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Research article

On fuzzy differential subordination associated with q-difference operator

Shujaat Ali Shah[1,*], **Ekram Elsayed Ali**[2,3], **Adriana Cătaş**[4,*] **and Abeer M. Albalahi**[2]

[1] Department of Mathematics and Statistics, Quaid-e-Awam University of Engineering, Science and Technology, Nawabshah 67450, Pakistan

[2] Department of Mathematics, College of Science, University of Ha'il, Ha'il 81451, Saudi Arabia

[3] Department of Mathematics and Computer Science, Faculty of Science, Port Said University, Port Said 42521, Egypt

[4] Department of Mathematics and Computer Science, University of Oradea 1 University St., 410087 Oradea, Romania

* **Correspondence:** Email: shahglike@yahoo.com, acatas@gmail.com.

Abstract: This article presents the link between the fuzzy differential subordination and the q-theory of functions. We use the fuzzy differential subordination to define certain subclasses of univalent functions associated with the q-difference operator. Certain inclusion results are proved, and invariance of the q-Bernardi integral operator for certain classes is discussed.

Keywords: analytic functions; q-difference operator; fuzzy differential subordination; the q-Srivastava-Attiya operator; the q-multiplier transformation
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1. Introduction

Let $\mathbf{A}(\Omega)$ denote the class of analytic functions $\mathfrak{f}(\varsigma)$ in the open unit disk $\Omega = \{\varsigma : |\varsigma| < 1\}$. The class A_n contains the functions $\mathfrak{f} \in \mathbf{A}(\Omega)$ having the series of the form

$$\mathfrak{f}(\varsigma) = \varsigma + a_{n+1}\varsigma^{n+1} + a_{n+2}\varsigma^{n+2} + ..., \quad (\varsigma \in \Omega). \tag{1.1}$$

For $n = 1$, we have $A_1 = A$, the class of normalized analytic functions in Ω. Let S, S^* and C denote the subclasses of A of univalent functions, starlike functions and convex functions, respectively. The class of Carathéodory functions is denoted by P. Let $\mathfrak{f}, g \in A$. Then, $\mathfrak{f} \prec g$ denotes the subordination of functions $\mathfrak{f}$ and g, defined as $\mathfrak{f}(\varsigma) = g(w(\varsigma))$, where $w(\varsigma)$ is a Schwartz function in Ω (see [1]). In [2, 3], the authors introduced and studied the concept of differential subordination. Fuzzy subordination and fuzzy differential subordination was first studied by G. I. Oros and Gh. Oros; see [4, 5]. Fuzzy differential subordination theory represents a generalization of the classical concept of differential subordination which emerged in recent years as a result of embedding the concept of fuzzy sets into geometric function theory. Several authors contributed in the study of fuzzy differential subordination; for examples, see [6–19]. Here, we give an overview of some useful basic concepts related to fuzzy differential subordination and q-calculus.

Definition 1.1. *[20] Let $Y \neq \phi$, and F is a mapping from Y to* $[0,1]$. *Then, F is called a fuzzy subset on Y.*

Alternatively, the fuzzy subset is also defined as the following.

Definition 1.2. *[20] A pair (I, F_I) is called a fuzzy subset on Y, where $F_I : Y \to [0,1]$ is the membership function of the fuzzy set (I, F_I), and $I = \{x \in Y : 0 < F_I(x) \leq 1\} = \sup(I, F_I)$ is the support of fuzzy set (I, F_I).*

Definition 1.3. *[20] Fuzzy subsets (I, F_I) and (J, F_J) of Y are equal if and only if $I = J$, whereas $(J, F_J) \subseteq (I, F_I)$ if and only if $F_I(x) \leq F_J(x)$, $x \in Y$.*

Definition 1.4. *[5] Let $\mathbb{D} \subset \mathbb{C}$ and $\varsigma_0 \in \mathbb{D}$ be a fixed point. Then, analytic function $\mathfrak{f}_1$ is fuzzy subordinate to the analytic function $\mathfrak{f}_2$ (written as $\mathfrak{f}_1 \prec_F \mathfrak{f}_2$ (or $\mathfrak{f}_1(\varsigma) \prec_F \mathfrak{f}_2(\varsigma)$)) if*

$$\mathfrak{f}_1(\varsigma_0) = \mathfrak{f}_2(\varsigma_0) \text{ and } F(\mathfrak{f}_1(\varsigma)) \leq F(\mathfrak{f}_2(\varsigma)), \varsigma \in \mathbb{D}.$$

Remark 1.1. *We can assume such a function $\Gamma_i : \mathbb{C} \to [0,1]$, $(i = 1,2,3,4)$, as any of the following.*

$$\Gamma_1(\varsigma) = \frac{|\varsigma|}{1+|\varsigma|}, \Gamma_2(\varsigma) = \frac{1}{1+|\varsigma|}, \Gamma_3(\varsigma) = |\cos|\varsigma||, \Gamma_4(\varsigma) = |\sin|\varsigma||.$$

Remark 1.2. *If $\mathbb{D} = \Omega$ in Definition 1.4, then the fuzzy subordination coincides with the classical subordination.*

In [21], the author studied the q-difference operator, which is defined by

$$\mathfrak{D}_q \mathfrak{f}(\varsigma) = \frac{\mathfrak{f}(\varsigma) - \mathfrak{f}(q\varsigma)}{(1-q)\varsigma}, \quad q \neq 1, \ \varsigma \neq 0, \tag{1.2}$$

for $q \in (0,1)$. Clearly, the q-difference operator becomes the well known differential operator for $q \to 1^-$.

For $n \in \mathbb{N} = \{1,2,3,..\}$ and $\varsigma \in \Omega$, we note that

$$\mathfrak{D}_q \left\{ \sum_{n=1}^{\infty} a_n \varsigma^n \right\} = \sum_{n=1}^{\infty} [n]_q \varsigma^{n-1}, \tag{1.3}$$

with

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + ... \tag{1.4}$$

Some important rules of $\mathfrak{D}_q$ are given as the following.

$$\mathfrak{D}_q\left(x\mathfrak{f}_1(\varsigma) \pm y\mathfrak{f}_2(\varsigma)\right) = x\mathfrak{D}_q\mathfrak{f}_1(\varsigma) \pm y\mathfrak{D}_q\mathfrak{f}_2(\varsigma)$$

$$\mathfrak{D}_q\left(\mathfrak{f}_1(\varsigma)\mathfrak{f}_2(\varsigma)\right) = \mathfrak{f}_1(q\varsigma)\mathfrak{D}_q\left(\mathfrak{f}_2(\varsigma)\right) + \mathfrak{f}_2(\varsigma)\mathfrak{D}_q\left(\mathfrak{f}_1(\varsigma)\right)$$

$$\mathfrak{D}_q\left(\frac{\mathfrak{f}_1(\varsigma)}{\mathfrak{f}_2(\varsigma)}\right) = \frac{\mathfrak{D}_q\left(\mathfrak{f}_1(\varsigma)\right)\mathfrak{f}_2(\varsigma) - \mathfrak{f}_1(\varsigma)\mathfrak{D}_q\left(\mathfrak{f}_2(\varsigma)\right)}{\mathfrak{f}_2(q\varsigma)\mathfrak{f}_2(\varsigma)}, \quad \mathfrak{f}_2(q\varsigma)\mathfrak{f}_2(\varsigma) \neq 0.$$

$$\mathfrak{D}_q\left(\log \mathfrak{f}_1(\varsigma)\right) = \frac{\ln q\,\mathfrak{D}_q\left(\mathfrak{f}_1(\varsigma)\right)}{(q-1)\,\mathfrak{f}_1(\varsigma)}.$$

In [22], the authors for the first time discussed some properties of the function theory in terms of q-theory.

Shah and Noor [23] generalized the Srivastava-Attiya operator and the multiplier transformation in terms of the q-calculus. Let $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $s \in \mathbb{C}$ when $|\varsigma| < 1$, and $\Re(s) > 1$ when $|\varsigma| = 1$. The q-Srivastava-Attiya operator $J_q^{s,b} : A \to A$ is given by

$$\begin{aligned} J_q^{s,b}\mathfrak{f}(\varsigma) &= \psi_q(s,b;\varsigma) * \mathfrak{f}(\varsigma) \\ &= \varsigma + \sum_{n=2}^{\infty}\left(\frac{[1+b]_q}{[n+b]_q}\right)^s a_n\varsigma^n, \end{aligned} \tag{1.5}$$

where

$$\psi_q(s,b;\varsigma) = \varsigma + \sum_{n=2}^{\infty}\left(\frac{[1+b]_q}{[n+b]_q}\right)^s \varsigma^n,$$

and " $*$ " denotes convolution (or Hadamard product).

It is noted that, if $q \to 1^-$, then the operator $J_{q,b}^s$ coincides with the Srivastava-Attiya operator; see [24]. Furthermore, this operator generalizes the q-Alexander operator, the q-Bernardi operator and the q-Libera operator; we refer to [25].

For $\mathfrak{f} \in A$, $b > -1$, $q \in (0,1)$ and real number s, the operator $I_q^{s,b} : A \to A$ is defined as follows:

$$I_q^{s,b}\mathfrak{f}(\varsigma) = \varsigma + \sum_{n=2}^{\infty}\left(\frac{[n+b]_q}{[1+b]_q}\right)^s a_n\varsigma^n. \tag{1.6}$$

This operator is known as the q-analogue of multiplier transformation. The operator $I_q^{s,b}$ generalizes a well-known Salagean q-differential operator; see [26].

We use (1.5) and (1.6) to obtain the following identities:

$$\varsigma\mathfrak{D}_q\left(J_q^{s+1,b}\mathfrak{f}(\varsigma)\right) = \left(1 + \frac{[b]_q}{q^b}\right)J_q^{s,b}\mathfrak{f}(\varsigma) - \frac{[b]_q}{q^b}J_q^{s+1,b}\mathfrak{f}(\varsigma). \tag{1.7}$$

$$\varsigma\mathfrak{D}_q\left(I_q^{s,b}\mathfrak{f}(\varsigma)\right) = \left(1 + \frac{[b]_q}{q^b}\right)I_q^{s+1,b}\mathfrak{f}(\varsigma) - \frac{[b]_q}{q^b}I_q^{s,b}\mathfrak{f}(\varsigma). \tag{1.8}$$

We use the notion of fuzzy subordination along with the q-difference operator to define the following classes:

Let T be the class of analytic and univalent convex functions $\mathfrak{g}(\varsigma)$ in Ω with $\mathfrak{g}(0) = 1$ and $Re(\mathfrak{g}(\varsigma)) > 0$ in Ω.

Definition 1.5. *Let $\mathfrak{f} \in A$, non-negative real s, $q \in (0,1)$, $0 \neq \tau \in \mathbb{C}$, $b \in \mathbb{N}$, and $\mathfrak{g}(\varsigma) \in T$. Then, $\mathfrak{f} \in FST_{q,\tau}(\mathfrak{g})$ if and only if*

$$1 + \frac{1}{\tau}\left(\frac{\varsigma\mathfrak{D}_q\mathfrak{f}(\varsigma)}{\mathfrak{f}(\varsigma)} - 1\right) \prec_F \mathfrak{g}(\varsigma),$$

and $\mathfrak{f} \in FCV_{q,\tau}(\mathfrak{g})$ if and only if

$$\varsigma\mathfrak{D}_q\mathfrak{f}(\varsigma) \in FST_{q,\tau}(\mathfrak{g}),$$

where $F : \mathbb{C} \to [0,1]$, and $\mathfrak{D}_q$ is the q-difference operator.

Next, some new classes are defined connecting the q-analogue of the linear operators discussed above with the classes introduced in the previous definition.

Definition 1.6. *Let $\mathfrak{f} \in A$, $b > -1$, s be a real and $0 \neq \tau \in \mathbb{C}$. Then,*

$$\mathfrak{f} \in FST_{q,\tau}^{s,b}(\mathfrak{g}) \text{ if and only if } J_q^{s,b}\mathfrak{f}(\varsigma) \in FST_{q,\tau}(\mathfrak{g}),$$

and

$$\mathfrak{f} \in FCV_{q,\tau}^{s,b}(\mathfrak{g}) \text{ if and only if } J_q^{s,b}\mathfrak{f}(\varsigma) \in FCV_{q,\tau}(\mathfrak{g}).$$

We note that

$$\mathfrak{f} \in FCV_{q,\tau}^{s,b}(\mathfrak{g}) \text{ if and only if } \varsigma\left(\mathfrak{D}_q\mathfrak{f}\right) \in FST_{q,\tau}^{s,b}(\mathfrak{g}). \tag{1.9}$$

Special cases:

(i) If $s = 0$, then $FST_{q,\tau}^{s,b}(\mathfrak{g}) = FST_{q,\tau}(\mathfrak{g})$, and $FCV_{q,\tau}^{s,b}(\mathfrak{g}) = FCV_{q,\tau}(\mathfrak{g})$.

(ii) If $q \to 1^-$ and $\tau = 1$, then the classes $FST_{q,\tau}^{s,b}(\mathfrak{g})$ and $FCV_{q,\tau}^{s,b}(\mathfrak{g})$ coincide with the classes $FST_b^s(\mathfrak{g})$ and $FC_b^s(\mathfrak{g})$ studied by Shah et al. [12].

(iii) If $s = 0$, $q \to 1^-$ and $\tau = 1$, then the classes $FST_{q,\tau}^{s,b}(\mathfrak{g})$ and $FCV_{q,\tau}^{s,b}(\mathfrak{g})$ coincide with the classes $FST(\mathfrak{g})$ and $FC(\mathfrak{g})$ studied by Shah et al. [12].

Definition 1.7. *Let $\mathfrak{f} \in \mathbf{A}$, $b > -1$, s be a real and $0 \neq \tau \in \mathbb{C}$. Then,*

$$\mathfrak{f} \in \widetilde{FST}_{q,\tau}^{s,b}(\mathfrak{g}) \text{ if and only if } I_q^{s,b}\mathfrak{f}(\varsigma) \in FST_{q,\tau}(\mathfrak{g}),$$

and

$$\mathfrak{f} \in \widetilde{FCV}_{q,\tau}^{s,b}(\mathfrak{g}) \text{ if and only if } I_q^{s,b}\mathfrak{f}(\varsigma) \in FCV_{q,\tau}(\mathfrak{g}).$$

It is obvious that

$$\mathfrak{f} \in \widetilde{FCV}_{q,\tau}^{s,b}(\mathfrak{g}) \text{ if and only if } \varsigma\left(\mathfrak{D}_q\mathfrak{f}\right) \in \widetilde{FST}_{q,\tau}^{s,b}(\mathfrak{g}). \tag{1.10}$$

2. Main results

Before the discussion of our main investigations, some required lemmas are given as the following.

Lemma 2.1. *[27] Let $p(\varsigma) = 1 + p_k(\varsigma) + ...$, $k \geq 1$, be analytic in Ω, and let $q(\varsigma) = 1 + c_k(\varsigma) + ...$, be analytic and univalent in $\overline{\Omega}$. If $p(\varsigma) \nprec q(\varsigma)$, then there exist a real number m ($m \geq 1$), $\varsigma_0 \in \Omega$ and $\xi_0 \in \partial\Omega$ such that*

(1) $p(|\varsigma| < |\varsigma_0|) \subset q(\Omega)$.

(2) $p(\varsigma_0) = q(\xi_0)$.

(3) $arg\left(\varsigma_0\mathfrak{D}_q p(\varsigma_0)\right) = arg\left(\xi_0\mathfrak{D}_q q(\xi_0)\right)$.

(4) $\left|\varsigma_0\mathfrak{D}_q p(\varsigma_0)\right| = m\left|\xi_0\mathfrak{D}_q q(\xi_0)\right|$.

Lemma 2.2. *Let $\xi, \lambda \in \mathbb{C}$ with $\xi \neq 0$, and let $\mathfrak{g}(\varsigma)$ be analytic in Ω with $\mathfrak{g}(0) = 1$ and*

$$\Re \{\xi \mathfrak{g}(\varsigma) + \lambda\} > 0. \tag{2.1}$$

If $p(\varsigma) = 1 + p_1\varsigma + p_2\varsigma^2 + ...$ is analytic in Ω, then

$$p(\varsigma) + \frac{\varsigma \mathfrak{D}_q p(\varsigma)}{\xi p(\varsigma) + \lambda} \prec_F \mathfrak{g}(\varsigma) \text{ implies } p(\varsigma) \prec_F \mathfrak{g}(\varsigma),$$

where $F : \mathbb{C} \to [0, 1]$.

Proof. We assume that all the functions under consideration are analytic in the closed disc $\overline{\Omega}$. Suppose on the contrary that $p(\varsigma) \nprec_F \mathfrak{g}(\varsigma)$. According to the Lemma 2.1, there exist a real number $m \geq 1$, $\varsigma_0 \in \Omega$ and $\xi_0 \in \partial\Omega$ such that

$$p(\varsigma_0) + \frac{\varsigma_0 \mathfrak{D}_q p(\varsigma_0)}{\xi p(\varsigma_0) + \lambda} = \mathfrak{g}(\xi_0) + \frac{\xi_0 \mathfrak{D}_q \mathfrak{g}(\xi_0)}{\xi \mathfrak{g}(\xi_0) + \lambda}. \tag{2.2}$$

From (2.1), we can write $arg\left|\xi \mathfrak{g}(\xi_0) + \lambda\right| < \frac{\pi}{2}$, and $\xi_0 \mathfrak{D}_q \mathfrak{g}(\xi_0)$ is in the direction of the outer normal to the convex domain $\mathfrak{g}(\Omega)$. Therefore,

$$p(\varsigma_0) + \frac{\varsigma_0 \mathfrak{D}_q p(\varsigma_0)}{\xi p(\varsigma_0) + \lambda} \notin \mathfrak{g}(\Omega).$$

This is the contradiction to the hypothesis. Hence, $p(\varsigma) \prec_F \mathfrak{g}(\varsigma)$.

Replace $p(\varsigma)$ by $p_\mu(\varsigma) = p(\mu\varsigma)$, and $\mathfrak{g}_\mu(\varsigma) = \mathfrak{g}(\mu\varsigma)$, $0 < \mu < 1$. Due to the theorem, we obtain $p_\mu(\varsigma) \prec_F \mathfrak{g}_\mu(\varsigma)$ for each μ. For $\mu \to 1$, we have $p(\varsigma) \prec_F \mathfrak{g}(\varsigma)$. □

2.1. Inclusion results

Theorem 2.1. *Let $F : \mathbb{C} \to [0, 1]$, $\mathfrak{g} \in T$, $q \in (0, 1)$, $s > 0$, $b \in \mathbb{N}$, and $0 \neq \tau \in \mathbb{C}$. Then,*

$$FST_{q,\tau}^{s,b}(\mathfrak{g}) \subset FST_{q,\tau}^{s+1,b}(\mathfrak{g}),$$

for

$$\Re\left\{\tau(\mathfrak{g}(\varsigma) - 1) + \left(1 + \epsilon_q\right)\right\} > 0, \text{ with } \epsilon_q = \frac{[b]_q}{q^b}.$$

Proof. Let $\mathfrak{f} \in FST_{q,\tau}^{s,b}(\mathfrak{g})$. For $p(\varsigma)$ analytic in Ω with $p(0) = 1$, we set

$$p(\varsigma) = \frac{1}{\tau}\left\{\frac{\varsigma \mathfrak{D}_q\left(J_q^{s+1,b}\mathfrak{f}(\varsigma)\right)}{J_q^{s+1,b}\mathfrak{f}(\varsigma)} - (1 - \tau)\right\}. \tag{2.3}$$

We use identity (1.7) and (2.3) to get

$$p(\varsigma) = \frac{1}{\tau}\left[\left\{\left(1 + \frac{[b]_q}{q^b}\right)\frac{J_q^{s,b}\mathfrak{f}(\varsigma)}{J_q^{s+1,b}\mathfrak{f}(\varsigma)} + \frac{[b]_q}{q^b}\right\} - (1 - \tau)\right],$$

and this implies

$$\left(1+\epsilon_q\right)\frac{J_q^{s,b}\mathfrak{f}(\varsigma)}{J_q^{s+1,b}\mathfrak{f}(\varsigma)}=\tau\left(p(\varsigma)-1\right)+\left(1+\epsilon_q\right),\quad\left(\text{for }\epsilon_q=\frac{[b]_q}{q^b}\right).$$

The q-logarithmic differentiation and (2.3) yields

$$\frac{1}{\tau}\left\{\frac{\varsigma\mathfrak{D}_q\left(J_q^{s,b}\mathfrak{f}(\varsigma)\right)}{J_q^{s,b}\mathfrak{f}(\varsigma)}-(1-\tau)\right\}=p(\varsigma)+\frac{\varsigma\mathfrak{D}_qp(\varsigma)}{\tau\left(p(\varsigma)-1\right)+\left(1+\epsilon_q\right)}.\tag{2.4}$$

Since $\mathfrak{f}\in FST_{q,\tau}^{s,b}(\mathfrak{g})$, from (2.3) we have

$$p(\varsigma)+\frac{\varsigma\mathfrak{D}_qp(\varsigma)}{\tau\left(p(\varsigma)-1\right)+\left(1+\epsilon_q\right)}\prec_F\mathfrak{g}(\varsigma),\tag{2.5}$$

for $\mathfrak{g}\in T$. If we assume

$$\Re\left\{\tau\left(\mathfrak{g}(\varsigma)-1\right)+\left(1+\epsilon_q\right)\right\}>0,$$

then by using Lemma 2.2 and (2.5), we obtain that $p(\varsigma)\prec_F\mathfrak{g}(\varsigma)$ implies $\mathfrak{f}\in FST_{q,\tau}^{s+1,b}(\mathfrak{g})$. □

Theorem 2.2. *Let* $F:\mathbb{C}\to[0,1]$, $\mathfrak{g}\in T$, $q\in(0,1)$, $s>0$, $b\in\mathbb{N}$, *and* $0\neq\tau\in\mathbb{C}$. *Then,*

$$FCV_{q,\tau}^{s,b}(\mathfrak{g})\subset FCV_{q,\tau}^{s+1,b}(\mathfrak{g}),$$

for

$$\Re\left\{\tau\left(\mathfrak{g}(\varsigma)-1\right)+\left(1+\epsilon_q\right)\right\}>0,\quad\left(\textit{for }\epsilon_q=\frac{[b]_q}{q^b}\right).$$

Proof. Let $\mathfrak{f}\in FCV_{q,\tau}^{s,b}(\mathfrak{g})$. Then, by (1.9), $\varsigma\mathfrak{D}_q\mathfrak{f}\in FST_{q,\tau}^{s,b}(\mathfrak{g})$. This implies, by using Theorem 2.1, $\varsigma\mathfrak{D}_q\mathfrak{f}\in FST_{q,\tau}^{s+1,b}(\mathfrak{g})$. Again by (1.9), we get $\mathfrak{f}\in FCV_{q,\tau}^{s+1,b}(\mathfrak{g})$. □

When $q\to1^-$ and $\tau=1$, we obtain the results proved in [12] and given as the following.

Corollary 2.1. *Let* $\mathfrak{g}(\varsigma)\in T$, $s>0$, *and* $b\in\mathbb{N}$. *Then,*

$$FST_b^s(\mathfrak{g})\subset FST_b^{s+1}(\mathfrak{g}),\ \textit{and}\ FCV_b^s(\mathfrak{g})\subset FCV_b^{s+1}(\mathfrak{g}).$$

The following inclusion results can easily be proved by using similar arguments as used before.

Theorem 2.3. *Let* $F:\mathbb{C}\to[0,1]$, $\mathfrak{g}\in T$, $q\in(0,1)$, $s>0$, $b\in\mathbb{N}$, *and* $0\neq\tau\in\mathbb{C}$. *Then,*

$$\widetilde{FST}_{q,\tau}^{s+1,b}(\mathfrak{g})\subset\widetilde{FST}_{q,\tau}^{s,b}(\mathfrak{g}),\ \textit{and}\ \widetilde{FCV}_{q,\tau}^{s+1,b}(\mathfrak{g})\subset\widetilde{FCV}_{q,\tau}^{s,b}(\mathfrak{g}),$$

for

$$\Re\left\{\tau\left(\mathfrak{g}(\varsigma)-1\right)+\left(1+\epsilon_q\right)\right\}>0,\ \textit{with}\ \epsilon_q=\frac{[b]_q}{q^b}.$$

2.2. Integral preserving property

Theorem 2.4. *Let $F : \mathbb{C} \to [0,1]$, $\mathfrak{g} \in T$, $q \in (0,1)$, $s > 0$, $b \in \mathbb{N}$ and $0 \neq \tau \in \mathbb{C}$, and let $\mathfrak{f} \in FST^{s,b}_{q,\tau}(\mathfrak{g})$. Then, $F_{q,b} \in FST^{s,b}_{q,\tau}(\mathfrak{g})$, where*

$$F_{q,b}(\varsigma) = \frac{[1+b]_q}{\varsigma^b} \int_0^{\varsigma} t^{b-1}\mathfrak{f}(t)\mathfrak{D}_q t, \tag{2.6}$$

for

$$\Re\left\{\tau\left(\mathfrak{g}(\varsigma)-1\right)+\left(1+[b]_q\right)\right\} > 0.$$

Proof. Let $\mathfrak{f} \in FST^{s,b}_{q,\tau}(\mathfrak{g})$. For $p(\varsigma)$ analytic in Ω with $p(0) = 1$, we set

$$p(\varsigma) = \frac{1}{\tau}\left\{\frac{\varsigma\mathfrak{D}_q\left(J_q^{s+1,b}\mathfrak{f}(\varsigma)\right)}{J_q^{s+1,b}\mathfrak{f}(\varsigma)} + \tau - 1\right\}. \tag{2.7}$$

From (2.6), we can write

$$\frac{\mathfrak{D}_q\left(\varsigma^b F_{q,b}(\varsigma)\right)}{[1+b]_q} = \varsigma^{b-1}\mathfrak{f}(\varsigma).$$

We use the product rule of the q-difference operator to get

$$\varsigma\mathfrak{D}_q F_{q,b}(\varsigma) = \left(1+\frac{[b]_q}{q^b}\right)\mathfrak{f}(\varsigma) - [b]_q F_{q,b}(\varsigma). \tag{2.8}$$

From (2.7), (2.8) and (1.5), we have

$$\left(1+\frac{[b]_q}{q^b}\right)\frac{J_q^{s,b}\mathfrak{f}(\varsigma)}{J_q^{s+1,b}F_{q,b}(\varsigma)} = \tau\left(p(\varsigma)-1\right)+\left(1+[b]_q\right).$$

We take q-logarithmic differentiation to get

$$p(\varsigma) + \frac{\varsigma\mathfrak{D}_q p(\varsigma)}{\tau\left(p(\varsigma)-1\right)+\left(1+[b]_q\right)} \prec_F \mathfrak{g}(\varsigma),$$

since $\mathfrak{f} \in FST^{s,b}_{q,\tau}(\mathfrak{g})$, and $\mathfrak{g} \in T$. If we assume

$$\Re\left\{\tau\left(\mathfrak{g}(\varsigma)-1\right)+\left(1+[b]_q\right)\right\} > 0,$$

on making use of Lemma 2.2, we obtain $p(\varsigma) \prec_F \mathfrak{g}(\varsigma)$, and this completes the proof. □

Remark 2.1. *(i) Using a similar technique, we can prove the above integral preserving property for the classes $FCV^{s,b}_{q,\tau}(\mathfrak{g})$, $\widetilde{FST}^{s,b}_{q,\tau}(\mathfrak{g})$ and $\widetilde{FCV}^{s,b}_{q,\tau}(\mathfrak{g})$.*

(ii) In particular, the classes $FST^{s}_{b}(\mathfrak{g})$ and $FCV^{s}_{b}(\mathfrak{g})$, defined in [12], preserve under the q-Bernardi integral operator.

3. Conclusions

In this article, the notion of a fuzzy subset is used to define certain subclasses of univalent functions associated with the q-difference operator. The applications of the q-Srivastava-Attiya operator and the q-multiplier transformation are discussed. On employing these operators, we introduced certain subclasses. Various important properties such as the inclusion relationship between the classes and the integral preserving properties are investigated. This article will inspire the scholars of this field for further investigations related to the notion of fuzzy subsets involving q-theory in the future.

Conflict of interest

The authors declare no conflict of interest.

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