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Research article

A novel Muth generalized family of distributions: Properties and applications to quality control

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Abstract: In this paper, we propose a novel family of distributions called the *odd Muth-G* distributions by using Transformed-Transformer methodology and study their essential properties. The distinctive feature of the proposed family is that it can provide numerous special models with significant applications in reliability analysis. The density of the new model is expressible in terms of linear combinations of generalized exponentials, a useful feature to extract most properties of the proposed family. Some of the structural properties are derived in the form of explicit expressions such as quantile function, moments, probability weighted moments and entropy. The model parameters are estimated following the method of maximum likelihood principle. Weibull is selected as a baseline to propose an odd Muth-Weibull distribution with some useful properties. In order to confirm that our results converge with minimized mean squared error and biases, a simulation study has been performed. Additionally, a plan acceptance sampling design is proposed in which the lifetime of an item follows an odd Muth-Weibull model by taking median lifetime as a quality parameter. Two real-life data applications are presented to establish practical usefulness of the proposed model with conclusive evidence that the model has enough flexibility to fit a wide panel of lifetime data sets.

Keywords: group acceptance sampling plan; maximum likelihood method; Muth-G family; quality parameter; T-X family
Mathematics Subject Classification: 62E15, 62E05, 62E10

1. Introduction

In real-life circumstances, there is always an element of uncertainty which always makes the applied researchers have jitters regarding the selection of an appropriate model. Thus, in order to be on the safe side, applied practitioners always prefer classical distributions such as the Exponential, Weibull or Gamma distribution. To add to the misery, theoreticians generally propose generalizations and modifications of such classical models in order to resolve discrepancies among them. There is an abundance of generalizations of such orthodox distributions as it the discretion of researchers to select the model which they want to explore both theoretically and in applied form.

Despite their importance in the literature, there are a number of distributions that have yet to be fully investigated. The functional complexity of the models may be the most plausible rationale, with the improvement of computational capabilities and numerical optimization techniques such as MATLAB, Python and the R language, this claim is easily refuted. In our perspective, the statistical literature should include these overlooked models that are seldom employed. For distributions that are not regularly discussed in the literature, the authors in [1] provided a comprehensive list. This motivated us to investigate such models or suggest long-overlooked generalizations based on such models. One such model is the Muth distribution, with the name pioneered by the authors in [1]. For a continuous univariate distribution, a random variable X is said to follow a Muth distribution such that $X \sim Muth(a)$ with the following distribution function:

$$G(x) = 1 - \exp\left\{a\,x - \frac{1}{a}[\exp(a\,x) - 1]\right\}, \qquad x > 0, \tag{1.1}$$

where parameter $a \in (0, 1)$. According to the authors in [2], it was Tessiér (1934) who initially studied this distribution in the context of an animal ageing mechanism. However, Muth (1977) pointed out instances where it appears as a good model to study the stochastic nature of the variable under consideration as compared to established models. In [3], the authors studied the scaled version of the Muth distribution and established its superiority over the existing distribution by using meteorology data. A few other works related to the Muth distribution have been esented in [2–7]. The reader is referred to [2], in which an excellent review of the Muth distribution in chronological order has been conducted by the authors.

In this article, we propose a generalization of the Muth distribution. Regarding the generalization of conventional models, the Transformed-transformer T-X approach, introducted in [8], is an integral part for the construction of generalized families of distributions. The distribution function (cdf) of the T–X family is defined by

$$F_{T-X}(x) = \int_a^{W\left(G(x;\xi)\right)} r(t)\,\mathrm{d}t = R\Big[W(G(x;\xi))\Big].$$

The pdf corresponding to (1) is

$$f_{T-X}(x) = r\left[W(G(x;\xi))\right]\frac{\mathrm{d}}{\mathrm{d}x}\,W(G(x;\xi))\,,$$

where the pdf of any baseline distribution is $g(x;\xi)$.

To the best of our knowledge, very few generalizations of the Muth distribution have been proposed in the literature. These include the Muth-G family by Almarashi and Elgarhy [9] using the T-X methodology, the Transmuted Muth-G class of distributions by [10] using quadratic rank transformation and the New Truncated Muth generated (NTM-G) family of distributions [31] in the context of a unit distribution. This further intrigued us to formulate a generalization of the Muth distribution via an odd random variable, denoted as odd Muth-G (OMG for short), in Eq (2.1) and study its mathematical properties. Similar generalizations based on odd ratios are odd Weibull-G in [11], odd generalized-exponential-G in [12], alternate odd generalized exponential-G in [13], odd gamma-G in [14], odd Lindley-G in [15], odd Burr-G in [16], odd power-Cauchy-G in [17], odd half-Cauchy in [18], odd additive Weibull-G [19], odd power-Lindley-G [20], odd Xgamma-G [21], etc. For a comprehensive review on generalized families, the reader is referred to [22, 23].

For a more apt background of the T-X approach, readers are referred to [8]. Further motivations to propose the OMG class include the following: The inverse distribution function, median, moment generating function and characteristic function of the Muth distribution are not mathematically tractable, though these properties exist for the OMG family; when the shape parameter $a \to 0$, the Muth distribution converges to the exponential family. Thus, there exists a relation between the OMG and exponential families such as exponentiated-G (EG) by [24] and exponentiated generalized-G (EGG) by [25]; the OMG class improves the flexibility of the tail properties of the baseline distribution in terms of improving the goodness of fit statistical criterion and the ability to fit symmetric as well as asymmetric real life phenomena; the Muth distribution is applied for the first time in the context of quality control, which is an integral part of reliability analysis.

The manuscript is structured as follows. In Section 2, the odd Muth-G family and its reliability properties are defined. The general properties of the proposed family are depicted in Section 3. Parameters estimation of the proposed family is illustrated in Section 4. A special model called odd Muth-Weibull (OMW) is presented in Section 5 with some essential properties. Section 6 is based on simulation analysis, while Section 7 showes the mathematical and numerical illustration of the group acceptance sampling plan (GASP). The application to real-life data is presented in Section 8. Section 9 ends the manuscript with some concluding remarks.

2. Layout of the proposed family

In this section, the odd Muth-G (OMG) family of distributions and its reliability properties are defined.

The following are the expressions of cdf, pdf, reliability function (rf), hazard rate function (hrf) and cumulative hazard rate function (chrf) of the OMG family, respectively:

$$\begin{aligned} F(x) &= \int_0^{\frac{G(x;\xi)}{\bar{G}(x;\xi)}} [\exp(a\,x) - a] \exp\left\{a\,x - \frac{1}{a}[\exp(a\,x) - 1]\right\} \mathrm{dx} \\ &= 1 - \exp\left\{\frac{a\,G(x;\xi)}{\bar{G}(x;\xi)} - \frac{1}{a}\left[\exp\left(\frac{a\,G(x;\xi)}{\bar{G}(x;\xi)}\right) - 1\right]\right\}, \end{aligned} \tag{2.1}$$

$$f(x) = \frac{g(x;\xi)}{\bar{G}(x;\xi)^2}\left\{\exp\left(\frac{a\,G(x;\xi)}{\bar{G}(x;\xi)}\right) - a\right\} \exp\left\{\frac{a\,G(x;\xi)}{\bar{G}(x;\xi)} - \frac{1}{a}\left[\exp\left(\frac{a\,G(x;\xi)}{\bar{G}(x;\xi)}\right) - 1\right]\right\}. \tag{2.2}$$

$$r(x) = \exp\left\{\frac{a\,G(x;\xi)}{\bar{G}(x;\xi)} - \frac{1}{a}\left[\exp\left(\frac{a\,G(x;\xi)}{\bar{G}(x;\xi)}\right) - 1\right]\right\}, \tag{2.3}$$

$$h(x) \;=\; \frac{g(x;\xi)}{\bar{G}(x;\xi)^2}\left\{\exp\left(\frac{a\,G(x;\xi)}{\bar{G}(x;\xi)}\right)-a\right\} \tag{2.4}$$

and

$$H(x) \;=\; \frac{a\,G(x;\xi)}{\bar{G}(x;\xi)}-\frac{1}{a}\left[\exp\left(\frac{a\,G(x;\xi)}{\bar{G}(x;\xi)}\right)-1\right]. \tag{2.5}$$

3. Proposed family and its properties

Here, we derive some basic properties of the OMG family.

3.1. The expression of quantile function

The following expression shows the quantile function (qf) of the OMG:

$$Q_X(u;a)=G^{-1}\left[1+\left\{\frac{1}{a}\log(1-u)-\frac{1}{a}W_{-1}\left(\frac{u-1}{a\exp(\frac{1}{a})}\right)-\frac{1}{a^2}\right\}^{-1}\right]^{-1}. \tag{3.1}$$

The above expression contains the Lambert-W function of the negative branch.

3.2. Densities expansion

In this section, we present a useful expansion for Eq (2.1) by using exponential series expansions, as

$$\mathrm{e}^{-\mathrm{b}\,\mathrm{z}}=\sum_{\mathrm{i}=0}^{\infty}\frac{(-1)^{\mathrm{i}}\mathrm{b}^{\mathrm{i}}}{\mathrm{i}!}\mathrm{z}^{\mathrm{i}}$$

and

$$\exp(b\,z)=\sum_{i=0}^{\infty}\frac{b^i}{i!}z^i.$$

By using the above exponential series on Eq (2.1), it reduces to

$$F(x) \;=\; \exp(\frac{1}{a})\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\frac{(-1)^{i+1}a^{j-i}(i+1)^j}{i!\,j!}G(x;\xi)^j\frac{1}{\left[\bar{G}(x;\xi)\right]^j}. \tag{3.2}$$

Using reciprocal power series expansion (see [26], p. 239) on Eq (3.2), we are given the following result:

$$\frac{1}{F(z)} \;=\; \sum_{k=0}^{\infty}L_k z^k,$$

where $L_0 = 1/b_0$ when $k = 0$, and

$$L_k=-\frac{1}{b_0}\sum_{m=1}^{k}b_m L_{k-m}, \qquad k\geq 1.$$

After incorporating results in Eq (3.2), the expression for linear representation will become

$$F(x) = \sum_{j=1}^{\infty} \sum_{k=0}^{j} \varpi_{j,k} G(x;\xi)^{j+k}, \tag{3.3}$$

where

$$\varpi_{j,k} = \sum_{i=0}^{\infty} \frac{(-1)^{i+1}}{i!\,j!} \exp(\frac{1}{a}) a^{j-i} (i+1)^j c_k,$$

$$c_k = \begin{cases} \frac{1}{b_0}, & \text{k=0}, \\ -\frac{1}{b_0} \sum_{m=1}^{k} b_m c_{k-m}, & \text{k}\geq 1, \end{cases}$$

and $b_k = (-1)^k \binom{j}{k}$.

The expression for the density function after taking the derivative of Eq (3.3) will become

$$f(x) = \sum_{j=1}^{\infty} \sum_{k=0}^{j} \varpi_{j,k}\, h_{j,\,k}, \tag{3.4}$$

where $h_{j,\,k} = (j+k) g(x)\, G(x;\xi)^{j+k-1}$ is a linear combination of the exp-G family, and one can obtain the various properties by taking into account Eq (3.4).

3.3. Moments

By using Eq (3.4), the rth ordinary or raw moment of the OMG family is given by

$$\mathbb{E}(X^r) = \sum_{j=1}^{\infty} \sum_{k=0}^{j} \varpi_{j,\,k}\, \mathbb{E}(Y_{j,\,k}^r). \tag{3.5}$$

By using Eq (3.5), one can get the actual moments and cumulants for X as

$$\mu_r = \sum_{s=0}^{n} (-1)^s \binom{r}{s} \mu_1'^s \mu_{r-s}' \qquad \text{and} \qquad \kappa_r = \mu_n' \sum_{s=1}^{r-1} \binom{r-1}{s-1} \kappa_s \mu_{r-s}'$$

respectively, where $\kappa_1 = \mu_1'$. By using the relationship between actual moments and ordinary moments, one can get the measures of skewness and kurtosis. The rth incomplete moment of OMG can be expressed as

$$I_r(x) = \int_0^t x^r f(x)\, dx, = \sum_{j=1}^{\infty} \sum_{k=0}^{j} \varpi_{j,\,k}\, \mathbb{E}(I_{j,\,k}^r(x)), \tag{3.6}$$

where $I_{j,\,k}^r(t) = \int_0^t x^r h(ij,\,k)\, d_x$, and the incomplete moments are vital in order to compute the well-known namely Bonferroni and Lorenz curves.

3.4. The expression of probability weighted moment

The expression of (r, q)th probability weighted moment (PWM) can be founded as

$$\rho_{r,q} = \int_0^\infty x^r F(x)^q f(x) dx. \tag{3.7}$$

Inserting Eqs (2.1) and (2.2) in Eq (3.7), and using the generalized binomial series expansion, $\rho_{r,q}$ can be expressed as

$$\begin{aligned}\rho_{r,q} &= \sum_{c=0}^{\infty} (-1)^c \binom{q}{c} \exp\left(\frac{(c+1)}{a}\right) \int_0^\infty x^r \frac{g(x;\xi)}{\bar{G}(x;\xi)^2} \left\{\exp\left(\frac{a\,G(x;\xi)}{\bar{G}(x;\xi)}\right) - a\right\} \exp\left(\frac{a(c+1)\,G(x;\xi)}{\bar{G}(x;\xi)}\right) \\ &\times \exp\left(-\frac{c+1}{a} \exp\left(\frac{a\,G(x;\xi)}{\bar{G}(x;\xi)}\right)\right) dx.\end{aligned}$$

Applying the power series expansion defined in Section 4 on Eq (3.8), it will become

$$\begin{aligned}\rho_{r,q} &= \sum_{c,i=0}^{\infty} \frac{(-1)^{c+i}}{i!} \binom{q}{c} \left(\frac{c+1}{a}\right)^i \exp\left(\frac{(c+1)}{a}\right) \int_0^\infty x^r \frac{g(x;\xi)}{\bar{G}(x;\xi)^2} \\ &\quad \underbrace{\left(\exp\left(a(c+i+2)\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]\right) - a\,\exp\left(a(c+i+1)\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]\right)\right)}_{A} dx.\end{aligned} \tag{3.8}$$

Applying a power series expansion on quantity A, it will reduce to

$$\begin{aligned}&\exp\left(a(c+i+2)\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]\right) - a\,\exp\left(a(c+i+1)\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]\right) \\ &\quad = \sum_{j=0}^{\infty} \frac{a^j}{j!} \left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^j \left\{(c+i+2)^j - a\,(c+i+1)^j\right\}.\end{aligned}$$

The expression for $\rho_{r,q}$ after incorporating the result of quantity A, can be expressed as

$$\begin{aligned}\rho_{r,q} &= \sum_{c,i,j=0}^{\infty} \frac{(-1)^{c+i} a^j}{i!} \binom{q}{c} \left(\frac{c+1}{a}\right)^i \mathrm{e}^{\frac{(c+1)}{a}} \left[(\mathrm{c}+\mathrm{i}+2)^{\mathrm{j}} - \mathrm{a}\,(\mathrm{c}+\mathrm{i}+1)^{\mathrm{j}}\right] \int_0^\infty \mathrm{x}^{\mathrm{r}}\, \mathrm{g}(\mathrm{x};\xi) \\ &\quad G(x;\xi)^j \bar{G}(x;\xi)^{-(j+2)} dx.\end{aligned} \tag{3.9}$$

Using generalized binomial series expansion on the above equation,

$$(1-z)^{-q} = \sum_{p=0}^{\infty} \frac{\Gamma[q+p]}{\Gamma[q]p!} z^p, \qquad q > 0.$$

After incorporating the result of the above equation, the expression for $\rho_{r,q}$ can be expressed as

$$\rho_{r,q} = \sum_{j,p=0}^{\infty} V_{j,p}\ (j+p+1) \int_0^\infty x^r g(x;\xi) G(x;\xi)^{j+p}\, dx. \tag{3.10}$$

Integrating (3.10), we can obtain the expression of PWMs, where

$$V_{j,p} = \sum_{c,i=0}^{\infty} \frac{(-1)^{c+i} a^{j-i} (c+1)^i (p+1)\,\Gamma[j+2+p]}{j!\,i!\,p!\,\Gamma[j+2](j+p+1)} \binom{c}{q} \exp\left(\frac{c+1}{a}\right) \left[(c+i+2)^j - a\,(c+i+1)^j\right].$$

3.5. Entropy

The entropy measure is important to underline the uncertainty variation of a *rv*; let X be a *rv* having pdf $f(x)$. The Rényi entropy can be found by the following expression:

$$I(\delta) = \frac{1}{1-\delta}\log\left[I(\delta)\right], \tag{3.11}$$

where $\delta > 0$, $\delta \neq 1$, and $I(\delta) = \int_{\Re} f^{\delta}(x)\,dx$.

Inserting Eq (2.2) in $f^{\delta}(x)$, gives

$$f^{\delta}(x) = \left[\frac{g(x;\xi)}{\bar{G}(x;\xi)^2}\left\{\exp\left(\frac{a\,G(x;\xi)}{\bar{G}(x;\xi)}\right)-a\right\}\exp\left\{\frac{a\,G(x;\xi)}{\bar{G}(x;\xi)}-\frac{1}{a}\left[\exp\left(\frac{a\,G(x;\xi)}{\bar{G}(x;\xi)}\right)-1\right]\right\}\right]^{\delta}.$$

Applying a power series yieldes

$$f^{\delta}(x) = \sum_{k,p=0}^{\infty} V_{k,p}\, g(x;\xi)^{\delta} G(x;\xi)^{k+p}\,dx. \tag{3.12}$$

After incorporating the result in Eq. (3.11), the expression for Rényi entropy will reduce to

$$I_{\delta}(f) = \frac{1}{1-\delta}\log\left[\sum_{k,p=0}^{\infty} V_{k,p}\int_{0}^{\infty} g(x;\xi)^{\delta} G(x;\xi)^{k+p}\,dx.\right], \tag{3.13}$$

where $V_{k,p} = \sum_{i,j=0}^{\infty}\frac{(-1)^{i+j+\delta}\Gamma(2\delta+k+p)}{i!\,j!\,p!\Gamma(2\delta+k)}\binom{\delta}{j}a^{\delta+k-j-1}(i+j+1)^{k}$.

4. Estimation

Here, we demonstrate the estimation of parameters by taking into account the maximum likelihood approach. The log-likelihood (LL) function $\ell(\mathbf{\Omega})$ for the vector of parameters $\Omega = (a,\,\xi)^{\top}$ can be expressed as

$$\begin{aligned} L(\Omega) &= \sum_{i=1}^{n}\log\left[g(x_i,\xi)\right] + \sum_{i=1}^{n}\left\{\frac{1-\exp\left(\frac{a\,G(x;\xi)}{\bar{G}(x;\xi)}\right)}{a} + \frac{aG(x_i,\xi)}{\bar{G}(x_i,\xi)}\right\} + \sum_{i=1}^{n}\log\left\{\exp\left(\frac{a\,G(x;\xi)}{\bar{G}(x;\xi)}\right)-a\right\} \\ &\quad -2\sum_{i=1}^{n}\log\left[\bar{G}(x_i,\xi)\right]. \end{aligned} \tag{4.1}$$

The first partial derivatives of Eq (4.1) with respect to a and ξ are

$$\begin{aligned} \frac{\partial L}{\partial a} &= \sum_{i=1}^{n}\left\{-\frac{1-\exp\left(\frac{a\,G(x;\xi)}{\bar{G}(x;\xi)}\right)}{a^2} - \frac{G(x_i,\xi)\exp\left(\frac{a\,G(x;\xi)}{\bar{G}(x;\xi)}\right)}{a\bar{G}(x_i,\xi)} + \frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}\right\} + \sum_{i=1}^{n}\left\{\frac{G(x_i,\xi)\exp\left(\frac{a\,G(x;\xi)}{\bar{G}(x;\xi)}\right)}{\bar{G}(x_i,\xi)} - 1\right\} \\ &\quad \times\left\{\exp\left(\frac{a\,G(x;\xi)}{\bar{G}(x;\xi)}\right)-a\right\}^{-1}, \end{aligned}$$

$$
\begin{aligned}
\frac{\partial L}{\partial \xi} &= \sum_{i=1}^{n} \frac{g_i^{\xi}}{g\left(x_i, \xi\right)} - 2\sum_{i=1}^{n} \frac{G_i^{\xi}}{\bar{G}\left(x_i, \xi\right)} + \sum_{i=1}^{n} \left[\exp\left(\frac{a\, G(x;\xi)}{\bar{G}(x;\xi)}\right) - a\right]^{-1} \left\{\frac{a \exp\left(\frac{a\, G(x;\xi)}{\bar{G}(x;\xi)}\right) G_i^{\xi}}{\bar{G}\left(x_i, \xi\right)} + a \right. \\
&\quad \left. \times \exp\left(\frac{a\, G(x;\xi)}{\bar{G}(x;\xi)}\right) \frac{G\left(x_i, \xi\right) G_i^{\xi}}{\left(\bar{G}\left(x_i, \xi\right)\right)^2}\right\} + \sum_{i=1}^{n} \left\{\frac{aG_i^{\xi}}{\bar{G}\left(x_i, \xi\right)} - \frac{1}{a}\left[\frac{aG_i^{\xi} \mathrm{e}^{\frac{aG(x_i,\xi)}{\bar{G}(x_i,\xi)}}}{\bar{G}\left(x_i, \xi\right)} + \frac{aG\left(x_i, \xi\right)\ \exp\left(\frac{a\, G(x;\xi)}{\bar{G}(x;\xi)}\right)\ G_i^{\xi}}{\left(\bar{G}\left(x_i, \xi\right)\right)^2}\right] \right. \\
&\quad \left. + \frac{aG\left(x_i, \xi\right) G_i^{\xi}}{\left(\bar{G}\left(x_i, \xi\right)\right)^2}\right\},
\end{aligned}
$$

where $g_i^{\xi} = \frac{\partial}{\partial \xi} g(x_i;\xi)$ and $G_i^{\xi} = \frac{\partial}{\partial \xi} G(x_i;\xi)$ are derivatives of column vectors of the same dimension of ξ.

5. The OMW distribution

Here we consider Weibull as a baseline model with cdf and pdf, respectively, given as $G(x;\xi) = 1 - \mathrm{e}^{-\alpha x^{\beta}}$ and $g(x;\xi) = \beta\,\alpha\, x^{\beta-1}\mathrm{e}^{-\alpha x^{\beta}}$, where $\alpha > 0$ is a scale, and $\beta > 0$ is a shape parameter. Then, the cdf, pdf, rf, hrf and chrf of the proposed OMW model, respectively, are given by

$$
F(x) = 1 - \exp\left\{a\left(\frac{1-\mathrm{e}^{-\alpha x^{\beta}}}{\mathrm{e}^{-\alpha x^{\beta}}}\right) - \frac{1}{a}\left[\exp\left(a\left(\frac{1-\mathrm{e}^{-\alpha x^{\beta}}}{\mathrm{e}^{-\alpha x^{\beta}}}\right)\right) - 1\right]\right\}, \tag{5.1}
$$

$$
f(x) = \beta\alpha x^{\beta-1}\mathrm{e}^{-\alpha x^{\beta}}\left[\exp\left(a\left(\frac{1-\mathrm{e}^{-\alpha x^{\beta}}}{\mathrm{e}^{-\alpha x^{\beta}}}\right)\right) - a\right]\exp\left\{a\left(\frac{1-\mathrm{e}^{-\alpha x^{\beta}}}{\mathrm{e}^{-\alpha x^{\beta}}}\right) - \frac{1}{a}\left[\exp\left(a\left(\frac{1-\mathrm{e}^{-\alpha x^{\beta}}}{\mathrm{e}^{-\alpha x^{\beta}}}\right)\right) - 1\right]\right\}, \tag{5.2}
$$

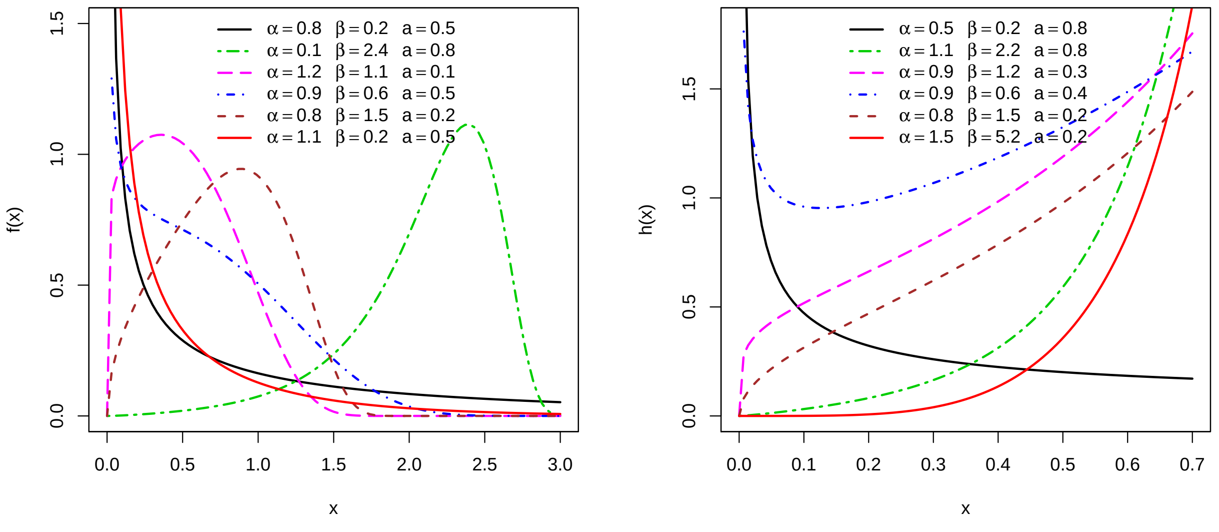
$$
r(x) = \exp\left\{a\left(\frac{1-\mathrm{e}^{-\alpha x^{\beta}}}{\mathrm{e}^{-\alpha x^{\beta}}}\right) - \frac{1}{a}\left[\exp\left(a\left(\frac{1-\mathrm{e}^{-\alpha x^{\beta}}}{\mathrm{e}^{-\alpha x^{\beta}}}\right)\right) - 1\right]\right\},
$$

$$
h(x) = \beta\alpha x^{\beta-1}\mathrm{e}^{-\alpha x^{\beta}}\left[\exp\left(a\left(\frac{1-\mathrm{e}^{-\alpha x^{\beta}}}{\mathrm{e}^{-\alpha x^{\beta}}}\right)\right) - a\right],
$$

and

$$
H(x)) = a\left(\frac{1-\mathrm{e}^{-\alpha x^{\beta}}}{\mathrm{e}^{-\alpha x^{\beta}}}\right) - \frac{1}{a}\left[\exp\left(a\left(\frac{1-\mathrm{e}^{-\alpha x^{\beta}}}{\mathrm{e}^{-\alpha x^{\beta}}}\right)\right) - 1\right].
$$

The graphical illustrations of pdf and hrf based on some selected parametric values of OMW are depicted in Figure 1 and reveal that the OMW model has flexibility in both pdf and hrf.



(a) (b)

Figure 1. Graphical illustrations of pdf (a) and hrf (b) of OMW model for some parametric values.

6. Properties of OMW model

First, we will derive a linear expression of the OMW density to get the useful mathematical properties of this new model.

Following Eq (3.4), the OMG density will become

$$f(x) = \sum_{j=1}^{\infty}\sum_{k=0}^{j} \varpi_{j,k}\alpha\beta(j+k)\, x^{\beta-1}\,\mathrm{e}^{-\alpha x^{\beta}}\left[1-\mathrm{e}^{-\alpha x^{\beta}}\right]^{j+k-1}, \tag{6.1}$$

$$f(x) = \sum_{p=0}^{\infty} v_p\, \pi(x;\alpha(p+1),\beta), \tag{6.2}$$

where $v_p = (-1)^p\binom{j+k-1}{p}\sum_{j=1}^{\infty}\sum_{k=0}^{j}\varpi_{j,k}(j+k)$, and $\pi(x;\alpha(p+1),\beta)$ is the Weibull density.

Several properties of the OMW model can be yielded by using Eq (6.2) because it is a linear combination of Weibull densities.

The qf of the OMW distribution is given as

$$Q_X(u) = \left[-\frac{1}{\alpha}\log\left\{1-\left[1+\left\{\frac{1}{a}\log(1-u)-\frac{1}{a}W_{-1}\left(\frac{u-1}{a\mathrm{e}^{\frac{1}{a}}}\right)-\frac{1}{a^2}\right\}^{-1}\right]^{-1}\right\}\right]^{\frac{1}{\beta}}. \tag{6.3}$$

The expression of rth moments is given by

$$\mu'_r = \Gamma\left(\frac{r}{\beta}+1\right)\sum_{p=0}^{\infty}\frac{v_p}{\alpha^{r/\beta}(p+1)^{r/\beta+1}}. \tag{6.4}$$

The graphical illustrations of skewness and kurtosis are depicted in Figure 2 for the OMW distribution.

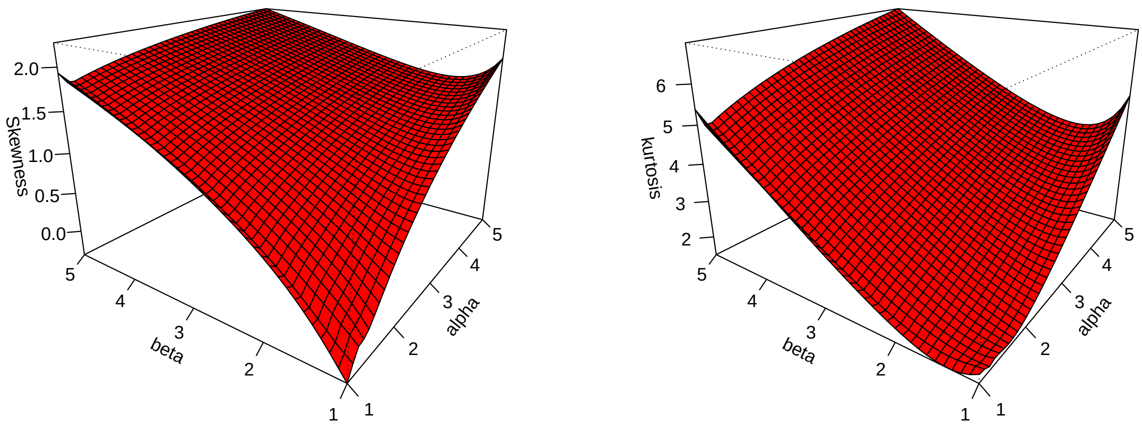


Figure 2. Graphical illustrations of Skewness and Kurtosis of OMW model at varying parametric values.

The expression for the rth incomplete moment can be written as

$$m_r(z) = \sum_{p=0}^{\infty}\frac{v_p}{\alpha^{r/\beta}(p+1)^{r/\beta}}\,\gamma\left(\tfrac{r}{\beta}+1,(p+1)\alpha\, z^{\beta}\right), \tag{6.5}$$

where $\gamma(s,x)=\int_0^{\infty} x^{s-1}\,\exp(-x)dx$.

The expression for the PWMs can be written as

$$\rho_{r,q} = \Gamma\left(\frac{r}{\beta}+1\right)\sum_{s=0}^{\infty}\frac{t_s}{\alpha^{r/\beta}(s+1)^{r/\beta+1}}, \tag{6.6}$$

where $t_s=(-1)^s\binom{j+p}{s}\sum_{j,p=0}^{\infty}V_{j,p}(j+p+1)$.

The expression for Rényi entropy can be written as

$$I_\delta(f)=\frac{1}{1-\delta}\log\left[\sum_{n=0}^{\infty}\omega_n\alpha^{\delta}\beta^{\delta-1}(\alpha(\delta+v))^{\frac{\delta-1}{\beta}-\delta}\Gamma\left(\frac{(\beta-1)\delta+1}{\beta}\right)\right], \tag{6.7}$$

where $\omega_n=(-1)^n\binom{k+p}{n}\sum_{k,p=0}^{\infty}V_{k,p}$.

6.1. Estimation

Let there be a sample of size n from the OMW model given in Eq (5.2). The LL function $\ell = \ell(\theta)$ for the vector of parameters $\theta = (\alpha, \beta, a)^{\top}$ is

$$
\begin{aligned}
\ell &= n \log(\alpha\beta) + (\beta - 1) \sum_{i=1}^{n} \log(x_i) + \alpha \sum_{i=1}^{n} x_i^{\beta} + \sum_{i=1}^{n} \log\left\{ \mathrm{e}^{a\left(1-\mathrm{e}^{-\alpha x_i^{\beta}}\right)\mathrm{e}^{\alpha x_i^{\beta}}} - a \right\} && (6.8)\\
&+ \sum_{i=1}^{n} \left\{ \frac{1 - \mathrm{e}^{a\left(1-\mathrm{e}^{-\alpha x_i^{\beta}}\right)\mathrm{e}^{\alpha x_i^{\beta}}}}{a} + \frac{a\left(1 - \mathrm{e}^{-\alpha x_i^{\beta}}\right)}{\mathrm{e}^{-\alpha x_i^{\beta}}} \right\}. && (6.9)
\end{aligned}
$$

Equation (6.8) can be easily maximized using the computational software R or Mathematica. The components of the score vector $U(\theta)$ are

$$
\begin{aligned}
U_{\alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{n} x_i^{\beta} + \sum_{i=1}^{n} \left\{ \frac{a x_i^{\beta} \mathrm{e}^{\alpha x_i^{\beta}} \mathrm{e}^{a\left(1-\mathrm{e}^{-\alpha x_i^{\beta}}\right)\mathrm{e}^{\alpha x_i^{\beta}}}}{\mathrm{e}^{a\left(1-\mathrm{e}^{-\alpha x_i^{\beta}}\right)\mathrm{e}^{\alpha x_i^{\beta}}} - a} \right\} - \sum_{i=1}^{n} x_i^{\beta} \mathrm{e}^{\alpha x_i^{\beta}} \left\{ \mathrm{e}^{a\left(1-\mathrm{e}^{-\alpha x_i^{\beta}}\right)\mathrm{e}^{\alpha x_i^{\beta}}} - a \right\},\\
U_{\beta} &= \frac{n}{\beta} + \sum_{i=1}^{n} \log(x_i) + \alpha \sum_{i=1}^{n} x_i^{\beta} \log(x_i) + \sum_{i=1}^{n} \left\{ \frac{a\alpha x_i^{\beta} \log(x_i)\, \mathrm{e}^{\alpha x_i^{\beta}} \mathrm{e}^{a\left(1-\mathrm{e}^{-\alpha x_i^{\beta}}\right)\mathrm{e}^{\alpha x_i^{\beta}}}}{\mathrm{e}^{a\left(1-\mathrm{e}^{-\alpha x_i^{\beta}}\right)\mathrm{e}^{\alpha x_i^{\beta}}} - a} \right\}\\
&- \sum_{i=1}^{n} \alpha x_i^{\beta} \mathrm{e}^{\alpha x_i^{\beta}} \log(x_i) \left\{ \mathrm{e}^{a\left(1-\mathrm{e}^{-\alpha x_i^{\beta}}\right)\mathrm{e}^{\alpha x_i^{\beta}}} - a \right\},\\
U_{a} &= \sum_{i=1}^{n} \left\{ \frac{\mathrm{e}^{\alpha x_i^{\beta}} \mathrm{e}^{a\left(1-\mathrm{e}^{-\alpha x_i^{\beta}}\right)\mathrm{e}^{\alpha x_i^{\beta}}} - \mathrm{e}^{a\left(1-\mathrm{e}^{-\alpha x_i^{\beta}}\right)\mathrm{e}^{\alpha x_i^{\beta}}} - 1}{\mathrm{e}^{a\left(1-\mathrm{e}^{-\alpha x_i^{\beta}}\right)\mathrm{e}^{\alpha x_i^{\beta}}} - a} \right\} + \sum_{i=1}^{n} \frac{1}{a^2} \left\{ \mathrm{e}^{a\left(1-\mathrm{e}^{-\alpha x_i^{\beta}}\right)\mathrm{e}^{\alpha x_i^{\beta}}} \right.\\
&\left. - a\left[\mathrm{e}^{a\left(1-\mathrm{e}^{-\alpha x_i^{\beta}}\right)\mathrm{e}^{\alpha x_i^{\beta}}} - a\mathrm{e}^{\alpha x_i^{\beta}} \right] \left(1 - \mathrm{e}^{-\alpha x_i^{\beta}}\right) - 1 \right\}.
\end{aligned}
$$

One can yield MLEs by setting these equations equal to zero and solving simultaneously.

7. Simulation study

This section is mainly based on simulation analysis, in order to understand the behavior of MLEs of the OMW distribution at varying sample sizes. We perform simulation analysis by considering N=1000 and n=50, 100, 200, 400, 500. Three sets of different parameter values are used to perform the simulation study: (1): $\alpha = 0.6, \beta = 0.09$ and $a = 0.7$; (2): $\alpha = 0.4, \beta = 0.2$ and $a = 0.5$; (3): $\alpha = 0.04, \beta = 0.02$ and $a = 0.05$. The simulation analysis biases, mean square errors (MSEs), coverage probability (CP) and average width (AW) show in Tables 1–3 that as sample size increases both biases and MSEs are reduced.

Table 1. Biases, MSEs, CPs and AW for set-1.

	n = 25			n = 50			n = 100		
	α	β	a	α	β	a	α	β	a
Bias	-0.045	0.028	-0.193	-0.031	0.017	-0.118	-0.020	0.010	-0.075
MSE	0.011	0.002	0.092	0.006	0.001	0.076	0.004	0.001	0.069
CP	0.940	0.990	1.000	0.94	0.950	0.940	0.910	0.870	0.830
AW	0.391	0.166	1.426	0.278	0.117	1.080	0.204	0.087	0.857

	n = 200			n = 400			n = 500		
	α	β	a	α	β	a	α	β	a
Bias	-0.015	0.007	-0.057	-0.012	0.006	-0.045	-0.009	0.004	-0.035
MSE	0.002	0.001	0.053	0.002	0.000	0.035	0.001	0.000	0.028
CP	0.870	0.830	0.800	0.870	0.840	0.810	0.890	0.850	0.820
AW	0.156	0.068	0.696	0.119	0.053	0.559	0.110	0.050	0.529

Table 2. Biases, MSEs, CPs and AW for set-2.

	n = 25			n = 50			n = 100		
	α	β	a	α	β	a	α	β	a
Bias	-0.031	0.033	-0.070	-0.022	0.022	-0.051	-0.016	0.015	-0.043
MSE	0.0110	0.006	0.064	0.007	0.003	0.059	0.005	0.003	0.057
CP	0.930	0.980	0.990	0.930	0.960	0.960	0.890	0.900	0.870
AW	0.442	0.319	1.436	0.349	0.243	1.163	0.269	0.184	0.914

	n = 200			n = 400			n = 500		
	α	β	a	α	β	a	α	β	a
Bias	-0.011	0.010	-0.033	-0.006	0.006	-0.013	-0.003	0.003	-0.005
MSE	0.004	0.002	0.046	0.003	0.001	0.033	0.002	0.001	0.030
CP	0.860	0.860	0.830	0.840	0.840	0.810	0.850	0.850	0.840
AW	0.214	0.145	0.745	0.169	0.114	0.599	0.157	0.105	0.559

Table 3. Biases, MSEs, CPs and AW for set-3.

	n = 25			n = 50			n = 100		
	α	β	a	α	β	a	α	β	a
Bias	0.027	-0.002	0.211	0.022	-0.002	0.163	0.013	-0.001	0.102
MSE	0.003	0.000	0.090	0.002	0.000	0.066	0.001	0.000	0.032
CP	0.980	0.960	0.970	0.980	0.960	0.960	0.990	0.980	0.980
AW	0.356	0.038	2.191	0.276	0.030	1.777	0.192	0.023	1.372

	n = 200			n = 400			n = 500		
	α	β	a	α	β	a	α	β	a
Bias	0.006	0.000	0.056	0.004	0.000	0.041	0.004	0.000	0.040
MSE	0.000	0.000	0.011	0.000	0.000	0.005	0.000	0.000	0.005
CP	0.990	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
AW	0.106	0.014	0.847	0.072	0.010	0.589	0.067	0.009	0.551

8. Group acceptance sampling plan (GASP)

This section is based on the illustration of GASP under the assumption that the lifetime distribution of an item follows an OMW model with known parameters α and β having cdf in Eq (8.2). In a GASP, say, n is the randomly selected sample size and distributed to g groups, and r for a preassigned time r items in a group are tested. If more than c failures occur in any group during the experiment time, the performed experiment is truncated. The reader is referred to Aslam et al. [29] and Khan and Alqarni [30] for a simple illustration of GASP and an application to real data. Designing the GASP reduced both the time and cost. Several lifetime traditional and extended models are used [10, 29, 33–35] in designing the GASP by taking into account the quality parameter as mean or median; usually, for skewed distributions median is preferable [29].

The GASP is simply the extension of the ordinary sampling plans i.e., the GASP reduces to the ordinary sampling plan by replacing $r = 1$, and thus $n = g$ [32].

GASP is based on the following process. First, select g (the number of groups) and allocate predefined r (group size) items to each group so that the sample of size of the lot will be $n = r \times g$. Second, select c and t_0 (the experiment time), respectively. Third, do experiment simultaneously for g groups and record the number of failures for each group. Finally, a conclusion is drawn either accepting or rejecting the lot. The lot is accepted if there no more than c failures occur in each and every group and otherwise the lot is rejected. The probability of accepting the lot is represented by the following expression:

$$p_{a(p)} = \left[\sum_{i=0}^{c} \binom{r}{i} p^i \left[1-p\right]^{r-i}\right]^g, \tag{8.1}$$

where the probability that an item in a group fails before t_0 is denoted by p and yielded by inserting (6.3) in (8.2). Let the lifetime of an item or product follow an OMW with known parameters α and β, with cdf given by for $t > 0$ for convenient we used $G(t) = 1 - \exp\left[-(t/\alpha)^\beta\right]$

$$F(t) = 1 - \exp\left\{\frac{a\left(1 - \mathrm{e}^{\left[-(\mathrm{t}/\alpha)^\beta\right]}\right)}{\mathrm{e}^{\left[-(\mathrm{t}/\alpha)^\beta\right]}} - \frac{1}{a}\left[\exp\left(\frac{a\left(1 - \mathrm{e}^{\left[-(\mathrm{t}/\alpha)^\beta\right]}\right)}{\mathrm{e}^{\left[-(\mathrm{t}/\alpha)^\beta\right]}}\right) - 1\right]\right\}, \tag{8.2}$$

the qf of OMW model using (3.1) is given by and if p=0.5 yielded the median of the lifetime distribution of the product is not smaller than the specified median value.

$$m = \alpha\left[-\log\left(1 - \left[\left[1 + \left\{\frac{1}{a}\log(1-u) - \frac{1}{a}W_{-1}\left(\frac{u-1}{a\exp(\frac{1}{a})}\right) - \frac{1}{a^2}\right\}^{-1}\right]^{-1}\right]\right)\right]^{1/\beta}, \tag{8.3}$$

and by taking η as

$$\eta = \left[-\log\left(1 - \left[\left[1 + \left\{\frac{1}{a}\log(1-u) - \frac{1}{a}W_{-1}\left(\frac{u-1}{a\exp(\frac{1}{a})}\right) - \frac{1}{a^2}\right\}^{-1}\right]^{-1}\right]\right)\right]^{1/\beta}, \tag{8.4}$$

Eq (8.3) is obtained by writing $\alpha = m/\eta$ and $t = a_1 m_0$. The ratio of a of product mean lifetime, the specified life time m/m_0 can be used to express the quality level of the product. Replacing the $\alpha = m/\eta$ and $t = m_0 a_1$ in Eq (8.2) yields the probability of failure, given by

$$p = 1 - \exp\left\{\frac{a\left(1 - e^{\left[-\left(\frac{a_1\eta}{m/m_0}\right)^{\beta}\right]}\right)}{e^{\left[-\left(\frac{a_1\eta}{m/m_0}\right)^{\beta}\right]}} - \frac{1}{a}\left[\exp\frac{a\left(1 - e^{\left[-\left(\frac{a_1\eta}{m/m_0}\right)^{\beta}\right]}\right)}{e^{\left[-\left(\frac{a_1\eta}{m/m_0}\right)^{\beta}\right]}} - 1\right]\right\}. \tag{8.5}$$

From Eq (8.2), for taking the values of a and β, p can be determined when a_1 and r_2 are specified, where $r_2 = m/m_0$. Here, we define the two failure probabilities, say, p_1 and p_2 corresponding to the consumer risk and producer risk, respectively. For given specific values of the parameters a , β, r_2, a_1, β^* and γ, we need to determine the values of c and g that simultaneously satisfy the following two equations:

$$p_{a\left(p_1|\frac{m}{m_0}=r_1\right)} = \left[\sum_{i=0}^{c}\binom{r}{i}p_1^i\left[1 - p_1\right]^{r-i}\right]^g \le \beta^*, \tag{8.6}$$

and

$$p_{a\left(p_2|\frac{m}{m_0}=r_2\right)} = \left[\sum_{i=0}^{c}\binom{r}{i}p_2^i\left[1 - p_2\right]^{r-i}\right]^g \ge 1 - \gamma, \tag{8.7}$$

where the mean ratio at consumer's risk and at producer's risk, respectively denoted by r_1 and r_2 and the probability of failure to be used in the above expression as follows

$$p_1 = 1 - \exp\left\{\frac{a\left(1 - e^{\left[-(a_1\eta)^{\beta}\right]}\right)}{e^{\left[-(a_1\eta)^{\beta}\right]}} - \frac{1}{a}\left[\exp\left(\frac{a\left(1 - e^{\left[-(a_1\eta)^{\beta}\right]}\right)}{e^{\left[-(a_1\eta)^{\beta}\right]}}\right) - 1\right]\right\}, \tag{8.8}$$

and

$$p_2 = 1 - \exp\left\{\frac{a\left(1 - e^{\left[-\left(\frac{a_1\eta}{r_2}\right)^{\beta}\right]}\right)}{e^{\left[-\left(\frac{a_1\eta}{r_2}\right)^{\beta}\right]}} - \frac{1}{a}\left[\exp\left(\frac{a\left(1 - e^{\left[-\left(\frac{a_1\eta}{r_2}\right)^{\beta}\right]}\right)}{e^{\left[-\left(\frac{a_1\eta}{r_2}\right)^{\beta}\right]}}\right) - 1\right]\right\}. \tag{8.9}$$

Tables 4 and 5 are based on arbitrary values of parameters to underline the effect of design parameters. When $r = 5$, from Table (4), with $\beta^* = 0.1$, a_1=0.5, r_2=4, there should be 185 items needed for testing (37*5=185). On the other hand, under the same condition when r=10, there should be 60 units tested. So, here, we should prefer when r=10, which significantly reduces the number of units that need to be tested.

Table 4. GASP under OMW, when $a = 0.5, \beta = 1$, showing minimum g and c.

		$r = 5$						$r = 10$					
		$a_1 = 0.5$			$a_1 = 1$			$a_1 = 0.5$			$a_1 = 1$		
β	r_2	g	c	p(a)	g	c	p(a)	g	c	p(a)	g	c	p(a)
0.25	2	–	–	–	44	4	0.9843	38	4	0.9604	3	5	0.9787
	4	22	2	0.984	3	2	0.9805	4	2	0.9703	1	3	0.9908
	6	5	1	0.9635	1	1	0.9698	4	2	0.9909	1	2	0.9826
	8	5	1	0.9794	1	1	0.9832	2	1	0.9655	1	2	0.9925
0.1	2	–	–	–	73	4	0.9742	321	5	0.9727	3	5	0.9647
	4	37	2	0.9732	4	2	0.9741	6	2	0.9558	2	3	0.9817
	6	37	2	0.9922	4	2	0.9928	6	2	0.9864	1	2	0.9826
	8	8	1	0.9673	2	1	0.9667	6	2	0.9942	1	2	0.9925
0.05	2	–	–	–	95	4	0.9665	417	5	0.9647	7	5	0.9509
	4	48	2	0.9653	5	2	0.9677	22	3	0.9875	2	3	0.9817
	6	48	2	0.9900	5	2	0.9910	8	2	0.9820	2	2	0.9654
	8	10	1	0.9593	2	1	0.9667	8	2	0.9923	2	2	0.9851
0.01	2	–	–	–	–	–	–	–	–	–	–	–	–
	4	627	3	0.9899	7	2	0.9551	34	3	0.9807	3	3	0.9727
	6	73	2	0.9848	7	2	0.9874	12	2	0.9731	2	2	0.9654
	8	73	2	0.9937	3	1	0.9504	12	2	0.9884	2	2	0.9851

Remark: A large sample size is required cells contains hyphens (–).

Table 5. GASP under OMW, when $a = 0.5, \beta = 1$, showing minimum g and c.

		$r = 5$						$r = 10$					
		$a_1 = 0.5$			$a_1 = 1$			$a_1 = 0.5$			$a_1 = 1$		
β	r_2	g	c	p(a)	g	c	p(a)	g	c	p(a)	g	c	p(a)
0.25	2	101	2	0.9611	3	2	0.9596	61	3	0.9846	1	3	0.9774
	4	12	1	0.9819	1	1	0.9888	4	1	0.9739	1	1	0.9552
	6	12	1	0.9935	1	1	0.9966	4	1	0.9906	1	1	0.9858
	8	12	1	0.9968	1	1	0.9985	4	1	0.9953	1	1	0.9934
0.1	2	–	–	–	12	3	0.9895	101	3	0.9746	2	3	0.9553
	4	20	1	0.9700	2	1	0.9778	6	1	0.9612	1	1	0.9555
	6	20	1	0.9893	2	1	0.9933	6	1	0.9859	1	1	0.9858
	8	20	1	0.9947	2	1	0.9970	6	1	0.9929	1	1	0.9934
0.05	2	–	–	–	15	3	0.9869	131	3	0.9672	2	3	0.9553
	4	26	1	0.9611	2	1	0.9778	28	2	0.9939	1	1	0.9552
	6	26	1	0.9861	2	1	0.9933	8	1	0.9812	1	1	0.9858
	8	26	1	0.9931	2	1	0.9970	8	1	0.9906	1	1	0.9934
0.01	2	–	–	–	23	3	0.9800	–	–	–	5	4	0.9829
	4	335	2	0.9936	3	1	0.9669	42	2	0.9908	2	2	0.9917
	6	40	1	0.9786	3	1	0.9899	12	1	0.9719	2	1	0.9717
	8	40	1	0.9894	3	1	0.9954	12	1	0.9859	2	1	0.9869

Remark: A large sample size is required cells contains hyphens (–).

9. Empirical investigation

The practical implementation of the proposed model is carried out in this section by considering two real-life data sets. The first and second data sets are taken from [27, 28], respectively.

Data 1: Aircraft Windshield Data:
The first real-life data set deals with the failure times of Aircraft Windshields. The data set is as follows: 0.0400, 1.8660, 2.3850, 3.4430, 0.3010, 1.8760, 2.4810, 3.4670, 0.3090, 1.8990, 2.6100, 3.4780, 0.5570, 1.9110, 2.6250, 3.5780, 0.9430, 1.9120, 2.6320, 3.5950, 1.0700, 1.9140, 2.6460, 3.6990, 1.1240, 1.9810, 2.6610, 3.7790, 1.2480, 2.0100, 2.6880, 3.9240, 1.2810, 2.0380, 2.820,3.0000, 4.0350, 1.2810, 2.0850, 2.8900, 4.1210, 1.3030, 2.0890, 2.9020, 4.1670, 1.4320, 2.0970, 2.9340, 4.2400, 1.4800, 2.1350, 2.9620, 4.2550, 1.5050, 2.1540, 2.9640, 4.2780, 1.5060, 2.1900, 3.0000, 4.3050, 1.5680, 2.1940, 3.1030, 4.3760, 1.6150, 2.2230, 3.1140, 4.4490, 1.6190, 2.2240, 3.1170, 4.4850, 1.6520, 2.2290, 3.1660, 4.5700, 1.6520, 2.3000, 3.3440, 4.6020, 1.7570, 2.3240, 3.3760, 4.6630.

Data 2: Fiber Strength Data:
The second real-life data set consists of 46 data points representing the strength of 15 cm glass fiber. The data set is as follows: 0.37, 0.40, 0.70, 0.75, 0.80 ,0.81 ,0.83, 0.86, 0.92, 0.92,0.94, 0.95, 0.98, 1.03, 1.06, 1.06, 1.08, 1.09, 1.10, 1.10,1.13, 1.14, 1.15, 1.17, 1.20, 1.20, 1.21, 1.22, 1.25, 1.28,1.28, 1.29, 1.29, 1.30, 1.35, 1.35, 1.37, 1.37, 1.38, 1.40,1.40, 1.42, 1.43, 1.51, 1.53, 1.61. The six well-known models exponentiated Weibull (EW), gamma Weibull (GaW) [36], Kumaraswamy Weibull (KwW), exponentiated generalized Weibull (EGW), beta Weibull (BW) and Weibull (W) are applied to these data sets.

The analysis of both data sets revealed that the proposed OMW model outperforms the comparative models, as per the least information criterion and higher P-values. The estimated parameters along with standard errors are depicted in Tables 6 and 8, whereas the accuracy measures are given in Tables 7 and 9. The graphical illustrations from Figures 3 and 4 are showing good agreement between the actual and fitted results.

The probability density functions of the comparative models are as follows:

$$
\begin{aligned}
f_{EW}(x) &= a\,\alpha\beta x^{\beta-1}\mathrm{e}^{-\alpha x^{\beta}}\left(1-\mathrm{e}^{-\alpha x^{\beta}}\right)^{a-1},\\
f_{GaW}(x) &= \frac{\alpha\beta}{\Gamma(a)}x^{\beta-1}\mathrm{e}^{a\,\alpha x^{\beta}}\left(1-\mathrm{e}^{-\alpha x^{\beta}}\right)^{a-1}\ \mathrm{e}^{-\left[\mathrm{e}^{\alpha x^{\beta}}-1\right]},\\
f_{BW}(x) &= \frac{\alpha\beta}{B(a,b)}x^{\beta-1}\mathrm{e}^{-b\alpha x^{\beta}}\left(1-\mathrm{e}^{-\alpha x^{\beta}}\right)^{a-1},\\
f_{KwW}(x) &= ab\alpha\beta x^{\beta-1}\mathrm{e}^{-\alpha x^{\beta}}\left(1-\mathrm{e}^{-(\alpha x)^{\beta}}\right)^{a-1}\left[1-\left(1-\mathrm{e}^{-(\alpha x)^{\beta}}\right)^{a}\right]^{b-1},\\
f_{EGW}(x) &= ab\alpha\beta x^{\beta-1}\mathrm{e}^{-a\,\alpha x^{\beta}}\left(1-\mathrm{e}^{-a\,\alpha x^{\beta}}\right)^{b-1},\\
f_{W}(x) &= 1-\mathrm{e}^{-\alpha x^{\beta}}.
\end{aligned}
$$

Table 6. Summary of the estimated parameters along with SEs of aircraft windshield data.

Distribution	α	β	a	b
OMW	0.3201	0.7953	0.9255	-
	(0.0477)	(0.1205)	(0.1147)	-
EW	0.0068	3.9182	0.4675	-
	(0.0053)	(0.5002)	(0.0947)	-
GaW	0.0813	2.3903	0.9938	-
	(0.1235)	(0.4308)	(0.8264)	-
BW	0.4328	2.8272	0.4044	0.0974
	(0.0059)	(0.0040)	(0.0015)	(0.0106)
KwW	0.0094	4.1470	0.4196	0.5545
	(0.0106)	(0.7399)	(0.1204)	(0.3833)
EGW	0.2123	3.2033	0.0987	0.6111
	(0.0880)	(0.0799)	(0.0416)	(0.0826)
W	0.0803	2.3932	-	-
	(0.0222)	(0.2099)	-	-

Table 7. Summary of the goodness of fit statistic for the aircraft windshield data.

Distribution	$\hat{\ell}$	AIC	CAIC	BIC	HQIC
OMW	127.9968	261.9935	262.2898	269.3215	264.9410
EW	129.1012	264.2024	264.4987	271.5303	267.1499
GaW	131.2885	268.5769	268.8732	275.9049	271.5244
BW	128.5337	265.0674	265.5674	274.8380	268.9974
KwW	128.9533	265.9066	266.4066	275.6773	269.8367
EGW	129.6350	267.2700	267.7700	277.0406	271.2000
W	131.2884	266.5769	266.7232	271.4622	266.5419

Table 8. Summary of the estimated parameters along with SEs of Fiber strength.

Distribution	α	β	a	b
OMW	0.5360	1.5825	0.9812	-
	(0.0651)	(0.4504)	(0.2323)	-
EW	0.0439	9.5690	0.3948	-
	(0.0778)	(3.8564)	(0.2175)	-
GaW	0.0247	10.2708	0.3663	-
	(0.0487)	(4.108)	(0.1977)	-
BW	9.0643	2.9154	2.2958	0.0723
	(0.0025)	(0.0025)	(1.0773)	(0.0110)
KwW	4.4880	4.3916	1.0727	0.0948
	(0.0025)	(0.0025)	(0.0991)	(0.0140)
EGW	12.3826	2.1211	0.1159	3.7782
	(0.0396)	(0.0084)	(0.0159)	(0.8978)
W	0.3450	5.1474	-	-
	(0.0784)	(0.6188)	-	-

Table 9. Summary of the goodness of fit statistic for the Fiber strength data.

Distribution	$\hat{\ell}$	AIC	CAIC	BIC	HQIC
OMW	1.9689	9.9378	10.5093	15.4237	11.9929
EW	2.0814	10.1627	10.7342	15.6487	12.2178
GaW	2.0673	10.1346	10.7061	15.6206	12.1897
BW	10.3151	28.6301	29.6057	35.9447	31.3702
KwW	3.7901	15.5803	16.5559	22.8949	18.3204
EGW	8.4238	24.8475	25.8231	32.1621	27.5876
W	3.3494	10.6988	10.9778	15.8561	12.0688

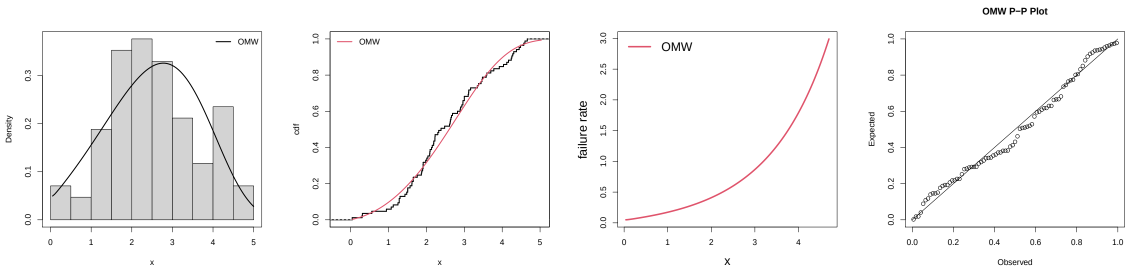


Figure 3. Plots of estimated density, estimated cdf, estimated hrf and P-P for the Aircraft Windshield Data.

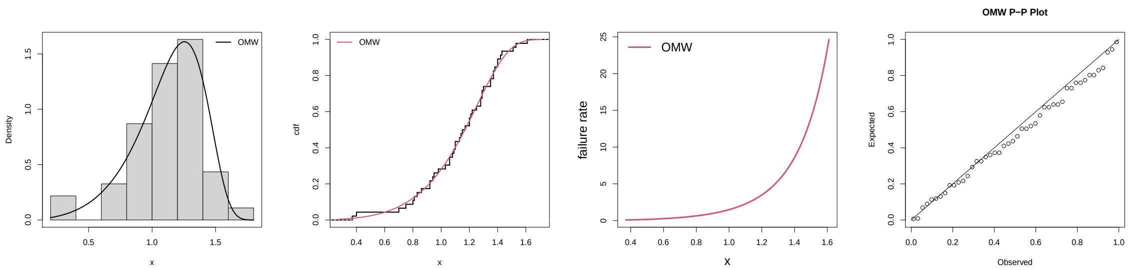


Figure 4. Plots of estimated density, estimated cdf, estimated hrf and P-P for the Fiber strength data.

When $r = 5$, from Table 10, with $\beta^* = 0.1$, a_1=0.5, r_2=4, there should be 860 items needed for testing (172*5=860). On the other hand, under the same condition, when r=10 there should be 240 units tested. So, here, we should prefer when r=10, which significantly reduces the number of units that need to be tested.

From Tables 11 and 12, when the true median life increases, the number of groups decreases, and operating characteristics values increases. For data 1 when $\beta^* = 0.05$, $a_1 = 1$, r=10, $\alpha = 0.3201$ and $\beta = 0.7953$ and for data 2 when $\beta^* = 0.05$, $a_1 = 1$, r=5, $\alpha = 0.5360$ and $\beta = 1.5825$ are the proposed GASP, when a lifetime of an item follows a OMW model.

Table 10. GASP based on MLEs aircraft windshield data.

		$r = 5$						$r = 10$					
		$a_1 = 0.5$			$a_1 = 1$			$a_1 = 0.5$			$a_1 = 1$		
β	r_2	g	c	p(a)	g	c	p(a)	g	c	p(a)	g	c	p(a)
0.25	2	624	3	0.9809	7	3	0.9846	28	3	0.9707	1	3	0.9501
	4	8	1	0.9705	1	1	0.9771	3	1	0.9536	1	2	0.9882
	6	8	1	0.9886	1	1	0.9923	3	1	0.9816	1	1	0.9684
	8	8	1	0.9941	1	1	0.9963	3	1	0.9903	1	1	0.9843
0.1	2	–	–	–	12	3	0.9737	46	3	0.9524	3	4	0.9708
	4	13	1	0.9526	2	1	0.9547	12	2	0.9901	1	2	0.9882
	6	13	1	0.9816	2	1	0.9846	4	1	0.9755	1	1	0.9684
	8	13	1	0.9904	2	1	0.9926	4	1	0.9871	1	1	0.9843
0.05	2	–	–	–	15	3	0.9672	303	4	0.9802	4	4	0.9612
	4	112	2	0.9917	2	1	0.9547	16	2	0.9869	2	2	0.9766
	6	17	1	0.9760	2	1	0.9846	5	1	0.9695	1	1	0.9784
	8	17	1	0.9874	2	1	0.9926	5	1	0.9838	1	1	0.9843
0.01	2	–	–	–	23	3	0.9501	466	4	0.9696	5	4	0.9517
	4	172	2	0.9873	7	1	0.9917	24	2	0.9803	2	2	0.9766
	6	26	1	0.9635	3	1	0.9770	8	1	0.9516	2	2	0.9952
	8	26	1	0.9808	3	1	0.9889	8	1	0.9743	2	1	0.9688

Remark: A large sample size is required cells contains hyphens (–).

Table 11. GASP based on MLEs fiber strength data.

		$r = 5$						$r = 10$					
		$a_1 = 0.5$			$a_1 = 1$			$a_1 = 0.5$			$a_1 = 1$		
β	r_2	g	c	p(a)	g	c	p(a)	g	c	p(a)	g	c	p(a)
0.25	2	97	1	0.9790	1	2	0.9857	24	1	0.977	1	2	0.994
	4	7	0	0.9724	1	0	0.9767	4	0	0.9685	1	0	0.9540
	6	7	0	0.9896	1	0	0.9920	4	0	0.9881	1	0	0.9841
	8	7	0	0.9930	1	0	0.996	4	0	0.9920	1	0	0.9920
0.1	2	160	1	0.9656	2	1	0.9715	40	1	0.9619	1	2	0.994
	4	12	0	0.9531	1	0	0.9767	6	0	0.9531	1	0	0.9540
	6	12	0	0.9822	1	0	0.9920	6	0	0.9822	1	0	0.9841
	8	12	0	0.9881	1	0	0.9960	6	0	0.9881	1	0	0.9920
0.05	2	208	1	0.9555	2	1	0.9715	52	1	0.9508	2	2	0.9881
	4	208	1	0.9987	1	0	0.9767	52	1	0.9985	1	0	0.9540
	6	15	0	0.9777	1	0	0.9920	8	0	0.9763	1	0	0.9841
	8	15	0	0.9851	1	0	0.9960	8	0	0.9841	1	0	0.9920
0.01	2	–	–	–	3	1	0.9576	771	2	0.9907	2	2	0.9881
	4	319	1	0.9980	2	0	0.9540	80	1	0.9977	1	0	0.9540
	6	23	0	0.9661	2	0	0.9841	12	0	0.9646	1	0	0.9841
	8	23	0	0.9773	2	0	0.9920	12	0	0.9763	1	0	0.9920

Remark: A large sample size is required cells contains hyphens (–).

Table 12. Proposed GASP under OMW model.

Data-1					Data-2				
r2	2	4	6	8	r2	2	4	6	8
g	4	2	1	1	g	2	1	1	1
OC	0.9612	0.9766	0.9784	0.9843	OC	0.9715	0.9767	0.992	0.996

10. Conclusions

We introduced the new odd Muth-G family of distributions with essential properties. A special model called the odd Muth-Weibull is presented with some useful properties. Further, a design of a group acceptance sampling plan is proposed under the OMW model by considering median life as a quality parameter. Real data application revealed that the proposed model yielded better fits compared to some commonly well known models.

Conflict of interest

The authors declare no conflict of interest.

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