s_1}^{q})^{\Omega_2}}, (\delta_{s_1})^{\Omega_2}, (F_{s_1})^{\Omega_2}\right), \\ \left(\sqrt[q]{1-(1-\lambda_1^{q})^{\Omega_2}}, (\mu_1)^{\Omega_2}, (\omega_1)^{\Omega_2}\right) \end{pmatrix}$$

$$\Omega_1\Gamma_1 \oplus \Omega_2\Gamma_1 = \begin{pmatrix} \left(\sqrt[q]{1-(1-\eth_{s_1}^{q})^{\Omega_1}}, (\delta_{s_1})^{\Omega_1}, (F_{s_1})^{\Omega_1}\right), \\ \left(\sqrt[q]{1-(1-\lambda_1^{q})^{\Omega_1}}, (\mu_1)^{\Omega_1}, (\omega_1)^{\Omega_1}\right) \end{pmatrix}$$

$$\oplus \begin{pmatrix} \left(\sqrt[q]{1-(1-\eth_{s_1}^{q})^{\Omega_2}}, (\delta_{s_1})^{\Omega_2}, (F_{s_1})^{\Omega_2}\right), \\ \left(\sqrt[q]{1-(1-\lambda_1^{q})^{\Omega_2}}, (\mu_1)^{\Omega_2}, (\omega_1)^{\Omega_2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} \sqrt[q]{1-(1-\eth_{s_1}^{q})^{\Omega_1}+1-(1-\eth_{s_1}^{q})^{\Omega_2}-(1-(1-\eth_{s_1}^{q})^{\Omega_1})(1-(1-\eth_{s_1}^{q})^{\Omega_2})}, \\ (\delta_{s_1})^{\Omega_1}(\delta_{s_1})^{\Omega_2}, (F_{s_1})^{\Omega_2}(F_{s_1})^{\Omega_2} \end{pmatrix}, \\ \begin{pmatrix} \sqrt[q]{1-(1-\lambda_1^{q})^{\Omega_1}+1-(1-\lambda_1^{q})^{\Omega_2}-(1-(1-\lambda_1^{q})^{\Omega_1})(1-(1-\lambda_2^{q})^{\Omega_2})}, \\ (\mu_1)^{\Omega_1}(\mu_1)^{\Omega_2}, (\omega_1)^{\Omega_1}(\omega_1)^{\Omega_2} \end{pmatrix} \end{pmatrix}; q \geq 1$$

$$= \begin{pmatrix} \left(\sqrt[q]{1-(1-\eth_{s_1}^{q})^{\Omega_1+\Omega_2}}, (\delta_{s_1})^{\Omega_1+\Omega_2}, (F_{s_1})^{\Omega_1+\Omega_2}\right), \\ \left(\sqrt[q]{1-(1-\lambda_1^{q})^{\Omega_1+\Omega_2}}, (\mu_1)^{\Omega_1+\Omega_2}, (\omega_1)^{\Omega_1+\Omega_2}\right) \end{pmatrix}$$

$$= (\Omega_1 \oplus \Omega_2)\Gamma_1.$$

Thus proved.

In praise of equality (7), we have

$$7)\ (\Gamma_1^{\Omega_1})^{\Omega_2} = \Gamma_1^{\Omega_1\Omega_2}$$

$$\Gamma_1^{\Omega_1} = \left(\left(\eth_{s_1}^{\Omega_1}, \delta_{s_1}^{\Omega_1}, \sqrt[q]{1-(1-F_{s_1}^{q})^{\Omega_1}}\right), \left(\lambda_1^{\Omega_1}, \mu_1^{\Omega_1}, \sqrt[q]{1-(1-\omega_1^{q})^{\Omega_1}}\right)\right)$$

$$(\Gamma_1^{\Omega_1})^{\Omega_2} = \left(\left(\eth_{s_1}^{\Omega_1}, \delta_{s_1}^{\Omega_1}, \sqrt[q]{1-(1-F_{s_1}^{q})^{\Omega_1}}\right), \left(\lambda_1^{\Omega_1}, \mu_1^{\Omega_1}, \sqrt[q]{1-(1-\omega_1^{q})^{\Omega_1}}\right)\right)^{\Omega_2}$$

$$= \left(\left(\left(\eth_{s_1}^{\Omega_1}\right)^{\Omega_2}, \left(\delta_{s_1}^{\Omega_1}\right)^{\Omega_2}, \left(\sqrt[q]{1-(1-F_{s_1}^{q})^{\Omega_1}}\right)^{\Omega_2}\right), \left(\left(\lambda_1^{\Omega_1}\right)^{\Omega_2}, \left(\mu_1^{\Omega_1}\right)^{\Omega_2}, \left(\sqrt[q]{1-(1-\omega_1^{q})^{\Omega_1}}\right)^{\Omega_2}\right)\right)$$

$$= \left(\eth_{s_1}^{\Omega_1\Omega_2}, \delta_{s_1}^{\Omega_1\Omega_2}, \left(\sqrt[q]{1-(1-F_{s_1}^{q})^{\Omega_1\Omega_2}}\right)\right), \left(\lambda_1^{\Omega_1\Omega_2}, \mu_1^{\Omega_1\Omega_2}, \left(\sqrt[q]{1-(1-\omega_1^{q})^{\Omega_1\Omega_2}}\right)\right)$$

$$= \Gamma_1^{\Omega_1\Omega_2}.$$

So, we obtain

$$(\Gamma_1^{\Omega_1})^{\Omega_2} = \Gamma_1^{\Omega_1\Omega_2}.$$

The proof of the remaining properties can be handled easily. □

4. Śpherical q-linear Diophantine fuzzy weighted aggregation operator

This section presented the list of novel algebraic norm based aggregation information under spherical q-linear Diophantine fuzzy sets. Also the score function is introduced for ranking the Sq-LDFNs.

Definition 13. *Let* $\Gamma = \{\wp, (\eth_s, \delta_s, F_s), (\lambda, \mu, \omega)\}$ *be an Śq-LDFN, then the transformation* $\mho_s : \acute{S}q - LDFN(U) \rightarrow [-1, 1]$ *is label as score function* (SF) *on U as shown*

$$\mho_{\Gamma_s} = \left[\frac{(\eth_s - \delta_s - F_s) + (\lambda^q - \mu^q - \omega^q)}{2}\right]; q \geq 1, \tag{4.1}$$

where Śq-LDFN(U) is a group of Śq-LDFNs on U.

Definition 14. *Let* Γ_{s_1} *and* Γ_{s_2} *be two Śq-LDFNs. Then*
(1). $\mho_{\Gamma_1} < \mho_{\Gamma_2}, \Gamma_1 < \Gamma_2$,
(2). $\mho_{\Gamma_1} = \mho_{\Gamma_2}, \Gamma_1 = \Gamma_2$.

Definition 15. *The transformation* $\Theta : \acute{S}q - LDFN(U) \longrightarrow [-1, 1]$ *manifest the quadratic score function* (QSF) *for Śq-LDNs* Γ_s *and can be exhibit as*

$$\Theta(\Gamma_s) = \left[\frac{\left(\eth_s^2 - \delta_s^2 - F_s^2\right) + \left((\lambda^q)^2 - (\mu^q)^2 - (\omega^q)^2\right)}{2}\right]; q \geq 1, \tag{4.2}$$

where Śq-LDFN(U) is a collection of Śq-LDFNs on U.

Definition 16. *Let* Γ_{s_1} *and* Γ_{s_2} *be two Śq-LDFNs. Then*
(1). $\Theta_{\Gamma_1} < \Theta_{\Gamma_2}, \Gamma_1 < \Gamma_2$,
(2). $\Theta_{\Gamma_1} = \Theta_{\Gamma_2}, \Gamma_1 = \Gamma_2$.

Definition 17. *An expectation score function* (ESF) *is represented by the mapping* $\vartheta_s : \acute{S}q - LDFN(U) \rightarrow [0, 1]$ *defined as:*

$$\vartheta_{\Gamma_s} = \vartheta(\Gamma_s) = \frac{1}{3}\left[\frac{(\eth_s - \delta_s - F_s + 2)}{2} + \frac{(\lambda^q - \mu^q - \omega^q + 2)}{2}\right]; q \geq 1, \tag{4.3}$$

where Śq-LDFN(U) is a group of Śq-LDFNs on U.

Definition 18. *Let* Γ_{s_1} *and* Γ_{s_2} *be two Śq-LDFNs. Then*
(1). $\vartheta(\Gamma_1) < \vartheta(\Gamma_2), \Gamma_1 < \Gamma_2$,
(2). $\vartheta(\Gamma_1) = \vartheta(\Gamma_2), \Gamma_1 = \Gamma_2$.

The generalizes form of (SF) is expectation score function (ESF). The range of (ESF) is $[0, 1]$ rather than the range $[-1, 1]$. Moreover, we define numerous aggregation techniques rely on (Sq-LDFNs).

Definition 19. *Let* $\Gamma_{\hat{h}} = \{(\eth_{s\hat{h}}, \delta_{s\hat{h}}, F_{s\hat{h}}), (\lambda_{\hat{h}}, \mu_{\hat{h}}, \omega_{\hat{h}})\} \left(\hat{h} \in N\right)$ *be the assemblage of Śq-LDFNs and* $\acute{S}q - LDFN\,(U)^n \longrightarrow \acute{S}q - LDFN\,(U)$, *then* $\acute{S}q - LDFWA$ *operator is defined as*

$$\acute{S}q - LDFWAA\,(\Gamma_1, \Gamma_2, ..., \Gamma_n) = \sum_{\hat{h}=1}^{n} \daleth_{\hat{h}} \cdot \Gamma_{s_{\hat{h}}},$$

where the set $\daleth = (\daleth_1, \daleth_2, ..., \daleth_n)^T$ *represented the weight vector for* $\Gamma_{\hat{h}} = \{(\eth_{s\hat{h}}, \delta_{s\hat{h}}, F_{s\hat{h}}), (\lambda_{\hat{h}}, \mu_{\hat{h}}, \omega_{\hat{h}})\}$ $\left(\hat{h} \in N\right)$ *such that* $\daleth > 0$ *along with* $\sum_{\hat{h}\in N} \daleth_{\hat{h}} = 1$.

Theorem 2. *For any* $\Gamma_{\hat{h}} = \{(\eth_{s\hat{h}}, \delta_{s\hat{h}}, F_{s\hat{h}}), (\lambda_{\hat{h}}, \mu_{\hat{h}}, \omega_{\hat{h}})\} \left(\hat{h} \in N\right)$ *assemblage of Śq-LDFNs over U along with weight vector* $\daleth = (\daleth_1, \daleth_2, ..., \daleth_n)^T$ *such that* $\daleth > 0$ *along with* $\sum_{\hat{h}\in N} \daleth_{\hat{h}} = 1$. *Then the Śpherical q-linear Diophantine fuzzy weighted averaging* $\left(\acute{S}q - LDFWA\right)$ *aggregation operator is define using the operational laws as follows:*

$$\acute{S}q - LDFWA\,(\Gamma_1, \Gamma_2, ..., \Gamma_n) = \sum_{\hat{h}=1}^{n} \daleth_{\hat{h}} \cdot \Gamma_{s_{\hat{h}}}$$

$$= \left[\begin{array}{c} \left(, \sqrt[q]{1 - \prod_{\hat{h}=1}^{n}(1 - \eth^q_{s_{\hat{h}}})^{\daleth_{\hat{h}}}}, \prod_{\hat{h}=1}^{n} (\delta_{s_{\hat{h}}})^{\daleth_{\hat{h}}}, \prod_{\hat{h}=1}^{n} (F_{s_{\hat{h}}})^{\daleth_{\hat{h}}}\right), \\ \left(\sqrt[q]{1 - \prod_{\hat{h}=1}^{n}(1 - \lambda^q_{\hat{h}})^{\daleth_{\hat{h}}}}, \prod_{\hat{h}=1}^{n} (\mu_{\hat{h}})^{\daleth_{\hat{h}}}, \prod_{\hat{h}=1}^{n} (\omega_{\hat{h}})^{\daleth_{\hat{h}}}\right) \end{array}\right]. \tag{4.4}$$

Proof. The inductive technique of mathematics may be used to prove this theorem. Therefore,

(1) . For $n = 2$, we have

$$\daleth_1\Gamma_{s_1} = \left[\begin{array}{c} \left(\sqrt[q]{1 - (1 - \eth^q_{s_1})^{\daleth_1}}, (\delta_{s_1})^{\daleth_1}, (F_{s_1})^{\daleth_1}\right), \\ \left(\sqrt[q]{1 - (1 - \lambda^q_1)^{\daleth_1}}, (\mu_1)^{\daleth_1}, (\omega_1)^{\daleth_1}\right) \end{array}\right]$$

$$\daleth_2\Gamma_{s_2} = \left[\begin{array}{c} \left(\sqrt[q]{1 - (1 - \eth^q_{s_2})^{\daleth_2}}, (\delta_{s_2})^{\daleth_2}, (F_{s_2})^{\daleth_2}\right), \\ \left(\sqrt[q]{1 - (1 - \lambda^q_2)^{\daleth_2}}, (\mu_2)^{\daleth_2}, (\omega_2)^{\daleth_2}\right) \end{array}\right].$$

Then

$$\acute{S}q - LDFWA\,(\Gamma_1, \Gamma_2) = \daleth_1\Gamma_{s_1} \oplus \daleth_2\Gamma_{s_2}$$

$$= \left[\begin{array}{c} \left(\begin{array}{c} \sqrt[q]{1 - (1 - \eth^q_{s_1})^{\daleth_1}}, \\ (\delta_{s_1})^{\daleth_1}, (F_{s_1})^{\daleth_1} \end{array}\right), \\ \left(\begin{array}{c} \sqrt[q]{1 - (1 - \lambda^q_1)^{\daleth_1}}, \\ (\mu_1)^{\daleth_1}, (\omega_1)^{\daleth_1} \end{array}\right) \end{array}\right] \oplus \left[\begin{array}{c} \left(\begin{array}{c} \sqrt[q]{1 - (1 - \eth^q_{s_2})^{\daleth 2}}, \\ (\delta_{s_2})^{\daleth_2}, (F_{s_2})^{\daleth_2} \end{array}\right), \\ \left(\begin{array}{c} \sqrt[q]{1 - (1 - \lambda^q_2)^{\daleth_2}}, \\ (\mu_2)^{\daleth_2}, (\omega_2)^{\daleth_2} \end{array}\right) \end{array}\right];$$

$$= \left[\begin{array}{c} \left(\sqrt[q]{\begin{array}{c} \left(\sqrt[q]{1-(1-\eth_{s_1}^q)^{\daleth_1}}\right)^q + \left(\sqrt[q]{1-(1-\eth_{s_2}^q)^{\daleth_2}}\right)^q - \\ \left(\sqrt[q]{1-(1-\eth_{s_1}^q)^{\daleth_1}}\right)^q \cdot \left(\sqrt[q]{1-(1-\eth_{s_2}^q)^{\daleth_2}}\right)^q \end{array}} \\ ,(\delta_{s_1})^{\daleth_1}(\delta_{s_2})^{\daleth_2},(F_{s_1})^{\daleth_1}(F_{s_2})^{\daleth_2} \right), \\ \left(\sqrt[q]{\begin{array}{c} \left(\sqrt[q]{1-(1-\lambda_{s_1}^q)^{\daleth_1}}\right)^q + \left(\sqrt[q]{1-(1-\lambda_{s_2}^q)^{\daleth_2}}\right)^q - \\ \left(\sqrt[q]{1-(1-\lambda_{s_1}^q)^{\daleth_1}}\right)^q \cdot \left(\sqrt[q]{1-(1-\lambda_{s_2}^q)^{\daleth_2}}\right)^q \end{array}} \\ ,(\mu_{s_1})^{\daleth_1}(\delta\mu_{s_2})^{\daleth_2},(\omega_{s_1})^{\daleth_1}(\omega_{s_2})^{\daleth_2} \right) \end{array} \right]$$

$$= \left[\begin{array}{c} \left(\sqrt[q]{1-(1-\eth_{s_1}^q)^{\daleth_1}(1-\eth_{s_2}^q)^{\daleth_2}},(\delta_{s_1})^{\daleth_1}(\delta_{s_2})^{\daleth_2},(F_{s_1})^{\daleth_1}(F_{s_2})^{\daleth_2}\right), \\ \left(\sqrt[q]{1-(1-\lambda_{s_1}^q)^{\daleth_1}(1-\lambda_{s_2}^q)^{\daleth_2}},(\mu_{s_1})^{\daleth_1}(\delta\mu_{s_2})^{\daleth_2},(\omega_{s_1})^{\daleth_1}(\omega_{s_2})^{\daleth_2}\right) \end{array} \right]$$

$$= \left[\begin{array}{c} \left(\sqrt[q]{1-\prod_{\hat{h}=1}^{2}(1-\eth_{s_{\hat{h}}}^q)^{\daleth_{\hat{h}}}},\prod_{\hat{h}=1}^{2}(\delta_{s_{\hat{h}}})^{\daleth_{\hat{h}}},\prod_{\hat{h}=1}^{2}(F_{s_{\hat{h}}})^{\daleth_{\hat{h}}}\right), \\ \left(\sqrt[q]{1-\prod_{\hat{h}=1}^{2}(1-\lambda_{s_{\hat{h}}}^q)^{\daleth_{\hat{h}}}},\prod_{\hat{h}=1}^{2}(\mu_{s_{\hat{h}}})^{\daleth_{\hat{h}}},\prod_{\hat{h}=1}^{2}(\omega_{s_{\hat{h}}})^{\daleth_{\hat{h}}}\right) \end{array} \right].$$

(2) . We assume that the equation is true for $n = \ell$, and it is demonstrated as follows.

$$= \left[\begin{array}{c} \left(\sqrt[q]{1-\prod_{\hat{h}=1}^{\ell}(1-\eth_{s_{\hat{h}}}^q)^{\daleth_{\hat{h}}}},\prod_{\hat{h}=1}^{\ell}(\delta_{s_{\hat{h}}})^{\daleth_{\hat{h}}},\prod_{\hat{h}=1}^{\ell}(F_{s_{\hat{h}}})^{\daleth_{\hat{h}}}\right), \\ \left(\sqrt[q]{1-\prod_{\hat{h}=1}^{\ell}(1-\lambda_{s_{\hat{h}}}^q)^{\daleth_{\hat{h}}}},\prod_{\hat{h}=1}^{\ell}(\mu_{s_{\hat{h}}})^{\daleth_{\hat{h}}},\prod_{\hat{h}=1}^{\ell}(\omega_{s_{\hat{h}}})^{\daleth_{\hat{h}}}\right) \end{array} \right].$$

(3) . Now, we have to prove for $n = \ell + 1$, then we have

$$\acute{S}q-LDFWA\left(\Gamma_{s_1},\Gamma_{s_{2,\ldots,}}\Gamma_{s_{\ell+1}}\right) = \daleth_1\Gamma_{s_1}\oplus\daleth_2\Gamma_{s_2}\oplus\ldots\oplus\daleth_\ell\Gamma_{s_\ell}\oplus\daleth_{\ell+1}\Gamma_{s_{\ell+1}}$$

$$\left[\begin{array}{c} \left(\sqrt[q]{1-\prod_{\hat{h}=1}^{\ell}(1-\eth_{s_{\hat{h}}}^q)^{\daleth_{\hat{h}}}},\prod_{\hat{h}=1}^{\ell}(\delta_{s_{\hat{h}}})^{\daleth_{\hat{h}}},\prod_{\hat{h}=1}^{\ell}(F_{s_{\hat{h}}})^{\daleth_{\hat{h}}}\right), \\ \left(\sqrt[q]{1-\prod_{\hat{h}=1}^{\ell}(1-\lambda_{s_{\hat{h}}}^q)^{\daleth_{\hat{h}}}},\prod_{\hat{h}=1}^{\ell}(\mu_{s_{\hat{h}}})^{\daleth_{\hat{h}}},\prod_{\hat{h}=1}^{\ell}(\omega_{s_{\hat{h}}})^{\daleth_{\hat{h}}}\right) \end{array} \right] \oplus$$

$$\left[\begin{array}{c} \left(\sqrt[q]{1-(1-\eth_{s_{\ell+1}}^q)^{\daleth_{\hat{h}}}},(\delta_{s_{\ell+1}})^{\daleth_{\hat{h}}},(F_{s_{\ell+1}})^{\daleth_{\hat{h}}}\right), \\ \left(\sqrt[q]{1-(1-\lambda_{s_{\ell+1}}^q)^{\daleth_{\hat{h}}}},(\mu_{s_{\ell+1}})^{\daleth_{\hat{h}}},(\omega_{s_{\ell+1}})^{\daleth_{\hat{h}}}\right) \end{array} \right]$$

$$= \left[\begin{array}{c} \left(\sqrt[q]{1-\prod_{\hat{h}=1}^{\ell+1}(1-\eth_{s_{\hat{h}}}^q)^{\daleth_{\hat{h}}}},\prod_{\hat{h}=1}^{\ell+1}(\delta_{s_{\hat{h}}})^{\daleth_{\hat{h}}},\prod_{\hat{h}=1}^{\ell+1}(F_{s_{\hat{h}}})^{\daleth_{\hat{h}}}\right), \\ \left(\sqrt[q]{1-\prod_{\hat{h}=1}^{\ell+1}(1-\lambda_{s_{\hat{h}}}^q)^{\daleth_{\hat{h}}}},\prod_{\hat{h}=1}^{\ell+1}(\mu_{s_{\hat{h}}})^{\daleth_{\hat{h}}},\prod_{\hat{h}=1}^{\ell+1}(\omega_{s_{\hat{h}}})^{\daleth_{\hat{h}}}\right) \end{array} \right].$$

Therefore, the equation is true for $n = \ell + 1$.
Hence proved. □

Definition 20. *Let* $\Gamma_{\hat{h}} = \{(\eth_{s\hat{h}}, \delta_{s\hat{h}}, F_{s\hat{h}}), (\lambda_{\hat{h}}, \mu_{\hat{h}}, \omega_{\hat{h}})\}$ $\left(\hat{h} \in N\right)$ *be the assemblage of Śq-LDFNs and* $\acute{S}q - LDFN(U)^n \longrightarrow \acute{S}q - LDFN(U)$*, then* $\acute{S}q - LDFOWA$ *operator is defined as*

$$\acute{S}q - LDFOWA(\Gamma_1, \Gamma_2, ..., \Gamma_n) = \sum_{\hat{h}=1}^{n} \daleth_{\hat{h}} \cdot \Gamma_{s_{\varsigma(\hat{h})}},$$

where the set $\daleth = (\daleth_1, \daleth_2, ..., \daleth_n)^T$ *represented the weight vector for* $\Gamma_{\hat{h}} = \{(\eth_{s\hat{h}}, \delta_{s\hat{h}}, F_{s\hat{h}}), (\lambda_{\hat{h}}, \mu_{\hat{h}}, \omega_{\hat{h}})\}$ $\left(\hat{h} \in N\right)$ *such that* $\daleth > 0$ *along with* $\sum_{\hat{h}\in N} \daleth_{\hat{h}} = 1$ *and* $\varsigma(1), \varsigma(2), ..., \varsigma(n)$ *be the permutation such that* $\varsigma\left(\hat{h}\right) < \varsigma\left(\hat{h} - 1\right)$.

Theorem 3. *For any* $\Gamma_{\hat{h}} = \{(\eth_{s\hat{h}}, \delta_{s\hat{h}}, F_{s\hat{h}}), (\lambda_{\hat{h}}, \mu_{\hat{h}}, \omega_{\hat{h}})\}$ $\left(\hat{h} \in N\right)$ *assemblage of Śq-LDFNs over U along with weight vector* $\daleth = (\daleth_1, \daleth_2, ..., \daleth_n)^T$ *such that* $\daleth > 0$ *along with* $\sum_{\hat{h}\in N} \daleth_{\hat{h}} = 1$. *Then the Śpherical q-linear Diophantine fuzzy ordered weighted averaging* $\left(\acute{S}q - LDFOWA\right)$ *aggregation operator is define using the operational laws as follows:*

$$\acute{S}q - LDFOWA(\Gamma_1, \Gamma_2, ..., \Gamma_n) = \sum_{\hat{h}=1}^{n} \daleth_{\hat{h}} \cdot \Gamma_{\varsigma(\hat{h})}$$

$$= \left[\begin{array}{c} \left(, \sqrt[q]{1 - \prod_{\hat{h}=1}^{n} (1 - \eth^{q}_{s_{\varsigma(\hat{h})}})^{\daleth_{\hat{h}}}}, \prod_{\hat{h}=1}^{n} (\delta_{s_{\varsigma(\hat{h})}})^{\daleth_{\hat{h}}}, \prod_{\hat{h}=1}^{n} (F_{s_{\varsigma(\hat{h})}})^{\daleth_{\hat{h}}}\right), \\ \left(\sqrt[q]{1 - \prod_{\hat{h}=1}^{n} (1 - \lambda^{q}_{\varsigma(\hat{h})})^{\daleth_{\hat{h}}}}, \prod_{\hat{h}=1}^{n} (\mu_{\varsigma(\hat{h})})^{\daleth_{\hat{h}}}, \prod_{\hat{h}=1}^{n} (\omega_{\varsigma(\hat{h})})^{\daleth_{\hat{h}}}\right) \end{array}\right]. \tag{4.5}$$

where $\varsigma(1), \varsigma(2), ..., \varsigma(n)$ *be the permutation such that* $\varsigma\left(\hat{h}\right) < \varsigma\left(\hat{h} - 1\right)$.

Proof. Prove is similar to proof of Theorem 2. □

Now, we presented some interesting properties that the averaging operators satisfy.

1). (Idempotency) Let $\Gamma_{\hat{h}} = \{(\eth_{s\hat{h}}, \delta_{s\hat{h}}, F_{s\hat{h}}), (\lambda_{\hat{h}}, \mu_{\hat{h}}, \omega_{\hat{h}})\}$ $\left(\hat{h} \in N\right)$ be the assemblage of Śq-LDFNs. If $\Gamma_1 = \Gamma_2 = ... = \Gamma_n = \Gamma$, then

$$\acute{S}q - LDFWA(\Gamma_1, \Gamma_2, ..., \Gamma_n) = \Gamma.$$

2). (Boundedness) Let $\Gamma_{\hat{h}} = \{(\eth_{s\hat{h}}, \delta_{s\hat{h}}, F_{s\hat{h}}), (\lambda_{\hat{h}}, \mu_{\hat{h}}, \omega_{\hat{h}})\}$ $\left(\hat{h} \in N\right)$ be the assemblage of Śq-LDFNs, with $\sigma_{s\hat{h}}$ is the refusal degree $\forall\ \Gamma_{\hat{h}}$. Then $\delta^*_{s\hat{h}} = \min\{\delta_{s\hat{h}}\}$, $F^*_{s\hat{h}} = \min\{F_{s\hat{h}}\}$, $\sigma^*_{s\hat{h}} = \min\{\sigma_{s\hat{h}}\}$ and $\mu^*_{\hat{h}}, = \min\{\mu_{\hat{h}}\}$, $\omega^*_{\hat{h}} = \min\{\omega_{\hat{h}}\}$, $\varpi^*_{\hat{h}} = \min\{\varpi_{\hat{h}}\}$, then, $\lambda \cdot \eth^*_{s\hat{h}} = 1 - \mu_{\hat{h}}\delta^*_{s\hat{h}} + \omega_{\hat{h}}F^*_{s\hat{h}} + \varpi_{\hat{h}}\sigma^*_{s\hat{h}}$

also $\delta'_{s\hat{h}} = \max\{\delta_{s\hat{h}}\}$, $F'_{s\hat{h}} = \max\{F_{s\hat{h}}\}$, $\sigma'_{s\hat{h}} = \max\{\sigma_{s\hat{h}}\}$ and $\mu'_{\hat{h}}, = \max\{\mu_{\hat{h}}\}$, $\omega'_{\hat{h}} = \max\{\omega_{\hat{h}}\}$, $\varpi'_{\hat{h}} = \max\{\varpi_{\hat{h}}\}$, then, $\lambda \cdot \eth'_{s\hat{h}} = 1 - \mu_{\hat{h}}\delta'_{s\hat{h}} + \omega_{\hat{h}}F'_{s\hat{h}} + \varpi_{\hat{h}}\sigma'_{s\hat{h}}$

$$\Gamma^* \leq \acute{S}q - LDFWA(\Gamma_1, \Gamma_2, ..., \Gamma_n) \leq \Gamma'.$$

Where $\Gamma'_{\hat{h}} = \left\{\left(\eth'_{s\hat{h}}, \delta'_{s\hat{h}}, F'_{s\hat{h}}\right), \left(\lambda'_{\hat{h}}, \mu'_{\hat{h}}, \omega'_{\hat{h}}\right)\right\}$ and $\Gamma^*_{\hat{h}} = \left\{\left(\eth^*_{s\hat{h}}, \delta^*_{s\hat{h}}, F^*_{s\hat{h}}\right), \left(\lambda^*_{\hat{h}}, \mu^*_{\hat{h}}, \omega^*_{\hat{h}}\right)\right\}$.

3). (Monotonicity) Let $\Gamma_{\hat{h}} = \{(\eth_{s\hat{h}}, \delta_{s\hat{h}}, F_{s\hat{h}}), (\lambda_{\hat{h}}, \mu_{\hat{h}}, \omega_{\hat{h}})\}$ and $\Gamma^{\circ}_{\hat{h}} = \{(\eth^{\circ}{}_{s\hat{h}}, \delta^{\circ}{}_{s\hat{h}}, F^{\circ}{}_{s\hat{h}}), (\lambda_{\hat{h}}{}^{\circ}, \mu_{\hat{h}}{}^{\circ}, \omega_{\hat{h}}{}^{\circ})\}\left(\hat{h} \in N\right)$ be two assemblage of Śq-LDFNs. If $\eth_{s\hat{h}} \leq \eth^{\circ}{}_{s\hat{h}}, \delta_{s\hat{h}} \leq \delta^{\circ}{}_{s\hat{h}}, F_{s\hat{h}} \leq F^{\circ}{}_{s\hat{h}}, \lambda_{\hat{h}} \leq \lambda_{\hat{h}}{}^{\circ}, \mu_{\hat{h}} \leq \mu_{\hat{h}}{}^{\circ}, \omega_{\hat{h}} \leq \omega_{\hat{h}}{}^{\circ}$,then

$$\acute{S}q - LDFWA(\Gamma_1, \Gamma_2, ..., \Gamma_n) \geq \acute{S}q - LDFWA(\Gamma^{\circ}_1, \Gamma^{\circ}_2, ..., \Gamma^{\circ}_n).$$

Definition 21. *Let* $\Gamma_{\hat{h}} = \{(\eth_{s\hat{h}}, \delta_{s\hat{h}}, F_{s\hat{h}}), (\lambda_{\hat{h}}, \mu_{\hat{h}}, \omega_{\hat{h}})\} \left(\hat{h} \in N\right)$ *be the assemblage of Śq-LDFNs and* $\acute{S}q - LDFN(U)^n \longrightarrow \acute{S}q - LDFN(U)$*, then* $\acute{S}q - LDFWG$ *operator is defined as*

$$\acute{S}q - LDFWG(\Gamma_1, \Gamma_2, ..., \Gamma_n) = \prod_{\hat{h}=1}^{n} \left(\Gamma_{s_{\hat{h}}}\right)^{\daleth_{\hat{h}}},$$

where the set $\daleth = (\daleth_1, \daleth_2, ..., \daleth_n)^T$ *represented the weight vector for* $\Gamma_{\hat{h}} = \{(\eth_{s\hat{h}}, \delta_{s\hat{h}}, F_{s\hat{h}}), (\lambda_{\hat{h}}, \mu_{\hat{h}}, \omega_{\hat{h}})\}$ $\left(\hat{h} \in N\right)$ *such that* $\daleth > 0$ *along with* $\sum_{\hat{h}\in N} \daleth_{\hat{h}} = 1$.

Theorem 4. *For any* $\Gamma_{\hat{h}} = \{(\eth_{s\hat{h}}, \delta_{s\hat{h}}, F_{s\hat{h}}), (\lambda_{\hat{h}}, \mu_{\hat{h}}, \omega_{\hat{h}})\} \left(\hat{h} \in N\right)$ *assemblage of Śq-LDFNs over U along with weight vector*
$\daleth = (\daleth_1, \daleth_2, ..., \daleth_n)^T$ *such that* $\daleth > 0$ *along with* $\sum_{\hat{h}\in N} \daleth_{\hat{h}} = 1$*. Then the Śpherical q-linear Diophantine fuzzy weighted averaging* $\left(\acute{S}q - LDFWA\right)$ *aggregation operator is define using the operational laws as follows:*

$$\acute{S}q - LDFWG(\Gamma_1, \Gamma_2, ..., \Gamma_n) = \prod_{\hat{h}=1}^{n} \left(\Gamma_{s_{\hat{h}}}\right)^{\daleth_{\hat{h}}}$$

$$= \left[\begin{array}{c} \left(, \prod_{\hat{h}=1}^{n} (\eth^q_{s_{\hat{h}}})^{\daleth_{\hat{h}}}, \prod_{\hat{h}=1}^{n} (\delta_{s_{\hat{h}}})^{\daleth_{\hat{h}}}, \sqrt[q]{1 - \prod_{\hat{h}=1}^{n} (1 - F_{s_{\hat{h}}})^{\daleth_{\hat{h}}}}\right), \\ \left(\prod_{\hat{h}=1}^{n} (\lambda^q_{\hat{h}})^{\daleth_{\hat{h}}}, \prod_{\hat{h}=1}^{n} (\mu_{\hat{h}})^{\daleth_{\hat{h}}}, \sqrt[q]{1 - \prod_{\hat{h}=1}^{n} (1 - \omega_{\hat{h}})^{\daleth_{\hat{h}}}}\right) \end{array}\right]. \tag{4.6}$$

Proof. The inductive technique of mathematics may be used to prove this theorem. Therefore,
(1) . For $n = 2$, we have

$$\left(\Gamma_{s_1}\right)^{\daleth_1} = \left[\begin{array}{c} \left((\eth^q_{s_1})^{\daleth 1}, (\delta_{s_1})^{\daleth_1}, \sqrt[q]{1 - (1 - F_{s_1})^{\daleth_1}}\right), \\ \left((\lambda^q_1)^{\daleth_1}, (\mu_1)^{\daleth_1}, \sqrt[q]{1 - (1 - \omega_1)^{\daleth_1}}\right) \end{array}\right]$$

$$\left(\Gamma_{s_2}\right)^{\daleth_2} = \left[\begin{array}{c} \left((\eth^q_{s_2})^{\daleth_2}, (\delta_{s_2})^{\daleth_2}, \sqrt[q]{1 - (1 - F_{s_2})^{\daleth_2}}\right), \\ \left((\lambda^q_2)^{\daleth_2}, (\mu_2)^{\daleth_2}, \sqrt[q]{1 - (1 - \omega_2)^{\daleth_2}}\right) \end{array}\right].$$

Then

$$\acute{S}q - LDFWG(\Gamma_1, \Gamma_2) = (\Gamma_1)^{\daleth_1} \oplus (\Gamma_2)^{\daleth_2}$$

$$= \left[\begin{array}{c} \left(\begin{array}{c} (\eth^q_{s_1})^{\daleth 1}, (\delta_{s_1})^{\daleth_1}, \\ \sqrt[q]{1 - (1 - F_{s_1})^{\daleth_1}} \end{array}\right), \\ \left(\begin{array}{c} (\lambda^q_1)^{\daleth_1}, (\mu_1)^{\daleth_1}, \\ \sqrt[q]{1 - (1 - \omega_1)^{\daleth_1}} \end{array}\right) \end{array}\right] \otimes \left[\begin{array}{c} \left(\begin{array}{c} (\eth^q_{s_2})^{\daleth_2}, (\delta_{s_2})^{\daleth_2}, \\ \sqrt[q]{1 - (1 - F_{s_2})^{\daleth_2}} \end{array}\right), \\ \left(\begin{array}{c} (\lambda^q_2)^{\daleth_2}, (\mu_2)^{\daleth_2}, \\ \sqrt[q]{1 - (1 - \omega_2)^{\daleth_2}} \end{array}\right) \end{array}\right];$$

$$= \left[\begin{array}{c} \left(\begin{array}{c} (\eth_{s_1})^{\daleth_1}(\eth_{s_2})^{\daleth_2}, (\delta_{s_1})^{\daleth_1}(\delta_{s_2})^{\daleth_2}, \\ \sqrt[q]{\begin{array}{c} \left(\sqrt[q]{1 - (1 - F^q_{s_1})^{\daleth_1}}\right)^q + \left(\sqrt[q]{1 - (1 - F^q_{s_2})^{\daleth_2}}\right)^q - \\ \left(\sqrt[q]{1 - (1 - F^q_{s_1})^{\daleth_1}}\right)^q \cdot \left(\sqrt[q]{1 - (1 - F^q_{s_2})^{\daleth_2}}\right)^q \end{array}} \end{array}\right), \\ \left(\begin{array}{c} (\lambda_{s_1})^{\daleth_1}(\lambda_{s_2})^{\daleth_2}, (\mu_{s_1})^{\daleth_1}(\delta\mu_{s_2})^{\daleth_2}, \\ \sqrt[q]{\begin{array}{c} \left(\sqrt[q]{1 - (1 - \omega^q_{s_1})^{\daleth_1}}\right)^q + \left(\sqrt[q]{1 - (1 - \omega^q_{s_2})^{\daleth_2}}\right)^q - \\ \left(\sqrt[q]{1 - (1 - \omega^q_{s_1})^{\daleth_1}}\right)^q \cdot \left(\sqrt[q]{1 - (1 - \omega^q_{s_2})^{\daleth_2}}\right)^q \end{array}} \end{array}\right) \end{array}\right]$$

$$= \left[\begin{array}{c} \left((\eth_{s_1})^{\daleth_1} (\eth_{s_2})^{\daleth_2}, (\delta_{s_1})^{\daleth_1} (\delta_{s_2})^{\daleth_2}, \sqrt[q]{1-(1-F_{s_1}^q)^{\daleth_1}(1-F_{s_2}^q)^{\daleth_2}} \right), \\ \left((\lambda_{s_1})^{\daleth_1} (\lambda_{s_2})^{\daleth_2}, (\mu_{s_1})^{\daleth_1} (\delta\mu_{s_2})^{\daleth_2}, \sqrt[q]{1-(1-\omega_{s_1}^q)^{\daleth_1}(1-\omega_{s_2}^q)^{\daleth_2}} \right) \end{array} \right]$$

$$= \left[\begin{array}{c} \left(\prod_{\hat{h}=1}^{2} (\eth_{s_{\hat{h}}})^{\daleth_{\hat{h}}}, \prod_{\hat{h}=1}^{2} (\delta_{s_{\hat{h}}})^{\daleth_{\hat{h}}}, \sqrt[q]{1-\prod_{\hat{h}=1}^{2}(1-F_{s_{\hat{h}}}^q)^{\daleth_{\hat{h}}}} \right), \\ \left(\prod_{\hat{h}=1}^{2} (\lambda_{s_{\hat{h}}})^{\daleth_{\hat{h}}}, \prod_{\hat{h}=1}^{2} (\mu_{s_{\hat{h}}})^{\daleth_{\hat{h}}}, \sqrt[q]{1-\prod_{\hat{h}=1}^{2}(1-\omega_{s_{\hat{h}}}^q)^{\daleth_{\hat{h}}}} \right) \end{array} \right].$$

(2) . We assume that the equation is true for $n = \ell$, and it is demonstrated as follows.

$$= \left[\begin{array}{c} \left(\prod_{\hat{h}=1}^{\ell} (\eth_{s_{\hat{h}}})^{\daleth_{\hat{h}}}, \prod_{\hat{h}=1}^{\ell} (\delta_{s_{\hat{h}}})^{\daleth_{\hat{h}}}, \sqrt[q]{1-\prod_{\hat{h}=1}^{\ell}(1-F_{s_{\hat{h}}}^q)^{\daleth_{\hat{h}}}} \right), \\ \left(\prod_{\hat{h}=1}^{\ell} (\lambda_{s_{\hat{h}}})^{\daleth_{\hat{h}}}, \prod_{\hat{h}=1}^{\ell} (\mu_{s_{\hat{h}}})^{\daleth_{\hat{h}}}, \sqrt[q]{1-\prod_{\hat{h}=1}^{\ell}(1-\omega_{s_{\hat{h}}}^q)^{\daleth_{\hat{h}}}} \right) \end{array} \right].$$

(3) . Now, we have to prove for $n = \ell + 1$, then we have

$$\acute{S}q - LDFWG\,(\Gamma_{s_1}, \Gamma_{s_2,\ldots,}\Gamma_{s_{\ell+1}}) = (\Gamma_1)^{\daleth_1} \oplus (\Gamma_2)^{\daleth_2} \oplus \ldots \oplus (\Gamma_\ell)^{\daleth_\ell} \oplus (\Gamma_{\ell+1})^{\daleth_{\ell+1}}$$

$$\left[\begin{array}{c} \left(\prod_{\hat{h}=1}^{\ell} (\eth_{s_{\hat{h}}})^{\daleth_{\hat{h}}}, \prod_{\hat{h}=1}^{\ell} (\delta_{s_{\hat{h}}})^{\daleth_{\hat{h}}}, \sqrt[q]{1-\prod_{\hat{h}=1}^{\ell}(1-F_{s_{\hat{h}}}^q)^{\daleth_{\hat{h}}}} \right), \\ \left(\prod_{\hat{h}=1}^{\ell} (\lambda_{s_{\hat{h}}})^{\daleth_{\hat{h}}}, \prod_{\hat{h}=1}^{\ell} (\mu_{s_{\hat{h}}})^{\daleth_{\hat{h}}}, \sqrt[q]{1-\prod_{\hat{h}=1}^{\ell}(1-\omega_{s_{\hat{h}}}^q)^{\daleth_{\hat{h}}}} \right) \end{array} \right] \otimes$$

$$\left[\begin{array}{c} \left((\eth_{s_{\ell+1}})^{\daleth_{\hat{h}}}, (\delta_{s_{\ell+1}})^{\daleth_{\hat{h}}}, \sqrt[q]{1-(1-F_{s_{\ell+1}}^q)^{\daleth_{\hat{h}}}} \right), \\ \left((\lambda_{s_{\ell+1}})^{\daleth_{\hat{h}}}, (\mu_{s_{\ell+1}})^{\daleth_{\hat{h}}}, \sqrt[q]{1-(1-\omega_{s_{\ell+1}}^q)^{\daleth_{\hat{h}}}} \right) \end{array} \right]$$

$$= \left[\begin{array}{c} \left(\prod_{\hat{h}=1}^{\ell+1} (\eth_{s_{\hat{h}}})^{\daleth_{\hat{h}}}, \prod_{\hat{h}=1}^{\ell+1} (\delta_{s_{\hat{h}}})^{\daleth_{\hat{h}}}, \sqrt[q]{1-\prod_{\hat{h}=1}^{\ell+1}(1-F_{s_{\hat{h}}}^q)^{\daleth_{\hat{h}}}} \right), \\ \left(\prod_{\hat{h}=1}^{\ell+1} (\lambda_{s_{\hat{h}}})^{\daleth_{\hat{h}}}, \prod_{\hat{h}=1}^{\ell+1} (\mu_{s_{\hat{h}}})^{\daleth_{\hat{h}}}, \sqrt[q]{1-\prod_{\hat{h}=1}^{\ell+1}(1-\omega_{s_{\hat{h}}}^q)^{\daleth_{\hat{h}}}} \right) \end{array} \right].$$

Therefore, the equation is true for $n = \ell + 1$.
Hence proved. □

Definition 22. *Let* $\Gamma_{\hat{h}} = \{(\eth_{s\hat{h}}, \delta_{s\hat{h}}, F_{s\hat{h}}), (\lambda_{\hat{h}}, \mu_{\hat{h}}, \omega_{\hat{h}})\}$ $\left(\hat{h} \in N\right)$ *be the assemblage of Śq-LDFNs and* $\acute{S}q - LDFN\,(U)^n \longrightarrow \acute{S}q - LDFN\,(U)$, *then* $\acute{S}q - LDFOWG$ *operator is defined as*

$$\acute{S}q - LDFOWG\,(\Gamma_1, \Gamma_2, ..., \Gamma_n) = \prod_{\hat{h}=1}^{n} \left(\Gamma_{\varsigma(\hat{h})}\right)^{\daleth_{\hat{h}}},$$

where the set $\daleth = (\daleth_1, \daleth_2, ..., \daleth_n)^T$ *represented the weight vector for* $\Gamma_{\hat{h}} = \{(\eth_{s\hat{h}}, \delta_{s\hat{h}}, F_{s\hat{h}}), (\lambda_{\hat{h}}, \mu_{\hat{h}}, \omega_{\hat{h}})\}$ $\left(\hat{h} \in N\right)$ *such that* $\daleth > 0$ *along with* $\sum_{\hat{h} \in N} \daleth_{\hat{h}} = 1$ *and* $\varsigma(1), \varsigma(2), ..., \varsigma(n)$ *be the permutation such that* $\varsigma\left(\hat{h}\right) < \varsigma\left(\hat{h} - 1\right)$.

Theorem 5. *For any* $\Gamma_{\hat{h}} = \{(\backepsilon_{s\hat{h}}, \delta_{s\hat{h}}, F_{s\hat{h}}), (\lambda_{\hat{h}}, \mu_{\hat{h}}, \omega_{\hat{h}})\} \left(\hat{h} \in N\right)$ *assemblage of Śq-LDFNs over U along with weight vector*

$\daleth = (\daleth_1, \daleth_2, ..., \daleth_n)^T$ *such that* $\daleth > 0$ *along with* $\sum_{\hat{h}\in N} \daleth_{\hat{h}} = 1$. *Then the Śpherical q-linear Diophantine fuzzy ordered weighted averaging* $\left(\acute{S}q - LDFOWG\right)$ *aggregation operator is define using the operational laws as follows:*

$$\acute{S}q - LDFOWG\,(\Gamma_1, \Gamma_2, ..., \Gamma_n) = \prod_{\hat{h}=1}^{n} \left(\Gamma_{\varsigma(\hat{h})}\right)^{\daleth_{\hat{h}}}$$

$$= \left[\begin{array}{c} \left(\prod_{\hat{h}=1}^{n} (\backepsilon_{s_{\varsigma(\hat{h})}})^{\daleth_{\hat{h}}}, \prod_{\hat{h}=1}^{n} (\delta_{s_{\varsigma(\hat{h})}})^{\daleth_{\hat{h}}}, \sqrt[q]{1 - \prod_{\hat{h}=1}^{n} (1 - F^q_{s_{\varsigma(\hat{h})}})^{\daleth_{\hat{h}}}}\right), \\ \left(\prod_{\hat{h}=1}^{n} (\lambda_{\varsigma(\hat{h})})^{\daleth_{\hat{h}}}, \prod_{\hat{h}=1}^{n} (\mu_{\varsigma(\hat{h})})^{\daleth_{\hat{h}}}, \sqrt[q]{1 - \prod_{\hat{h}=1}^{n} (1 - \omega^q_{\varsigma(\hat{h})})^{\daleth_{\hat{h}}}}\right) \end{array}\right], \tag{4.7}$$

where $\varsigma(1), \varsigma(2), ..., \varsigma(n)$ *be the permutation such that* $\varsigma\left(\hat{h}\right) < \varsigma\left(\hat{h} - 1\right)$.

Proof. Prove is similar to proof of Theorem 4. □

Now, we presented some interesting properties that the averaging operators satisfy.

1). (Idempotent) Let $\Gamma_{\hat{h}} = \{(\backepsilon_{s\hat{h}}, \delta_{s\hat{h}}, F_{s\hat{h}}), (\lambda_{\hat{h}}, \mu_{\hat{h}}, \omega_{\hat{h}})\} \left(\hat{h} \in N\right)$ be the assemblage of Śq-LDFNs. If $\Gamma_1 = \Gamma_2 = ... = \Gamma_n = \Gamma$, then

$$\acute{S}q - LDFWG\,(\Gamma_1, \Gamma_2, ..., \Gamma_n) = \Gamma.$$

2). (Boundedness) Let $\Gamma_{\hat{h}} = \{(\backepsilon_{s\hat{h}}, \delta_{s\hat{h}}, F_{s\hat{h}}), (\lambda_{\hat{h}}, \mu_{\hat{h}}, \omega_{\hat{h}})\} \left(\hat{h} \in N\right)$ be the assemblage of Śq-LDFNs, with $\sigma_{s\hat{h}}$ is the refusal degree $\forall\ \Gamma_{\hat{h}}$. Then $\delta^*_{s\hat{h}} = \min\{\delta_{s\hat{h}}\}$, $F^*_{s\hat{h}} = \min\{F_{s\hat{h}}\}$, $\sigma^*_{s\hat{h}} = \min\{\sigma_{s\hat{h}}\}$ and $\mu^*_{\hat{h}}, = \min\{\mu_{\hat{h}}\}$, $\omega^*_{\hat{h}} = \min\{\omega_{\hat{h}}\}$, $\varpi^*_{\hat{h}} = \min\{\varpi_{\hat{h}}\}$, then, $\lambda \cdot \backepsilon^*_{s\hat{h}} = 1 - \mu_{\hat{h}}\delta^*_{s\hat{h}} + \omega_{\hat{h}} F^*_{s\hat{h}} + \varpi_{\hat{h}}\sigma^*_{s\hat{h}}$

also $\delta'_{s\hat{h}} = \max\{\delta_{s\hat{h}}\}$, $F'_{s\hat{h}} = \max\{F_{s\hat{h}}\}$, $\sigma'_{s\hat{h}} = \max\{\sigma_{s\hat{h}}\}$ and $\mu'_{\hat{h}}, = \max\{\mu_{\hat{h}}\}$, $\omega'_{\hat{h}} = \max\{\omega_{\hat{h}}\}$, $\varpi'_{\hat{h}} = \max\{\varpi_{\hat{h}}\}$, then, $\lambda \cdot \backepsilon'_{s\hat{h}} = 1 - \mu_{\hat{h}}\delta'_{s\hat{h}} + \omega_{\hat{h}} F'_{s\hat{h}} + \varpi_{\hat{h}}\sigma'_{s\hat{h}}$

$$\Gamma^* \le \acute{S}q - LDFWG(\Gamma_1, \Gamma_2, ..., \Gamma_n) \le \Gamma'.$$

Where $\Gamma'_{\hat{h}} = \left\{\left(\backepsilon'_{s\hat{h}}, \delta'_{s\hat{h}}, F'_{s\hat{h}}\right), \left(\lambda'_{\hat{h}}, \mu'_{\hat{h}}, \omega'_{\hat{h}}\right)\right\}$ and $\Gamma^*_{\hat{h}} = \left\{\left(\backepsilon^*_{s\hat{h}}, \delta^*_{s\hat{h}}, F^*_{s\hat{h}}\right), \left(\lambda^*_{\hat{h}}, \mu^*_{\hat{h}}, \omega^*_{\hat{h}}\right)\right\}$.

3). (Monotonicity) Let $\Gamma_{\hat{h}} = \{(\backepsilon_{s\hat{h}}, \delta_{s\hat{h}}, F_{s\hat{h}}), (\lambda_{\hat{h}}, \mu_{\hat{h}}, \omega_{\hat{h}})\}$ and $\Gamma^\circ_{\hat{h}} = \{(\backepsilon^\circ{}_{s\hat{h}}, \delta^\circ{}_{s\hat{h}}, F^\circ{}_{s\hat{h}}), (\lambda_{\hat{h}}{}^\circ, \mu_{\hat{h}}{}^\circ, \omega_{\hat{h}}{}^\circ)\}$ $\left(\hat{h} \in N\right)$ be two assemblage of Śq-LDFNs. If $\backepsilon_{s\hat{h}} \le \backepsilon^\circ{}_{s\hat{h}}$, $\delta_{s\hat{h}} \le \delta^\circ{}_{s\hat{h}}$, $F_{s\hat{h}} \le F^\circ{}_{s\hat{h}}$, $\lambda_{\hat{h}} \le \lambda_{\hat{h}}{}^\circ$, $\mu_{\hat{h}} \le \mu_{\hat{h}}{}^\circ$, $\omega_{\hat{h}} \le \omega_{\hat{h}}{}^\circ$, then

$$\acute{S}q - LDFWG(\Gamma_1, \Gamma_2, ..., \Gamma_n) \ge \acute{S}q - LDFWG(\Gamma^\circ_1, \Gamma^\circ_2, ..., \Gamma^\circ_n).$$

5. Mathematical modeling

This section consists of an algorithm based on the proposed list of novel aggregation operators under spherical q-linear Diophantine fuzzy (Sq-LDF) information to tackle the real world decision making problems. The set $N_g = \left\{N_{g_1}, N_{g_2}, N_{g_3}, ..., N_{g_m}\right\}$ contains the numbers of alternatives while $\eth = \{\eth_1, \eth_2, \eth_3, ..., \eth_n\}$ represents the numbers of attributes. Let $\daleth = (\daleth_1, \daleth_2, ..., \daleth_n)^T$ be the weight vector met the criteria as $\daleth_{\hat{h}} > 0$, $\sum_{\hat{h}\in N} \daleth_{\hat{h}} = 1$.

Consider the Sq-LDF decision matrix $DM = ((\text{Ɔ}_s, \delta_s, F_s), (\lambda_s, \mu_s, \omega_s))_{(gm\times n)}$, where Ɔ_s is the membership degree, δ_s is the neutral degree, F_s is the non-membership degree and λ_s, μ_s & ω_s are the control factors for which the alternative $\left(N_{gm}\right)$ satisfies the $(\eth_n)$ attribute provided by the decision-makers, such that $\text{Ɔ}_s, F_s, \lambda_s, \mu_s \in [0, 1]$ as

$$0 \leq \lambda_s^q \text{Ɔ}_s + \mu_s \delta_s + \omega_s^q F_s \leq 1, (q = 1, 2, 3, ..., t).$$

The algorithm contain the follows steps.

Step-1 Collect the considered expert information using the novel notion of Sq-LDFS. The decision makers group is represented by $DM = \{DM_1, DM_2, DM_3, ..., DM_i\}$. Sq-LDFNs are used to calculate individual DM's interests. Consequently, use the decision information defined in decision matrix, that are Śq-LDFNs and shown in the form of $DM_1, DM_2, DM_3, DM_4, DM_5, ..., DM_i$ along with the weight vector ℸ.

Step-2 Expert evaluation information required in standard Śq-LDF numbers: Prior to beginning the computations, the input data must be normalized in order to achieve the best result. It is therefore possible to standardized the Sq-LDF information.

$$DM^s = \begin{cases} \begin{pmatrix} \left(\text{Ɔ}_{sq}, \delta_{sq}, F_{sq}\right), \\ \left(\lambda_{sq}, \mu_{sq}, \omega_{sq}\right) \end{pmatrix} & if \quad \text{data is benefit type,} \\ \begin{pmatrix} \left(F_{sq}, \delta_{sq}, \text{Ɔ}_{sq}\right), \\ \left(\omega_{sq}, \mu_{sq}, \lambda_{sq}\right) \end{pmatrix} & if \quad \text{data is cost type.} \end{cases}$$

Step-3 Evaluation of resultant weight vector as follows:

$$\text{C} = \left(\frac{1}{q}\sum_{\partial=1}^{q} {}^{\partial}\text{C}_1, \frac{1}{q}\sum_{\partial=1}^{q} {}^{\partial}\text{C}_2, \frac{1}{q}\sum_{\partial=1}^{q} {}^{\partial}\text{C}_3, ..., \frac{1}{q}\sum_{\partial=1}^{q} {}^{\partial}\text{C}_m\right)^T.$$

Step-4 Using the proposed list of aggregation operators under Śq-LDFSs to compute the integrated (combined) aggregated value for individual attribute ð along with their weight vector.

Step-5 Utilizing the concept of graded functions, score function, quadratic function and expectation score functions to evaluate the scores for individual attribute ð from aggregated expert values.

Step-6 Individual rank the attribute based on the values of the score function, quadratic score function and expectation score functions.

Step-7 The attribute that get highest score has the highest ranking and must be selected for the final selection.

Step-8 End.

The flow chart of the algorithm is presented in Figure 3:

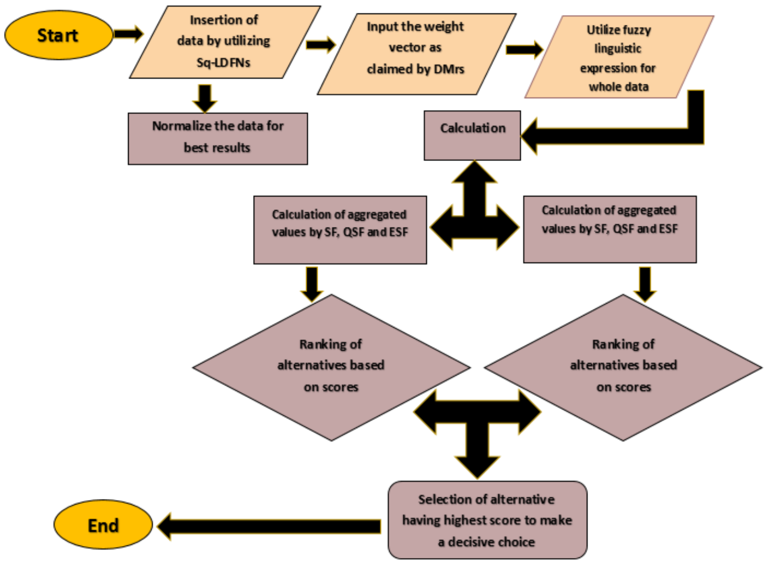

Figure 3. Flow chart for the proposed algorithm.

5.1. Numerical illustration of the proposed algorithm

This part of paper is the implementation of the proposed algorithm to tackle the uncertainty in selection of top-ranked university among five universities under five attributes.

Case Study: Let $\wp = \{\wp_1, \wp_2, \wp_3, \wp_4, \wp_5\}$ be a set of five universities (alternatives). We evaluate the best university under the list of five attributes set $J = \{J_1, J_2, J_3, J_4, J_5\}$ as follows.

$J_1 = \{$Shows Academic Staffs$\}$,
$J_2 = \{$shows projects for Culturure and Community representation$\}$,
$J_3 = \{$shows Library$\}$,
$J_4 = \{$shows Scientific Research$\}$,
$J_5 = \{$shows National and International Scientific Activities for students best learning$\}$.

Step-1 The expert information in the form of Sq-LDFSs as follows in Tables 2 and 3.

Table 2. Sq-LDF expert evaluation information.

	$\eth_1$	$\eth_2$	$\eth_3$
N_{g_1}	$\begin{pmatrix}(.85,.24,.45),\\(.25,.34,.18)\end{pmatrix}$	$\begin{pmatrix}(.73,.31,.48),\\(.34,.11,.23)\end{pmatrix}$	$\begin{pmatrix}(.63,.45,.38),\\(.41,.28,.11)\end{pmatrix}$
N_{g_2}	$\begin{pmatrix}(.77,.41,.52),\\(.34,.21,.22)\end{pmatrix}$	$\begin{pmatrix}(.82,.51,.43),\\(.13,.25,.21)\end{pmatrix}$	$\begin{pmatrix}(.58,.43,.41),\\(.31,.23,.15)\end{pmatrix}$
N_{g_3}	$\begin{pmatrix}(.95,.41,.38),\\(.41,.25,.18)\end{pmatrix}$	$\begin{pmatrix}(.77,.62,.43),\\(.31,.25,.21)\end{pmatrix}$	$\begin{pmatrix}(.86,.41,.38),\\(.41,.23,.17)\end{pmatrix}$
N_{g_4}	$\begin{pmatrix}(.82,.41,.38),\\(.41,.21,.11)\end{pmatrix}$	$\begin{pmatrix}(.91,.61,.53),\\(.38,.21,.22)\end{pmatrix}$	$\begin{pmatrix}(.73,.61,.48),\\(.25,.31,.18)\end{pmatrix}$
N_{g_5}	$\begin{pmatrix}(.73,.61,.53),\\(.41,.21,.18)\end{pmatrix}$	$\begin{pmatrix}(.83,.51,.68),\\(.31,.21,.15)\end{pmatrix}$	$\begin{pmatrix}(.73,.61,.58),\\(.41,.23,.16)\end{pmatrix}$

Table 3. Sq-LDF expert evaluation information.

	$\eth_4$	$\eth_5$
N_{g_1}	$\begin{pmatrix}(.81,.41,.32),\\(.31,.23,.31)\end{pmatrix}$	$\begin{pmatrix}(.78,.17,.45),\\(.33,.12,.27)\end{pmatrix}$
N_{g_2}	$\begin{pmatrix}(.78,.45,.31),\\(.51,.11,.18)\end{pmatrix}$	$\begin{pmatrix}(.83,.21,.43),\\(.72,.13,.14)\end{pmatrix}$
N_{g_3}	$\begin{pmatrix}(.89,.38,.46),\\(.46,.32,.11)\end{pmatrix}$	$\begin{pmatrix}(.83,.21,.38),\\(.51,.18,.17)\end{pmatrix}$
N_{g_4}	$\begin{pmatrix}(.83,.63,.47),\\(.38,.21,.17)\end{pmatrix}$	$\begin{pmatrix}(.76,.58,.43),\\(.31,.23,.33)\end{pmatrix}$
N_{g_5}	$\begin{pmatrix}(.81,.32,.58),\\(.38,.31,.14)\end{pmatrix}$	$\begin{pmatrix}(.93,.21,.41),\\(.41,.21,.13)\end{pmatrix}$

Step-2 The consider expert information is benefit type so we do not need to standardized the information.

Step-3 Assume that the decision-maker have a weight vector with the following values:
expert 1 viewpoint ${}^{1}\complement = (0.1, 0.042, .045, .02, 0.01)^T$,
expert 2 viewpoint ${}^{2}\complement = (0.09, .02, .02, .01, .005)^T$,
expert 3 viewpoint ${}^{3}\complement = (0.08, .06, .05, .04, .007)^T$,
expert 4 viewpoint ${}^{3}\complement = (0.07, .03, .04, .01, .04)^T$,
expert 5 viewpoint ${}^{3}\complement = (0.11, .058, .044, .02, .044)^T$,
yielding final WV $=(0.45, .210, .200, .100, .040)^T$, $\sum_{z=1}^{5} \complement_z = 1$.

Step-4 Now we'll utilize the proposed aggregation operators Śq-LDFWA, Śq-LDFOWA, Sq-LDFGA and Sq-LDFOWG to compute the integrated (combine) Sq-LDF data computed in Table 4.

Table 4. Sq-LDF aggregated information.

	Śq-LDFWA		*Śq-LDFOWA*
N_{g1}	$((.866, .327, .444), (.339, .270, .179))$	N_{g1}	$((.901, .372, .412), (.380, .251, .166))$
N_{g2}	$((.792, .432, .470), (.314, .169, .217))$	N_{g2}	$((.812, .431, .480), (.322, .166, .218))$
N_{g3}	$((.711, .457, 402), (.382, .259, .137))$	N_{g3}	$((.775, .446, 402), (.383, .247, .152))$
N_{g4}	$((.827, .426, .364), (.411, .211, .205))$	N_{g4}	$((.821, .418, .367), (.456, .182, .176))$
N_{g5}	$((.811, .211, .427), (.517, .144, .212))$	N_{g5}	$((.862, .206, .416), (.612, .155, .153))$
	Śq-LDFWGA		*Śq-LDFOWGA*
N_{g1}	$((.843, .327, .094), (.316, .270, .005))$	N_{g1}	$((.873, .372, .402), (.363, .251, .004))$
N_{g2}	$((.777, .432, .111), (.275, .169, .005))$	N_{g2}	$((.790, .431, .447), (.280, .166, .005))$
N_{g3}	$((.673, .457, .070), (.368, .259, .002))$	N_{g3}	$((.728, .446, .383), (.369, .247, .002))$
N_{g4}	$((.821, .426, .060), (.383, .221, .002))$	N_{g4}	$((.813, .418, .390), (.434, .182, .007))$
N_{g5}	$((.804, .211, .079), (.425, .144, .005))$	N_{g5}	$((.849, .206, .397), (.540, .155, .004))$

Step-5 Now the scores are computed as follows in Tables 5–8.

Table 5. The Score detail under Sq-LDFWA Operator.

Score function	$\mho_{N_{g1}}$	$\mho_{N_{g2}}$	$\mho_{N_{g3}}$	$\mho_{N_{g4}}$	$\mho_{N_{g5}}$
	0.054	-0.047	-0.056	0.045	0.149
Ranking sequence	$N_{g_5} > N_{g_1} > N_{g4} > N_{g_2} > N_{g_3}$				
Quardratic score function	$\Theta_{N_{g1}}$	$\Theta_{N_{g2}}$	$\Theta_{N_{g3}}$	$\Theta_{N_{g4}}$	$\Theta_{N_{g5}}$
	0.224	0.110	0.070	0.188	0.224
Ranking sequence	$N_{g_5} > N_{g_1} > N_{g4} > N_{g_2} > N_{g_3}$				
Expectation *score function*	$\vartheta_{N_{g1}}$	$\vartheta_{N_{g2}}$	$\vartheta_{N_{g3}}$	$\vartheta_{N_{g4}}$	$\vartheta_{N_{g5}}$
	0.351	0.318	0.315	0.348	0.383
Ranking sequence	$N_{g_5} > N_{g_1} > N_{g4} > N_{g_2} > N_{g_3}$				

Table 6. The Score detail under Sq-LDFOWA Operator.

Score function	$\mho_{N_{g1}}$	$\mho_{N_{g2}}$	$\mho_{N_{g3}}$	$\mho_{N_{g4}}$	$\mho_{N_{g5}}$
	0.076	-0.040	-0.018	0.060	0.231
Ranking sequence	$N_{g_5} > N_{g_1} > N_{g4} > N_{g_2} > N_{g_3}$				
Quardratic score function	$\Theta_{N_{g1}}$	$\Theta_{N_{g2}}$	$\Theta_{N_{g3}}$	$\Theta_{N_{g4}}$	$\Theta_{N_{g5}}$
	0.250	0.118	0.116	0.185	0.278
Ranking sequence	$N_{g_5} > N_{g_1} > N_{g4} > N_{g_2} > N_{g_3}$				
Expectation *score function*	$\vartheta_{N_{g1}}$	$\vartheta_{N_{g2}}$	$\vartheta_{N_{g3}}$	$\vartheta_{N_{g4}}$	$\vartheta_{N_{g5}}$
	0.358	0.319	0.326	0.353	0.405
Ranking sequence	$N_{g_5} > N_{g_1} > N_{g4} > N_{g_2} > N_{g_3}$				

Table 7. The Score detail under Sq-LDFWG Operator.

Score function	$\mho_{N_{g1}}$ 0.217	$\mho_{N_{g2}}$ 0.125	$\mho_{N_{g3}}$ 0.090	$\mho_{N_{g4}}$ 0.191	$\mho_{N_{g5}}$ 0.293
Ranking sequence	$N_{g_5} > N_{g_1} > N_{g4} > N_{g_2} > N_{g_3}$				
Quardratic score function	$\Theta_{N_{g1}}$ 0.202	$\Theta_{N_{g2}}$ 0.094	$\Theta_{N_{g3}}$ 0.042	$\Theta_{N_{g4}}$ 0.173	$\Theta_{N_{g5}}$ 0.212
Ranking sequence	$N_{g_5} > N_{g_1} > N_{g4} > N_{g_2} > N_{g_3}$				
Expectation *score function*	$\vartheta_{N_{g1}}$ 0.678	$\vartheta_{N_{g2}}$ 0.645	$\vartheta_{N_{g3}}$ 0.640	$\vartheta_{N_{g4}}$ 0.673	$\vartheta_{N_{g5}}$ 0.704
Ranking sequence	$N_{g_5} > N_{g_1} > N_{g4} > N_{g_2} > N_{g_3}$				

Table 8. The Score detail under Sq-LDFOWG Operator.

Score function	$\mho_{N_{g1}}$ 0.065	$\mho_{N_{g2}}$ -0.035	$\mho_{N_{g3}}$ -0.033	$\mho_{N_{g4}}$ 0.040	$\mho_{N_{g5}}$ 0.200
Ranking sequence	$N_{g_5} > N_{g_1} > N_{g4} > N_{g_2} > N_{g_3}$				
Quardratic score function	$\Theta_{N_{g1}}$ 0.228	$\Theta_{N_{g2}}$ 0.101	$\Theta_{N_{g3}}$ 0.087	$\Theta_{N_{g4}}$ 0.167	$\Theta_{N_{g5}}$ 0.266
Ranking sequence	$N_{g_5} > N_{g_1} > N_{g4} > N_{g_2} > N_{g_3}$				
Expectation *score function*	$\vartheta_{N_{g1}}$ 0.685	$\vartheta_{N_{g2}}$ 0.646	$\vartheta_{N_{g3}}$ 0.652	$\vartheta_{N_{g4}}$ 0.677	$\vartheta_{N_{g5}}$ 0.730
Ranking sequence	$N_{g_5} > N_{g_1} > N_{g4} > N_{g_2} > N_{g_3}$				

Step-6 The ranking reslts of the considered alternatives are as follows in Tables 9 and 10:

Table 9. Ranking.

Developed operators	Score	Quardratic score
Sq-LDFWA	$N_{g_5} > N_{g_1} > N_{g4} > N_{g_2} > N_{g_3}$	$N_{g_5} > N_{g_1} > N_{g4} > N_{g_2} > N_{g_3}$
Sq-LDFOWA	$N_{g_5} > N_{g_1} > N_{g4} > N_{g_2} > N_{g_3}$	$N_{g_5} > N_{g_1} > N_{g4} > N_{g_2} > N_{g_3}$
Sq-LDFWG	$N_{g_5} > N_{g_1} > N_{g4} > N_{g_2} > N_{g_3}$	$N_{g_5} > N_{g_1} > N_{g4} > N_{g_2} > N_{g_3}$
Sq-LDFOWG	$N_{g_5} > N_{g_1} > N_{g4} > N_{g_2} > N_{g_3}$	$N_{g_5} > N_{g_1} > N_{g4} > N_{g_2} > N_{g_3}$

Table 10. Ranking.

Developed operators	Expectation score
Sq-LDFWA	$N_{g_5} > N_{g_1} > N_{g4} > N_{g_2} > N_{g_3}$
Sq-LDFOWA	$N_{g_5} > N_{g_1} > N_{g4} > N_{g_2} > N_{g_3}$
Sq-LDFWG	$N_{g_5} > N_{g_1} > N_{g4} > N_{g_2} > N_{g_3}$
Sq-LDFOWG	$N_{g_5} > N_{g_1} > N_{g4} > N_{g_2} > N_{g_3}$

Step-7 The above tables shows that $\wp_5$ is the best alternative (Due to the fact that the scoring functions for the data in Tables 5–8 for quadratic score function are not identical, hence they are sufficient to identify the optimal choice).

Step-8 End.

6. Comparative analysis and discussion

This section compare the proposed Śq-LDF aggregation operators with the existing methodology in the literature described in [42], demonstrating their ability to handle physical-world decision making problems under complex uncertainty. Because of the qth power, this notion is impressive in that it covers the valuation spaces of IFSs, SFSs, q-ROFSs, LDFSs and SLDFSs.

Cogency and integrity of the propose method: Our approach is adaptable and suited for all forms of input data. The proposed model is effective in dealing with uncertainty. This technique covers the areas of IFS, SFS, q-ROFS, LDFS and SLDFS with the addition of qth degree of CFs. By increasing the qth degree of factors, more membership, neutral and non-membership space is created, as well as the physical layout. We can utilize our strategy in a diversity of situations. we're using it in selection of required best university. The proposed Śq-LDFS may be simply modified to produce a variety of outputs.

Score function impact: We generalized and then implemented the previously described three types of score functions consists of three membership grades with their related control parameters, SF, QSF and ESF. Allow for a somewhat variable result because each SF has its own observation and ordering techniques. Tables 4–7 shows that the SF, ESF and QSF rankings differ slightly from one another. However, it's worth noting that the end result from both algorithms is practically identical for all scoring functions.

Aggregation versatility with various inputs and outputs: Due to qth power of control factors and the three membership grades increases grade space and can differ build on the scenarios in MADM approaches, this approach is significantly more versatile than others. It can also be utilized for a variety of input and output informations where the spherical and q-linear Diophantine fuzzy sets fail to fulfill their requirements.

Comparison of the suggested method to existing approaches and its superiority: Because Sq-LDFS handles qth simulations, it takes up a lot of space when compared to IFSs, SFSs, q-ROFSs, LDFSs and SLDFSs. [3] introduced q-LDFSs with additional qth degree, although q-LDFSs have some limitations and cannot handle the problem related to three grades. We expand the concept of q-LDFSs by merging it with SFS and proposed Sq-LDFSs to fill this knowledge gap. The SLDFS only serves for $q = 1$, but the strategy we propose serves for $q \geq 1$.

The proposed approach and MADM difficulties are inextricably linked. Tables 11–13 shows the comparison of aggregation operation values based on the Sq-LDFWA, Sq-LDFOWA, Sq-LDFWGA and Sq-LDFOWGA and their graphical representation is in Figures 4 and 5.

Table 11. Analysis of score using existing methods.

	$\mho_{N_{g1}}$	$\mho_{N_{g2}}$	$\mho_{N_{g3}}$	$\mho_{N_{g4}}$	$\mho_{N_{g5}}$
$Sq-LDFWA$	0.054	−0.047	−0.056	0.045	0.149
$Sq-LDFOWA$	0.076	−0.040	−0.018	0.060	0.231
$Sq-LDFWG$	0.217	0.125	0.090	0.191	0.293
$Sq-LDFOWG$	0.065	−0.035	−0.033	0.040	0.200
$\acute{S}LDFWA$ $(existing)$	−0.034	−0.109	−0.024	−0.023	−0.160
$\acute{S}LDFWG$ $(existing)$	−0.043	−0.117	−0.027	−0.920	−0.184

Table 12. Analysis of Quardratic score using existing methods.

	$\mho_{N_{g1}}$	$\mho_{N_{g2}}$	$\mho_{N_{g3}}$	$\mho_{N_{g4}}$	$\mho_{N_{g5}}$
$Sq-LDFWA$	0.224	0.110	0.070	0.188	0.224
$Sq-LDFOWA$	0.254	0.122	0.122	0.187	0.290
$Sq-LDFWG$	0.298	0.203	0.121	0.246	0.300
$Sq-LDFOWG$	0.232	0.120	0.093	0.171	0.273
$\acute{S}LDFWA$ $(existing)$	0.175	0.100	0.270	0.147	0.037
$\acute{S}LDFWG$ $(existing)$	0.158	0.085	0.237	0.122	0.008

Table 13. Analysis of Expectation score using existing methods.

	$\mho_{N_{g1}}$	$\mho_{N_{g2}}$	$\mho_{N_{g3}}$	$\mho_{N_{g4}}$	$\mho_{N_{g5}}$
$Sq-LDFWA$	0.351	0.318	0.315	0.348	0.383
$Sq-LDFOWA$	0.359	0.320	0.327	0.353	0.410
$Sq-LDFWG$	0.406	0.375	0.363	0.397	0.431
$Sq-LDFOWG$	0.355	0.322	0.322	0.347	0.400
$\acute{S}LDFWA$ $(existing)$	0.655	0.630	0.675	0.644	0.613
$\acute{S}LDFWG$ $(existing)$	0.652	0.628	0.667	0.636	0.605

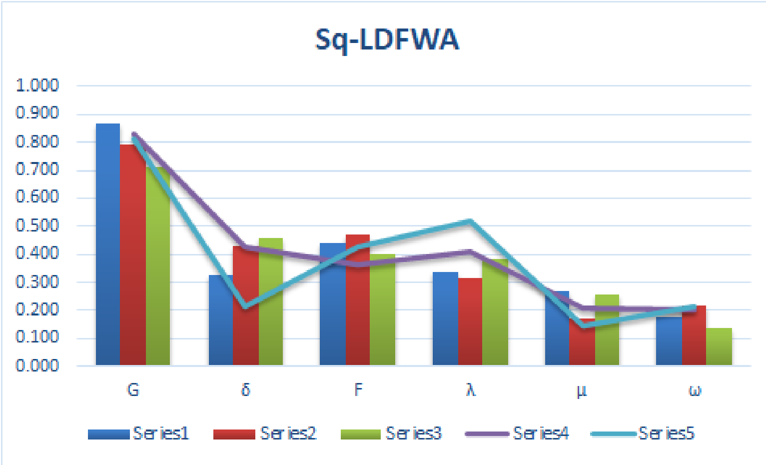

Figure 4. Ranking related to Sq-LDFWA.

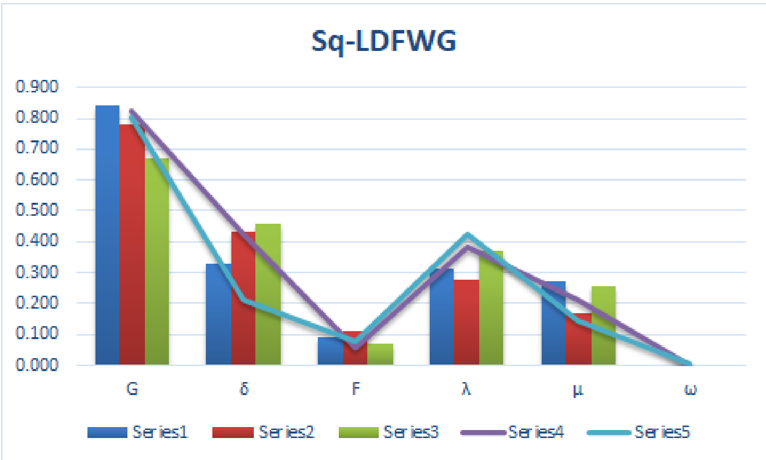

Figure 5. Ranking related to Sq-LDFWG.

We make comparison of SLDF weighted and geometric operators with Sq-LDF weighted and geometric operators. The Tables 11–13 shows the better results than the results of SLDFSs.

The fuzzy set theory, whose popularity has increased ever since Zadeh introduced it, is supported by the work in this paper. Specific academics expanded fuzzy set theory and the most significant of which are IFSs, PyFSs and SFSs. In order to address uncertainty in real-life problems are challenging to resolve utilizing fuzzy models. In 2019 the structure of Pythagorean fuzzy sets extended by [3] to initiate the concept of q-LDFS, in which they introduced the role of control factors, which hold the condition $0 \leq \lambda^q \eth + \mu^q F \leq 1, \forall \wp \in U$ and $0 \leq \lambda^q + \mu^q \leq 1$. However the total sum of degrees with scalar product with control factors provided by DM may be greater than one, i.e. $\lambda^q + \mu^q > 1$, opposing the q-LDFS constraint. Consequently, q-LDFS, SFSs and SLDFSs are restricted to meet his target in terms of degrees, information are as follows in Table 14:

Table 14. Detail on comparative study of Sq-LDFS to existing techniques.

Family of sets	Remarks	Framework
FS	Incapable to deal with the degree of non-membership F	$\times$
IFS	Cann't fulfill the condition $\eth + F > 1$	$\times$
q-ROFS	Incapable to treat smaller"q" power in this state ,$\eth^q + F^q > 1$ for $\eth = 1 = F$	$\times$
SFS	Incapable for this state, $\eth + \delta + F > 1$ for $\eth = \delta = F = 1$	$\times$
SLDFS	This collection take into account the of condition $0 \leq \lambda\eth + \mu\delta + \omega F \leq 1$, as well as the influence of control factors. SLDF operators are the sole existing method,which we compare to our suggested method.	$\times$

It limits the MADM and has an impact on the best decision. We offer the innovative idea of the Sq-LDFS, which is capable of dealing with these circumstances and eliminate these contradiction.

7. Conclusions

The manuscript briefly demonstrated how the Sq-LDFS framework extends all existing theories and provides a strong foundation with no limitations. The formal definition of Sq-LDFS was stated which is generalization of q-linear Diophantine fuzzy set by merging it with spherical fuzzy set to enhance the memberships space. Under the Sq-LDF context, set theoretical operations were introduced, and several aggregation operators were established. Some interesting properties of the proposed aggregation operators were explored. Furthermore, a MADM technique based on suggested aggregating operators and scoring functions was established. A case study was offered to demonstrate how the suggested strategy be used. As a limitation on our study, we only take into account five alternatives in order to demonstrate the veracity of the suggested strategy. The suggested technique works where the SFSs, q-linear Diophantine fuzzy set did not work. The SLDFS only works for q=1 but Sq-LDFS works for $q \geq 1$. The methodology of the suggested technique may be converted into a computer program, allowing us to conduct our research for a small number of qualities and alternatives while using huge data and taking into account additional factors. Future research goals include investigating additional aggregating operators like Hamacher and Bonferroni, similarity and distance metrics, and extending the suggested operators to the Archimedean norm.

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Conflict of interest

The authors declare that they have no conflict of interest regarding the publication of the research article.

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