Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(3): 6543–6558.
DOI: 10.3934/math.2023330
Received: 18 September 2022
Revised: 24 December 2022
Accepted: 26 December 2022
Published: 05 January 2023

Research article

Three nonnegative solutions for Sturm-Liouville BVP and application to the complete Sturm-Liouville equations

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Abstract: The main purpose of this manuscript is to investigate the Sturm-Liouville BVP for non-autonomous Lagrangian systems. Under the suitable assumptions, we establish an existence theorem for three nonnegative solutions via Bonanno-Candito's three critical point theory. As an application in the complete Sturm-Liouville equations with Sturm-Liouville BVC, we get an existence theorem of three nonnegative solutions. Meanwhile, we give three examples to show the correctness of our results.

Keywords: Sturm-Liouville BVC; Lagrangian systems; critical points; index theory; complete Sturm-Liouville equations
Mathematics Subject Classification: 34B15, 34B24, 35B38, 58E05, 70H05

1. Introduction and main results

This manuscript discusses the existence of nonnegative solutions for the following Sturm-Liouville boundary value problems (BVP for short) in non-autonomous Lagrangian systems:

$$\begin{cases} -(B(t)y'(t))' + P(t)y(t) = \mu\nabla_y V(t,y), \text{ a.e. } t \in [0,1], \\ y(0)\cos\theta_1 - B(0)y'(0)\sin\theta_1 = 0, \\ y(1)\cos\theta_2 - B(1)y'(1)\sin\theta_2 = 0, \end{cases} \tag{1.1}$$

where $B(t) = \text{diag}\{(b_1(t),\cdots,b_n(t)\} \in C^1([0,1],\mathcal{L}_d(\mathbf{R}^n))$ with $b_j(t) \in C^1([0,1],\mathbf{R})$, $j = 1,\cdots,n$, and $B(t)$ is a positive definite diagonal matrix for $t \in [0,1]$. $P(t) = \text{diag}\{(p_1(t),\cdots, p_n(t)\} \in L^\infty([0,1],\mathcal{L}_d(\mathbf{R}^n))$, $\nabla_y V(t,y)$ is the gradient of $V(t,y)$ for $y \in \mathbf{R}^n$, and $\mu > 0, \theta_1 \in [0,\pi), \theta_2 \in (0,\pi]$. We make the following assumption:

(H_0) $V : [0,1]\times\mathbf{R}^n \to \mathbf{R}$ is measurable in t for every $y \in \mathbf{R}^n$ and continuously differentiable in y for a.e. $t \in [0,1]$. If $n = 1$, then the function $\nabla_y V(t,y) = v(t,y) : [0,1]\times\mathbf{R} \to \mathbf{R}$ is L^1-Carathéodory; if $n \geq 2$, then

$$|\nabla_y V(t,y)| \leq C(1 + |y|^\gamma),\ \forall y \in \mathbf{R}^n,\ \text{a.e. } t \in [0,1],$$

where $C > 0$ and $\gamma > 0$.

Taking $n = 1, \theta_1 = \theta_2 = \frac{\pi}{2}$, the problem (1.1) is the Neumann BVP for Sturm-Liouville equations

$$\begin{cases} -(b(t)y'(t))' + p(t)y(t) = \mu v(t, y), \text{ a.e. } t \in [0, 1], \\ y'(1) = y'(0) = 0, \end{cases} \tag{1.2}$$

which implies that the problem (1.2) is a special case of (1.1).

For the equilibrium problems of strings, columns, beams etc. in mathematical physics, the Neumann BVP has played an important role. Hence, many researchers have paid attention to it in recent years (see [5, 7, 13–15, 17–20] and the references therein). When $b(t) > 0, b(t) \in C^1([0, 1]), p(t) \geq 0$ and $p(t) \in C([0, 1])$, using the fixed point theorems, the authors of [14, 15, 20] have investigated the multiplicity of solutions for the problem (1.2). In particular, when $b(t) > 0, b(t) \in C^1([0, 1]), p(t) \geq 0$ and $p(t) \in C([0, 1])$, using the critical point theorems of [1, 3, 4, 6], Bonanno-D'Aguì [5] and Bonanno-Iannizzotto-Marras [7] established a three-nonnegative-solutions result and a two-positive-solutions result for the problem (1.2), respectively. Meanwhile, the authors of [5, 7] gave the application of these results in the complete Sturm-Liouville equations

$$\begin{cases} -(b(t)y'(t))' + q(t)y'(t) + r(t)y(t) = \mu g(t, y(t)), \ t \in [0, 1], \\ y'(1) = y'(0) = 0, \end{cases} \tag{1.3}$$

where $q(t) > 0, r(t) > 0, b(t) \in C^1([0, 1])$ and $q(t), r(t) \in C([0, 1])$.

In addition, in [2, 9, 10], with the aid of the three-critical-points theorem of [1, 4], Averna-Giovannelli-Tornatore established a three-solutions result for the mixed BVP

$$\begin{cases} -(b(t)y'(t))' + p(t)y(t) = \mu v(t, y), \ t \in [0, 1], \\ y'(1) = y(0) = 0, \end{cases} \tag{1.4}$$

where $b(t) \in C^1([0, 1])$, $\operatorname{essinf}_{t\in[0,1]} b(t) > 0$, $p(t) \in C([0, 1])$, and $\operatorname{essinf}_{t\in[0,1]} p(t) \geq 0$, . Meanwhile, the authors of [2, 9, 10] also gave the application of the result in the complete Sturm-Liouville equations:

$$\begin{cases} -y'' + y' + y = \mu g(y(t)), \ t \in [0, 1], \\ y'(1) = y(0) = 0. \end{cases} \tag{1.5}$$

Clearly, taking $n = 1, \theta_1 = 0, \theta_2 = \frac{\pi}{2}$, the problem (1.1) becomes the problem (1.4).

In this manuscript, we are interested in the function $p(t) \in L^\infty([0, 1])$ and without the assumption of $p(t) \geq 0$ for the n-dimensional the problems (1.2) and (1.4). To this end, we reconsider in the framework of the problem (1.1) some theorems proved in [2, 5, 7]. With the aid of index theory, we construct a variational construction. Then using Bonanno-Candito's three critical point theory obtained in [4], we give some new criteria to have at least three nonnegative solutions for the problem (1.1). As a direct application, we obtain the corresponding results for the complete Sturm-Liouville equations meeting Sturm-Liouville boundary value conditions (BVC for short). Furthermore, we give three examples to show the correctness of the obtained conclusions and to indicate that these results unify and sharply improve some recent results.

Now, for all $A(t) \in L^\infty([0, 1], \mathcal{L}_s(\mathbf{R}^n))$, we use the index $(i^B_{\theta_1,\theta_2}(A), \nu^B_{\theta_1,\theta_2}(A)) \in \mathbf{N}\times\mathbf{N}$ in [11] to express our primary results.

Theorem 1.1. *Assume that there are $A_0(t), A(t) \in L^\infty([0,1], \mathcal{L}_d(\mathbf{R}^n))$ that satisfy $P(t) = -A_0(t) + A(t)$, $i^B_{\theta_1,\theta_2}(A_0) = 0, \nu^B_{\theta_1,\theta_2}(A_0) \neq 0$, where $A(t)$ is a positive definite diagonal matrix for $t \in [0,1]$, i.e., there exist $\bar{s} \geq \underline{s} > 0$ such that $\bar{s}|y|^2 \geq (A(t)y, y) \geq \underline{s}|y|^2$ for all $y \in \mathbf{R}^n, t \in [0,1]$.*

Suppose $V(t,y)$ satisfies (H_0) *and the following:*

(H_1) *$V(t,0) = 0$, and $\nabla_y V(t,y) \geq 0$, i.e. $\frac{\partial V}{u_i} \geq 0,\ \forall i = 1, \cdots, n$, where $y = (u_1, u_2, \cdots, u_n)$.*

(H_2) *There are $c_0 > 0, b_0 > 0$ and $y_0 \in \ker(\Lambda - \mathrm{A}_0)$ with $\frac{c_0}{k}\sqrt{\frac{2}{\underline{s}}} < \|y_0\|_{L_2} < \frac{b_0}{k\sqrt{2\bar{s}}}$, such that*

$$\frac{\int_0^1 \sup_{|y| \leq c_0} V(t,y)dt}{c_0^2} < \frac{2\int_0^1 V(t,y_0)dt}{3\bar{s}k^2\|y_0\|^2_{L^2}} \tag{1.6}$$

and

$$\frac{\int_0^1 \sup_{|y \leq b_0} V(t,y)dt}{b_0^2} < \frac{\int_0^1 V(t,y_0)dt}{3\bar{s}k^2\|y_0\|^2_{L^2}}, \tag{1.7}$$

for all $y \in \mathbf{R}^n$, where $(\Lambda y - A_0 y)(t) = -(B(t)y')'(t) - A_0(t)y(t)$, $k = \delta_0(\min\{1, \underline{s}\})^{\frac{-1}{2}}$ and $\delta_0 > 0$ is the compact embedding constant of $Z \hookrightarrow L^\infty$(or see (3.2)*).*
Then, for each $\mu \in \left(\frac{3\bar{s}\|y_0\|^2_{L^2}}{4\int_0^1 V(t,y_0)dt}, \min\{\frac{c_0^2}{2k^2\alpha}, \frac{b_0^2}{2k^2\beta}\}\right)$, the problem (1.1) *has at least three nonnegative solutions y_i with $|y_i| < b_0$ for $i = 1,2,3$, where $\alpha = \int_0^1 \sup_{|y| \leq c_0} V(t,y)dt, \beta = \int_0^1 \sup_{|y| \leq b_0} V(t,y)dt$, and the nonnegative solutions $y(t) = (u_1(t), \cdots, u_n(t))$ means that $u_i(t) \geq 0,\ \forall i = 1,2,\cdots,n$.*

Next, as an application, we consider the complete Sturm-Liouville equations

$$\begin{cases} -(b(t)y'(t))' + q(t)y'(t) + r(t)y(t) = \mu g(t, y(t)), \text{ a.e. } t \in [0,1], \\ y(0)\cos\theta_1 - e^{-Q(0)}b(0)y'(0)\sin\theta_1 = 0, \\ y(1)\cos\theta_2 - e^{-Q(1)}b(1)y'(1)\sin\theta_2 = 0, \end{cases} \tag{1.8}$$

where $\mu > 0, \theta_1 \in [0,\pi), \theta_2 \in (0,\pi], Q'(t) = \frac{q(t)}{b(t)}$, $b(t) \in C^1([0,1], \mathbf{R})$ with $\operatorname{ess\,inf}_{[0,1]} b(t) > 0$, and $q(t), r(t) \in L^\infty([0,1], \mathbf{R})$ with $\operatorname{meas}\{t \in [0,1] : q(t) \neq 0\} > 0$.

Theorem 1.2. *Assume that there is $A_0(t) \in L^\infty([0,1], \mathbf{R})$ such that $i^{B_1}_{\theta_1,\theta_2}(A_0) = 0, \nu^{B_1}_{\theta_1,\theta_2}(A_0) \neq 0$, and $\underline{s}_0 = \operatorname{ess\,inf}_{[0,1]}\{e^{-Q(t)}r(t) + A_0(t)\} > 0$, where $B_1(t) = b(t)e^{-Q(t)}$.*

Suppose the function $g : [0,1] \times \mathbf{R} \to \mathbf{R}$ is L^1-Carathéodory and satisfies the following:

(G_1) *For all $t \in [0,1]$ and $y \in \mathbf{R}$, $g(t,y) \geq 0$.*

(G_2) *There exist $c_0 > 0, b_0 > 0$ and $y_0 \in \ker(\Lambda_1 - \mathrm{A}_0)$ with $\frac{c_0}{k_0}\sqrt{\frac{2}{\underline{s}_0}} < \|y_0\|_{L^2} < \frac{b_0}{k_0\sqrt{2\bar{s}_0}}$, such that*

$$\frac{\int_0^1 \sup_{|y| \leq c_0} \int_0^y g(t,\xi)d\xi dt}{c_0^2} < \frac{2\int_0^1 \int_0^{y_0} g(t,\xi)d\xi dt}{3\bar{s}_0 k_0^2 \|y_0\|^2_{L^2}}$$

and

$$\frac{\int_0^1 \sup_{|y| \leq b_0} \int_0^y g(t,\xi)d\xi dt}{b_0^2} < \frac{\int_0^1 \int_0^{y_0} g(t,\xi)d\xi dt}{3\bar{s}_0 k_0^2 \|y_0\|^2_{L^2}},$$

for all $y \in \mathbf{R}$, *where* $(\Lambda_1 y - A_0 y)(t) = -(B_1(t)y')'(t) - A_0(t)y(t)$, $k_0 = \delta_0(\min\{1, \underline{s}_0\})^{\frac{-1}{2}}$ *and* $\bar{s}_0 = \operatorname{ess\,sup}_{[0,1]}\{e^{-Q(t)}r(t) + A_0(t)\}$.

Then, for each $\mu \in \left(\frac{3\bar{s}_0\|y_0\|^2_{L^2}}{4\int_0^1 e^{-Q(t)}\int_0^{y_0} g(t,\xi)d\xi dt}, \min\left\{\frac{c_0^2}{2k_0^2\alpha_1}, \frac{b_0^2}{2k_0^2\beta_1}\right\}\right)$, *the problem* (1.8) *has at least three nonnegative solutions* y_i *with* $|y_i| < b_0$ *for* $i = 1, 2, 3$, *where*

$$\alpha_1 = \int_0^1 e^{-Q(t)} \sup_{|y|\le c_0} \int_0^y g(t,\xi)d\xi dt \ \ and\ \ \beta_1 = \int_0^1 e^{-Q(t)} \sup_{|y|\le b_0} \int_0^y g(t,\xi)d\xi dt.$$

The organization of this manuscript is as follows. The main content of Section 2 is recalling the three critical points theorem in [4] and some conclusions about the index theory of the second order linear Lagrangian systems in [11, 12]. In Section 3, we construct a variational construction for (1.1) in Z and give the proof of Theorems 1.1 and 1.2. Some helpful corollaries and some examples are given in Section 4 to show the validity of our results. Meanwhile, we emphasize that our results unify and sharply improve the correlative results of [2, 5, 7, 17, 18] via some remarks.

2. Preliminaries

First, we recall the three critical points theorem in [4] and some results about the index theory of linear second order Lagrangian systems in [11, 12] in order to prove Theorems 1.1 and 1.2.

Lemma 2.1. ([4]) *Let* Z *be a reflexive real Banach space. Assume that two functionals* $\Phi, \Psi : Z \to \mathbf{R}$ *are continuously Gâteaux differentiable,* Φ *is coercive and convex with* Φ' *having a continuous inverse on* Z^*, *and* Ψ' *is compact, such that*
(1) $\Phi(\theta) = \Psi(\theta) = \inf_Z \Phi = 0$.
(2) *For each* $\mu > 0$, *if* y_1, y_2 *are two local minima of* $\Phi - \mu\Psi$ *with* $\Psi(y_1) \ge 0$ *and* $\Psi(y_2) \ge 0$, *then* $\inf_{\vartheta\in[0,1]} \Psi(\vartheta y_1 + (1-\vartheta)y_2) \ge 0$.

If there are $\rho_1 > 0, \rho_2 > 0$ *and* $\tilde{y} \in Z$ *with* $2\rho_1 < \Phi(\tilde{y}) < \frac{\rho_2}{2}$, *such that*

(i) $\frac{\sup_{y\in\Phi^{-1}((-\infty,\rho_1))}\Psi(y)}{\rho_1} < \frac{2\Psi(\tilde{y})}{3\Phi(\tilde{y})}$

and

(ii) $\frac{\sup_{y\in\Phi^{-1}((-\infty,\rho_2))}\Psi(y)}{\rho_2} < \frac{\Psi(\tilde{y})}{3\Phi(\tilde{y})}$,

then, for each $\mu \in \left(\frac{3\Phi(\tilde{y})}{2\Psi(\tilde{y})}, \min\left\{\frac{\rho_1}{\sup_{y\in\Phi^{-1}((-\infty,\rho_1))}\Psi(y)}, \frac{\rho_2}{2\sup_{y\in\Phi^{-1}((-\infty,\rho_2))}\Psi(y)}\right\}\right)$, $\Phi - \mu\Psi$ *has at least three distinct critical points which lie in* $\Phi^{-1}((-\infty, \rho_2))$.

The index theory in [11, 12] is designed to address the classification problem of $L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$ associated with the Lagrangian system

$$-(B(t)y')' - A(t)y = 0, \tag{2.1}$$

$$y(0)\cos\theta_1 - B(0)y'(0)\sin\theta_1 = 0, \tag{2.2}$$

$$y(1)\cos\theta_2 - B(1)y'(1)\sin\theta_2 = 0, \tag{2.3}$$

where $\theta_1 \in [0,\pi), \theta_2 \in (0,\pi], A(t) \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n)) = \{A(t) = (a_{jk})_{n\times n} | a_{jk}(t) = a_{kj}(t), t \in [0,1], a_{jk}(t) \in L^\infty([0,1])\}$, and $B(t) \in C^1([0,1], \mathcal{L}_s(\mathbf{R}^n))$ with $B(t)$ is a positive definite matrix for $t \in [0,1]$.

Let $Y = L^2([0,1],\mathbf{R}^n)$. Set $(\Lambda y)(t) = -(B(t)y')'(t)$, and

$$D(\Lambda) = \{y \in H^2([0,1],\mathbf{R}^n) : y \text{ satisfies } (2.2, 2.3)\}.$$

In Section 2.3 of [11], it has been proved the operator Λ is self-adjoint, and $\sigma(\Lambda) = \sigma_d(\Lambda)$ is bounded from below, where $\sigma_d(\Lambda) = \{\eta \in \mathbf{R} : \eta \text{ is the point spectrum of } \Lambda\}$. Then, for all $x, y \in Z$, we consider the bilinear form

$$L(x,y) = \int_0^1 (B(t)x'(t), y'(t))dt + (x(0), y(0))\gamma(\theta_1) - (x(1), y(1))\gamma(\theta_2), \tag{2.4}$$

where $(\cdot,\cdot)$ is the commonly used inner product in $\mathbf{R}^n$, $\gamma(t) = \cot t$ as $t \in (0,\pi), \gamma(t) = 0$ as $t = 0$ or $t = \pi$, and

$$Z = \begin{cases} \{y \in H^1([0,1],\mathbf{R}^n) | y'(1) = y'(0) = 0\}, & \theta_1 = \theta_2 = \frac{\pi}{2}; \\ \{y \in H^1([0,1],\mathbf{R}^n) | y'(1) = y(0) = 0\}, & \theta_1 = 0, \theta_2 = \frac{\pi}{2}; \\ \{y \in H^1([0,1],\mathbf{R}^n) | y(1) = y(0) = 0\}, & \theta_1 = 0, \theta_2 = \pi; \\ \{y \in H^1([0,1],\mathbf{R}^n) | y(1) = 0\}, & \theta_1 = 0, \theta_2 \in (0,\pi); \\ \{y \in H^1([0,1],\mathbf{R}^n) | y(0) = 0\}, & \theta_1 \in (0,\pi), \theta_2 = \pi; \\ H^1([0,1],\mathbf{R}^n), & \theta_1, \theta_2 \in (0,\pi). \end{cases} \tag{2.5}$$

Similarly to the proof of Proposition 1.17 in [8], it can be proved Z is a Hilbert space. Furthermore, $Z = D(|\Lambda|^{\frac{1}{2}})$ is capable of being equipped with an equivalent norm

$$||y||_Z = \left(\int_0^1 [|y(t)|^2 + y'(t)|^2]dt\right)^{\frac{1}{2}}, \quad \forall\, y \in Z,$$

which implies that two embedded mappings $Z \hookrightarrow L^\infty$ and $Z \hookrightarrow L^2 = Y$ are compact.

For any $A(t) \in L^\infty([0,1],\mathcal{L}_s(\mathbf{R}^n))$, define $\psi^{B,A}_{\theta_1,\theta_2}(x,y)$ as follows:

$$\psi^{B,A}_{\theta_1,\theta_2}(x,y) = L(x,y) - \int_0^1 (A(t)x(t), y(t))dt, \quad \forall\, x, y \in Z. \tag{2.6}$$

Proposition 2.2. ([11]) *For any $A(t) \in L^\infty([0,1],\mathcal{L}_s(\mathbf{R}^n))$, the space*

$$Z = Z^-(A) \oplus Z^0(A) \oplus Z^+(A)$$

is a $\psi^{B,A}_{\theta_1,\theta_2}$-orthogonal decomposition, where $\psi^{B,A}_{\theta_1,\theta_2}$ is negative definite, null and positive definite on $Z^-(A), Z^0(A)$ and $Z^+(A)$, respectively. Particularly, $Z^-(A)$ and $Z^0(A)$ are finitely dimensional.

Definition 2.3. ([11]) *For any $A(t) \in L^\infty([0,1],\mathcal{L}_s(\mathbf{R}^n))$, we define*

$$i^B_{\theta_1,\theta_2}(A) = \sum_{\lambda<0} \nu^B_{\theta_1,\theta_2}(A + \lambda I_n), \quad \nu^B_{\theta_1,\theta_2}(A) = \dim\ker(\Lambda - A).$$

Proposition 2.4. ([12]) *For any $A \in L^\infty([0,1],\mathcal{L}_s(\mathbf{R}^n))$, $Z^0(A)$ is the solution subspace of the systems 2.1 and 2.3, and*

$$i^B_{\theta_1,\theta_2}(A) = \dim Z^-(A), \quad \nu^B_{\theta_1,\theta_2}(A) = \dim Z^0(A).$$

$i^B_{\theta_1,\theta_2}(A)$ and $\nu^B_{\theta_1,\theta_2}(A)$ are called the index and nullity of A with respect to $\psi^{B,A}_{\theta_1,\theta_2}(\cdot,\cdot)$, respectively.

Proposition 2.5. ([16]) *For any $A \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$, for $y = y_1 + y_2 \in Z$, if $y_1 \in Z^-(A)$ and $y_2 \in Z^+(A)$, then $(-\psi^{B,A}_{\theta_1,\theta_2}(y_1, y_1))^{\frac{1}{2}} + (\psi^{B,A}_{\theta_1,\theta_2}(y_2, y_2))^{\frac{1}{2}}$ is an equivalent norm on Z.*

Remark 1. ([12]) If $B(t) \equiv I_n, c \in \mathbf{R}$, one has

$$\begin{aligned}
&\nu^{I_n}_{0,\frac{\pi}{2}}(cI_n) = n \text{ as } c = (\frac{1}{2} + k)^2\pi^2,\\
&\nu^{I_n}_{0,\frac{\pi}{2}}(cI_n) = 0 \text{ as } c \neq (\frac{1}{2} + k)^2\pi^2 \text{ for } k \in \mathbf{N},\\
&i^{I_n}_{0,\frac{\pi}{2}}(cI_n) = 0 \text{ as } c \leq \frac{\pi^2}{4},\\
&i^{I_n}_{0,\frac{\pi}{2}}(cI_n) = (k+1)n \text{ as } c \in \left((\frac{1}{2} + k)^2\pi^2, (\frac{1}{2} + k + 1)^2\pi^2\right),\\
&\nu^{I_n}_{\frac{\pi}{2},\frac{\pi}{2}}(cI_n) = n \text{ as } c = k^2\pi^2,\ \nu^{I_n}_{\frac{\pi}{2},\frac{\pi}{2}}(cI_n) = 0 \text{ as } c \neq k^2\pi^2 \text{ for } k \in \mathbf{N},\\
&i^{I_n}_{\frac{\pi}{2},\frac{\pi}{2}}(cI_n) = 0 \text{ as } c \leq 0,\ i^{I_n}_{\frac{\pi}{2},\frac{\pi}{2}}(cI_n) = (k+1)n \text{ as } c \in \left(k^2\pi^2, (k+1)^2\pi^2\right).
\end{aligned}$$

Remark 2. Since $C^1([0,1], \mathcal{L}_d(\mathbf{R}^n)) \subset C^1([0,1], \mathcal{L}_s(\mathbf{R}^n)), L^\infty([0,1], \mathcal{L}_d(\mathbf{R}^n)) \subset L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$, for $B(t) \in C^1([0,1], \mathcal{L}_d(\mathbf{R}^n)), A(t) \in L^\infty([0,1], \mathcal{L}_d(\mathbf{R}^n))$, the above index theories also hold.

3. Proof of the main results

From the assumptions $i^B_{\theta_1,\theta_2}(A_0) = 0$ and $\nu^B_{\theta_1,\theta_2}(A_0) \neq 0$, we can see that $\min\sigma(\Lambda - A_0) = 0$ via Definition 2.3 and Proposition 2.4. Hence, we define another inner product:

$$\langle x, y\rangle = L(x,y) - \int_0^1 (A_0(t)x(t), y(t))dt + \int_0^1 (x(t), y(t))dt, \quad \forall x, y \in Z,$$

where the corresponding norm is defined as

$$\|y\| = (L(y,y) - (A_0 y, y)_{L^2} + \|y\|^2_{L^2})^{\frac{1}{2}}, \quad \forall y \in Z.$$

By Proposition 2.5, we can see that $\|\cdot\|$ is equivalent to $\|\cdot\|_Z$. For all $y \in Z$, put

$$\|y\|_S = \left(L(y,y) - \int_0^1 (A_0(t)y(t), y(t))dt + \int_0^1 (A(t)y(t), y(t))dt\right)^{\frac{1}{2}}.$$

Since $A(t)$ is a positive definite matrix, for all $y \in Y$ there are $\bar{s} \geq \underline{s} > 0$ such that $\bar{s}\|y\|^2_{L^2} \geq (Ay, y)_{L^2} \geq \underline{s}\|y\|^2_{L^2}$. Thus, one has

$$\min\{1, \underline{s}\}\|y\|^2 \leq \|y\|^2_S \leq \max\{1, \bar{s}\}\|y\|^2. \tag{3.1}$$

Let $\|\cdot\|_\infty$ be the norm of $L^\infty([0,1], \mathbf{R}^n)$. By the compactness of the embedded mappings $Z \hookrightarrow L^2 = Y$ and $Z \hookrightarrow L^\infty$, we know that there is $\delta_0 > 0$ such that

$$|y| \leq \|y\|_\infty \leq \delta_0\|y\| \leq k\|y\|_S \tag{3.2}$$

for all $y \in Z$, where $k = \delta_0(\min\{1, \underline{s}\})^{\frac{-1}{2}}$.

Next, we prove that Theorems 1.1 and 1.2. To this end, set

$$I(y)=\frac{\|y\|_S^2}{2}-\mu\int_0^1 V(t,y)dt,\quad \forall y\in Z. \tag{3.3}$$

By (H_0), it is not difficult to prove that I is continuously differentiable in Z, and

$$I'(y)x=L(y,x)+\int_0^1(P(t)y,x)dt-\mu\int_0^1(\nabla_y V(t,y),x)dt,$$

for all $x,y\in Z$. Similar to the proof of Proposition 2.3.3 (1) in [11], it can easily be proved that the critical points of I are the solutions of the problem (1.1), and we leave out the details here.

Proof of Theorem 1.1. For each $y\in Z$, set

$$\Phi(y)=\frac{\|y\|_S^2}{2},\ \ \Psi(y)=\int_0^1 V(t,y)dt.$$

Obviously, the critical points of $\Phi-\mu\Psi$ in Z correspond to the solutions of (1.1).

From (3.1), the compactness of the embedding $Z\hookrightarrow L^2=Y$ and the condition (H_0), we can see that Φ is coercive, convex and continuously G$\hat{a}$teaux differentiable, and Ψ is continuously G$\hat{a}$teaux differentiable with Ψ' being compact. Meanwhile, $V(t,0)=0$ implies that (1) of Lemma 2.1 is valid.

Next, we show that Φ' has a continuous inverse on Z^*. Noting (3.1), for all $x,y\in Z$, we have

$$\langle\Phi'(y)-\Phi'(x),y-x\rangle=\|y-x\|_S^2\geq\min\{1,\underline{s}\}\|y-x\|^2,$$

which means that Φ' is uniformly monotone on Z^*. With the aid of standard arguments, we can ensure that Φ' is also hemicontinuous and coercive on Z^*. Moreover, using Theorem 26. A of [21], it is easy to show that Φ' has a continuous inverse on Z^*.

Put $\tilde{y}=y_0$ and $\rho_1=\frac{1}{2}(\frac{c_0}{k})^2$. By (3.2), we know that $\{y\in Z:\Phi(y)<\rho_1\}\subset\{y\in Z:|y|\leq c_0\}$, which implies that

$$\sup_{\Phi(y)<\rho_1}\Psi(y)=\sup_{\Phi(y)<\rho_1}\int_0^1 V(t,y)dt\leq\int_0^1\sup_{|y|\leq c_0}V(t,y)dt.$$

Taking into account that $y_0\in\ker(\Lambda-A_0)$, from Propositions 2.2 and 2.4, we obtain

$$\begin{aligned}\Phi(y_0)&=\frac{1}{2}L(y_0,y_0)+\frac{1}{2}\int_0^1(P(t)y_0,y_0)dt\\&=\frac{1}{2}\psi_{\theta_1,\theta_2}^{B,A_0}(y_0,y_0)+\frac{1}{2}\int_0^1(A(t)y_0,y_0)dt\\&=\frac{1}{2}\int_0^1(A(t)y_0,y_0)dt,\end{aligned}$$

which shows that

$$\frac{1}{2}\underline{s}\|y_0\|_{L^2}^2\leq\Phi(y_0)\leq\frac{1}{2}\bar{s}\|y_0\|_{L^2}^2. \tag{3.4}$$

Noticing that

$$\frac{\sup_{\Phi(y)<\rho_1}\Psi(y)}{\rho_1}\leq\frac{2k^2\int_0^1\sup_{|y|\leq c_0}V(t,y)dt}{c_0^2}$$

and

$$\frac{2\Psi(y_0)}{3\Phi(y_0)} = \frac{4\int_0^1 V(t, y_0)dt}{3\int_0^1 (A(t)y_0, y_0)dt} \geq \frac{4\int_0^1 V(t, y_0)dt}{3\bar{s}\|y_0\|_{L^2}^2},$$

by (1.6) of the condition (H_2), we obtain

$$\frac{\sup_{y\in\Phi^{-1}((-\infty,\rho_1))}\Psi(y)}{\rho_1} < \frac{2\Psi(y_0)}{3\Phi(y_0)},$$

which means that (i) of Lemma 2.1 holds.

Again, put $\rho_2 = \frac{1}{2}(\frac{b_0}{k})^2$. As above, we can see that

$$\frac{\sup_{\Phi(y)<\rho_2}\Psi(y)}{\rho_2} \leq \frac{2k^2\int_0^1 \sup_{|y|\leq b_0} V(t, y)dt}{b_0^2}$$

and

$$\frac{\Psi(y_0)}{3\Phi(y_0)} \geq \frac{2\int_0^1 V(t, y_0)dt}{3\bar{s}\|y_0\|_{L^2}^2}.$$

By (1.7) of (H_2), we have

$$\frac{\sup_{y\in\Phi^{-1}((-\infty,\rho_2))}\Psi(y)}{\rho_2} < \frac{\Psi(y_0)}{3\Phi(y_0)},$$

which means that (ii) of Lemma 2.1 holds.

Moreover, from $\frac{c_0}{k}\sqrt{\frac{2}{\underline{s}}} < \|y_0\|_{L^2} < \frac{b_0}{k\sqrt{2\bar{s}}}$ and (3.4), we obtain $2\rho_1 < \Phi(y_0) < \frac{\rho_2}{2}$.

Finally, we prove that (2) of Lemma 2.1 holds. For each $\mu > 0$, if y_1 and y_2 are two local minima of $\Phi - \mu\Psi$ with $\Psi(y_1) \geq 0$ and $\Psi(y_2) \geq 0$, then y_1 and y_2 are two critical points of $\Phi - \mu\Psi$, which implies that y_1 and y_2 are two solutions of the problem (1.1). Taking into account that $\nabla_y V(t, y) \geq 0$, we have $y_1(t) \geq 0, y_2(t) \geq 0$ via the following Lemma 3.1. Then, it follows that $(1-\vartheta)y_1 + \vartheta y_2 \geq 0, \forall\vartheta \in [0, 1]$. Hence,

$$\inf_{\vartheta\in[0,1]}\Psi((1-\vartheta)y_1 + \vartheta y_2) = \inf_{\vartheta\in[0,1]}\int_0^1\int_0^1 (\nabla_y V(t, \xi((1-\vartheta)y_1 + \vartheta y_2)), (1-\vartheta)y_1 + \vartheta y_2)d\xi dt \geq 0$$

via $\nabla_y V(t, y) \geq 0$ and $V(t, 0) = 0$.

By Lemma 2.1, for each

$$\mu \in \left(\frac{3\Phi(y_0)}{2\Psi(y_0)}, \min\{\frac{\rho_1}{\sup_{y\in\Phi^{-1}((-\infty,\rho_1))}\Psi(y)}, \frac{\rho_2}{2\sup_{y\in\Phi^{-1}((-\infty,\rho_2))}\Psi(y)}\}\right),$$

$I = \Phi - \mu\Psi$ has at least three distinct critical points y_i with $\Phi(y_i) < \rho_2$ for $i = 1, 2, 3$ in Z. Hence, for each

$$\mu \in \left(\frac{3\bar{s}\|y_0\|_{L^2}^2}{4\int_0^1 V(t,y_0)dt}, \min\{\frac{c_0^2}{2k^2\int_0^1 \sup_{|y|\leq c_0} V(t,y)dt}, \frac{b_0^2}{2k^2\int_0^1 \sup_{|y|\leq b_0} V(t,y)dt}\}\right),$$

by (3.2) and the following Lemma 3.1, we know that the problem (1.1) has at least three nonnegative solutions y_i with $|y_i| < b_0$ for $i = 1, 2, 3$ in Z. The proof is complete. □

Lemma 3.1. *Let* $\nu^B_{\theta_1,\theta_2}(A_0) \neq 0, i^B_{\theta_1,\theta_2}(A_0) = 0$ *and* $\nabla_y V(t, y) \geq 0$. *If* $y \in Z$ *is a solutions of the problem* (1.1)*, then* y *is nonnegative.*

Proof. Set $y^-(t) = -\min\{0, y(t)\} = (-\min\{0, u_1(t)\}, \cdots, -\min\{0, u_n(t)\})$. Evidently, $y^-(t) \in Z$. Since $b(t) \in C^1([0, 1], \mathcal{L}_d(\mathbf{R}^n)), P(t) \in L^\infty([0, 1], \mathcal{L}_d(\mathbf{R}^n))$,

$$(P(t)y, y^-) = -(P(t)y^-, y^-), L(y, y^-) = -L(y^-, y^-).$$

Taking into account that $y \in Z$ is a solution of the problem (1.1), we choose $y = y^-(t)$. By (3.2), $i^B_{\theta_1,\theta_2}(A_0) = 0$ and Proposition 2.2, we have

$$\begin{aligned} 0 &\leq \mu \int_0^1 (\nabla_y V(t, y), y^-)dt \\ &= L(y, y^-) + \int_0^1 (P(t)y, y^-)dt \\ &= -L(x^-, x^-) - \int_0^1 (P(t)y^-, y^-)dt \\ &= -\|y^-\|^2_S \leq -\frac{\delta_0^2}{k^2}\|y^-\|^2 \leq 0. \end{aligned}$$

That is, $y^- = 0$ a.e. in $[0, 1]$, and y is nonnegative. □

Proof of Theorem 1.2. In consideration of $\frac{q(t)}{b(t)}$ being Lebesgue integrable in $[0, 1]$, we set the function $Q(t)$ satisfying $Q'(t) = \frac{q(t)}{b(t)}$ a.e. in $[0, 1]$. Consider the following problem:

$$\begin{cases} -(e^{-Q(t)}b(t)y'(t))' + r(t)y(t)e^{-Q(t)} = \mu g(t, y(t))e^{-Q(t)}, \\ y(0)\cos\theta_1 - e^{-Q(0)}b(0)y'(0)\sin\theta_1 = 0, \\ y(1)\cos\theta_2 - e^{-Q(1)}b(1)y'(1)\sin\theta_2 = 0, \end{cases} \tag{3.5}$$

for a.e. $t \in [0, 1]$. We can prove that the solutions of (3.5) are also the solutions of (1.8). Thus, setting $B_1(t) = b(t)e^{-Q(t)}, P(t) = r(t)e^{-Q(t)}$ and $V(t, y) = \int_0^y g(t, \xi)e^{-Q(t)}d\xi$, the validity of the conditions of Theorem 1.1 can be proved. Hence, from Theorem 1.1, the conclusion follows. □

4. Corollaries and examples

As can be seen from (2.5) of Section 2, if $\theta_1 = \theta_2 = \frac{\pi}{2}$; $\theta_1 = 0, \theta_2 = \frac{\pi}{2}$; or $\theta_1 = 0, \theta_2 = \pi$ in Theorems 1.1 and 1.2, we immediately obtain the corresponding the existence results for the n-dimensional Neumann, mixed or two point BVP.

First, we discuss the Neumann BVP as follows:

$$\begin{cases} -(B(t)y'(t))' + P(t)x(t) = \mu\nabla_y V(t, y), \text{ a.e. } t \in [0, 1], \\ y'(1) = y'(0) = 0, \end{cases} \tag{4.1}$$

where $B(t) \in C^1([0,1], \mathcal{L}_d(\mathbf{R}^n))$ and $P(t) \in L^\infty([0,1], \mathcal{L}_d(\mathbf{R}^n))$ with $P(t)$ and $B(t)$ positive definite for $t \in [0,1]$. After a simple calculation, we know that $\ker(\Lambda) = \mathbf{R}^n$, $i^B_{\frac{\pi}{2},\frac{\pi}{2}}(0) = 0$, and $\nu^B_{\frac{\pi}{2},\frac{\pi}{2}}(0) \neq 0$. Set $A_0 = 0$, $P(t) = A(t)$. Therefore, we immediately obtain Corollary 4.1 through Theorem 1.1.

Corollary 4.1. *Assume that $V(t,y)$ satisfies* (H_0), (H_1). *If there are $c_0 > 0, b_0 > 0$ and $y_0 \in \mathbf{R}^n$ with $\frac{c_0}{k}\sqrt{\frac{2}{\underline{s}}} < |y_0| < \frac{b_0}{k\sqrt{2\bar{s}}}$, such that $V(t,y)$ satisfies* (H_2), *then the conclusion of Theorem* 1.1 *is still valid.*

Remark 3. In Corollary 4.1, if $n = 1$, we have $\ker(\Lambda) = \mathbf{R}$. Thus, Corollary 4.1 can be reduced to Theorem 3.4 in [5] via some simple calculations. However, we still need to point out that Corollary 4.1 generalizes Theorem 3.4 in [5] as $n = 1$ in two aspects. First, Corollary 4.1 requires $\nabla_y V(t,y) = v(t,y)$ being an L^1-Carathéodory function instead of continuous in $t \in [0,1]$ and $y \in \mathbf{R}$; second Corollary 4.1 requires $P(t) \in L^\infty_{[0,1]}$ instead of $P(t) \in C([0,1])$.

Next, we give an example of (4.1).

Example 1. Let $\alpha(t) \in C([0,1], \mathbf{R}^+)$ with $\int_0^1 \alpha(t)dt > 0$. Consider

$$\begin{cases} -y''(t) + P(t)y(t) = \lambda \nabla_y V(t,y), \\ y'(0) = y'(1) = 0, \end{cases} \tag{4.2}$$

where $n = 2$, $P(t) = \mathrm{diag}\{p_1(t), p_2(t)\}$ with

$$p_1(t) = p_2(t) = \begin{cases} 1, & t \in [0, \frac{1}{2}], \\ 2, & t \in (\frac{1}{2}, 1], \end{cases}$$

and

$$\begin{aligned} & V(t,y) = V(t,u_1,u_2) \\ = & \begin{cases} \alpha(t)(u_1+u_2), & |u_1+u_2| \le 1, \\ \alpha(t)[\frac{(u_1+u_2)^{11}}{11} + \frac{10}{11}], & 1 < |u_1+u_2| \le 2, \\ \alpha(t)[2^{10}(u_1+u_2) + \frac{10(1-2^{11})}{11}], & 2 < |u_1+u_2| \le 300, \\ \alpha(t)[\frac{2^{10}}{5\times 300^4}(u_1+u_2)^5 + 240\times 2^{10} & \\ +\frac{10(1-2^{11})}{11}], & 300 < |u_1+u_2|. \end{cases} \end{aligned}$$

Clearly, $\underline{s} = 1$, $\bar{s} = 2$. For $y \in Z$, by

$$\begin{aligned} |y(t)| & \le |\int_{t_1}^t y'(s)ds| + |y(t_1)| \le \int_0^1 |y'(s)|ds + |y(t_1)| \\ & \le \left(\int_0^1 |y'(s)|^2 ds\right)^{\frac{1}{2}} + \left(\int_0^1 |y(s)|^2 ds\right)^{\frac{1}{2}}, \end{aligned}$$

we have

$$\max_{t\in[0,1]} |y(t)| \le \sqrt{2}\|y\|,$$

which implies that $\delta_0 = \sqrt{2}$ and $k = \sqrt{2}$. We easily check that $V(t,y)$ satisfies (H_0) and (H_1). By $\ker(\Lambda) = \mathbf{R}^2$, taking $y_0 = (1,1)$, $c_0 = \frac{\sqrt{2}}{2}$ and $b_0 = 150\sqrt{2}$, we have $\frac{1}{2} = \frac{c_0}{k}\sqrt{\frac{2}{\underline{s}}} < |y_0| = \sqrt{2} < \frac{b_0}{k\sqrt{2\bar{s}}} = 75$,

$$\begin{aligned}\frac{\int_0^1 \sup_{|y|\le c_0} V(t,y)dt}{c_0^2} &= \sqrt{2}\int_0^1 \alpha(t)dt \\ < \quad \frac{\left(\frac{2^{11}}{11}+\frac{10}{11}\right)\int_0^1 \alpha(t)dt}{12} &= \frac{2\int_0^1 V(t,y_0)dt}{3\bar{s}k^2|y_0|^2}\end{aligned}$$

and

$$\begin{aligned}\frac{\int_0^1 \sup_{|y|\le b_0} V(t,y)dt}{b_0^2} &= \frac{\left(2^{10}\times 300+\frac{10(1-2^{11})}{11}\right)\int_0^1 \alpha(t)dt}{2\times 150^2} \\ < \quad \frac{\left(\frac{2^{11}}{11}+\frac{10}{11}\right)\int_0^1 \alpha(t)dt}{24} &= \frac{\int_0^1 V(t,y_0)dt}{3\bar{s}k^2|y_0|^2}\end{aligned}$$

which show that (1.6) and (1.7) of (H_2) hold. After a simple calculation, from Corollary 4.1, we know that for each $\mu \in (\frac{33}{1034\int_0^1 \alpha(t)dt}, \frac{12375}{335873\int_0^1 \alpha(t)dt})$, the problem (4.2) has at least three positive solutions y_i such that $|y_i| < 150\sqrt{2}$ for all $t \in [0,1]$, $i = 1,2,3$.

Remark 4. The inability of Theorem 3.4 in [5] to apply to Example 1 is because the assumption of $P(t) \in C([0,1],\mathbf{R}))$ in Theorem 3.4 is necessary. In addition, by Remark 4.5 of [5], it is not difficult to find that Theorem 1 of [17] and [18] also cannot be applied to Example 1. Therefore, Corollary 4.1 unifies and sharply improves the prior results.

Now, we discuss the complete Sturm-Liouville equation

$$\begin{cases} -(b(t)y'(t))' + q(t)y'(t) + r(t)y(t) = \lambda g(t,y(t)), \text{ a.e. } t \in [0,1], \\ y'(1) = y'(0) = 0. \end{cases} \tag{4.3}$$

Assume that $q(t), r(t) \in L^\infty_{[0,1]}$, $b(t) \in C^1([0,1])$ satisfy $\text{meas}\{t \in [0,1] : q(t) \neq 0\} > 0$, $\text{ess}\inf_{[0,1]} b(t) > 0$ and $\text{ess}\inf_{[0,1]} r(t) > 0$. By a simple calculation, we know that $\ker(\Lambda_1) = \mathbf{R}$, $i^{B_1}_{\frac{\pi}{2},\frac{\pi}{2}}(0) = 0$ and $\nu^{B_1}_{\frac{\pi}{2},\frac{\pi}{2}}(0) \neq 0$. Therefore, from Theorem 1.2 we immediately obtain Corollary 4.2.

Corollary 4.2. *Assume that L^1-Carathéodory function $g : [0,1]\times\mathbf{R} \to \mathbf{R}$ satisfies* (G_1). *If there exist $c_0 > 0, b_0 > 0$ and $x_0 \in \mathbf{R}$ with $\frac{c_0}{k_0}\sqrt{\frac{2}{\underline{s}_0}} < |y_0| < \frac{b_0}{k_0\sqrt{2\bar{s}_0}}$, such that g satisfies* (G_2), *then the conclusion of Theorem* 1.2 *still holds true, where $\bar{s}_0 = \text{ess}\sup_{[0,1]}\{e^{-Q(t)}r(t)\}$, $\underline{s}_0 = \text{ess}\inf_{[0,1]}\{e^{-Q(t)}r(t)\}$, and $k_0 = \delta_0(\min\{1,\underline{s}_0\})^{\frac{-1}{2}}$.*

Example 2. Let $r(t) \in L^\infty_{[0,1]}$ with $\sup_{t\in[0,1]} r(t) = 2$, $\inf_{t\in[0,1]} r(t) = 1$, and

$$g(t,y) = \begin{cases} t, & y \le 1, \\ ty^{12}, & 1 < y \le 2, \\ t2^{12}, & 2 < y \le 2^{14}, \\ th(y), & 2^{14} < y, \end{cases}$$

where $h(y) \geq 0$ is an arbitrary L^1-Carathéodory function. We consider

$$\begin{cases} -y''(t) + y'(t) + r(t)y(t) = \lambda g(t, y(t)), \\ y'(1) = y'(0) = 0. \end{cases} \tag{4.4}$$

Clearly, $\underline{s}_0 = \frac{1}{e}, \bar{s}_0 = 2, k_0 = \sqrt{2e}$, and ($G_1$) holds. Noticing that $\ker(\Lambda_1) = \mathbf{R}$, put $c_0 = \frac{1}{2e}, b_0 = 2^{14}, y_0 = 2$, and we have $\frac{1}{2e} = \frac{c_0}{k_0}\sqrt{\frac{2}{\underline{s}_0}} < 2 = |y_0| < \frac{b_0}{k_0\sqrt{2\bar{s}_0}} = \frac{2^{13}}{\sqrt{2e}}$,

$$\begin{aligned} &\frac{\int_0^1 \sup_{|y|\leq c_0} \int_0^y g(t,\xi)d\xi dt}{c_0^2} = e \\ < \ &\frac{2051}{156e} = \frac{2\int_0^1 \int_0^{y_0} g(t,\xi)d\xi dt}{3\bar{s}_0 k_0^2 |y_0|^2} \end{aligned}$$

and

$$\begin{aligned} &\frac{\int_0^1 \sup_{|y|\leq b_0} \int_0^y g(t,\xi)d\xi dt}{b_0^2} = \frac{\frac{1}{2}\left(1 + \frac{2^{13}}{13} - \frac{1}{13} + 2^{12}(2^{14}-2)\right)}{2^{28}} \\ < \ &\frac{2^{27}}{2^{29}} = \frac{1}{4} < \frac{2051}{312e} = \frac{\int_0^1 \int_0^{y_0} g(t,\xi)d\xi dt}{3\bar{s}_0 k_0^2 \|y_0\|_{L^2}^2}, \end{aligned}$$

implying that (G_2) holds. Hence, by a simple calculation and Corollary 4.2, we know that for each $\mu \in (\frac{39e}{4102(e-2)}, \frac{1}{8e(e-2)})$, the problem (4.4) admits at least three positive solutions y_i such that $|y_i| < 2^{14}$ for $t \in [0, 1], i = 1, 2, 3$.

Remark 5. Since the proof in Theorem 1.1 in [7] requires $r(t) \in C([0, 1], \mathbf{R}))$, Theorem 1.1 in [7] cannot be used to study Example 2. Moreover, since $g(t, y)$ is not the autonomous case, we know that Corollary 4.3 in [5] also cannot be used to study Example 2. These show that Corollary 4.2 improves the prior results.

Next, we consider the following 1-dimensional mixed BVP:

$$\begin{cases} -y''(t) + p(t)y(t) = \lambda v(t, y), \text{ a.e. } t \in [0, 1], \\ y'(1) = y(0) = 0, \end{cases} \tag{4.5}$$

where $p(t) \in L^\infty_{[0,1]}$. By Remark 1 and a simple calculation, we know that $i^{I_1}_{0,\frac{\pi}{2}}(\frac{\pi^2}{4}) = 0$, $\nu^{I_1}_{0,\frac{\pi}{2}}(\frac{\pi^2}{4}) \neq 0$ and $\ker(\Lambda - \frac{\pi^2}{4}) = \{\mu \sin(\frac{\pi}{2}t) : \mu \in \mathbf{R}\}$. Set $A_0 = \frac{\pi^2}{4}, p(t) = -\frac{\pi^2}{4} + A(t)$. From Theorem 1.1, we have Corollary 4.3.

Corollary 4.3. *Assume that* $A(t) = p(t) + \frac{\pi^2}{4} > 0$ *for* $t \in [0, 1]$, $V(t, y) = \int_0^y v(t, \xi)d\xi$ *satisfies* (H_0), (H_1), *and there exist three constants* $c_0 > 0, b_0 > 0$ *and* $\mu_0 \in \mathbf{R}$ *with* $\frac{c_0}{k}\sqrt{\frac{2}{\underline{s}}} < \|y_0\|_{L_2} = \|\mu_0 \sin(\frac{\pi}{2}t)\|_{L_2} < \frac{b_0}{k\sqrt{2\bar{s}}}$, *such that* ($H_2$) *holds. Then, the conclusion of Theorem* 1.1 *is still valid.*

Remark 6. Since $-\frac{\pi^2}{4} < p(t) < 0$ and $p(t) \in L^\infty([0, 1], \mathbf{R})$ are allowed in Corollary 4.3, our result generalizes Theorem 3.1 in [2]. Here is an example of Corollary 4.3 to illustrate its validity.

Example 3. Consider problem (4.5). Let $p(t) = \begin{cases} -\frac{\pi^2}{8}, & t \in [0, \frac{1}{2}], \\ -\frac{\pi^2}{16}, & t \in (\frac{1}{2}, 1], \end{cases}$ and

$$V(t,y) = \begin{cases} ty, & y \le 1, \\ t[\frac{y^{11}}{11} + \frac{10}{11}], & 1 < y \le 2, \\ t[2^{10}y + \frac{10(1-2^{11})}{11}], & 2 < y \le 2^{20}, \\ t[\frac{2^{10}}{5\times 2^{80}}y^5 + \frac{2^{32}}{5} + \frac{10(1-2^{11})}{11}], & 2^{20} < y. \end{cases}$$

Clearly, $A(t) = \begin{cases} \frac{\pi^2}{8}, & t \in [0, \frac{1}{2}], \\ \frac{3\pi^2}{16}, & t \in (\frac{1}{2}, 1], \end{cases}$ and $\underline{s} = \frac{\pi^2}{8}, \bar{s} = \frac{3\pi^2}{16}$. For $y \in Z$, by

$$|y(t)| \le |\int_0^t y'(s)ds| \le \left(\int_0^1 |y'(s)|^2 ds\right)^{\frac{1}{2}},$$

we have

$$\max_{t\in[0,1]} |y(t)| \le \|y\|,$$

which implies that $\delta_0 = 1$ and $k = 1$. Obviously, $V(t, y)$ satisfies (H_0) and (H_1). By $\ker(\Lambda - A_0) = \{\mu \sin(\frac{\pi}{2}t) : \mu \in \mathbf{R}\}$, taking $\mu_0 = 2, c_0 = 1$ and $b_0 = 2^{20}$, we have $\frac{4}{\pi} = \frac{c_0}{k}\sqrt{\frac{2}{\underline{s}}} < \|\mu_0 \sin(\frac{\pi}{2}t)\|_{L^2} = \sqrt{2} < \frac{b_0}{k\sqrt{2\bar{s}}} = \frac{2^{22}}{\pi\sqrt{6}}$,

$$\begin{aligned} & \frac{\int_0^1 \sup_{|y|\le c_0} V(t,y)dt}{c_0^2} = \frac{1}{2} \\ < \ & \frac{240\sqrt{3}}{11\pi^2} < \frac{80}{81\pi^2} \times \frac{(\sqrt{3})^{11}}{11} \\ < \ & \frac{32}{9\pi^2}\int_{\frac{2}{3}}^1 t\left(\frac{(\sqrt{3})^{11}}{11} + \frac{10}{11}\right)dt \\ < \ & \frac{32}{9\pi^2}\left(\int_0^{\frac{1}{3}} 2t\sin(\frac{\pi}{2}t)dt + \int_{\frac{1}{3}}^1 t\left(\frac{(2\sin(\frac{\pi}{2}t))^{11}}{11} + \frac{10}{11}\right)dt\right) \\ = \ & \frac{2\int_0^1 V(t,y_0)dt}{3\bar{s}k^2\|\mu_0 \sin(\frac{\pi}{2}t)\|_{L^2}} \end{aligned}$$

and

$$\begin{aligned} & \frac{\int_0^1 \sup_{|y|\le b_0} V(t,y)dt}{b_0^2} = \frac{\left(2^{30} + \frac{10(1-2^{11})}{11}\right)\int_0^1 tdt}{2^{41}} \\ < \ & \frac{1}{2^{11}} < \frac{120\sqrt{3}}{11\pi^2} < \frac{\int_0^1 V(t,y_0)dt}{3\bar{s}k^2\|\mu_0 \sin(\frac{\pi}{2}t)\|_{L^2}} \end{aligned}$$

showing that (1.6) and (1.7) of (H_2) hold. After a simple calculation, from Corollary 4.3, we know that for each $\mu \in (\frac{9\pi^2}{32\int_0^1 V(t,2\sin(\frac{\pi}{2}t)dt}, 1)$, the problem (4.5) has at least three nonnegative solutions y_i such that $|y_i| < 2^{20}$ for all $t \in [0,1], i = 1,2,3$.

Finally, we discuss the complete Sturm-Liouville equation

$$\begin{cases} -y'' + y' + r(t)y(t) = \lambda g(t, y(t)), \text{ a.e. } t \in [0,1], \\ y(0) = y'(1) = 0, \end{cases} \tag{4.6}$$

where $r(t) \in L^\infty([0,1], \mathbf{R})$ with $\operatorname{ess\,inf}_{[0,1]} r(t) > 0$. By a simple calculation, we know that there exists $A_0 > 0$ such that $i^{B_1}_{0,\frac{\pi}{2}}(A_0) = 0, \nu^{B_1}_{0,\frac{\pi}{2}}(A_0) \neq 0$, where $B_1(t) = e^{-t}$. Thus, by Theorem 1.2, we obtain Corollary 4.4.

Corollary 4.4. *Assume that* $\underline{s}_0 = \operatorname{ess\,inf}_{[0,1]}\{r(t)e^{-Q(t)} + A_0\} > 0$, *and the function* $g : [0,1] \times \mathbf{R} \to \mathbf{R}$ *is* L^1*-Carathéodory and satisfies* (G_1). *If there exist* $c_0 > 0, b_0 > 0$ *and* $y_0 \in \ker(\Lambda_1 - \mathrm{A}_0)$ *with* $\frac{c_0}{k_0}\sqrt{\frac{2}{\underline{s}_0}} < \|y_0\|_{L^2} < \frac{b_0}{k_0\sqrt{2\bar{s}_0}}$, *such that g satisfies* (G_2), *then the conclusion of Theorem* 1.2 *still holds true, where* $\bar{s}_0 = \operatorname{ess\,sup}_{[0,1]}\{e^{-Q(t)}r(t) + A_0\}$.

Remark 7. Since the proof in Theorem 4.1 in [2] requires $r(t), g(t,y) \in C([0,1], \mathbf{R}))$, Corollary 4.4 is a new conclusion.

5. Conclusions

For the equilibrium problems of strings, columns, beams, etc. in mathematical physics, the nonnegative solutions of the Neumann BVP and mixed BVP have played an important role. For the Neumann BVP and mixed BVP, there are many works reported on the existence of three nonnegative solutions. However, the conditions needed to obtain the results are relatively strong. In this paper, we reconsider, in the framework of the Sturm-Liouville BVP for the non-autonomous Lagrangian systems, these problems. With the aid of index theory, using Bonanno-Candito's three critical point theory, we give some new criteria to have at least three nonnegative solutions for the Sturm-Liouville BVP. As some direct applications, we obtain the corresponding results for the Neumann BVP, mixed BVP and the complete Sturm-Liouville equations meeting Sturm-Liouville BVC. The conditions of the theorems in this paper are clearly weaker than those found in other papers. For more details, see Examples 1–3 and Remarks 3–7 of this paper.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (Grant No.12071219) and the Natural Science Foundation of the Jiangsu Higher Education Institution of China (No. 21KJB110026).

Conflict of Interest

All authors declare no conflicts of interest in this paper.

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