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Research article

On some new results in a pursuit differential game with many pursuers and one evader

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Abstract: We study a simple motion differential game with many pursuers and one evader with equal capabilities in $\mathbb{R}^n$. The control functions of the players are subject to the Grönwall-type constraints. If the state of the evader coincides with the state of a pursuer, then the game is considered completed. If the state of the evader does not coincide with the state of any pursuer at all times, then we say that evasion is possible. We show that pursuit can be completed if a condition on the convex hull of the initial states of the pursuers is satisfied.

Keywords: differential game; Grönwall constraints; pursuer; evader; strategy
Mathematics Subject Classification: Primary 05C57, Secondary 91A43

1. Introduction

The notion of differential games was introduced in the works of Isaacs [14], Pontryagin [19] and Krasovskii [16]. The theory was developed further by Azamov [1, 2], Chikrii [4], Friedman [5], Petrosyan [18], Pshenichnyi [20], Satimov [22] and Subbotin [24].*

In terms of pursuit evasion games, several studies were devoted to differential games with many players with geometric constraints on the controls of the players. Ivanov [15] studied a relay

*Note that the above class of games is distinct from the class of pursuit evasion games on graphs (see Ibragimov and Luckraz [13] and Luckraz [17], for instance).

pursuit-evasion game problem with m, where $m \geq n + 1$, pursuers and one evader with equal capabilities. The differential game was considered on a closed bounded convex subset of R^n under geometric constraints on the controls of the players. It was shown that the differential game can be completed in time $O(n^3)$. For a simple motion differential game with many pursuers and one evader with equal capabilities, Pshenichnii [20] obtained necessary and sufficient conditions for the completion of pursuit by considering the convex hull of initial states of pursuers.

Chernousko and Zak [3] studied an evasion differential game with a high speed evader and several pursuers. They proved that the evader can avoid capture by moving in a given neighborhood of a ray. Similar differential game problems with multiple pursuers and a single evader have been addressed by the works of [7, 9, 25, 26].

Garcia et al. [8] studied a border-defense differential game of n pursuers and n evaders. The purpose of the group of evaders is to reach the boundary of the constraint. If the evaders are captured before reaching the boundary, then they minimize their distance from the boundary. The cases $n = m$ and $n > m$ were analyzed.

A few studies were devoted to the differential games with many players and with integral constraints. One such study was a linear differential game considered by Satimov et al. [23], where sufficient conditions for the completion of pursuit were obtained by considering the control of the resources of the players. More recently, Ibragimov, Satimov and Yu [11] extended this result to a pursuit differential game with many pursuers and many evaders and showed that if the total energy of the pursuers is greater than the total energy of the evaders, then pursuit can be completed. Ibragimov et al. [10] showed that if the total energy of pursuers is less or equal to the total energy of evaders, then evasion is possible. For the case of many pursuers and one evader, this work was extended to a linear evasion differential game by Ibragimov et al. [12].

In the work of Samatov et al. [21], for a differential game with Grönwall-type constraints on the controls of players, the optimal strategies of the players were constructed, and an equation for the optimal pursuit time was obtained for the case of one pursuer and one evader. Furthermore, it was shown that a strategy of parallel approach is optimal for the pursuer. According to this strategy, for any time $t \geq 0$, the straight line passing through the state of the pursuer and that of the evader remains parallel to the straight line passing through their initial states.

The present research focuses on a simple motion pursuit-evasion differential game with many players with Grönwall-type constraints on the players' controls (see [6]). Our result can be seen as an extension of the result by Samatov et al. [21], who considered the case of one pursuer and one evader with Grönwall-type constraints. In the present paper, we prove that if $y_0 \in \text{intconv}\{x_{10}, ..., x_{m0}\}$, where $x_{10}, ..., x_{m0}$ and y_0 are the initial positions of pursuers and evader, respectively, then pursuit can be completed in the differential game, described by (2.1)–(2.4) below, for some finite time T. Moreover, if $y_0 \notin \text{intconv}\{x_{10}, ..., x_{m0}\}$, then evasion is possible in the differential game. We obtain a formula for the guaranteed pursuit time T, and we construct explicit strategies for the pursuers in the pursuit differential game and a strategy for the evader in the evasion differential game.

2. Statement of problem

Let the dynamics of pursuers x_i and evader y be described by the following equations:

$$\dot{x}_i = u_i, \quad x_i(0) = x_{i0}, \ i = 1, ..., m, \tag{2.1}$$

$$\dot{y} = v, \quad y(0) = y_0, \tag{2.2}$$

where $x_i, y, x_{i0}, y_0 \in \mathbb{R}^n$, $x_{i0} \neq y_0$, $i = 1, ..., m$.

Definition 1. *Borel measurable functions $u_i(t)$, $i \in \{1, 2, ..., m\}$, $t \geq 0$, and $v(t)$, $t \geq 0$, that satisfy the constraints*

$$|u_i(t)|^2 \leq \rho_i^2 + 2k \int_0^t |u_i(s)|^2 ds, \quad t \geq 0, \tag{2.3}$$

$$|v(t)|^2 \leq \sigma^2 + 2k \int_0^t |v(s)|^2 ds, \quad t \geq 0, \tag{2.4}$$

are called admissible controls of the pursuer x_i, $i \in \{1, \ldots, m\}$, and evader y, respectively, where ρ_i, σ, k are given positive numbers.

The constraints (2.3) and (2.4) are called the Grönwall-type constraints.

Remark 1. *It can be shown that inequalities (2.3) and (2.4) imply, respectively, that $|u_i(t)| \leq \rho_i e^{kt}$ and $|v(t)| \leq \sigma e^{kt}$, but not vice versa. Also, note that if $|u_i(t)| = \rho_i e^{kt}$ and $|v(t)| = \sigma e^{kt}$, then constraints (2.3) and (2.4) are satisfied.*

We consider a pursuit differential game where pursuers apply strategies and evader uses an arbitrary admissible control. We give a definition for the strategies of the pursuers as follows.

Definition 2. *We call the functions*

$$(t, x_i, y, v) \to U_i(t, x_i, y, v), \quad i = 1, 2, ..., m,$$

strategies of pursuers x_i, $i = 1, 2, ..., m$, if, for $u_i = U_i(t, x_i, y, v)$, $i = 1, 2, ..., m$, and for any admissible control $v = v(t)$ of the evader, the initial value problem (2.1) has a unique solution $(x_1(t), x_2(t), ..., x_m(t), y(t))$, $t \geq 0$, and along this solution the following inequalities hold:

$$|U_i(t, x_i(t), y(t), v(t))|^2 \leq \rho_i^2 + 2k \int_0^t |U_i(s, x_i(s), y(s), v(s))|^2 ds, \quad t \geq 0.$$

Definition 3. *If, for some number $T > 0$ and any initial states of players $x_{10}, x_{20}, ..., x_{m0}, y_0$, there exist strategies of pursuers such that, for any control of evader, we have $x_i(\tau) = y(\tau)$ at some $0 < \tau \leq T$ and $i \in \{1, 2, ..., m\}$, then we say that pursuit is completed in the game (2.1)–(2.4) for the time T. Also, the time T is called a guaranteed pursuit time.*

The pursuers try to complete the game as early as possible. In other words, pursuers try to minimize the guaranteed pursuit time.

Also, we consider an evasion differential game. We define the strategy of the evader as follows.

Definition 4. *We call the function*

$$(t, x_1, x_2, ..., x_m, y) \to V(t, x_1, x_2, ..., x_m, y),$$

strategy of evader y, if, for $v = V(t, x_1, x_2, ..., x_m, y)$*, and for any admissible controls* $u_i = u_i(t)$ *of the pursuers, the initial value problem (2.1) has a unique solution* $(x_1(t), x_2(t), ..., x_m(t), y(t))$*,* $t \geq 0$*, and along this solution the following inequality holds:*

$$|V(t, x_1(t), x_2(t), ..., x_m(t), y(t))|^2 \leq \sigma^2 + 2k \int_0^t |V(s, x_1(s), x_2(s), ..., x_m(s), y(s))|^2 ds, \quad t \geq 0.$$

Definition 5. *If there exists a strategy of evader such that, for any controls of pursuers,* $x_i(t) \neq y(t)$ *for all* $t \geq 0$ *and* $i = 1, 2, ..., m$*, then we say that evasion is possible in differential game (2.1)–(2.4).*

In the present paper, we consider the case where $\rho_i = \sigma$, $i = 1, 2, ..., m$, and therefore, all pursuers and evader have equal capabilities.

Problem 1. *Find a condition of completion of pursuit and a guaranteed pursuit time in differential game (2.1)–(2.4).*

Problem 2. *Find a condition of evasion in differential game (2.1)–(2.4).*

3. Main result

In this section, we formulate the main results of the paper. Let $\text{intconv}\{x_{10}, ..., x_{m0}\}$ denote the interior of the convex hull of points $x_{10}, ..., x_{m0}$. To prove the main theorems, we need the following well-known statements from functional analysis on the separation of convex sets.

Proposition 1. *The point* y_0 *belongs to the set* $\text{intconv}\{x_{10}, ..., x_{m0}\}$ *if and only if, for any vector p,* $|p| = 1$*, there is a number* $i \in \{1, 2, ..., m\}$ *such that* $(p, e_i) < 0$*, where* $e_i = (y_0 - x_{i0})/|y_0 - x_{i0}|$.

Proposition 2. *The point* y_0 *does not belong to the set* $\text{intconv}\{x_{10}, ..., x_{m0}\}$ *if and only if there is a vector* p_0*,* $|p_0| = 1$*, such that* $(p_0, e_i) \geq 0$ *for all* $i = 1, ..., m$.

3.1. Pursuit differential game

For the pursuit differential game, we prove the following statement.

Theorem 1. *If* $y_0 \in \text{intconv}\{x_{10}, ..., x_{m0}\}$*, then pursuit can be completed in differential game (2.1)–(2.4).*

Proof. To prove the theorem, we construct strategies for the pursuers as follows:

$$U_i = v - (v, e_i)e_i + e_i \sqrt{\sigma^2 e^{2kt} - |v|^2 + (v, e_i)^2}, \quad i = 1, 2, ..., m. \tag{3.1}$$

For this strategy, we have $|U_i(t)| = \sigma e^{kt}$, and therefore, as noted in Remark 1, the constructed strategy satisfies the constraints (2.3). Hence, the strategy (3.1) is admissible.

Next, we show that pursuit can be completed for some finite time T. Using the strategy (3.1) we obtain

$$
\begin{aligned}
y(t) - x_i(t) &= y_0 + \int_0^t v(s)\,ds - x_{i0} - \int_0^t U_i(s)\,ds \\
&= y_0 - x_{i0} - e_i \int_0^t \left(\sqrt{\sigma^2 e^{2ks} - |v(s)|^2 + (v(s), e_i)^2} - (v(s), e_i)\right) ds \\
&= d_i(t) e_i,
\end{aligned}
$$

where

$$
d_i(t) = |y_0 - x_{i0}| - \int_0^t \left(\sqrt{\sigma^2 e^{2ks} - |v(s)|^2 + (v(s), e_i)^2} - (v(s), e_i)\right) ds.
$$

We estimate

$$
\begin{aligned}
d(t) &= \sum_{i=1}^m d_i(t) \\
&= \sum_{i=1}^m |y_0 - x_{i0}| - \int_0^t \sum_{i=1}^m \left(\sqrt{\sigma^2 e^{2ks} - |v(s)|^2 + (v(s), e_i)^2} - (v(s), e_i)\right) ds. \qquad (3.2)
\end{aligned}
$$

Putting $\bar{v} = e^{-kt} v$, we have $|\bar{v}| = |e^{-kt} v| \leq \sigma$, and

$$
\sum_{i=1}^m \left(\sqrt{\sigma^2 e^{2ks} - |v(s)|^2 + (v(s), e_i)^2} - (v(s), e_i)\right)
$$

$$
= e^{ks} \sum_{i=1}^m \left(\sqrt{\sigma^2 - |\bar{v}|^2 + (\bar{v}, e_i)^2} - (\bar{v}, e_i)\right). \qquad (3.3)
$$

We consider the following function:

$$
f(\bar{v}) = \sum_{i=1}^m \left(\sqrt{\sigma^2 - |\bar{v}|^2 + (\bar{v}, e_i)^2} - (\bar{v}, e_i)\right), \quad |\bar{v}| \leq \sigma.
$$

Observe that the function $f(\bar{v})$ is continuous on the compact set $|\bar{v}| \leq \sigma$, and so $f(\bar{v})$ attains its minimum at some $\bar{v} = \bar{v}_0$, $|\bar{v}_0| \leq \sigma$. Let

$$
\min_{|\bar{v}| \leq \sigma} f(\bar{v}) = f(\bar{v}_0) = \alpha. \qquad (3.4)
$$

Since

$$
\sqrt{\sigma^2 - |\bar{v}|^2 + (\bar{v}, e_i)^2} - (\bar{v}, e_i) \geq 0,
$$

$f(\bar{v}) \geq 0$, $|\bar{v}| \leq \sigma$. Consequently, $\alpha \geq 0$.

Next, we prove that $\alpha > 0$. Assume the contrary, and let $\alpha = 0$. Then, $f(\bar{v}_0) = \alpha = 0$, that is,

$$\sum_{i=1}^{m} \left(\sqrt{\sigma^2 - |\bar{v}_0|^2 + (\bar{v}_0, e_i)^2} - (\bar{v}_0, e_i) \right) = 0. \tag{3.5}$$

Note that $\bar{v}_0 \neq 0$, since otherwise we get

$$\alpha = \sum_{i=1}^{m} \left(\sqrt{\sigma^2 - |0|^2 + (0, e_i)^2} - (0, e_i) \right) = m\sigma > 0,$$

which contradicts our assumption $\alpha = 0$.

Next, since

$$\sqrt{\sigma^2 - |\bar{v}_0|^2 + (\bar{v}_0, e_i)^2} - (\bar{v}_0, e_i) \geq 0,$$

we obtain from (3.5) that

$$\sqrt{\sigma^2 - |\bar{v}_0|^2 + (\bar{v}_0, e_i)^2} = (\bar{v}_0, e_i), \quad i = 1, 2, ..., m.$$

This implies that $(\bar{v}_0, e_i) \geq 0$ for all $i = 1, 2, ..., m$. By Proposition 2 this means that $y_0 \notin \text{intconv}\{x_{10}, ..., x_{m0}\}$. This contradicts the hypothesis of the theorem. Thus, $\alpha > 0$. We obtain then from (3.2) that

$$\begin{aligned} d(t) &\leq \sum_{i=1}^{m} |y_0 - x_{i0}| - \alpha \int_0^t e^{ks} ds \\ &= \sum_{i=1}^{m} |y_0 - x_{i0}| - \frac{\alpha}{k} \cdot (e^{kt} - 1). \end{aligned}$$

Since the right hand side of this equation is equal to zero at

$$T = \frac{1}{k} \ln \left(1 + \frac{k}{\alpha} \sum_{i=1}^{m} |y_0 - x_{i0}| \right),$$

we have $d(\tau) = 0$ at some $\tau \in [0, T]$. This implies that $d_{i_0}(\tau_{i_0}) = 0$ at some $\tau_{i_0} \in [0, \tau]$ and $i_0 \in \{1, 2, ...m\}$, and therefore $x_{i_0}(\tau_{i_0}) = y(\tau_{i_0})$, that is, pursuit is completed. The proof of the theorem is complete. □

3.2. Evasion differential game

For the evasion differential game, we prove the following statement.

Theorem 2. *If* $y_0 \notin \text{intconv}\{x_{10}, ..., x_{m0}\}$, *then evasion is possible in differential game (2.1)–(2.4).*

Proof. We construct a strategy for the evader. By Proposition 2, there is a vector p_0, $|p_0| = 1$, such that $(p_0, e_i) \geq 0$ for all $i = 1, 2, ..., m$. For the strategy of evader, we set

$$V(t) = \sigma e^{kt} p_0, \quad t \geq 0.$$

Since $|V(t)| = \sigma e^{kt}$, as noted in Remark 1, the above strategy of the evader is admissible. We have, for any $i = 1, 2, ...m$,

$$\begin{aligned} |y(t) - x_i(t)| &= \left| y_0 + \int_0^t V(s)ds - x_{i0} - \int_0^t u_i(s)ds \right| \\ &\geq \left| y_0 + \int_0^t \sigma e^{ks} p_0 ds - x_{i0} \right| - \left| \int_0^t u_i(s)ds \right| \\ &\geq \left| y_0 - x_{i0} + \frac{\sigma}{k}(e^{kt} - 1)p_0 \right| - \int_0^t |u_i(s)|\, ds \\ &\geq \sqrt{|y_0 - x_{i0}|^2 + 2\frac{\sigma}{k}(e^{kt} - 1)(y_0 - x_{i0}, p_0) + \frac{\sigma^2}{k^2}(e^{kt} - 1)^2} - \int_0^t \sigma e^{ks} ds. \end{aligned} \tag{3.6}$$

Since $(y_0 - x_{i0}, p_0) = |y_0 - x_{i0}|(p_0, e_i) \geq 0$, (3.6) implies that

$$|y(t) - x_i(t)| \geq \sqrt{|y_0 - x_{i0}|^2 + \frac{\sigma^2}{k^2}(e^{kt} - 1)^2} - \frac{\sigma}{k}(e^{kt} - 1) > 0.$$

Hence, $y(t) \neq x_i(t)$, for all $t \geq 0$ and $i = 1, 2, ..., m$. The proof of the theorem is complete. □

4. Conclusions and discussion

We have studied a simple motion pursuit-evasion differential game with many players in $\mathbb{R}^n$, where the control functions of the players are subject to Grönwall-type constraints. We used some ideas from the work of [20], where a simple motion pursuit-evasion differential game with geometric constraints was studied. Our contribution to the class of differential games with Grönwall-type constraints on the control functions of the players can be summarized as follows. We have proved that if $y_0 \in \text{intconv}\{x_{10}, ..., x_{m0}\}$, then pursuit can be completed in the differential game (2.1)–(2.4) for some finite time T, whereas if $y_0 \notin \text{intconv}\{x_{10}, ..., x_{m0}\}$, then evasion is possible. We have also obtained a formula for the guaranteed pursuit time T. Furthermore, we constructed explicit strategies for the pursuers in the pursuit differential game and a strategy for the evader in the evasion differential game.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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