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Research article

Existence and concentration of solutions for a Kirchhoff-type problem with sublinear perturbation and steep potential well

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Abstract: In this paper, we consider the following nonlinear Kirchhoff-type problem with sublinear perturbation and steep potential well

$$\begin{cases} -\Big(a+b\displaystyle\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)\Delta u+\lambda V(x)u=f(x,u)+g(x)|u|^{q-2}u \ \text{ in } \mathbb{R}^3,\\ u\in H^1(\mathbb{R}^3),\end{cases}$$

where a and b are positive constants, $\lambda>0$ is a parameter, $1<q<2$, the potential $V\in C(\mathbb{R}^3,\mathbb{R})$ and $V^{-1}(0)$ has a nonempty interior. The functions f and g are assumed to obey a certain set of conditions. The existence of two nontrivial solutions are obtained by using variational methods. Furthermore, the concentration behavior of solutions as $\lambda\to\infty$ is also explored.

Keywords: Kirchhoff-type problem; steep potential well; variational methods
Mathematics Subject Classification: 35J20, 35J60, 35B40

1. Introduction

In the present paper, we investigate the existence and concentration of the solutions to a class of nonlinear Kirchhoff-type problem

$$\begin{cases} -\Big(a+b\displaystyle\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)\Delta u+\lambda V(x)u=f(x,u)+g(x)|u|^{q-2}u \ \text{ in } \mathbb{R}^3,\\ u\in H^1(\mathbb{R}^3),\end{cases} \tag{1.1}$$

where $a,b>0$ are constants, $\lambda>0$ is a real parameter, $1<q<2$, $f\in C(\mathbb{R}^3\times\mathbb{R},\mathbb{R})$ and the potential V satisfies the following conditions:

(V_1) $V(x) \in C(\mathbb{R}^3)$, $V(x) \geq 0$ on $\mathbb{R}^3$, and there exists $V_0 > 0$ such that the set $\{V < V_0\} = \{x \in \mathbb{R}^3 | V(x) < V_0\}$ is nonempty and

$$|\{V < V_0\}|^{\frac{2}{3}} < \mathfrak{C},$$

where $|\cdot|$ is the Lebesgue measure and $\mathfrak{C}$ is the best constant for the embedding of $\mathcal{D}^{1,2}(\mathbb{R}^3)$ in $L^6(\mathbb{R}^3)$;

(V_2) $\Omega_* = \mathrm{int} V^{-1}(0)$ is nonempty and has a smooth boundary with $\overline{\Omega}_* = V^{-1}(0)$;

(V_3) there exists an open set $\Omega \subset \Omega_*$ such that $V(x) \equiv 0$ for all $x \in \overline{\Omega}$.

These kinds of hypotheses were first put forward by Bartsch and Wang [1] in their research on the nonlinear Schrödinger equations, and it has attracted the attention of several researchers, e.g., see [2,3,9,12,13,16,27]. We note that the conditions (V_1) and (V_2) imply that λV represents a potential well with the bottom $V^{-1}(0)$, and that its depth is controlled by λ. In view of this, we can expect to find the solutions which are concentrated at the bottom of the potential V as the depth goes to infinity.

In recent years, the elliptic problem

$$\begin{cases} -\Big(a + b \displaystyle\int_\Omega |\nabla u|^2 dx\Big)\Delta u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases} \tag{1.2}$$

has been widely studied by many researchers, where $a, b > 0$, $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain and $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$. Problem (1.2) is often referred to as a nonlocal problem on account of the presence of the term $\big(\int_\Omega |\nabla u|^2 dx\big)\Delta u$, we do not know

$$\Big(\int_\Omega |\nabla u_n|^2 dx\Big)^2 \to \Big(\int_\Omega |\nabla u|^2 dx\Big)^2$$

from $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, which implies that (1.2) is no longer a pointwise identity. This phenomenon causes some mathematical difficulties, but at the same time, it makes the research of (1.2) particularly interesting. Moreover, problem (1.2) has a profound and interesting physical context, which is related to the stationary analogue of the equation

$$u_{tt} - \Big(a + b \int_\Omega |\nabla u|^2 dx\Big)\Delta u = f(x,u),$$

where u denotes the displacement, $f(x,u)$ is the external force, a is the initial tension and b is related to the inherent characteristics of the string (such as the Young's modulus). This hyperbolic equation generalizes the following equation:

$$\rho \frac{\partial^2 u}{\partial t^2} - \Big(\frac{p_0}{h} + \frac{E}{2L}\int_0^L \Big|\frac{\partial^2 u}{\partial x^2}\Big| dx\Big)\frac{\partial^2 u}{\partial x^2} = 0,$$

which is proposed by Kirchhoff in [4] as an extension of the classical D'Alembert's wave equations for free vibrations of elastic strings. This model takes into account the changes in length of the string produced by transverse vibrations. For more mathematical and physical background on the Kirchhoff-type problem, we refer the readers to [5,6] and the references therein.

More recently, many researchers have been devoted to investigations into the Kirchhoff-type problem defined in the whole space $\mathbb{R}^3$, i.e., the following problem:

$$\begin{cases} -\Big(a+b\displaystyle\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)\Delta u+V(x)u=f(x,u)\ \text{ in }\mathbb{R}^3,\\ u\in H^1(\mathbb{R}^3), \end{cases} \tag{1.3}$$

where $V\in C(\mathbb{R}^3,\mathbb{R})$ is a potential function and $f\in C(\mathbb{R}^3\times\mathbb{R},\mathbb{R})$. There have been a lot of studies on the solutions of (1.3) under many different assumptions. See, for example, in [7], Li et al. obtained a positive solution for (1.3) by using the cut-off technique and monotone method. In [8], Li and Ye proved the existence of ground state solutions in the case of $f(x,u)=|u|^{p-1}u$ and $2<p\leq 3$. Later Ye [10] obtained a positive high-energy solution with superlinear nonlinearities by using the Nehari-Pohozaev manifold. For more results about Kirchhoff-type problems, readers can see [11, 14, 15, 17] and the references therein.

Very recently, Du et al. [18] considered the following Kirchhoff-type problem:

$$\begin{cases} -\Big(a+b\displaystyle\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)\Delta u+\lambda V(x)u=f(x,u)\ \text{ in }\mathbb{R}^3,\\ u\in H^1(\mathbb{R}^3), \end{cases} \tag{1.4}$$

where $a,b,\lambda>0$ are parameters and the potential V satisfies the conditions (V_1) and (V_2). They showed that the existence and asymptotic behavior of ground state solutions when $f(x,u)$ behaved similar to $|u|^{p-2}u$ with $4<p<6$. In [19], the authors obtained the existence of nontrivial solutions for the case of $f(x,u)=|u|^{p-2}u$ with $4\leq p<6$. After that, Zhang and Du [20] obtained the positive solutions for b small and λ large by combining the truncation technique and the parameter-dependent compactness lemma when $f(x,u)=|u|^{p-2}u$ and $2<p<4$. Furthermore, Sun and Wu [21] proved the existence of generate multiple solutions when $f(x,u)$ was replaced by $f(x)|u|^{p-2}u$. Zhou and Zhu [22] got the existence and asymptotic behavior of ground-state solutions to (1.4) with general convolution nonlinearity. In particular, Choudhuri [23] obtained the existence of infinitely many solutions to a p-Kirchhoff-type problem without the Ambrosetti-Rabinowitz condition.

Motivated by the works mentioned above, the purpose of this paper was to study the existence and concentration of solutions for problem (1.1) with a sublinear perturbation term and steep potential well, which is different from the papers above. In addition, as is well known, this type of problem is characterized by its lack of compactness due to the embedding fails, which prevents us from using the variational methods in a standard way. We will construct some inequalities in order to apply them to recover compactness as λ is considered to be large enough.

Before stating our results we need to introduce some notations and conditions.

Throughout this paper, we denote by $|\cdot|_p$ the usual norm of the space $L^p(\mathbb{R}^3)$, $1\leq p\leq\infty$, $C_1,C_2,...$ stand for different positive constants and $g^{\pm}=\sup\{\pm g,0\}$. For any $x\in\mathbb{R}^3$ and $r>0$, $B_r(x):=\{y\in\mathbb{R}^3:|y-x|<r\}$. Moreover, if we take a subsequence of a sequence $\{u_n\}$, we shall denote it again as $\{u_n\}$. We use $o(1)$ to denote any quantity which tends to zero when $n\to\infty$.

Considering that $F(x,s)=\int_0^s f(x,\tau)d\tau$, which is the primitive of f, we suppose the following hypotheses:

(f_1) $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, where $f(x,s)s > 0$ for all $s \neq 0$ and $f(x,s) = 0$ for all $s \leq 0$, and it satisfies

$$\lim_{s \to 0^+} \frac{f(x,s)}{s} = 0 \text{ and } \lim_{s \to +\infty} \frac{f(x,s)}{s} = +\infty$$

uniformly for $x \in \mathbb{R}^3$;

(f_2) there exist $\gamma_1 > 0$ and $p \in (2,6)$ such that

$$f(x,s) \leq \gamma_1(1 + s^{p-1}) \text{ for all } (x,s) \in \mathbb{R}^3 \times \mathbb{R};$$

(f_3) there exists $\gamma_2 \in \left(0, \frac{\mathcal{S}\min\{a,1\}}{4\mathfrak{C}_2^2(\mathcal{S}+1)}\right)$ such that

$$F(x,s) - \frac{1}{4} f(x,s)s \leq \gamma_2 s^2 \text{ for all } (x,s) \in \mathbb{R}^3 \times \mathbb{R},$$

where $\mathcal{S}$ and $\mathfrak{C}_2$ are positive constants (see Remark 2.2).

Remark 1.1. Obviously, (f_1) and (f_2) imply that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$0 \leq F(x,s) \leq \varepsilon|s|^2 + C_\varepsilon |s|^p, \quad f(x,s)s \leq \frac{\varepsilon}{2}|s|^2 + \frac{C_\varepsilon}{p}|s|^p \tag{1.5}$$

for all $(x,s) \in \mathbb{R}^3 \times \mathbb{R}$. An example of nonlinearity in $f(x,s)$ satisfying the hypotheses (f_1)–(f_3) is given by

$$f(x,s) = \begin{cases} h_1(x)s^3 + h_2(x)[s - \ln(s+1)], & \text{if } (x,s) \in \mathbb{R}^3 \times (0, +\infty), \\ 0, & \text{if } (x,s) \in \mathbb{R}^3 \times (-\infty, 0], \end{cases}$$

where for each $i = \{1,2\}$, h_i is some positive continuous bounded function.

The sublinear perturbation g is given by the following condition:

(g_1) $g \in L^{\frac{2}{2-q}}(\mathbb{R}^3)$ and there exists an open set $\Omega_g \subset \Omega$ such that $g > 0$ on Ω_g. In addition, there exists $\gamma_3 > 0$ independent of ε such that $\gamma_3 > C_\varepsilon$ for $d(p,q) := \frac{(p-2)^{p-2}(2-q)^{2-q}}{(p-q)^{p-q}}$ and $K := \frac{(\min\{a,1\})^{p-q}}{2^{p-q}\left[\gamma_3 \mathfrak{C}_p^p \left(1+\frac{1}{\mathcal{S}}\right)^{\frac{p}{2}}\right]^{2-q}}$,

with g satisfying

$$|g^+|_{\frac{2}{2-q}} < \frac{q[d(p,q)K]^{\frac{1}{p-2}}}{\mathfrak{C}_2^q \left(1 + \frac{1}{\mathcal{S}}\right)^{\frac{q}{2}}},$$

where C_ε and $\mathfrak{C}_p$ are given by Remarks 1.1 and 2.2 respectively.

Now we may state our main results as follows:

Theorem 1.2. Suppose that (V_1)–(V_3), (f_1)–(f_3) and (g_1) hold. Then, there exist $\widehat{b} > 0$ and $\Lambda_* > 0$ such that $b \in (0, \widehat{b})$, problem (1.1) has at least two solutions for all $\lambda > \Lambda_*$.

On the concentration of nontrivial solutions we have the following result.

Theorem 1.3. Let $u_\lambda^{(1)}$ and $u_\lambda^{(2)}$ be two solutions of problem (1.1) given by Theorem 1.2 and $\Omega = \Omega_*$. Then, $u_\lambda^{(1)} \to u_0^{(1)}$ and $u_\lambda^{(2)} \to u_0^{(2)}$ in E_λ as $\lambda \to \infty$, where $u_0^{(1)} \neq u_0^{(2)} \in H_0^1(\Omega)$ are two nontrivial solutions of

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u) + g(x)|u|^{q-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.6}$$

The outline of this paper is as follows. In Section 2, some definitions and preliminary results are introduced. In Sections 3 and 4, we prove Theorems 1.2 and 1.3.

2. Preliminaries

In this section, we will establish the variational framework for problem (1.1) and prove some useful lemmas. We recall the definition of the Hilbert space $H^1(\mathbb{R}^3)$ endowed with the standard scalar product and norm

$$\langle u, v\rangle_{H^1} = \int_{\mathbb{R}^3} (\nabla u \nabla v + uv)dx, \quad \|u\|_{H^1} = \langle u, u\rangle_{H^1}^{\frac{1}{2}}.$$

$\mathfrak{C}$ denotes the best Sobolev constant

$$\mathfrak{C} := \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 dx}{(\int_{\mathbb{R}^3} |u|^6 dx)^{\frac{1}{3}}},$$

where $\mathcal{D}^{1,2}(\mathbb{R}^3) := \{u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}$ is the Sobolev space with the norm

$$\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)} = \Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)^{\frac{1}{2}}.$$

Let

$$E = \Big\{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 dx < +\infty\Big\}$$

be equipped with the inner product and norm

$$\langle u, v\rangle = \int_{\mathbb{R}^3} (\nabla u \nabla v + V(x)uv)dx, \ \ \|u\| = \langle u, u\rangle^{\frac{1}{2}}.$$

For $\lambda > 0$, we also need the following inner product and norm

$$\langle u, v\rangle_\lambda = \int_{\mathbb{R}^3} (\nabla u \nabla v + \lambda V(x)uv)dx, \quad \|u\|_\lambda = \langle u, u\rangle_\lambda^{\frac{1}{2}}.$$

If $\lambda \geq 1$, then $\|u\| \leq \|u\|_\lambda$. Set $E_\lambda = (E, \|u\|_\lambda)$, we have the following results.

Lemma 2.1. Suppose that $V(x)$ satisfies (V_1). Then, E is continuously embedded in $H^1(\mathbb{R}^3)$.

Proof. From the condition (V_1) and the Sobolev inequality, we have

$$\begin{aligned}
\int_{\mathbb{R}^3} u^2 dx &= \int_{\{V<V_0\}} u^2 dx + \int_{\{V\geq V_0\}} u^2 dx \\
&\leq |\{V < V_0\}|^{\frac{2}{3}}\Big(\int_{\mathbb{R}^3} |u|^6 dx\Big)^{\frac{1}{3}} + \frac{1}{V_0}\int_{\mathbb{R}^3} V(x)u^2 dx \\
&\leq \frac{|\{V < V_0\}|^{\frac{2}{3}}}{\mathfrak{C}}\Big(\int_{\mathbb{R}^3} u^2 dx + \int_{\mathbb{R}^3} |\nabla u|^2 dx\Big) + \frac{1}{V_0}\int_{\mathbb{R}^3} V(x)u^2 dx,
\end{aligned}$$

which implies that

$$\begin{aligned}
\int_{\mathbb{R}^3} u^2 dx &\leq \frac{\mathfrak{C}}{\mathfrak{C} - |\{V < V_0\}|^{\frac{2}{3}}}\Big(\frac{|\{V < V_0\}|^{\frac{2}{3}}}{\mathfrak{C}} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{V_0}\int_{\mathbb{R}^3} V(x)u^2 dx\Big) \\
&\leq \frac{\max\{|\{V < V_0\}|^{\frac{2}{3}}, \frac{\mathfrak{C}}{V_0}\}}{\mathfrak{C} - |\{V < V_0\}|^{\frac{2}{3}}} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)dx. \qquad (2.1)
\end{aligned}$$

This show that

$$\|u\|_{H^1}^2 \le \Big(1+\frac{\max\{|\{V<V_0\}|^{\frac{2}{3}},\frac{\mathfrak{C}}{V_0}\}}{\mathfrak{C}-|\{V<V_0\}|^{\frac{2}{3}}}\Big)\|u\|^2, \tag{2.2}$$

which implies that E is continuously embedded in $H^1(\mathbb{R}^3)$. This ends the proof. □

Remark 2.2. (i) Let

$$\mathcal{S}=\frac{\mathfrak{C}-|\{V<V_0\}|^{\frac{2}{3}}}{|\{V<V_0\}|^{\frac{2}{3}}} \quad \text{and} \quad \Lambda=\frac{\mathfrak{C}}{V_0|\{V<V_0\}|^{\frac{2}{3}}}.$$

For all $\lambda \ge \Lambda$, by using the same conditions and techniques in (2.1) and (2.2), we obtain

$$\int_{\mathbb{R}^3}|u|^2dx \le \frac{1}{\mathcal{S}}\|u\|_\lambda^2 \quad \text{and} \quad \|u\|_{H^1}^2 \le \Big(1+\frac{1}{\mathcal{S}}\Big)\|u\|_\lambda^2.$$

(ii) The embedding $E_\lambda \hookrightarrow L^p(\mathbb{R}^3)$ is continuous for $p\in[2,6]$, and $E_\lambda \hookrightarrow L^p_{loc}(\mathbb{R}^3)$ is compact for $p\in[2,6)$, namely, for all $u\in E_\lambda$ and $\lambda\ge\Lambda$, there are constants $\mathfrak{C}_p$ such that

$$|u|_p \le \mathfrak{C}_p\|u\|_{H^1} \le \mathfrak{C}_p\Big(1+\frac{1}{\mathcal{S}}\Big)^{\frac{1}{2}}\|u\|_\lambda. \tag{2.3}$$

Considering problem (1.1), we have the energy functional $I_\lambda : E_\lambda \to \mathbb{R}$ given by

$$I_\lambda(u)=\frac{1}{2}\Big(a\int_{\mathbb{R}^3}|\nabla u|^2dx+\int_{\mathbb{R}^3}\lambda Vu^2dx\Big)+\frac{b}{4}\Big(\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)^2-\int_{\mathbb{R}^3}F(x,u)dx-\frac{1}{q}\int_{\mathbb{R}^3}g(x)|u|^qdx.$$

Moreover, for all $u,v\in E_\lambda$, it is easy to see that $I_\lambda\in C^1(E_\lambda,\mathbb{R})$ and

$$\langle I_\lambda'(u),v\rangle=\Big(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)\int_{\mathbb{R}^3}\nabla u\nabla vdx+\int_{\mathbb{R}^3}\lambda Vuvdx-\int_{\mathbb{R}^3}f(x,u)vdx-\int_{\mathbb{R}^3}g(x)|u|^{q-2}uvdx.$$

Hence, if $u\in E_\lambda$ is a critical point of I_λ, then u is a solution of problem (1.1).

Next, we give the following variant of the mountain pass theorem (see [24]) where we consider the Cerami condition. Let X be a Banach space and $I\in C^1(X,\mathbb{R})$. We recall that a sequence $\{u_n\}\subset X$ is said to be a Cerami sequence (in short $(Ce)_c$ sequence) at the level $c\in\mathbb{R}$ if $I(u_n)\to c$ and $(1+\|u_n\|_X)\|I'(u_n)\|_{X^*}\to 0$, where X^* denotes the dual space of X.

Lemma 2.3. Let X be a real Banach space. Suppose that $I\in C^1(X,\mathbb{R})$, $I(0)=0$ and

(A_1) there exist $\alpha,\rho>0$ such that $I(u)\ge\alpha$ provided $\|u\|_X=\rho$;

(A_2) there exists $e\in X$ with $\|e\|_X>\rho$ such that $I(e)<0$.

Define

$$c:=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}I(\gamma(t)),$$

where

$$\Gamma=\{\gamma\in C([0,1],X):\gamma(0)=0,\gamma(1)=e\}.$$

Then, there exists a sequence $\{u_n\}\subset X$ such that

$$I(u_n)\to c\ge\alpha \quad \text{and} \quad (1+\|u_n\|_X)\|I'(u_n)\|_{X^*}\to 0.$$

3. Proof of Theorem 1.2

In the next lemma we check that I_λ satisfies the mountain pass geometry introduced in Lemma 2.3.
Lemma 3.1. Suppose that (V_1), (V_2), (f_1) and (f_2) are satisfied. In addition, there exists $\widehat{b} > 0$ such that $b \in (0, \widehat{b})$. Then, I_λ satisfies Lemma 2.3 of (A_1) and (A_2) for all $\lambda \geq \Lambda$.
Proof. We can use the condition (g_1), (1.5), (2.3) and the Hölder inequality to obtain

$$\begin{aligned} I_\lambda(u) =& \frac{1}{2}\Big(a \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} \lambda V u^2 dx\Big) + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)^2 - \int_{\mathbb{R}^3} F(x,u)dx - \frac{1}{q}\int_{\mathbb{R}^3} g(x)|u|^q dx \\ \geq& \frac{1}{2}\Big(a \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} \lambda V u^2 dx\Big) - \varepsilon \int_{\mathbb{R}^3} u^2 dx - C_\varepsilon \int_{\mathbb{R}^3} |u|^p dx - \frac{1}{q}\int_{\mathbb{R}^3} g(x)|u|^q dx \\ \geq& \frac{\min\{a,1\}}{2}\|u\|_\lambda^2 - \varepsilon \mathfrak{C}_2^2\Big(1+\frac{1}{\mathcal{S}}\Big)\|u\|_\lambda^2 - C_\varepsilon \mathfrak{C}_p^p\Big(1+\frac{1}{\mathcal{S}}\Big)^{\frac{p}{2}}\|u\|_\lambda^p - \frac{|g^+|_{\frac{2}{2-q}}}{q}\Big(\int_{\mathbb{R}^3}|u|^2 dx\Big)^{\frac{q}{2}} \\ \geq& \Big[\frac{\min\{a,1\}}{2} - \varepsilon \mathfrak{C}_2^2\Big(1+\frac{1}{\mathcal{S}}\Big)\Big]\|u\|_\lambda^2 - \gamma_3 \mathfrak{C}_p^p\Big(1+\frac{1}{\mathcal{S}}\Big)^{\frac{p}{2}}\|u\|_\lambda^p - \frac{\mathfrak{C}_2^q\Big(1+\frac{1}{\mathcal{S}}\Big)^{\frac{q}{2}}|g^+|_{\frac{2}{2-q}}}{q}\|u\|_\lambda^q \\ :=& \Big[\frac{\min\{a,1\}}{2} - \varepsilon \mathfrak{C}_2^2\Big(1+\frac{1}{\mathcal{S}}\Big)\Big](\|u\|_\lambda^2 - P\|u\|_\lambda^p - Q\|u\|_\lambda^q). \end{aligned}$$

Let $B = \frac{\mathfrak{C}_2^q\left(1+\frac{1}{\mathcal{S}}\right)^{\frac{q}{2}}|g^+|_{\frac{2}{2-q}}}{q}$ and $D = \gamma_3 \mathfrak{C}_p^p\left(1+\frac{1}{\mathcal{S}}\right)^{\frac{p}{2}}$. Again from the condition (g_1), we see that $\frac{\min\{a,1\}}{2} > \left(\frac{B^{p-2}D^{2-q}}{d(p,q)}\right)^{\frac{1}{p-q}}$, then we can take $\varepsilon \in \left(0, \frac{\mathcal{S}\min\{a,1\}}{2\mathfrak{C}_2^2(\mathcal{S}+1)} - \frac{\mathcal{S}}{\mathfrak{C}_2^2(\mathcal{S}+1)}\left(\frac{B^{p-2}D^{2-q}}{d(p,q)}\right)^{\frac{1}{p-q}}\right)$, and by Lemma 3.1 in [25], it is easy to see that there is $t_P > 0$ such that, for $\rho := t_P = \|u\|_\lambda$,

$$I_\lambda(u) \geq \alpha := \Big[\frac{\min\{a,1\}}{2} - \varepsilon \mathfrak{C}_2^2\Big(1+\frac{1}{\mathcal{S}}\Big)\Big]\Psi(t_P) > 0,$$

where $\Psi(t) = t^2 - Pt^p - Qt^q, P, Q > 0$, which finishes the proof (A_1).

In order to prove (A_2), notice from the assumption (f_1) that

$$\lim_{u\to+\infty} \frac{F(x,u)}{u^2} = +\infty.$$

So, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $F(x,u) > \frac{u^2}{\varepsilon}$ for all $u > \delta$. Let $R_\varepsilon = \frac{\delta^2}{\varepsilon}$, then $F(x,u) > \frac{u^2}{\varepsilon} - \frac{\delta^2}{\varepsilon}$. Next, let $0 < \varphi \in C_0^\infty(\mathbb{R}^3)$ be fixed, we have

$$\int_{\mathbb{R}^3} \frac{F(x,t\varphi)}{t^2} dx \geq \frac{1}{\varepsilon}\int_{supp(\varphi)} |\varphi(x)|^2 dx - \int_{supp(\varphi)} \frac{\delta^2}{\varepsilon t^2} dx.$$

This implies

$$\lim_{t\to+\infty} \int_{\mathbb{R}^3} \frac{F(x,t\varphi)}{t^2} dx \geq \frac{1}{\varepsilon}\int_{supp(\varphi)} |\varphi(x)|^2 dx, \tag{3.1}$$

since ε is arbitrary, by (3.1) we obtain

$$\lim_{t\to+\infty} \int_{\mathbb{R}^3} \frac{F(x,t\varphi)}{t^2} dx = +\infty.$$

Thus, using Fatou's Lemma, we have that

$$\limsup_{t\to+\infty}\frac{I_{\lambda,0}(t\varphi)}{t^2}\le\frac{\max\{a,1\}}{2}\|\varphi\|_\lambda^2-\int_{\mathbb{R}^3}\liminf_{t\to+\infty}\Big[\frac{F(x,t\varphi)}{t^2}+\frac{g(x)|\varphi|^q}{qt^{2-q}}\Big]dx<0,$$

where $I_{\lambda,0}(u)=I_\lambda(u)$ for $b=0$. Therefore, if $\lim_{t\to+\infty}I_{\lambda,0}(t\varphi)=-\infty$, then there exists $e=t\varphi\in E_\lambda$ with $\|e\|_\lambda>\rho$ such that $I_{\lambda,0}(e)<0$. Since $\lim_{b\to0^+}I_\lambda(e)=I_{\lambda,0}(e)$, we see that there exists $\widehat{b}>0$ such that $I_\lambda(e)<0$ for all $b\in(0,\widehat{b})$. This ends the proof. □

Let $I_\lambda(u)|_{H_0^1(\Omega)}$ be a restriction of I_λ on $H_0^1(\Omega)$, that is

$$I_\lambda(u)|_{H_0^1(\Omega)}=\frac{a}{2}\int_\Omega|\nabla u|^2dx+\frac{b}{4}\Big(\int_\Omega|\nabla u|^2dx\Big)^2-\int_\Omega F(x,u)dx-\frac{1}{q}\int_\Omega g(x)|u|^qdx \tag{3.2}$$

for all $u\in H_0^1(\Omega)$. Define

$$c_\lambda=\inf_{\gamma\in\Gamma_\lambda}\max_{t\in[0,1]}I_\lambda(\gamma(t))\quad\text{and}\quad\widetilde{c}=\inf_{\gamma\in\widetilde{\Gamma}}\max_{t\in[0,1]}I_\lambda|_{H_0^1(\Omega)}(\gamma(t)),$$

where

$$\Gamma_\lambda=\{\gamma\in C([0,1],E_\lambda):\gamma(0)=0,I_\lambda(\gamma(1))<0\}\ \text{ and}$$

$$\widetilde{\Gamma}=\{\gamma\in C([0,1],H_0^1(\Omega)):\gamma(0)=0,I_\lambda|_{H_0^1(\Omega)}(\gamma(1))<0\}.$$

Indeed, it is easily seen that $\widetilde{c}$ is independent of λ. Furthermore, if the conditions (f_1) and (f_2) hold, then by the proofs of Lemma 3.1, we can conclude that $I_\lambda|_{H_0^1(\Omega)}(u)$ satisfies the hypothesis of the mountain pass theorem as in Lemma 2.3. Since $H_0^1(\Omega)\subset E_\lambda$ for all $\lambda>0$, one has $0<\alpha\le c_\lambda\le\widetilde{c}$ for all $\lambda\ge\Lambda$. Now, we can take $M>\widetilde{c}$. Thus

$$0<\alpha\le c_\lambda\le\widetilde{c}<M\ \text{ for all }\ \lambda\ge\Lambda. \tag{3.3}$$

In view of Lemmas 2.3 and 3.1 there exists $\{u_n\}\subset E_\lambda$ such that

$$I_\lambda(u_n)\to c_\lambda\ \text{ and }\ (1+\|u_n\|_\lambda)\|I'_\lambda(u_n)\|_{E_\lambda^*}\to0, \tag{3.4}$$

where c_λ is given by (3.3).

Lemma 3.2. Suppose that (V_1), (V_2) and (f_1)–(f_3) are satisfied. Then, the sequence $\{u_n\}$ defined by (3.4) is bounded in E_λ for all $\lambda\ge\Lambda$.

Proof. By using the condition (f_3) and (2.3), we have

$$\begin{aligned}c_\lambda+o(1)=&I_\lambda(u_n)-\frac{1}{4}\langle I'_\lambda(u_n),u_n\rangle\\=&\frac{1}{4}\Big(a\int_{\mathbb{R}^3}|\nabla u_n|^2dx+\int_{\mathbb{R}^3}\lambda Vu_n^2dx\Big)-\int_{\mathbb{R}^3}\Big[F(x,u_n)-\frac{1}{4}f(x,u_n)u_n\Big]dx\\&-\Big(\frac{1}{q}-\frac{1}{4}\Big)\int_{\mathbb{R}^3}g(x)|u_n|^qdx\end{aligned}$$

$$\geq \frac{\min\{a,1\}}{4}\|u_n\|_\lambda^2 - \gamma_2\int_{\mathbb{R}^3}|u_n|^2dx - \frac{(4-q)|g^+|_{\frac{2}{2-q}}}{4q}\Big(\int_{\mathbb{R}^3}|u_n|^2dx\Big)^{\frac{q}{2}}$$

$$\geq \Big[\frac{\min\{a,1\}}{4} - \gamma_2\mathfrak{C}_2^2\Big(1+\frac{1}{S}\Big)\Big]\|u_n\|_\lambda^2 - \frac{\mathfrak{C}_2^q(4-q)\Big(1+\frac{1}{S}\Big)^{\frac{q}{2}}|g^+|_{\frac{2}{2-q}}}{4q}\|u_n\|_\lambda^q,$$

we can deduce that $\{u_n\}$ is bounded in E_λ for n large enough. This ends the proof. □

We are now ready to give the following compactness conditions for I_λ.

Lemma 3.3. Suppose that (V_1), (V_2), (g_1), (f_1) and (f_2) are satisfied. Then, there exist positive constants $\beta, C > 0$ such that I_λ satisfies the $(Ce)_c$ condition in E_λ for all $\lambda \geq \max\Big\{\Lambda, \frac{8C^2}{\beta V_0}\Big\}$.

Proof. Let $\{u_n\}$ be a $(Ce)_c$ sequence. By Lemma 3.2, we see that, up to a subsequence, $\{u_n\}$ is bounded in E_λ. Passing to a subsequence again if necessary, we may assume that there exist $u \in E_\lambda$ and $A \in \mathbb{R}$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } E_\lambda, \\ u_n \to u & \text{strongly in } L^p_{loc}(\mathbb{R}^3), \text{ for all } p \in [2,6), \\ u_n \to u & \text{a.e. in } \mathbb{R}^3, \end{cases} \tag{3.5}$$

and

$$A^2 = \lim_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_n|^2dx \geq \int_{\mathbb{R}^3}|\nabla u|^2dx. \tag{3.6}$$

Set $v_n = u_n - u$. By the condition (g_1), (3.5) and the Brezis-Lieb Lemma [26], one has

$$\int_{\mathbb{R}^3}|\nabla v_n|^2dx = \int_{\mathbb{R}^3}|\nabla u_n|^2dx - \int_{\mathbb{R}^3}|\nabla u|^2dx + o(1), \tag{3.7}$$

$$A^2 + o(1) = \int_{\mathbb{R}^3}|\nabla u_n|^2dx = \int_{\mathbb{R}^3}|\nabla v_n|^2dx + \int_{\mathbb{R}^3}|\nabla u|^2dx + o(1) \tag{3.8}$$

and

$$o(1) = \int_{\mathbb{R}^3}g(x)|v_n|^qdx = \int_{\mathbb{R}^3}g(x)|u_n|^qdx - \int_{\mathbb{R}^3}g(x)|u|^qdx + o(1). \tag{3.9}$$

Define

$$\Phi_\lambda(u) = \frac{1}{2}\Big(a\int_{\mathbb{R}^3}|\nabla u|^2dx + \int_{\mathbb{R}^3}\lambda V u^2dx\Big) + \frac{b}{2}A^2\int_{\mathbb{R}^3}|\nabla u|^2dx - \int_{\mathbb{R}^3}F(x,u)dx - \frac{1}{q}\int_{\mathbb{R}^3}g(x)|u|^qdx.$$

Now, we claim that $\Phi_\lambda'(u) = 0$. Indeed, from $I_\lambda'(u_n) \to 0$, we have

$$(a+bA^2)\int_{\mathbb{R}^3}\nabla u\nabla v dx + \int_{\mathbb{R}^3}\lambda V uv dx - \int_{\mathbb{R}^3}f(x,u)v dx - \int_{\mathbb{R}^3}g(x)|u|^{q-2}uv dx = 0$$

for any $v \in E_\lambda$, which implies that $\Phi_\lambda'(u) = 0$. Next we prove that $u_n \to u$ strongly in E_λ. Then, $\{v_n\}$ satisfies exactly one of the following conditions:

(B_1) $\displaystyle\lim_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_r(y)}|v_n|^2dx > 0;$

(B_2) $\displaystyle\lim_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_r(y)}|v_n|^2dx = 0.$

Suppose that the case (B_1) holds. Then, there exists a constant $\beta > 0$ such that

$$\limsup_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_r(y)} |v_n|^2 dx = \beta > 0. \tag{3.10}$$

In view of the weakly lower semi-continuity of the norm, we have

$$\|u\|_\lambda \leq \liminf_{n\to\infty} \|u_n\|_\lambda. \tag{3.11}$$

Since the sequence $\{u_n\}$ is bounded in E_λ, there exists a positive constant C (independent of λ) such that

$$\limsup_{n\to\infty} \|u_n\|_\lambda \leq C. \tag{3.12}$$

Then, by (3.11) and (3.12), one has

$$\limsup_{n\to\infty} \|v_n\|_\lambda = \limsup_{n\to\infty} \|u_n - u\|_\lambda \leq 2C. \tag{3.13}$$

Define

$$A_R := \{x \in \mathbb{R}^3 \setminus B_R : V(x) \geq V_0\} \quad \text{and} \quad F_R := \{x \in \mathbb{R}^3 \setminus B_R : V(x) < V_0\}.$$

Then, we can take $\lambda \geq \frac{8C^2}{\beta V_0}$, one gets

$$\int_{A_R} v_n^2 dx \leq \frac{1}{\lambda V_0} \int_{A_R} \lambda V v_n^2 dx \leq \frac{\|v_n\|_\lambda^2}{\lambda V_0}. \tag{3.14}$$

Applying (3.13) and (3.14) leads to

$$\limsup_{n\to\infty} \int_{A_R} v_n^2 dx = \limsup_{n\to\infty} \frac{\|v_n\|_\lambda^2}{\lambda V_0} \leq \frac{\beta}{2}.$$

By using the condition (V_1), $|F_R| \to 0$ as $R \to \infty$. Combining (2.3) and the Hölder inequality, we get

$$\int_{F_R} v_n^2 dx \leq |F_R|^{\frac{p-2}{p}} \Big(\int_{F_R} v_n^p dx \Big)^{\frac{2}{p}} \leq \mathfrak{C}_p^2 \Big(1 + \frac{1}{\mathcal{S}}\Big) \|v_n\|_\lambda^2 |F_R|^{\frac{p-2}{p}},$$

which implies that

$$\begin{aligned}\limsup_{n\to\infty} \int_{F_R} v_n^2 dx &\leq \mathfrak{C}_p^2 \Big(1 + \frac{1}{\mathcal{S}}\Big) \limsup_{n\to\infty} \|v_n\|_\lambda^2 |F_R|^{\frac{p-2}{p}} \\ &\leq 4C^2 \mathfrak{C}_p^2 \Big(1 + \frac{1}{\mathcal{S}}\Big) |F_R|^{\frac{p-2}{p}} \to 0\end{aligned}$$

for any $p \in [2.6)$ as $R \to \infty$. From $R \to \infty$ and $v_n \to 0$ in $L^p_{loc}(\mathbb{R}^3)$ with $p \in [2, 6)$, we have

$$\beta = \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_r(y)} |v_n|^2 dx \leq \limsup_{n\to\infty} \int_{\mathbb{R}^3} |v_n|^2 dx$$

$$
\begin{aligned}
&= \limsup_{n\to\infty}\Big(\int_{B_R}|v_n|^2dx + \int_{B_R^c}|v_n|^2dx\Big)\\
&= \limsup_{n\to\infty}\Big(\int_{A_R}|v_n|^2dx + \int_{F_R}|v_n|^2dx\Big)\\
&\le \frac{\beta}{2},
\end{aligned}
$$

which contradicts (3.10), where $B_R^c := \{x \in \mathbb{R}^3 : |x| \ge R\}$. Thus, if the case (B_2) holds, by the Lions Lemma [26], $v_n \to 0$ in $L^p(\mathbb{R}^3)$ for any $p \in (2, 6)$. Then, using (1.5) and the Lebesgue dominated convergence theorem gives

$$
\int_{\mathbb{R}^3} f(x, u_n)u_n dx = \int_{\mathbb{R}^3} f(x, u)u dx + o(1). \tag{3.15}
$$

It follows from (3.6)–(3.9) and (3.15) that

$$
\begin{aligned}
o(1) =&\langle I_\lambda'(u_n), u_n\rangle\\
=&a\int_{\mathbb{R}^3}|\nabla u_n|^2dx + \int_{\mathbb{R}^3}\lambda V u_n^2dx + b\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2dx\Big)^2\\
&-\int_{\mathbb{R}^3} f(x,u_n)u_ndx - \int_{\mathbb{R}^3} g(x)|u_n|^qdx - \langle \Phi_\lambda'(u), u\rangle\\
\ge& \min\{a,1\}\|v_n\|_\lambda^2 + bA^4 - bA^2\int_{\mathbb{R}^3}|\nabla u|^2dx + o(1)\\
\ge& \min\{a,1\}\|v_n\|_\lambda^2 + o(1),
\end{aligned}
$$

which implies that $v_n \to 0$ strongly in E_λ. This ends the proof. □

The proof of Theorem 1.2. Under the assumptions of (V_1)–(V_3), (f_1)–(f_3) and (g_1), according to Lemmas 2.3, 3.1 and 3.2, for each $b \in (0, \widehat{b})$, set

$$
\Lambda_* := \max\Big\{\frac{\mathfrak{C}}{V_0|\{V < V_0\}|^{\frac{2}{3}}}, \frac{8C^2}{\beta V_0}\Big\},
$$

then for all $\lambda > \Lambda_*$, there exists the $(Ce)_{c_\lambda}$ sequence $\{u_n\}$ for I_λ on E_λ. Then, by Lemma 3.3 and $0 < c_\lambda \le \widetilde{c} < M$, we can obtain that there exist a subsequence $\{u_n\}$ and $u_\lambda^{(1)} \in E_\lambda$ such that $u_n \to u_\lambda^{(1)}$ strongly in E_λ. Furthermore, $I_\lambda(u_\lambda^{(1)}) = c_\lambda \ge \alpha > 0$ and $u_\lambda^{(1)}$ is a nontrivial solution for problem (1.1).

The second solution of problem (1.1) will be constructed by using local minimization. Now, we show that there exists $\phi \in E_\lambda$ such that $I_\lambda(l\phi) < 0$ for all $l > 0$ small enough. Using the condition (g_1) and (3.2), take $\phi \in H_0^1(\Omega)$ with $\int_\Omega g(x)|\phi|^q dx > 0$, we have, for all $l > 0$ small enough,

$$
I_\lambda(l\phi) \le \frac{l^2}{2}\int_\Omega a|\nabla\phi|^2dx + \frac{bl^4}{4}\Big(\int_\Omega|\nabla\phi|^2dx\Big)^2 - \frac{l^q}{q}\int_\Omega g(x)|\phi|^qdx < 0. \tag{3.16}
$$

It follows from Lemma 3.3 and (3.16) that the minimum of the (weakly lower semi-continuous) functional I_λ on any closed ball in E_λ with a center 0 and radius $R < \rho$ satisfying $I_\lambda(u) \ge 0$ for all $u \in E_\lambda$ with $\|u\|_\lambda = R$ is achieved in the corresponding open ball and thus yields a nontrivial solution

$u_\lambda^{(2)}$ of problem (1.1) satisfying $I_\lambda(u_\lambda^{(2)}) < 0$ and $\|u_\lambda^{(2)}\|_\lambda < R$. In addition, (3.16) implies that there exist $l_0 > 0$ and $\varrho < 0$ independent of λ such that $I_\lambda(l_0\phi) = \varrho$ and $\|l_0\phi\|_\lambda < R$. Then, we can conclude that

$$I_\lambda(u_\lambda^{(2)}) \le \varrho < 0 < \alpha \le c_\lambda = I_\lambda(u_\lambda^{(1)}).$$

The proof is finished.

4. Proof of Theorem 1.3

In this section, we investigate the concentration of solutions for $\lambda \to \infty$.

Proof of Theorem 1.3. Let $u_\lambda^{(i)}, i = 1, 2$ be the nontrivial solutions of problem (1.1) obtained in Theorem 1.2. For any sequence $\lambda_n \to \infty$, let $u_n^{(i)} := u_{\lambda_n}^{(i)}$ be the critical points of I_{λ_n}, namely, $I'_{\lambda_n}(u_n^{(i)}) = 0$ and

$$I_{\lambda_n}(u_n^{(2)}) \le \varrho < 0 < \alpha \le c_{\lambda_n} = I_{\lambda_n}(u_n^{(1)}) < M. \tag{4.1}$$

Then, similar to the proof in Lemma 3.2, we have

$$\begin{aligned} M + o(1) >& I_{\lambda_n}(u_n^{(i)}) - \frac{1}{4}\langle I'_{\lambda_n}(u_n^{(i)}), u_n^{(i)}\rangle \\ \ge& \Big[\frac{\min\{a, 1\}}{4} - \gamma_2\mathfrak{C}_2^2\Big(1 + \frac{1}{\mathcal{S}}\Big)\Big]\|u_n^{(i)}\|_{\lambda_n}^2 - \frac{\mathfrak{C}_2^q(4-q)\Big(1+\frac{1}{\mathcal{S}}\Big)^{\frac{q}{2}}|g^+|_{\frac{2}{2-q}}}{4q}\|u_n^{(i)}\|_{\lambda_n}^q, \end{aligned}$$

then, there exists constant $C_1 > 0$ independent of λ_n such that

$$\|u_n^{(i)}\|_{\lambda_n} \le C_1. \tag{4.2}$$

Hence $u_n^{(i)}$ is bounded in E_λ. Thus, we assume that $u_n^{(i)} \rightharpoonup u_0^{(i)}$ weakly in E_λ and $u_n^{(i)} \to u_0^{(i)}$ strongly in $L^p_{loc}(\mathbb{R}^3)$ for $p \in [2, 6)$. Now, we show that $u_n^{(i)} \to u_0^{(i)}$ strongly in $L^p(\mathbb{R}^3)$ for $p \in [2, 6)$.

Recall the definition of A_R and F_R in Lemma 3.3, and that $|F_R| \to 0$ as $R \to \infty$ by the condition (V_1). Then, for $\lambda_n \to \infty$, one has

$$\int_{A_R}(u_n^{(i)})^2dx \le \frac{1}{\lambda_nV_0}\int_{A_R}\lambda_nV(u_n^{(i)})^2dx \le \frac{C_1}{\lambda_nV_0} \to 0. \tag{4.3}$$

Thus, by the Hölder and Sobolev inequalities, (2.3), (4.2) and (4.3), we obtain

$$\begin{aligned} \int_{B_R^c}(u_n^{(i)})^pdx =& \Big(\int_{B_R^c}|u_n^{(i)}|^6dx\Big)^{\frac{p-2}{4}}\Big(\int_{B_R^c}|u_n^{(i)}|^2dx\Big)^{\frac{6-p}{4}} \\ \le& \mathfrak{C}^{\frac{3(2-p)}{4}}\Big(\int_{\mathbb{R}^3}|\nabla u_n^{(i)}|^2dx\Big)^{\frac{3(p-2)}{4}}\Big(\int_{A_R}(u_n^{(i)})^2dx + \int_{F_R}(u_n^{(i)})^2dx\Big)^{\frac{6-p}{4}} \\ \le& \mathfrak{C}^{\frac{3(2-p)}{4}}\|u_n^{(i)}\|_{\lambda_n}^{\frac{3(p-2)}{2}}\Big[\frac{C_1}{\lambda_nV_0} + |F_R|^{\frac{p-2}{p}}\Big(\int_{F_R}(u_n^{(i)})^pdx\Big)^{\frac{2}{p}}\Big]^{\frac{6-p}{4}} \\ \le& C_2\Big(\frac{C_1}{\lambda_nV_0} + C_3|F_R|^{\frac{p-2}{p}}\Big)^{\frac{6-p}{4}} \to 0 \end{aligned}$$

as $\lambda_n \to \infty$. Then, we infer that

$$\int_{B_R^c} \left| |u_n^{(i)}|^p - |u_0^{(i)}|^p \right| dx \leq \int_{B_R^c} |u_n^{(i)}|^p dx + \int_{B_R^c} |u_0^{(i)}|^p dx \to 0$$

as $R \to \infty$. Since $u_n^{(i)} \to u_0^{(i)}$ strongly in $L^p_{loc}(\mathbb{R}^3)$ for $p \in [2, 6)$, we have

$$\int_{|x|<R} |u_n^{(i)}|^p dx - \int_{|x|<R} |u_0^{(i)}|^p dx \to 0.$$

Therefore, $u_n^{(i)} \to u_0^{(i)}$ strongly in $L^p(\mathbb{R}^3)$ for $p \in [2, 6)$. Set $w_n^{(i)} := u_n^{(i)} - u_0^{(i)}$, use a similar argument to the proof in Lemma 3.3, we can claim that $w_n^{(i)} \to 0$ strongly in E_λ.

Thus, by (4.2) and Fatou's lemma, we have

$$\int_{\mathbb{R}^3} V(x)(u_0^{(i)})^2 dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^3} V(x)(u_n^{(i)})^2 dx \leq \liminf_{n\to\infty} \frac{\|u_n^{(i)}\|^2_{\lambda_n}}{\lambda_n} = 0.$$

Hence, $u_0^{(i)} = 0$ a.e. in $\mathbb{R}^3 \setminus \Omega$ and $u_0^{(i)} \in H_0^1(\Omega)$ by the condition (V_2). Given $u_0^{(i)} \in H_0^1(\Omega)$, we obtain

$$\Big(a + b\int_\Omega |\nabla u_0^{(i)}|^2 dx\Big) \int_\Omega \nabla u_0^{(i)} \nabla v^{(i)} dx = \int_\Omega f(x, u_0^{(i)}) v^{(i)} dx + \int_\Omega g(x) |u_0^{(i)}|^{q-2} u_0^{(i)} v^{(i)} dx$$

for any $v^{(i)} \in H_0^1(\Omega)$. Finally, it follows from (4.1) that

$$\frac{a}{2}\int_\Omega |\nabla u_0^{(1)}|^2 dx + \frac{b}{4}\Big(\int_\Omega |\nabla u_0^{(1)}|^2 dx\Big)^2 - \int_\Omega F(x, u_0^{(1)}) dx - \frac{1}{q}\int_\Omega g(x)|u_0^{(1)}|^q dx \geq \alpha > 0$$

and

$$\frac{a}{2}\int_\Omega |\nabla u_0^{(2)}|^2 dx + \frac{b}{4}\Big(\int_\Omega |\nabla u_0^{(2)}|^2 dx\Big)^2 - \int_\Omega F(x, u_0^{(2)}) dx - \frac{1}{q}\int_\Omega g(x)|u_0^{(2)}|^q dx \leq \varrho < 0,$$

which implies that $u_0^{(i)} \neq 0$ and $u_0^{(1)} \neq u_0^{(2)}$. This completes the proof. □

5. Conclusions

In this paper, two nontrivial solutions are obtained for a Kirchhoff-type problem by using variational methods. Furthermore, the concentration behavior of solutions as $\lambda \to \infty$ is also explored. The results obtained in this paper are slightly different from previous works [7, 8, 10, 18–22]. They may not have considered the existence and concentration of the solutions for Kirchhoff-type problems with sublinear perturbation and steep potential well. Therefore, the results of this paper expand the previous work to a certain extent.

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Conflict of interest

The authors declare no conflict of interest.

References

1. T. Bartsch, Z. Q. Wang, Existence and multiplicity results for superlinear elliptic problems on $\mathbb{R}^N$, *Commun. Part. Differ. Eq.*, **20** (1995),1725–1741. https://doi.org/10.1080/03605309508821149
2. T. Bartsch, A. Pankov, Z. Q. Wang, Nonlinear Schrödinger equations with steep potential well, *Commun. Contemp. Math.*, **3** (2001), 549–569. https://doi.org/10.1142/S0219199701000494
3. Y. H. Ding, A. Szulkin, Bound states for semilinear Schrödinger equations with sign-changing potential, *Calc. Var. Partial Dif.*, **29** (2007), 397–419. https://doi.org/10.1007/s00526-006-0071-8
4. G. Kirchhoff, *Mechanik*, Leipzig: Teubner, 1883.
5. A. Arosio, S. Panizzi, On the well-posedness of the Kirchhoff string, *Trans. Am. Math. Soc.*, **348** (1996), 305–330.
6. M. Chipot, B. Lovat, Some remarks on non local elliptic and parabolic problems, *Nonlinear Anal.*, **30** (1997), 4619–4627. https://doi.org/10.1016/S0362-546X(97)00169-7
7. Y. H. Li, F. Y. Li, Existence of a positive solution to Kirchhoff type problems without compactness conditions, *J. Differ. Equ.*, **253** (2012), 2285–2294. https://doi.org/10.1016/j.jde.2012.05.017
8. G. B. Li, H. Y. Ye, Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in $\mathbb{R}^3$, *J. Differ. Equ.*, **257** (2014), 566–600. https://doi.org/10.1016/j.jde.2014.04.011
9. Z. P. Wang, H. S. Zhou, Positive solutions for nonlinear Schrödinger equations with deepening potential well, *J. Eur. Math. Soc.*, **11** (2009), 545–573. https://doi.org/10.4171/JEMS/160
10. H. Y. Ye, Positive high energy solution for Kirchhoff equation in $\mathbb{R}^3$ with superlinear nonlinearities via Nehari-Pohozaev manifold, *Discret. Contin. Dyn. Syst.*, **35** (2015), 3857–3877. https://doi.org/10.3934/dcds.2015.35.3857
11. Z. S. Liu, S. J. Guo, Positive solutions for asymptotically linear Schrödinger-Kirchhoff-type equations, *Math. Meth. Appl. Sci.*, **37** (2014), 571–580. https://doi.org/10.1002/mma.2815
12. Y. S. Jiang, H. S. Zhou, Schrödinger-Poisson system with steep potential well, *J. Differ. Equ.*, **251** (2011), 582–608. http://dx.doi.org/10.1016/j.jde.2011.05.006
13. L. G. Zhao, H. D. Liu, F. K. Zhao, Existence and concentration of solutions for the Schrödinger-Poisson equations with steep well potential, *J. Differ. Equ.*, **255** (2013), 1–23. https://doi.org/10.1016/j.jde.2013.03.005
14. H. Zhang, F. B. Zhang, Ground states for the nonlinear Kirchhoff type problems, *J. Math. Anal. Appl.*, **423** (2015), 1671–1692. http://dx.doi.org/10.1016/j.jmaa.2014.10.062
15. Z. J. Guo, Ground states for Kirchhoff equations without compact condition, *J. Differ. Equ.*, **259** (2015), 2884–2902. http://dx.doi.org/10.1016/j.jde.2015.04.005
16. C. T. Ledesma, Existence and concentration of solutions for a non-linear fractional Schrödinger equation with steep potential well, *Commun. Pur. Appl. Anal.*, **15** (2016), 535–547. https://doi.org/10.3934/cpaa.2016.15.535

17. Y. H. Li, Q. Geng, The existence of nontrivial solution to a class of nonlinear Kirchhoff equations without any growth and Ambrosetti-Rabinowitz, *Appl. Math. Lett.*, **96** (2019), 153–158. http://dx.doi.org/10.1016/j.aml.2019.04.027

18. M. Du, L. X. Tian, J. Wang, F. B. Zhang, Existence of ground state solutions for a super-biquadratic Kirchhoff-type equation with steep potential well, *Appl. Anal.*, **95** (2016), 627–645. https://doi.org/10.1080/00036811.2015.1022312

19. D. Q. Zhang, G. Q. Chai, W. M. Liu, Existence and concentration of solutions for the nonlinear Kirchhoff type equations with steep well potential, *Bound. Value Probl.*, **2017** (2017), 142. https://doi.org/10.1186/s13661-017-0875-9

20. F. B. Zhang, M. Du, Existence and asymptotic behavior of positive solutions for Kirchhoff type problems with steep potential well, *J. Differ. Equ.*, **269** (2020), 10085–10106. https://doi.org/10.1016/j.jde.2020.07.013

21. J. T. Sun, T. F. Wu, Steep potential well may help Kirchhoff type equations to generate multiple solutions, *Nonlinear Anal.*, **190** (2020), 111609. https://doi.org/10.1016/j.na.2019.111609

22. L. Zhou, C. X. Zhu, Existence and asymptotic behavior of ground state solutions to Kirchhoff-type equations of general convolution nonlinearity with a steep potential well, *Mathematics*, **10** (2022), 812. https://doi.org/10.3390/math10050812

23. D. Choudhuri, Existence and Hölder regularity of infinitely many solutions to a p-Kirchhoff type problem involving a singular nonlinearity without the Ambrosetti-Rabinowitz (AR) condition, *Z. Angew. Math. Phys.*, **72** (2021), 36. https://doi.org/10.1007/s00033-020-01464-9

24. M. Schechter, A variation of the mountain pass lemma and applications, *J. London Math. Soc.*, **44** (1991), 491–502. https://doi.org/101112/jlms/s2-44.3.491

25. J. T. Sun, T. F. Wu, Multiplicity and concentration of homoclinic solutions for some second order Hamiltonian systems, *Nonlinear Anal.*, **114** (2015), 105–115. https://doi.org/10.1016/j.na.2014.11.009

26. M. Willem, *Minimax theorems*, Boston: Birkhäuser, 1996.

27. R. K. Giri, D. Choudhuri, S. Pradhan, Existence and concentration of solutions for a class of elliptic PDEs involving p-biharmonic operator, *Mat. Vestn.*, **70** (2018), 147–154.

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