Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(3): 6590–6608.
DOI: 10.3934/math.2023333
Received: 29 July 2022
Revised: 29 November 2022
Accepted: 12 December 2022
Published: 05 January 2023

Research article

An inertially constructed projection based hybrid algorithm for fixed point and split null point problems

Yasir Arfat[1,2], **Poom Kumam**[1,2,3], **Supak Phiangsungnoen**[4,5,*], **Muhammad Aqeel Ahmad Khan**[6,*] **and Hafiz Fukhar-ud-din**[7]

[1] Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bangkok 10140, Thailand

[2] Center of Excellence in Theoretical and Computational Science (TaCS-CoE), King Mongkut's University of Technology Thonburi (KMUTT), Bangkok 10140, Thailand

[3] China Medical University Hospital, China Medical University, Taichung 40402, Taiwan

[4] Department of General Education (Mathematics), Faculty of Liberal Arts, Rajamangala University of Technology Rattanakosin, Samphanthawong, Bangkok, Thailand

[5] Institute of Research and Development, Rajamangala University of Technology Rattanakosin, Nakhon Pathom 73170, Thailand

[6] Department of Mathematics, COMSATS University Islamabad, Lahore 54000, Pakistan

[7] Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur 63100, Pakistan

* **Correspondence:** supak.pia@rmutr.ac.th, itsakb@hotmail.com.

Abstract: In this paper, we posit a framework for the investigation of the fixed point problems (FPP) involving an infinite family of $\Bbbk$-demicontractive operators and the split common null point problems (SCNPP) in Hilbert spaces. We employ an accelerated variant of the hybrid shrinking projection algorithm for the construction of a common solution associated with the FPP and SCNPP. Theoretical results comprise strong convergence characteristics under suitable sets of constraints as well as numerical results are established for the underlying algorithm. Applications to signal processing as well as various abstract problems are also incorporated.

Keywords: hybrid algorithm; inertial extrapolation; strong convergence; fixed point problem; demicontractive operator; null point problem
Mathematics Subject Classification: 47H05, 47H10, 47J25, 49M30, 54H25

1. Introduction

Convex optimization problem (COP) is considered very important in the current literature as it covers a diverse range of problems with possible applications in signal processing, image processing and machine learning. As a consequence, the tremendous progress in studying the COP has led the emergence of a theory of convex optimization and a useful interface linking various branches of sciences.

Monotone operator theory is a prominent framework for various nonlinear problems and closely related with the theory of convex optimization. One of the fundamental themes in monotone operator theory is to compute zeros of the (maximal-) monotone operators. The importance of this concept stems from the fact that the sub-differential operator associated with a proper, convex and lower semicontinuous (PCLS) function is not only a maximal monotone operator but also solves the convex minimization problem. It is remarked that most of the practical phenomenon can be reformulated as zero point problem. This formalism includes variational inequalities, evolution equations, complementarity problems and inclusions [12].

The class of split feasibility problems (SFP) has an extraordinary utility and broad applicability in medical image reconstruction, signal processing and computerized tomography [15, 17, 18, 21]. Some interesting and crucial results regarding the SFP with areas of feasible applications are established in [16, 19, 20]. The first prototype strategies for computing the optimal solution of the split common null point problem (SCNPP) can be found in [16]. Since then, different variants of these strategies have been proposed and analyzed for SCNPP and other instances of SFP [19, 20, 29].

Another useful formalism for modelling various nonlinear phenomenon is the fixed point problem (FPP) of the operator under consideration. Most of the problems in diverse areas such as mathematical economics, variational inequality theory, control theory and game theory can be reformulated in terms of FPP. It is remarked that various nonlinear fixed point operators play equivalent important role in COP. In 2015, Takahashi et al. [31] investigated a unified formalism of null point problem and FPP in Hilbert spaces. Since then, FPP associated with different nonlinear operators are jointly investigated with (split common-) null point problem in this domain. It is therefore natural to investigate FPP associated with an infinite family of operators jointly with SCNPP in Hilbert spaces.

A variety of strategies combining iterative optimization algorithms and fixed point algorithms have been introduced and analyzed to construct an optimal solution of the SCNPP and FPP. Each strategy has certain shortcomings in terms of convergence characteristic and/or rate of convergence. The hybrid shrinking projection algorithm is a well-known strategy for the strong convergence characteristic whereas the inertial extrapolation technique, essentially due to [27] and see also [1–11, 32], enhances the rate of convergence of the algorithm under consideration.

Our main contributions in this ongoing fruitful research direction are as follows:

(1) We posit a framework to jointly investigate SCNPP and FPP associated with an infinite family of operators in Hilbert spaces. For the case of an infinite family of fixed point operators, we exploit the construction of an auxiliary operator defined in [28, 34];

(2) We employ an algorithmic approach combining the hybrid shrinking projection algorithm with the inertial extrapolation technique to construct the common optimal solution of the problems as mentioned in item (1);

(3) We establish the strong convergence analysis of the proposed algorithm by employing the suitable

constraints in accordance with the standard techniques and tools in the current literature;
(4) We posit different frameworks, as an application of the framework mentioned in item (1), for various instances of SFP in Hilbert spaces;
(5) Last but not least, we incorporate an appropriate numerical example for the demonstration of the framework as well as the applicability of the proposed algorithm for signal recovery problem.

2. Preliminaries

Throughout the rest of the sections, the triplet $(\Xi_1, <\cdot,\cdot>, \|\cdot\|)$ indicates the real Hilbert space with the conventional notations of the inner product and the norm and $A_1 \subseteq \Xi_1 \times \Xi_1$ denotes a set-valued operator with the usual definitions of $dom(A_1)$, $gra(A_1)$ and $zer(A_1)$. We use $\to$ (resp. $\rightharpoonup$) to represent the symbol for strong convergence (resp. weak convergence). The set of reals and natural numbers are symbolized as $\mathbb{R}$ and $\mathbb{N}$, respectively.

Recall from the celebrated monograph [12] that the set-valued operator A_1 is called monotone if $\langle r-s, u-v\rangle \geq 0,\ \forall (r,u),\ (s,v) \in gra(A_1)$. In addition, A_1 is coined as maximal monotone operator provided that the graph of A_1 could not be properly contained in the graph of any other monotone operator. Let $m > 0$, then the resolvent operator of A_1 is defined as $J_m^{A_1} = (Id+mA_1)^{-1}$, where Id denotes the identity operator. In this connection, $J_m^{A_1}$ is well-defined, single-valued and firmly nonexpansive operator.

Let $T : H \to H$ be an operator defined on a nonempty subset H of Ξ_1. We use $Fix(T) = \{p \in H \mid p = Tp\}$ to denote the set of fixed points of the operator T. The metric projection operator $\Pi_H : \Xi_1 \to H$ associated with the nonempty closed convex subset H of Ξ_1 is defined as $\Pi_H(u) = argmin_{v\in H}\|u-v\|$. It is well-known that the operator Π_H is firmly nonexpansive and satisfies $\langle u-\Pi_H u, \Pi_H u - v\rangle \geq 0, \forall\ u \in \Xi_1, v \in H$. Recall that the operator T is known as $\mathbb{k}$-demicontractive [24] provided that $\mathbb{k} \in [0,1)$ such that

$$\|Tq-p\|^2 \leq \|q-p\|^2 + \mathbb{k}\|q-Tq\|^2,\ \forall q \in H, p \in Fix(T).$$

The class of $\mathbb{k}$-demicontractive operators has been studied extensively in various instances of fixed point problems in Hilbert spaces. However, we are concerned with the fixed point problem of an infinite family of $\mathbb{k}$-demicontractive operators in Hilbert spaces via the following construction of auxiliary operator S_k:

$$\begin{aligned}
Q_{k,k+1} &= Id,\\
Q_{k,k} &= \beta_k T'_k Q_{k,k+1} + (1-\beta_k)Id,\\
Q_{k,k-1} &= \beta_{k-1} T'_{k-1} Q_{k,k} + (1-\beta_{k-1})Id,\\
&\ \vdots\\
Q_{k,m} &= \beta_m T'_m Q_{k,m+1} + (1-\beta_m)Id,\\
&\ \vdots\\
Q_{k,2} &= \beta_2 T'_2 Q_{k,3} + (1-\beta_2)Id,\\
S_k = Q_{k,1} &= \beta_1 T'_1 Q_{k,2} + (1-\beta_1)Id,
\end{aligned}$$

where $0 \leq \beta_m \leq 1$ and $T'_m = \alpha x + (1+\alpha)T_m x$ for all $x \in H$ with T_m being $\mathbb{k}$-demicontractive operator and $\alpha \in [\mathbb{k}, 1)$. It is well-known in the context of operator S_k that each T'_m is nonexpansive and the limit

$\lim_{k\to\infty} Q_{k,m}$ exists. Moreover

$$Sx = \lim_{k\to\infty} S_k = \lim_{k\to\infty} Q_{k,1}, \ \forall\ x \in H.$$

This implies that $Fix(S) = \bigcap_{k=1}^{\infty} Fix(S_k)$ [28, 34].

We now finally introduce the formalism of the proposed problem.

Let $A_1 \subseteq \Xi_1 \times \Xi_1$ and $A_2 \subseteq \Xi_2 \times \Xi_2$ be maximal monotone operators such that the domain of A_1 is the subset of H and let $J_m^{A_1}$ and $J_m^{A_2}$ be the resolvents of A_1 and A_2, respectively, for $m > 0$. Let $h : \Xi_1 \to \Xi_2$ be a bounded linear operator and let h^* be the adjoint operator of h. Let S_k be the S-operator such that $\Gamma := \Omega \cap Fix(S) \neq \emptyset$, where $\Omega := \{\hat{p} \in A_1^{-1}(0) : h\hat{p} \in A_2^{-1}(0)\}$ indicates the SCNPP [16]. We aim to find

$$\hat{p} \in \Gamma. \tag{2.1}$$

The following crucial results are required in the sequel:

Lemma 2.1. *[14] Let $U : H \to H$ be an operator defined on a nonempty closed convex subset H of a real Hilbert space Ξ_1 and let (p_k) be a sequence in H. If $p_k \rightharpoonup p$ and if $(Id - U)p_k \to 0$, then $p \in Fix(U)$.*

Lemma 2.2. *Let $\mu, \nu \in \Xi_1$ and $\theta \in \mathbb{R}$ then*

(i) $\|\mu + \nu\|^2 \leq \|\mu\|^2 + 2\langle \nu, \mu + \nu\rangle$;
(ii) $\|\mu - \nu\|^2 \leq \|\mu\|^2 - \|\nu\|^2 - 2\langle \mu - \nu, \nu\rangle$;
(iii) $\|\theta\mu + (1-\theta)\nu\|^2 = \theta\|\mu\|^2 + (1-\theta)\|\nu\|^2 - \theta(1-\theta)\|\mu - \nu\|^2$.

Lemma 2.3. *[34] Let H be a nonempty closed and convex subset of a real Hilbert space Ξ_1 and let $T : H \to H$ be a $\Bbbk$-demicontractive operator with $\Bbbk \in [0, 1)$. Let $\alpha \in [\Bbbk, 1)$ and set $T' = (1-\alpha)Id + \alpha T$, then $T' : H \to H$ is a nonexpansive operator such that $Fix(T') = Fix(T)$.*

Lemma 2.4. *[26] Let H be a nonempty closed convex subset of a real Hilbert space Ξ_1. For every $p, q, t \in \Xi_1$ and $\gamma \in \mathbb{R}$, the set*

$$D = \{\nu \in H : \|q - \nu\|^2 \leq \|p - \nu\|^2 + \langle t, \nu\rangle + \gamma\},$$

is closed and convex.

Lemma 2.5. *[34] Let H be a nonempty closed and convex subset of a real Hilbert space Ξ_1 and let (T_m') be a sequence of nonexpansive operators such that $\bigcap_{k=1}^{\infty} Fix(T_k') \neq \emptyset$ and $0 \leq \beta_m \leq b < 1$. Then for a bounded subset K of H, we have*

$$\lim_{k\to\infty} \sup_{x\in K} \|Sx - S_k x\| = 0.$$

3. Algorithm and convergence analysis

We start with the architecture of the algorithm for the construction of an optimal solution of the problem (2.1).

Theorem 3.1. *The sequence (p_k) generated by the Algorithm 1, under the following control conditions,*

Algorithm 1 Inertially constructed hybrid algorithm (Algo.1)

Initialization: Choose arbitrarily, $p_0,\ p_1 \in H$, set $k \geq 1$ and nonincreasing sequence $\alpha_k, \beta_k \subset (0,1)$, $\theta_k \subset [0,1)$, $m_k \in (0,\infty)$ and $\delta \in (0, \frac{2}{\|h\|^2})$ such that $\|h\|^2 = L$ is the spectral radius of h^*h. Choose the inertial parameter

$$\theta_k = \begin{cases} \min\{\frac{\nu_k}{\|p_k - p_{k-1}\|}, \theta\} & if \quad p_k \neq p_{k-1}; \\ \theta & \text{otherwise}, \end{cases}$$

where $\{\nu_k\}$ is a positive sequence such that $\sum_{k=1}^{\infty} \nu_k < \infty$ and $\theta \in [0,1)$.

Iterative Steps: Given $p_k \in \Xi_1$, calculate e_k, $\bar{w}_k$ and x_k as follows:

Step 1. Compute

$$\begin{cases} e_k = p_k + \theta_k(p_k - p_{k-1}); \\ \bar{w}_k = (1-\alpha_k)e_k + \alpha_k S_k e_k; \\ x_k = (1-\beta_k)\bar{w}_k + \beta_k(J_{m_k}^{A_1}(\bar{w}_k + \delta h^*(J_{m_k}^{A_2} - Id)h\bar{w}_k)). \end{cases}$$

The algorithm aborts if $x_k = \bar{w}_k = e_k = p_k$ and p_k is the required approximation.
Otherwise,

Step 2. Compute

$$\begin{aligned} H_{k+1} &= \{z \in H_k : \|x_k - z\|^2 \leq \|p_k - z\|^2 + \theta_k^2\|p_k - p_{k-1}\|^2 + 2\theta_k\langle p_k - z, p_k - p_{k-1}\rangle\}, \\ p_{k+1} &= \Pi_{H_{k+1}} p_1, \ \forall\, k \geq 1, \end{aligned}$$

set $k =: k+1$ and go back to **Step 1.**

(C1) $\sum_{k=1}^{\infty} \theta_k \|p_k - p_{k-1}\| < \infty$;
(C2) $0 < a \leq \liminf_{k\to\infty} \alpha_k \leq \limsup_{k\to\infty} \alpha_k \leq a^*$;
(C3) $\liminf_{k\to\infty} \beta_k > 0$;
(C4) $\liminf_{k\to\infty} m_k > 0$;

converges strongly to an element $p^* \in \Gamma$.

Proof. **Step 1.** The Algorithm 1 is well-defined.

Recall that the set Γ is closed and convex whereas the closedness and the convexity of H_{k+1}, for each $k \geq 1$, follows from Lemma 2.4. Let $p^* \in \Gamma$, then recalling the Algorithm 1, we have

$$\begin{aligned} \|e_k - p^*\|^2 &= \|(p_k - p^*) + \theta_k(p_k - p_{k-1})\|^2 \\ &\leq \|p_k - p^*\|^2 + \theta_k^2\|p_k - p_{k-1}\|^2 + 2\theta_k\langle p_k - p^*, p_k - p_{k-1}\rangle. \end{aligned} \tag{3.1}$$

Also from Algorithm 1 and Lemma 2.3, we have

$$\begin{aligned} \|\bar{w}_k - p^*\|^2 &= \|(1-\alpha_k)e_k + \alpha_k S_k e_k - p^*\|^2 \\ &\leq (1-\alpha_k)\|e_k - p^*\|^2 + \alpha_k\|S_k e_k - p^*\|^2 - \alpha_k(1-\alpha_k)\|(Id - S_k)e_k\|^2 \\ &\leq (1-\alpha_k)\|e_k - p^*\|^2 + \alpha_k\|e_k - p^*\|^2 - \alpha_k(1-\alpha_k)\|(Id - S_k)e_k\|^2 \\ &= \|e_k - p^*\|^2 - \alpha_k(1-\alpha_k)\|(Id - S_i)e_k\|^2 \\ &\leq \|p_k - p^*\|^2 + \theta_k^2\|p_k - p_{k-1}\|^2 + 2\theta_k\langle p_k - p^*, p_k - p_{k-1}\rangle. \end{aligned} \tag{3.2}$$

Further, we obtain

$$\begin{aligned}\|x_k - p^*\|^2 &= \|(1-\beta_k)(\bar{w}_k - p^*) + \beta_k(J_{m_k}^{A_1}(\bar{w}_k + \delta h^*(J_{m_k}^{A_2} - Id)h\bar{w}_k) - p^*)\|^2 \\ &\le (1-\beta_k)\|\bar{w}_k - p^*\|^2 + \beta_k\|J_{m_k}^{A_1}(\bar{w}_k + \delta h^*(J_{m_k}^{A_2} - Id)h\bar{w}_k) - p^*\|^2. \end{aligned} \tag{3.3}$$

Recalling the nonexpansivity of $J_{m_k}^{A_1}$, we obtain

$$\begin{aligned}&\|J_{m_k}^{A_1}(\bar{w}_k + \delta h^*(J_{m_k}^{A_2} - Id)h\bar{w}_k) - J_{m_k}^{A_1}p^*\|^2 \\ \le&\|\bar{w}_k + \delta h^*(J_{m_k}^{A_2} - Id)h\bar{w}_k - p^*\|^2 \\ \le&\|\bar{w}_k - p^*\|^2 + \delta^2\|h^*(J_{m_k}^{A_2} - Id)h\bar{w}_k\|^2 + 2\delta\langle\bar{w}_k - p^*, h^*(J_{m_k}^{A_2} - Id)h\bar{w}_k\rangle \\ \le&\|\bar{w}_k - p^*\|^2 + \delta^2\|h\|^2\|(J_{m_k}^{A_2} - Id)h\bar{w}_k\|^2 + 2\delta\langle h\bar{w}_k - hp^*, (J_{m_k}^{A_2} - Id)h\bar{w}_k)\rangle. \end{aligned} \tag{3.4}$$

Denote $\lambda_k = 2\delta\langle h\bar{w}_k - hp^*,\ (J_{m_k}^{A_2} - Id)h\bar{w}_k\rangle$ and recalling the firm nonexpansivity of $J_{m_k}^{A_2}$, we get

$$\begin{aligned}\lambda_k &= 2\delta\langle h\bar{w}_k - hp^* + (J_{m_k}^{A_2}(h\bar{w}_k) - h\bar{w}_k) - (J_{m_k}^{A_2}(h\bar{w}_k) - h\bar{w}_k), J_{m_k}^{A_2}(h\bar{w}_k) - h\bar{w}_k\rangle \\ &= 2\delta(\langle h\bar{w}_k - hp^* + J_{m_k}^{A_2}(h\bar{w}_k) - h\bar{w}_k, J_{m_k}^{A_2}(h\bar{w}_k) - h\bar{w}_k\rangle \\ &- \langle J_{m_k}^{A_2}(h\bar{w}_k) - h\bar{w}_k, J_{m_k}^{A_2}(h\bar{w}_k) - h\bar{w}_k\rangle) \\ &= 2\delta(\langle J_{m_k}^{A_2}(h\bar{w}_k) - hp^*, J_{m_k}^{A_2}(h\bar{w}_k) - h\bar{w}_k\rangle - \|J_{m_k}^{A_2}(h\bar{w}_k) - h\bar{w}_k\|^2) \\ &\le -2\delta\|(J_{m_k}^{A_2} - Id)h\bar{w}_k)\|^2. \end{aligned} \tag{3.5}$$

The estimate (3.3) implies after recalling the estimates (3.4) and (3.5)

$$\begin{aligned}\|x_k - p^*\|^2 &\le (1-\beta_k)\|\bar{w}_k - p^*\|^2 + \beta_k(\|\bar{w}_k - p^*\|^2 + \delta^2\|h\|^2\|(J_{m_k}^{A_2} - Id)h\bar{w}_k)\|^2 - 2\delta\|(J_{m_k}^{A_2} - Id)h\bar{w}_k\|^2), \\ &= (1-\beta_k)\|\bar{w}_k - p^*\|^2 + \beta_k(\|\bar{w}_k - p^*\|^2 - \delta(2 - \delta\|h\|^2)\|(J_{m_k}^{A_2} - Id)h\bar{w}_k\|^2) \\ &\le (1-\beta_k)\|\bar{w}_k - p^*\|^2 + \beta_k\|\bar{w}_k - p^*\|^2 \\ &\le \|p_k - p^*\|^2 + \theta_k^2\|p_k - p_{k-1}\|^2 + 2\theta_k\langle p_k - p^*, p_k - p_{k-1}\rangle. \end{aligned} \tag{3.6}$$

The above estimate (3.6) indicates the inclusion $\Gamma \subset H_{k+1}$. Summarising the above stated facts, this infers that the Algorithm 1 is well-defined.

Step 2. The limit $\lim_{k\to\infty}\|p_k - p_1\|$ exists.

Note that $\|p_{k+1} - p_1\| \le \|p - p_1\|$, for all $p \in H_{k+1}$ by employing the fact that $p_{k+1} = \Pi_{H_{k+1}}p_1$. This infers that $\|p_{k+1} - p_1\| \le \|p^* - p_1\|$, for all $p^* \in \Gamma \subset H_{k+1}$ and consequently establishes the boundedness of $(\|p_k - p_1\|)$. Also from $p_k = \Pi_{H_k}p_1$, we have that

$$\|p_k - p_1\| \le \|p_{k+1} - p_1\|.$$

The above approximation infers that the sequence $(\|p_k - p_1\|)$ is non-decreasing, therefore, we have

$$\lim_{k\to\infty}\|p_k - p_1\| \text{ exists.} \tag{3.7}$$

Step 3. Prove that $q \in \Gamma$.

The following crucial estimates are required to prove the claim:

$$
\begin{aligned}
\|p_{k+1}-p_k\|^2 &= \|p_{k+1}-p_1+p_1-p_k\|^2 \\
&= \|p_{k+1}-p_1\|^2+\|p_k-p_1\|^2-2\langle p_k-p_1, p_{k+1}-p_1\rangle \\
&= \|p_{k+1}-p_1\|^2+\|p_k-p_1\|^2-2\langle p_k-p_1, p_{k+1}-p_k+p_k-p_1\rangle \\
&= \|p_{k+1}-p_1\|^2-\|p_k-p_1\|^2-2\langle p_k-p_1, p_{k+1}-p_k\rangle \\
&\le \|p_{k+1}-p_1\|^2-\|p_k-p_1\|^2 .
\end{aligned}
$$

By employing the lim sup, and recalling the estimate (3.7), the above estimate implies that $\limsup_{k\to\infty}\|p_{k+1}-p_k\|^2=0$. That is

$$\lim_{k\to\infty}\|p_{k+1}-p_k\|=0. \tag{3.8}$$

Recalling the definition of (e_k) and the condition (C1), we have

$$\lim_{k\to\infty}\|e_k-p_k\|=\lim_{k\to\infty}\theta_k\|p_k-p_{k-1}\|=0. \tag{3.9}$$

As an easy inference of the approximates (3.8) and (3.9), we get

$$\lim_{k\to\infty}\|e_k-p_{k+1}\|=0. \tag{3.10}$$

Since $p_{k+1}\in H_{k+1}$, we have

$$\|x_k-p_{k+1}\|\le\|p_k-p_{k+1}\|+\theta_k\|p_k-p_{k-1}\|+2\theta_k\langle p_k-p_{k+1}, p_k-p_{k-1}\rangle.$$

Recalling the estimate (3.8) and the condition (C1), the above estimate implies that

$$\lim_{k\to\infty}\|x_k-p_{k+1}\|=0. \tag{3.11}$$

Recalling the estimates (3.8), (3.11) and the following triangular inequality:

$$\|x_k-p_k\|\le\|x_k-p_{k+1}\|+\|p_{k+1}-p_k\|,$$

we get

$$\lim_{k\to\infty}\|x_k-p_k\|=0. \tag{3.12}$$

Consider the estimate (3.6) in the following variation:

$$
\begin{aligned}
a(1-a^*)\|(Id-S_k)e_k\|^2 \le (\|p_k-p^*\|+\|x_k-p^*\|)\|p_k-x_k\|+\theta_k^2\|p_k-p_{k-1}\|^2 \\
+2\theta_k\|p_k-p^*\|\|p_k-p_{k-1}\|.
\end{aligned}
$$

Letting $k\to\infty$ and recalling the conditions (C1)–(C2) and the estimate (3.12), we have

$$\lim_{k\to\infty}\|(Id-S_k)e_k\|=0. \tag{3.13}$$

The estimate (3.13) also implies that

$$\lim_{k\to\infty}\|\bar{w}_k-e_k\|=\lim_{k\to\infty}a^*\|(Id-S_k)e_k\|=0. \tag{3.14}$$

Recalling the estimates (3.8), (3.14) and the following triangular inequality:

$$\|\bar{w}_k - p_k\| \le \|\bar{w}_k - p_{k+1}\| + \|p_{k+1} - p_k\|,$$

we get

$$\lim_{k\to\infty} \|\bar{w}_k - p_k\| = 0. \tag{3.15}$$

Now recalling the estimates (3.4), (3.5) and Lemma 2.2, we have

$$\begin{aligned}
\|x_k - p^*\|^2 &= \|(1-\beta_k)\bar{w}_k + \beta_k(J^{A_1}_{m_k}(\bar{w}_k + \delta h^*(J^{A_2}_{m_k} - Id)h\bar{w}_k)) - p^*)\|^2 \\
&\le (1-\beta_k)\|\bar{w}_k - p^*\|^2 + \beta_k(\|\bar{w}_k - p^*\|^2 - \delta(2-\delta\|h\|^2)\|(J^{A_2}_{m_k} - Id)h\bar{w}_k\|^2) \\
&\le \|\bar{w}_k - p^*\|^2 - \beta_k\delta(2-\delta\|h\|^2)\|(J^{A_2}_{m_k} - Id)h\bar{w}_k\|^2 \\
&\le \|e_k - p^*\|^2 - \beta_k\delta(2-\delta\|h\|^2)\|(J^{A_2}_{m_k} - Id)h\bar{w}_k\|^2 \\
&\le \|p_k - p^*\|^2 + 2\theta_k\langle p_k - p_{k-1}, e_k - p^*\rangle - \beta_k\delta(2-\delta\|h\|^2)\|(J^{A_2}_{m_k} - Id)h\bar{w}_k\|^2.
\end{aligned} \tag{3.16}$$

Rearranging the estimate (3.16), we have

$$\begin{aligned}
&\beta_k\delta(2-\delta\|h\|^2)\|(J^{A_2}_{m_k} - Id)h\bar{w}_k\|^2 \le \|p_k - p^*\|^2 - \|x_k - p^*\|^2 + 2\theta_k\langle p_k - p_{k-1}, e_k - p^*\rangle \\
&\le (\|p_k - p^*\| + \|e_k - p^*\|)\,\|p_k - e_k\| + 2\theta_k\langle p_k - p_{k-1}, e_k - p^*\rangle.
\end{aligned} \tag{3.17}$$

Now recalling the conditions (C1), (C3), the estimate (3.9) and $\delta \in (0, \frac{2}{\|h\|^2})$, the estimate (3.17) implies that

$$\lim_{k\to\infty} \|(J^{A_2}_{m_k} - Id)h\bar{w}_k\| = 0. \tag{3.18}$$

Recalling the estimates (3.4) and (3.5), we obtain

$$\begin{aligned}
\|J^{A_1}_{m_k}(\bar{w}_k + \delta h^*(J^{A_2}_{m_k} - Id)h\bar{w}_k) - J^{A_1}_{m_k}p^*\|^2 &\le \|\bar{w}_k + \delta h^*(J^{A_2}_{m_k} - Id)h\bar{w}_k - p^*\|^2 \\
&\le \|\bar{w}_k - p^*\|^2.
\end{aligned} \tag{3.19}$$

Denote $\xi_k = J^{A_1}_{m_k}(\bar{w}_k + \delta h^*(J^{A_2}_{m_k} - Id)h\bar{w}_k)$ and recalling the estimate (3.19), it follows that

$$\begin{aligned}
\|\xi_k - p^*\|^2 &= \|J^{A_1}_{m_k}\bar{w}_k + \delta h^*(J^{A_2}_{m_k} - Id)h\bar{w}_k) - J^{A_1}_{m_k}p^*\|^2 \\
&\le \langle J^{A_1}_{m_k}(\bar{w}_k + \delta h^*(J^{A_2}_{m_k} - Id)h\bar{w}_k) - J^{A_1}_{m_k}p^*, \bar{w}_k + \delta h^*(J^{A_2}_{m_k} - Id)h\bar{w}_k - p^*\rangle \\
&= \langle \xi_k - p^*, \bar{w}_k + \delta h^*(J^{A_2}_{m_k} - Id)h\bar{w}_k - p^*\rangle \\
&= \frac{1}{2}(\|\xi_k - p^*\|^2 + \|\bar{w}_k + \delta h^*(J^{A_2}_{m_k} - Id)h\bar{w}_k - p^*\|^2 - \|\xi_k - \bar{w}_k - \delta h^*(J^{A_2}_{m_k} - Id)h\bar{w}_k\|^2) \\
&\le \frac{1}{2}(\|\xi_k - p^*\|^2 + \|\bar{w}_k - p^*\|^2 - \|\xi_k - \bar{w}_k - \delta h^*(J^{A_2}_{m_k} - Id)h\bar{w}_k\|^2) \\
&= \frac{1}{2}(\|\xi_k - p^*\|^2 + \|\bar{w}_k - p^*\|^2 - \|\xi_k - \bar{w}_k\|^2 - \delta^2\|h^*(J^{A_2}_{m_k} - Id)h\bar{w}_k\|^2 \\
&+ 2\delta\langle \xi_k - \bar{w}_k, h^*(J^{A_2}_{m_k} - Id)h\bar{w}_k\rangle) \\
&\le \frac{1}{2}(\|\xi_k - p^*\|^2 + \|\bar{w}_k - p^*\|^2 - \|\xi_k - \bar{w}_k\|^2 - \delta^2\|h^*(J^{A_2}_{m_k} - Id)h\bar{w}_k\|^2 \\
&+ 2\delta\|\xi_k - \bar{w}_k\|\|h^*(J^{A_2}_{m_k} - Id)h\bar{w}_k\|).
\end{aligned} \tag{3.20}$$

That is

$$\|\xi_k - p^*\|^2 \leq \|\bar{w}_k - p^*\|^2 - \|\xi_k - \bar{w}_k\|^2 + 2\delta\|\xi_k - \bar{w}_k\|\|h^*(J_{m_k}^{A_2} - Id)h\bar{w}_k\|. \tag{3.21}$$

As a consequence, we have

$$\begin{aligned}\|x_k - p^*\|^2 &\leq (1-\beta_k)\|\bar{w}_k - p^*\|^2 + \beta_k\|\xi_k - p^*\|^2 \\ &\leq (1-\beta_k)\|\bar{w}_k - p^*\|^2 + \beta_k(\|\bar{w}_k - p^*\|^2 - \|\xi_k - \bar{w}_k\|^2 \\ &\quad + 2\delta\|\xi_k - \bar{w}_k\|\|h^*(J_{m_k}^{A_2} - Id)h\bar{w}_k\|).\end{aligned} \tag{3.22}$$

The estimate (3.22), gives that

$$\beta_k\|\xi_k - \bar{w}_k\|^2 \quad\leq\quad \|\bar{w}_k - p^*\|^2 - \|x_k - p^*\|^2 \; - 2\beta_k\delta\|\xi_k - \bar{w}_k\|\|h^*(J_{m_k}^{A_2} - Id)h\bar{w}_k\|). \tag{3.23}$$

Recalling the estimate (3.18) and the condition (C3), we have

$$\lim_{k\to\infty} \|\xi_k - \bar{w}_k\| = 0. \tag{3.24}$$

Reasoning as above, by recalling the definition of (e_k), the condition (C1) and the estimate (3.24), we get

$$\lim_{k\to\infty} \|\xi_k - p_k\| = 0. \tag{3.25}$$

Note that the existence of a convergent subsequence (p_{k_j}) of (p_k) such that $p_{k_j} \rightharpoonup q \in \Xi_1$ as $j \to \infty$ follows from the boundedness of (p_k). This also infers that $\xi_{k_j} \rightharpoonup q$ and $\bar{w}_{k_j} \rightharpoonup q$ as $j \to \infty$. To establish the claim, we first prove that $q \in \Omega$.

Let $(u, v) \in gra(A_1)$. Since $\xi_{k_j} = J_{m_{k_j}}^{A_1}(\bar{w}_{k_j} + \delta h^*(J_{m_{k_j}}^{A_2} - Id)h\bar{w}_{k_j})$, therefore, we have

$$\bar{w}_{k_j} + \delta h^*(J_{m_{k_j}}^{A_2} - Id)h\bar{w}_{k_j} \in \xi_{k_j} + m_{k_j}A_1(\xi_{k_j}).$$

This implies that

$$\frac{1}{m_{k_j}}(\bar{w}_{k_j} - \xi_{k_j}) + \frac{1}{m_{k_j}}\delta h^*(J_{m_{k_j}}^{A_2} - Id)h\bar{w}_{k_j} \in A_1(\xi_{k_j}).$$

Recalling the monotonicity of A_1, we have

$$\langle u - \xi_{k_j}, v - (\frac{1}{m_{k_j}}(\bar{w}_{k_j} - \xi_{k_j}) + \frac{1}{m_{k_j}}(\delta h^*(J_{m_{k_j}}^{A_2} - Id)h\bar{w}_{k_j}))\rangle \geq 0.$$

The above estimate infers that

$$\begin{aligned}\langle u - \xi_{k_j}, v\rangle &\geq \langle u - \xi_{k_j}, \frac{1}{m_{k_j}}(\bar{w}_{k_j} - \xi_{k_j}) + \frac{1}{m_{k_j}}(\delta h^*(J_{m_{k_j}}^{A_2} - Id)h\bar{w}_{k_j})\rangle \\ &= \langle u - \xi_{k_j}, \frac{1}{m_{k_j}}(\bar{w}_{k_j} - \xi_{k_j})\rangle + \langle u - \xi_{k_j}, \frac{1}{m_{k_j}}(\delta h^*(J_{m_{k_j}}^{A_2} - Id)h\bar{w}_{k_j})\rangle.\end{aligned} \tag{3.26}$$

Since $\xi_{k_j} \rightharpoonup q$, we obtain $\lim_{j\to\infty}\langle u - \xi_{k_j}, v\rangle = \langle u - q, v\rangle$. Now utilizing the estimates (3.18), (3.24) and (3.26), we have $\langle u - q, v\rangle \geq 0$. This implies that $0 \in A_1 q$.

Recalling the facts that (i) h is a bounded linear operator implies that $h\bar{w}_{k_j} \rightharpoonup hq$ as $j \to \infty$, (ii) $J_{m_k}^{A_2}$ is a nonexpansive operator with $Id - J_{m_k}^{A_2}$ being demiclosed at the origin (Lemma 2.1), we also obtain that $0 \in A_2(hq)$. Hence $q \in \Omega$.

We now show that $q \in Fix(S)$. Observe that

$$\begin{aligned}\|e_k - Se_k\| &\le \|e_k - S_k e_k\| + \|S_k e_k - Se_k\| \\ &\le \|e_k - S_k e_k\| + \sup_{x\in K} \|S_k x - Sx\|.\end{aligned}$$

Recalling the estimate (3.13) and Lemma 2.5, the above estimate implies that $\lim_{k\to\infty} \|e_k - Se_k\| = 0$. This together with the fact that $e_{k_j} \rightharpoonup q$ implies, with the help of Lemma 2.1, that $q \in Fix(S) = \bigcap_{k=1}^{\infty} Fix(S_k)$.

Step 4. Prove that $p_k \to p^* = \Pi_\Gamma p_1$.

Let $p^* = \Pi_\Gamma p_1$. As $p_{k+1} = \Pi_{H_{k+1}} p_1$ and $\Gamma \subset H_{k+1}$, therefore, we have

$$\|p_{k+1} - p_1\| \le \|p^* - p_1\|.$$

Also

$$\|p_1 - p^*\| \le \|p_1 - q\| \le \liminf_{j\to\infty} \left\|p_1 - p_{k_j}\right\| \le \limsup_{k\to\infty} \left\|p_1 - p_{k_j}\right\| \le \|p_1 - p^*\|.$$

That is

$$\lim_{j\to\infty} \left\|p_{k_j} - p_1\right\| = \|q - p_1\| = \|p^* - p_1\|.$$

This implies that $\lim_{k\to\infty} p_k = q = p^* = \Pi_\Gamma p_1$ and hence completes the proof. □

If we take $A_2 = 0$ and $\Xi_1 = \Xi_2$, then the problem (2.1) reduces to find a point of the following problem:

$$\hat{p} \in \Gamma := \{p \in A_1^{-1}(0) \cap Fix(S)\}.$$

Hence the following result is an easy consequence of the Theorem 3.1:

Corollary 3.1. *Assume that $\Gamma \neq \emptyset$. Then the sequence (p_k)*

$$\begin{cases} e_k = p_k + \theta_k(p_k - p_{k-1}); \\ \bar{w}_k = (1-\alpha_k)e_k + \alpha_k S_k e_k; \\ x_k = (1-\beta_k)\bar{w}_k + \beta_k J_{m_k}^{A_1}(\bar{w}_k); \\ H_{k+1} = \{z \in H_k : \|x_k - z\|^2 \le \|p_k - z\|^2 + \theta_k^2\|p_k - p_{k-1}\|^2 + 2\theta_k\langle p_k - z, p_k - p_{k-1}\rangle\}; \\ p_{k+1} = \Pi_{H_{k+1}} p_1, \forall\, k \ge 1; \end{cases} \tag{3.27}$$

generated by (3.27), under the control conditions (C1)–(C4), converges strongly to an element $p^* = \Pi_\Gamma p_1$.

4. Applications

In this section, we posit different frameworks, as an application of the framework established in Section 3.

4.1. Split feasibility problems

The classical SFP, essentially due to Censor and Elfving [18], aims to find $\hat{p} \in \omega := H \cap h^{-1}(G) = \{\bar{q} \in H : h\bar{q} \in G\}$, where $H \subset \Xi_1$ and $G \subset \Xi_2$ are nonempty, closed and convex subsets of Ξ_1 and Ξ_2, respectively. For the implementation of the Theorem 3.1, we recall the indicator operator of a nonempty, closed and convex subset H of Ξ_1 as

$$\Phi_H(p^*) := \begin{cases} 0, & p^* \in H; \\ \infty, & otherwise. \end{cases}$$

It has been established that the subdifferential $\partial\Phi_H$ satisfies the maximal monotonicity provided that the operator Φ_H is proper, convex and lower semicontinuous. Also $\partial\Phi_G = \mathcal{N}(\mu, H)$, where $\mathcal{N}(\mu, H)$ is the normal cone of H at μ. Utilizing this fact, we conclude that the resolvent operator of $\partial\Phi_H$ is the metric projection operator of Ξ_1 onto H. We, therefore, arrive at the following variant of the problem (2.1):

Corollary 4.1. *Assume that $\Gamma = \omega \cap Fix(S) \neq \emptyset$. Then the sequence (p_k)*

$$\begin{cases} e_k = p_k + \theta_k(p_k - p_{k-1}); \\ \bar{w}_k = (1 - \alpha_k)e_k + \alpha_k S_k e_k; \\ x_k = (1 - \beta_k)\bar{w}_k + \beta_k(\Pi_H(\bar{w}_k + \delta h^*(\Pi_G - Id)h\bar{w}_k)); \\ H_{k+1} = \{\|x_k - z\|^2 \leq \|p_k - z\|^2 + \theta^2\|p_k - p_{k-1}\|^2 + 2\theta_k\langle p_k - z, p_k - p_{k-1}\rangle\}; \\ p_{k+1} = \Pi_{H_{k+1}} p_1, \ \forall\, k \geq 1, \end{cases} \tag{4.1}$$

generated by (4.1), under the control conditions (C1)–(C4), converges strongly to an element $p^* = \Pi_\Gamma p_1$.

4.2. Split equilibrium problems

The equilibrium problem from [13] aims to compute a point $p^* \in H$ such that

$$f(p^*, \bar{y}) \geq 0, \ \text{ for all } \bar{y} \in H, \tag{4.2}$$

where $f : H \times H \to \mathbb{R}$ is a bifunction satisfying,

(A1) $f(p^*, p^*) = 0$ for all $p^* \in H$;
(A2) f is monotone, i.e., $f(p^*, q^*) + f(q^*, p^*) \leq 0$ for all $p^*, q^* \in H$;
(A3) for each $p^*, q^*, t^* \in H$, $\limsup_{x\to 0} f(xt^* + (1 - x)p^*, q^*) \leq f(p^*, q^*)$;
(A4) for each $p^* \in H$, $q^* \mapsto h(p^*, q^*)$ is convex and lower semi-continuous.

The set EP(f) denotes the set of all solutions associated with the equilibrium problem (4.2). Recall the following auxiliary results for the equilibrium problem:

Lemma 4.1. *[25] Let H be a nonempty closed convex subset of a real Hilbert space Ξ_1 and let $f : H \times H \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). For $s > 0$ and $p^* \in \Xi_1$, there exists $t^* \in H$ such that*

$$f(t^*, q^*) + \frac{1}{s}\langle q^* - t^*, t^* - p^*\rangle \geq 0, \ \forall q^* \in H.$$

Moreover, define an operator $U_s^f : \Xi_1 \to H$ by

$$U_s^f(p^*) = \left\{t^* \in H : f(t^*, q^*) + \frac{1}{s}\langle q^* - t^*, t^* - p^*\rangle \geq 0, \ \forall q^* \in H\right\}.$$

Lemma 4.2. *[30] Let H be a nonempty closed convex subset of Ξ_1 and let $f : H \times H \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let $A_f : \Xi_1 \to 2^{\Xi_1}$ be a multivalued operator defined as:*

$$A_f(p^*) = \begin{cases} \{q^* \in \Xi_1 : f(p^*, t^*) \geq \langle t^* - p^*, q^* \rangle, \ \forall t^* \in H\}, \text{ if } p^* \in H, \\ \emptyset, \text{ if } p^* \notin H. \end{cases}$$

Recall that the operator A_f is a maximal monotone operator with domain $D(A_f) \subset C$ and $EP(f) = A_f^{-1}0$. Moreover, $U_s^f = (Id + sA_f)^{-1}$ for $s > 0$, i.e., U_s^f is the resolvent of A_f. We, therefore, arrive at the following variant of the problem (2.1):

Corollary 4.2. *Assume that $\Gamma = EP(f_1) \cap h^{-1}(EP(f_2)) \cap Fix(S) \neq \emptyset$. Then the sequence (p_k)*

$$\begin{cases} e_k = p_k + \theta_k(p_k - p_{k-1}); \\ \bar{w}_k = (1 - \alpha_k)e_k + \alpha_k S_k e_k; \\ x_k = (1 - \beta_k)\bar{w}_k + \beta_k(U_s^{f_1}(\bar{w}_k + \delta h^*(U_s^{f_2} - Id)h\bar{w}_k)); \\ H_{k+1} = \{\|x_k - z\|^2 \leq \|p_k - z\|^2 + \theta_k^2\|p_k - p_{k-1}\|^2 + 2\theta_k\langle p_k - z, p_k - p_{k-1}\rangle\}; \\ p_{k+1} = \Pi_{H_{k+1}} p_1, \ \forall\, k \geq 1, \end{cases} \tag{4.3}$$

generated by (4.3), under the control conditions (C1)-(C4), converges strongly to an element $p^* = \Pi_\Gamma p_1$.

4.3. Split optimization problems

Let $g : \Xi_1 \to (-\infty, \infty]$ be a PCLS function, then the set of minimizer associated with g is defined as

$$\text{argmin } g := \{p^* \in \Xi_1 : g(p^*) \leq g(q^*), \text{ for all } q^* \in \Xi_1\}.$$

Recall that the ∂g of PCLS function g is a maximal monotone operator and the corresponding resolvent operator of ∂g is called the proximity operator (see [22]). Hence argmin $g = (\partial g)^{-1}0$. We, therefore, arrive at the following variant of the problem (2.1).

Corollary 4.3. *Assume that $\Gamma = \{x \in \arg\min g_1 : hx \in \arg\min g_2\} \cap Fix(S) \neq \emptyset$. Then the sequence (p_k)*

$$\begin{cases} e_k = p_k + \theta_k(p_k - p_{k-1}); \\ \bar{w}_k = (1 - \alpha_k)e_k + \alpha_k S_k e_k; \\ x_k = (1 - \beta_k)\bar{w}_k + \beta_k(J_{m_k}^{\partial g_1}(\bar{w}_k + \delta h^*(J_{m_k}^{\partial g_2} - Id)h\bar{w}_k)); \\ H_{k+1} = \{\|x_k - z\|^2 \leq \|p_k - z\|^2 + \theta_k^2\|p_k - p_{k-1}\|^2 + 2\theta_k\langle p_k - z, p_k - p_{k-1}\rangle\}; \\ p_{k+1} = \Pi_{H_{k+1}} p_1, \ \forall\, k \geq 1, \end{cases} \tag{4.4}$$

generated by (4.4), under the control conditions (C1)–(C4), converges strongly to an element $p^* = \Pi_\Gamma p_1$.

5. Numerical experiment and results

In this section, we present an example that characterizes the performance of our algorithm.

Example 5.1. Let $\Xi_1 = \Xi_2 = (\mathbb{R}, \langle\cdot,\cdot\rangle, |\cdot|)$ where $\langle p, q\rangle = pq$. Consider the operators $h, A_1, A_2 : \mathbb{R} \to \mathbb{R}$ are defined as $h(p) = 3p$, $A_1 p = 2p$ and $A_2 p = 3p$, respectively. It is evident from the definition that A_1, A_2 are maximal monotone operators such that $\Omega := \{p^* \in A_1^{-1}0 : hp^* \in A_2^{-1}0\} = 0$ and h is a bounded linear operator on $\mathbb{R}$ with the adjoint operator h^* such that $||h|| = ||h^*|| = 3$. Let the sequence of operators $S_k : \mathbb{R} \to \mathbb{R}$ be defined by

$$S_k(p) = \begin{cases} -\frac{p}{k}, \, p \in [0, \infty); \\ p, \, p \in (-\infty, 0). \end{cases}$$

Then S_k is an infinite family of $\frac{1-k^2}{(1+k)^2}$-demicontractive operators with $\bigcap_{k=1}^{\infty} Fix(S_k) = \{0\}$. Hence $\Gamma = \Omega \cap \bigcap_{k=1}^{\infty} Fix(S_k) = 0$. We use the following initialization parameters for the computation of the Algorithm 1: $\theta = 0.5$, $\alpha_k = \frac{1}{100k+1}$, $\beta_k = \frac{k}{100k+1}$, $\delta = \frac{1}{8}$, $L = 3$ and m=0.02. Also observe that

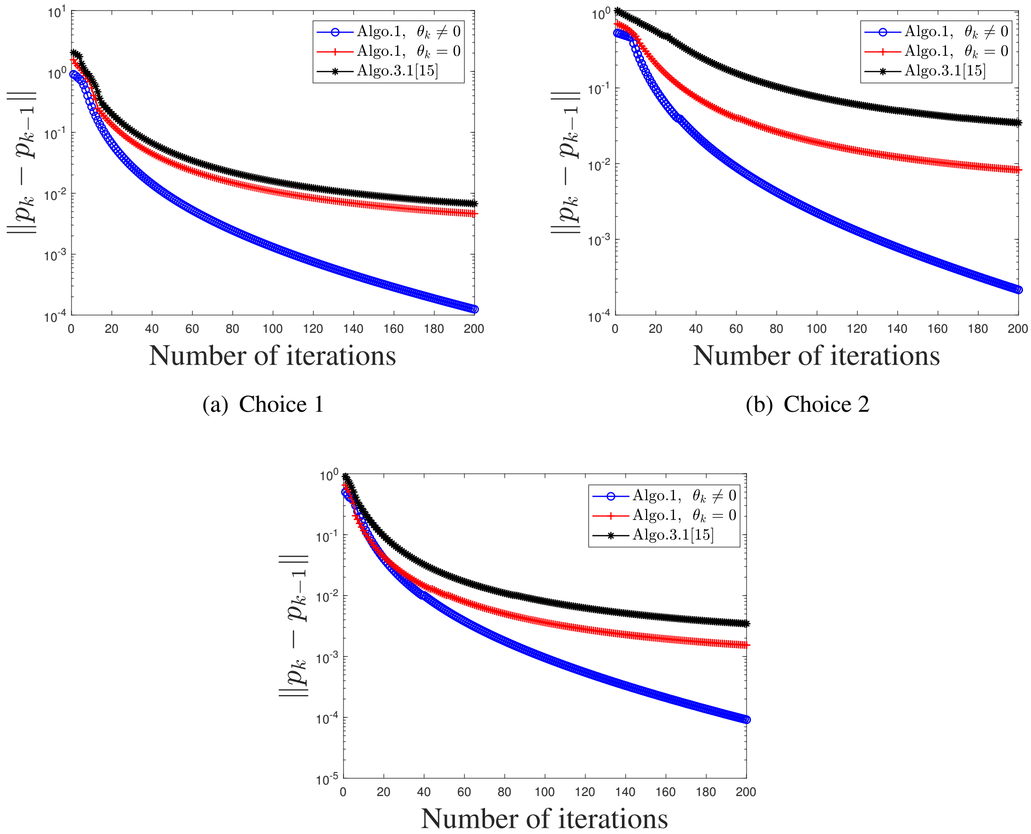
$$\begin{cases} \min\{\frac{1}{k^2||p_k - p_{k-1}||}, 0.5\}, & if \ p_k \neq p_{k-1}; \\ 0.5, & \text{otherwise.} \end{cases}$$

Let $Error = E_k = ||p_k - p_{k-1}|| < 10^{-5}$ denote the stopping criteria. The performance of the Algorithm 1 (i.e., Algo.1, $\theta_k \neq 0$) is compared with the non-inertial variant of the Algorithm 1 (i.e., Algo.1, $\theta_k = 0$) and Algo. 3.1 [16]. For different choices of the initial inputs p_0 and p_1, the values of Algo.1 are summarized in the following table:
Choice A. Choose $x_0 = (5)$, $x_1 = (2)$,
Choice B. Choose $x_0 = (4.2)$, $x_1 = (1.5)$,
Choice C. Choose $x_0 = (-7)$, $x_1 = (4)$.

Table 1. Computed Data Representation for the Algorithm 1 and Algo. 3.1(Byrne et. al [16]).

	No. of Iterations			CPU Time(Sec)		
	Choice 1	Choice 2	Choice 3	Choice 1	Choice 2	Choice 3
Algo.1,$\theta_k \neq 0$,	1787	1401	1920	0.064561	0.049590	0.057585
Algo.1,$\theta_k = 0$,	1819	1620	2008	0.071565	0.063519	0.076661
Algo. 3.1 [16]	7620	6598	8958	0.284874	0.687913	0.912241

The error plotting E_k against Algorithm 1 and Algorithm 3.1 [16], for each choices in Table 1, has shown in Figure 1.



(a) Choice 1

(b) Choice 2

(c) Choice 3

Figure 1. Comparison of Algorithm 1 (i.e., Algo.1, $\theta_k \neq 0$), Algorithm 1 (i.e., Algo.1, $\theta_k = 0$) and Algorithm 3.1 [16].

It is evident from Figure 1 that Algorithm 1 outperforms its noninertial variant and Algorithm 3.1 [16] with respect to the computation of error, CPU time and the number of iterations.

5.1. Signal processing

The mathematical model of the signal recovery problem in an under-determined linear equation system is expressed as follows:

$$\zeta = h\nu + \rho, \tag{5.1}$$

where $\nu \in \mathbb{R}^D$ denotes the original unknown signal to be recovered, $\zeta \in \mathbb{R}^P$ denotes the observed signal distort via the bounded linear matrix operator $h : \mathbb{R}^D \to \mathbb{R}^P, (P < D)$ and the noise ρ. One can define a natural convex constrained optimization-theoretic formulation of (5.1) via the following well known LASSO problem [33]:

$$\min_{\nu \in \mathbb{R}^D} \left\{ \frac{1}{2} \|h\nu - \zeta\|^2 \right\} \text{ subject to } \|\nu\|_1 \leq c, \ \forall\, c > 0. \tag{5.2}$$

The set of solutions of the ℓ_1-minimization problem (5.1) is equivalent to the set of solutions of (5.2) under certain control conditions on the matrix h [15]. The ℓ_1-norm based regularization problems are widely applicable in signal and image processing.

Set $\Gamma = H \cap h^{-1}(G) \neq \emptyset$ with $H = \{\nu \mid \|\nu\|_1 \leq c\}$ and $G = \{\zeta\}$. The experiment is conducted under the matrix $h^{D\times P}$ whose elements are generated from independently distributed normal distributions having 0 as mean and 1 as variance. The sparse vector ν, having $t = spikes$ nonzero elements, is generated via uniform distribution in $[-2, 2]$. The following iterative regularization method, often known as the Richardson method (or the Landweber method) [23], is generally employed to solve the problem (5.2):

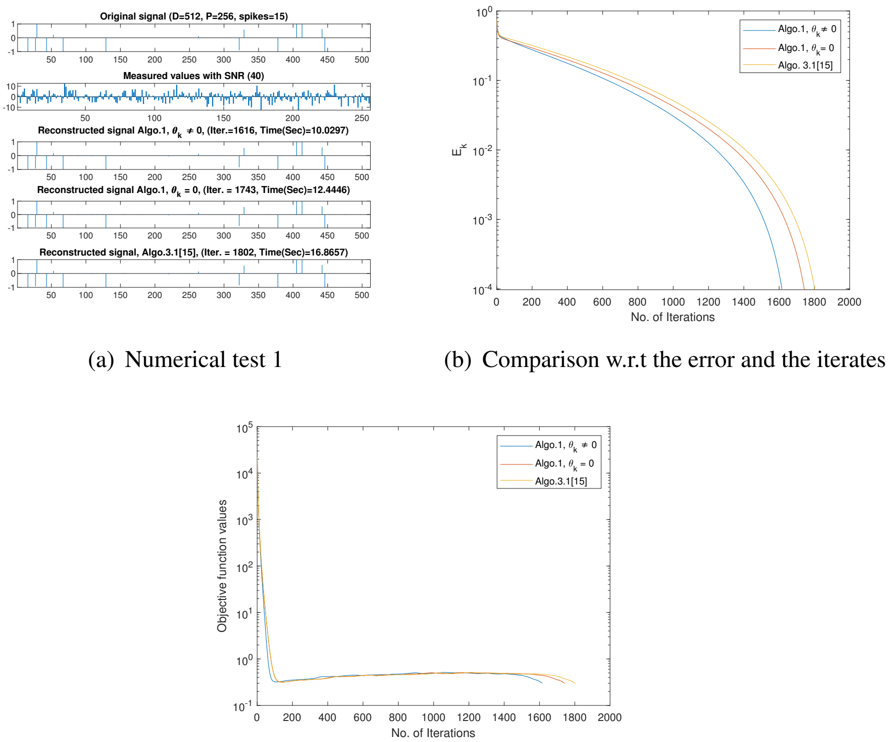
$$\nu_{k+1} = \nu_k + \eta h^T(\zeta - h\nu_k), \tag{5.3}$$

where η, the step size, is assumed to be constant. The algorithm (5.3) converges for $0 < \eta < \frac{2}{\epsilon_{\max}^2}$, where $\epsilon_{\max}$ is the maximum singular value of h. The initial points ν_0, ν_1 are chosen randomly. We use the mean squared error indicator to examine the performance of the algorithm for image restoration, i.e., $E_k = \frac{1}{N}\|\nu_k - \nu^*\| < 10^{-4}$, where ν^* is the approximation of the signal ν. The computation of the observed signal ζ is carried out by employing the Gaussian noise associated with the signal-to-noise ratio (SNR=40). Also set $m_k = \frac{1.85}{\|h\|^2}$, $\alpha_k = \frac{k}{100k+1}$, $\beta_k = \frac{1}{15k+1}$, $\delta = 0.04$, $c = t - 0.002$, $\mu = 0$ and θ_k=0.5.

Performance Test 1: Fix $D = 512$, $P = 256$ and $spikes = 15$.

Performance Test 2: Fix $D = 1024$, $P = 512$ and $spikes = 35$.

It is clear from the Figures 2 and 3 that the Algorithm 1 outperforms its variants and Algorithm 3.1 [16] for the signal recovery problem as well as exhibits fast convergence characteristic with regards to the error and number iterations.

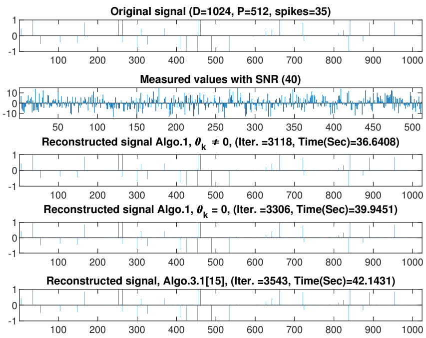


(a) Numerical test 1

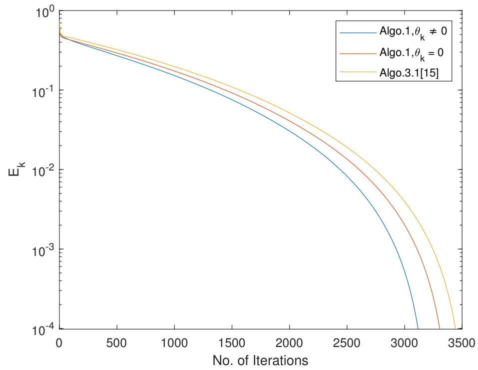
(b) Comparison w.r.t the error and the iterates

(c) Comparison w.r.t the objective function and the iterates

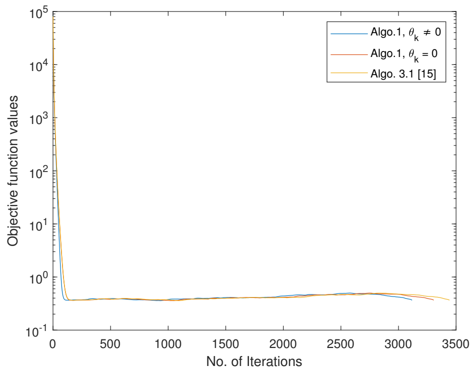
Figure 2. Comparison for the performance test 1.



(a) Numerical test 2



(b) Comparison w.r.t the error and the iterates



(c) Comparison w.r.t the objective function and the iterates

Figure 3. Comparison for the performance test 2.

6. Conclusions

In this paper, we have posited a framework for the investigation of the SCNPP and the FPP associated with an infinite family of $\Bbbk$-demicontractive operators in Hilbert spaces. The common optimal solution of the problem is then constructed via an inertial hybrid projection algorithm under the suitable set of constraints. We have incorporated an appropriate numerical example for the demonstration of the framework as well as for the applicability of our algorithm. We found that our algorithm outperforms its variants and Algorithm 3.1 [16]. We have also discussed various instances of the proposed formalism and can pave a way for an important future research direction in the theories of SCNPP and FPP.

Acknowledgments

The authors Y. Arfat and P. Kumam acknowledge the financial support provided by the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), KMUTT. Moreover, this research project is supported by Thailand Science Research and Innovation (TSRI) Basic Research Fund: Fiscal year 2021 under project number 64A306000005. The author Y. Arfat was supported by the Petchra

Pra Jom Klao Ph.D Research Scholarship from King Mongkut's University of Technology Thonburi, Thailand (Grant No.16/2562). Finally, the corresponding author Supak Phiangsungnoen acknowledge the financial support provided by institute of Research and Development, Rajamangala University of Technology Rattanakosin (Fundamental Fund Grant No.FRB6620/25).

Availability of data and material

Data sharing not applicable to this article as no data-sets were generated or analysed during the current study.

Conflict of interest

The authors declare no conflict of interest.

References

1. F. Alvarez, H. Attouch, An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping, *Set-valued Anal.*, **9** (2001), 3–11. http://doi.org/10.1023/A:1011253113155
2. Y. Arfat, O. S. Iyiola, M. A. A. Khan, P. Kumam, W. Kumam, K. Sitthithakerngkiet, Convergence analysis of the shrinking approximants for fixed point problem and generalized split common null point problem, *J. Inequal. Appl.*, **2022** (2022), 67.
3. Y. Arfat, P. Kumam, M. A. A. Khan, Y. J. Cho, A hybrid steepest-descent algorithm for convex minimization over the fixed point set of multivalued mappings, *Carpathian J. Math.*, **39** (2023), 303–314. https://doi.org/10.37193/CJM.2023.01.21
4. Y. Arfat, P. Kumam, M. A. A. Khan, O. S. Iyiola, Multi-inertial parallel hybrid projection algorithm for generalized split null point problems, *J. Appl. Math. Comput.*, **68** (2022), 3179–3198. http://doi.org/10.1007/s12190-021-01660-4
5. Y. Arfat, M. A. A. Khan, P. Kumam, W. Kumam, K. Sitthithakerngkiet, Iterative solutions via some variants of extragradient approximants in Hilbert spaces, *AIMS Mathematics*, **7** (2022), 13910–13926. https://doi.org/10.3934/math.2022768
6. Y. Arfat, P. Kumam, M. A. A. Khan, P. S. Ngiamsunthorn, An accelerated variant of the projection based parallel hybrid algorithm for split null point problems, *Topol. Methods Nonlinear Anal.*, **1** (2022), 1–18. https://doi.org/10.12775/TMNA.2022.015
7. Y. Arfat, P. Kumam, M. A. A. Khan, P. S. Ngiamsunthorn, An accelerated Visco-Cesaro means Tseng type splitting method for fixed point and monotone inclusion problems, *Carpathian J. Math.*, **38** (2022), 281–297.
8. Y. Arfat, P. Kumam, M. A. A. Khan, P. S. Ngiamsunthorn, An inertial extragradient algorithm for equilibrium and generalized split null point problems, *Adv. Comput. Math.*, **48** (2022), 53.
9. Y. Arfat, P. Kumam, M. A. A. Khan, P. S. Ngiamsunthorn, Parallel shrinking inertial extragradient approximants for pseudomonotone equilibrium, fixed point and generalized split null point problem, *Ric. Mat.*, 2021. https://doi.org/10.1007/s11587-021-00647-4

10. Y. Arfat, P. Kumam, M. A. A. Khan, P. S. Ngiamsunthorn, Shrinking approximants for fixed point problem and generalized split null point problem in Hilbert spaces, *Optim. Lett.*, **16** (2022), 1895–1913. https://doi.org/10.1007/s11590-021-01810-4

11. Y. Arfat, P. Kumam, M. A. A. Khan, P. S. Ngiamsunthorn, A. Kaewkhao, A parallel hybrid accelerated extragradient algorithm for pseudomonotone equilibrium, fixed point, and split null point problems, *Adv. Differ. Equ.*, **364** (2021), 364. http://doi.org/10.1186/s13662-021-03518-2

12. H. H. Bauschke, P. L. Combettes, Convex analysis and monotone operator theory in Hilbert spaces, In: *CMS Books in Mathematics*, New York: Springer, 2011.

13. E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Stud.*, **63** (1994), 123–145.

14. F. E. Browder, Nonexpansive nonlinear operators in a Banach space, *Proc. Natl. Acad. Sci.*, **54** (1965), 1041–1044. https://doi.org/10.1073/pnas.54.4.1041

15. C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, *Inverse Problems*, **20** (2004), 103–120. https://doi.org/10.1088/0266-5611/20/1/006

16. C. Byrne, Y. Censor, A. Gibali, S. Reich, The split common null point problem, *J. Nonlinear Convex Anal.*, **13** (2012), 759–775. https://doi.org/10.48550/arXiv.1108.5953

17. Y. Censor, T. Bortfeld, B. Martin, A. Trofimov, A unified approach for inversion problems in intensity-modulated radiation therapy, *Phys. Med. Biol.*, **51** (2006), 2353–2365. https://doi.org/10.1088/0031-9155/51/10/001

18. Y. Censor, T. Elfving, A multiprojection algorithm using Bregman projections in a product space, *Numer. Algorithms*, **8** (1994), 221–239.

19. Y. Censor, A. Gibali, S. Reich, Algorithms for the split variational inequality problem, *Numer. Algorithms*, **59** (2012), 301–323.

20. Y. Censor, A. Segal, The split common fixed point problem for directed operators, *J. Convex Anal.*, **26** (2010), 55007. https://doi.org/10.1088/0266-5611/26/5/055007

21. P. L. Combettes, The convex feasibility problem in image recovery, *Adv. Imaging Electron Phys.*, **95** (1996), 155–270. https://doi.org/10.1016/S1076-5670(08)70157-5

22. P. L. Combettes, J. C. Pesquet, Proximal splitting methods in signal processing, In: *Springer optimization and its applications*, **49** (2011), 185–212.

23. J. Duchi, S. Shalev-Shwartz, Y. Singer, T. Chandra, Efficient projections onto the l_1-ball for learning in high dimensions, In: *Proceedings of the 25th international conference on machine learning*, Helsinki, 2008.

24. T. L. Hicks, J. D. Kubicek, On the Mann iteration process in a Hilbert space, *J. Math. Anal. Appl.*, **59** (1977), 498–504. https://doi.org/10.1016/0022-247X(77)90076-2

25. Z. Ma, L. Wang, S-S. Chang, W. Duan, Convergence theorems for split equality mixed equilibrium problems with applications, *Fixed Point Theory Appl.*, **2015** (2015), 31. http://doi.org/10.1186/s13663-015-0281-x

26. C. Martinez-Yanes, H. K. Xu, Strong convergence of CQ method for fixed point iteration processes, *Nonlinear Anal.*, **64** (2006), 2400–2411. https://doi.org/10.1016/j.na.2005.08.018

27. B. T. Polyak, Some methods of speeding up the convergence of iteration methods, *USSR Comput. Math. Math. Phys.*, **4** (1964), 1–17. https://doi.org/10.1016/0041-5553(64)90137-5
28. W. Takahashi, K. Shimoji, Convergence theorems for nonexpansive mappings and feasibility problems, *Math. Comput. Model.*, **32** (2000), 1463–1471. https://doi.org/10.1016/S0895-7177(00)00218-1
29. S. Takahashi, W. Takahashi, The split common null point problem and the shrinking projection method in Banach spaces, *Optimization*, **65** (2016), 281–287. https://doi.org/10.1080/02331934.2015.1020943
30. S. Takahashi, W. Takahashi, M. Toyoda, Strong convergence theorem for maximal monotone operators with nonlinear mappings in Hilbert spaces, *J. Optim. Theory Appl.*, **147** (2010), 27–41. http://doi.org/10.1007/s10957-010-9713-2
31. W. Takahashi, H. K. Xu, J. C. Yao, Iterative methods for generalized split feasibility problems in Hilbert spaces, *Set Valued Var. Anal.*, **23** (2015), 205–221. http://doi.org/10.1007/s11228-014-0285-4
32. B. Tan, Z. Zhou, S. X. Li, Viscosity-type inertial extragradient algorithms for solving variational inequality problems and fixed point problems, *J. Appl. Math. Comput.*, **68** (2022), 1387-1411. https://doi.org/10.1007/S12190-021-01576-Z
33. R. Tibshirani, Regression shrinkage and selection via lasso, *J. R. Stat. Soc. Ser. B Stat. Methodol.*, **58** (1996), 267–288. https://doi.org/10.1111/j.2517-6161.1996.tb02080.x
34. S. Wang, A general iterative method for obtaining an infinite family of strictly pseudo-contractive mappings in Hilbert spaces, *Appl. Math. Lett.*, **24** (2011), 901–907. https://doi.org/10.1016/j.aml.2010.12.048