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*Research article*

## **Alternating direction method for the fixed point problem of set-valued mappings with second-order cone double constraints**

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**Abstract:** This paper focuses on solving a class of equilibrium problems, namely, the fixed point problem of set-valued mappings with second-order cone double constraints. Under certain conditions, the variational inequality form of the fixed point problem of set-valued mappings with second-order cone double constraints is obtained by using the generalized saddle point theory three times. The alternating direction method is used to solve the fixed point problem of set-valued mappings with second-order cone double constraints, and the global convergence of the algorithm is proved. Finally, numerical results of solving five examples with an inexact alternating direction method are given, and the feasibility and effectiveness of the algorithm are demonstrated by comparing with other algorithms.

**Keywords:** second-order cone; double constraints; set-valued mapping; fixed point; alternating direction method; global convergence

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### **1. Introduction**

A set-valued mapping is a set-to-set correspondence. Problems in many fields can be abstracted as set-valued mapping models, such as sub-differential mapping of convex functions, optimal solution mapping for optimization problems with parametric variables, optimal approximation mapping, parametric mapping in cybernetics and normal dual mapping in functional analysis. These are common examples that can be abstracted into a set-valued mapping model, which shows that it is very meaningful to carry out research on this; see [1–8].

The fixed point problem of set-valued mappings with double constraints appears frequently in many fields, such as equilibrium constraint problems [9–12], n-person games [13,14] and bi-level programming problems [12,15–17]. Such problems are different from simple single-objective optimization problems. The constraint conditions are often more complex and diverse, and they are relatively difficult to deal with.

At present, there are relatively few studies on the fixed point problem of set-valued mappings with double constraints. In [18], the fixed point problem of set-valued mappings with double constraints is discussed, and the properties of symmetric and skew symmetric functions are given. The differential feedback control gradient-type method for solving equilibrium problems with double constraints is innovatively proposed, and its global convergence is proved. However, Antipin [18] only gives the theoretical part. Wang and Zhang [19] used the augmented Lagrange function of the generalized saddle point problem to construct a numerical algorithm on the basis of Antipin's idea. The study [19] proved that the algorithm has global convergence, and some numerical examples are given. However, numerical examples are not well integrated with practical applications.

Motivated by the above and by [20–22], this paper uses the projection idea to give a class of inexact alternating direction methods to solve the fixed point problem of set-valued mappings with second-order cone double constraints, and it obtains very good results.

In this paper, we introduce some basic knowledge, properties, definitions and theorems in Section 2. In Section 3, we use the generalized saddle point theory three times to obtain the variational inequality form of the fixed point problem of set-valued mappings with second-order cone double constraints. We also solve the variational inequality problem with the exact alternate method and prove the convergence of the algorithm in Section 4. We use the inexact alternating direction method to solve the variational inequality problem and prove the convergence of the algorithm in Section 5. At last, in Section 6, we solve certain types of complex mathematical optimization problems and give examples to illustrate the feasibility and effectiveness of the inexact alternating direction method.

## 2. Preliminaries

We will give some basic definitions and properties in this section, which can be found in [18–21].

**Definition 2.1.** [18] A function  $g$  from  $\mathbb{R}^n \times \mathbb{R}^n$  to  $\mathbb{R}^m$  is called symmetric onto  $\Omega_0 \times \Omega_0$  if it obeys the equality

$$g(v, w) = g(w, v), \quad \forall w \in \Omega_0, \forall v \in \Omega_0.$$

Symmetric functions have the following properties:

**Property 2.1.** [18] The matrices of gradient-restrictions of vector symmetric functions  $g(v, w)$  with respect to variables  $v$  and  $w$  onto the diagonal of the square  $\Omega_0 \times \Omega_0$  are identical. That is,

$$\nabla_w^T g(v, v) = \nabla_v^T g(v, v), \quad \forall v \in \Omega_0.$$

Note that the gradient-restrictions  $\nabla_w^T g(v, w)|_{v=w}$  and  $\nabla_v^T g(v, w)|_{v=w}$  are the gradients of the function  $g(v, v)$ , and “ $\nabla$ ” is a differential operator.

**Property 2.2.** [18] The operator  $2\nabla_w g(v, w)|_{v=w}$  is potential, i.e.,  $2\nabla_w^T g(v, v) = \nabla^T g(v, v)$ .

**Definition 2.2.** [18] A function  $\Phi(v, w)$  from  $\mathbb{R}^n \times \mathbb{R}^n$  to  $\mathbb{R}$  is called skew symmetric onto  $\Omega_0 \times \Omega_0$  if it obeys the equality

$$\Phi(w, w) - \Phi(w, v) - \Phi(v, w) + \Phi(v, v) \geq 0, \quad \forall v \in \Omega_0, \forall w \in \Omega_0.$$

**Definition 2.3.** [19] If the differentiable map  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is convex, then for any  $x, y \in \mathbb{R}^n$ ,

$$\langle \nabla f(x), y - x \rangle \leq f(y) - f(x) \leq \langle \nabla f(y), y - x \rangle.$$

**Definition 2.4.** [20] Assuming that  $\Omega \subseteq \mathbb{R}^n$  is a non-empty closed convex set, any projection of  $x \in \mathbb{R}^n$  on  $\Omega$  is defined as

$$\Pi_{\Omega}(x) = \operatorname{argmin}_{y \in \Omega} \{\|y - x\|\}.$$

**Definition 2.5.** [21] If  $\mathcal{K}^n \subseteq \mathbb{R}^n$  ( $n \geq 1$ ) satisfies

$$\mathcal{K}^n = \{(x_1; x_2) | x_1 \in \mathbb{R}, x_2 \in \mathbb{R}^{n-1}, x_1 \geq \|x_2\|\},$$

it is called an  $n$ -dimensional second-order cone, also called an ice cream cone.

**Definition 2.6.** [21] If  $x = (x_1; x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ , then its projection on the second-order cone is

$$\Pi_{\mathcal{K}}(x) = \begin{cases} \frac{1}{2} \left( 1 + \frac{x_1}{\|x_2\|} \right) (\|x_2\|; x_2), & \text{if } |x_1| < \|x_2\|, \\ x, & \text{if } \|x_2\| \leq x_1, \\ 0, & \text{if } \|x_2\| \leq -x_1. \end{cases}$$

**Definition 2.7.** [20] (The non-expansive property of projection)  $\Omega$  is a closed convex set;

$$\|\Pi_{\Omega}(u) - \Pi_{\Omega}(v)\| \leq \|u - v\|, \quad \forall u, v \in \Omega.$$

**Definition 2.8.**  $\Pi$  is an orthogonal projection if and only if there is a subspace  $W$  of  $V$  such that  $\Pi$  maps all elements in  $V$  to  $W$ , and  $\Pi$  is an identity transformation on  $W$ .

### 3. The fixed point of set-valued mappings with second-order cone double constraints

The fixed point problem of set-valued mappings with second-order cone double constraints is as follows: Find  $v^* \in \Omega$  satisfying

$$v^* \in \operatorname{argmin}\{\Phi(v^*, w) | -g(v^*, w) \in \mathcal{K}, w \in \Omega\}, \quad (3.1)$$

where  $\Phi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $g : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^m$  are two maps,  $\mathcal{K} = \mathcal{K}^{m_1} \times \mathcal{K}^{m_2} \times \dots \times \mathcal{K}^{m_p}$ ,  $m_1, m_2, \dots, m_p \geq 1$ , and  $m_1 + m_2 + \dots + m_p = m$ ,  $\mathcal{K}^{m_i}$ ,  $i = 1, 2, \dots, p$ , is an  $m_i$ -dimensional second-order cone.  $\Omega \subseteq \mathbb{R}^n$  is a non-empty closed convex set. Assume that  $\Phi(v, w)$  and each component function of vector-valued function  $g(v, w)$  are convex in  $w \in \Omega$  for any  $v \in \Omega$ . It is assumed that the solution mapping  $w(v) \equiv \operatorname{argmin}\{\Phi(v, w) | -g(v, w) \in \mathcal{K}, w \in \Omega\}$  is well-defined for any  $v \in \Omega$ , and the solution set of the original problem is  $\Omega^* = \{v^* \in \Omega | v^* \in w(v^*)\} \subset \Omega$ .  $\Omega^*$  is non-empty, which can be obtained from the continuity of  $\Phi(v, w)$  and the convex property of  $\Phi(v, w)$  in  $w \in \Omega$  for any  $v \in \Omega$ ; see [25].

Under the condition that the objective function  $\Phi(v, w)$  is a skew symmetric function and the constraint condition  $g(v, w)$  is a symmetric function, the original problem (3.1) is transformed into the form of a variational inequality by using the generalized saddle point three times.

Assuming  $v^* \in \Omega$  is the solution of the original problem (3.1), then

$$\Phi(v^*, v^*) \leq \Phi(v^*, w), \quad -g(v^*, w) \in \mathcal{K}, \quad \forall w \in \Omega. \quad (3.2)$$

Now, let  $\Psi(v, w) = \Phi(v, w) - \Phi(v, v)$ , and then (3.2) can be rewritten as

$$\Psi(v^*, w) \geq 0, \quad -g(v^*, w) \in \mathcal{K}, \quad \forall w \in \Omega. \quad (3.3)$$

If  $\Phi(v, w)$  is a skew symmetric function, according to Definition 2.2, we have that

$$\Phi(w, w) - \Phi(v, w) - \Phi(w, v) + \Phi(v, v) \geq 0, \quad \forall w \in \Omega, \quad \forall v \in \Omega.$$

Setting  $v^* = v$  in the above inequality, we obtain

$$\Phi(w, w) - \Phi(v^*, w) - \Phi(w, v^*) + \Phi(v^*, v^*) \geq 0, \quad \forall w \in \Omega,$$

which is

$$\Phi(w, w) - \Phi(w, v^*) \geq \Phi(v^*, w) - \Phi(v^*, v^*), \quad \forall w \in \Omega.$$

Combining the above inequality with (3.3), we can get

$$\Phi(w, w) - \Phi(w, v^*) \geq \Phi(v^*, w) - \Phi(v^*, v^*) \geq 0, \quad \forall w \in \Omega.$$

Therefore,

$$\Phi(w, w) - \Phi(w, v^*) \geq 0, \quad -g(v^*, w) \in \mathcal{K}, \quad \forall w \in \Omega.$$

If  $g(v, w)$  is a symmetric function, according to Definition 2.1, the above inequality can be transformed into

$$\Phi(w, w) - \Phi(w, v^*) \geq 0, \quad -g(w, v^*) \in \mathcal{K}, \quad \forall w \in \Omega,$$

which is

$$\Phi(v, v) - \Phi(v, v^*) \geq 0, \quad -g(v, v^*) \in \mathcal{K}, \quad \forall v \in \Omega.$$

The above inequality implies that

$$\Psi(v, v^*) \leq 0, \quad -g(v, v^*) \in \mathcal{K}, \quad \forall v \in \Omega. \quad (3.4)$$

Comparing (3.3) and (3.4), we can see that  $(v^*, v^*)$  is the saddle point of the function  $\Psi(v, w)$  on the symmetric set  $\{-g(v, w) \in \mathcal{K} | w, v \in \Omega \times \Omega\}$ , that is,

$$\begin{cases} \Psi(v, v^*) \leq \Psi(v^*, v^*) \leq \Psi(v^*, w), \\ -g(v, v^*) \in \mathcal{K}, \quad -g(v^*, w) \in \mathcal{K}, \quad \forall w, v \in \Omega. \end{cases} \quad (3.5)$$

Obviously, inequality (3.5) is a generalized saddle point problem. On the one hand, according to Definition 2.3, if the function  $\Phi$  is convex differentiable in  $w \in \Omega$  for any  $v$ , then the original problem (3.1) can be described equivalently as follows: Find  $v^* \in \Omega$  satisfying

$$\langle \nabla_w \Phi(v^*, v^*), w - v^* \rangle \geq 0, \quad -g(v^*, w) \in \mathcal{K}, \quad \forall w \in \Omega. \quad (3.6)$$

On the other hand, since the function  $\Phi$  is skew symmetric,  $\forall w, v \in \Omega$ , the monotonicity of  $\nabla_w \Phi(v, w)|_{v=w}$  is as follows:

$$\langle \nabla_w \Phi(v, v), w - v \rangle \leq \Phi(v, w) - \Phi(v, v) \leq \langle \nabla_w \Phi(w, w), w - v \rangle,$$

which implies that

$$\langle \nabla_w \Phi(w, w) - \nabla_w \Phi(v, v), w - v \rangle \geq 0, \quad \forall w, v \in \Omega. \quad (3.7)$$

Setting  $v^* = v$  in (3.7), we get

$$\langle \nabla_w \Phi(w, w) - \nabla_w \Phi(v^*, v^*), w - v^* \rangle \geq 0, \quad \forall w \in \Omega. \quad (3.8)$$

Combining (3.8) with (3.6), we can obtain

$$\langle \nabla_w \Phi(w, w), w - v^* \rangle \geq 0, \quad -g(v^*, w) \in \mathcal{K}, \quad \forall w \in \Omega.$$

Since  $g(v, w)$  is a symmetric function, the above inequality can be rewritten as

$$\langle \nabla_w \Phi(v, v), v^* - v \rangle \leq 0, \quad -g(v, v^*) \in \mathcal{K}, \quad \forall v \in \Omega. \quad (3.9)$$

Comparing (3.6) with (3.9), we can see that  $(v^*, v^*)$  is the saddle point of the function  $\langle \nabla_w \Phi(v, v), w - v \rangle$  on the symmetric set  $\{-g(v, w) \in \mathcal{K} | w, v \in \Omega \times \Omega\}$ . Hence,

$$\begin{cases} \langle \nabla_w \Phi(v, v), v^* - v \rangle \leq \langle \nabla_w \Phi(v^*, v^*), v^* - v^* \rangle \leq \langle \nabla_w \Phi(v^*, v^*), w - v^* \rangle, \\ -g(v, v^*) \in \mathcal{K}, \quad -g(v^*, w) \in \mathcal{K}, \quad \forall w, v \in \Omega. \end{cases} \quad (3.10)$$

Obviously, inequality (3.10) is a generalized saddle point problem. The Lagrange function of (3.10) has the following form:

$$L(v, w, \lambda) = \langle \nabla_w \Phi(v, v), w - v \rangle + \langle \lambda, g(v, w) \rangle, \quad (3.11)$$

where  $\lambda \in \mathcal{K}$ .

If the feasible region  $\{w | -g(v^*, w) \in \mathcal{K}, w \in \Omega\}$  satisfies the regular condition at  $w = v^*$ , for example, the Slater condition holds; then, inequality (3.10) is a convex optimization problem on  $w$ . Let  $v^*$  be a feasible solution to the problem (3.10), and we have  $\lambda^* \in \Lambda(v^*)$  such that the function  $L(v^*, w, \lambda)$  satisfies

$$L(v^*, v^*, \lambda) \leq L(v^*, v^*, \lambda^*) \leq L(v^*, w, \lambda^*), \quad \forall w \in \Omega, \quad \forall \lambda \in \mathcal{K}. \quad (3.12)$$

Obviously, inequality (3.12) is a generalized saddle point problem.

For the left side of (3.12):

$$\langle \nabla_w \Phi(v^*, v^*), v^* - v^* \rangle + \langle \lambda, g(v^*, v^*) \rangle \leq \langle \nabla_w \Phi(v^*, v^*), v^* - v^* \rangle + \langle \lambda^*, g(v^*, v^*) \rangle, \quad \forall \lambda \in \mathcal{K},$$

which implies

$$\langle \lambda^* - \lambda, g(v^*, v^*) \rangle \geq 0, \quad \forall \lambda \in \mathcal{K}.$$

For the right side of (3.12):

$$\langle \nabla_w \Phi(v^*, v^*), v^* - v^* \rangle + \langle \lambda^*, g(v^*, v^*) \rangle \leq \langle \nabla_w \Phi(v^*, v^*), w - v^* \rangle + \langle \lambda^*, g(v^*, w) \rangle, \quad \forall w \in \Omega,$$

which implies

$$\begin{aligned} \langle \nabla_w \Phi(v^*, v^*), w - v^* \rangle + \langle \lambda^*, g(v^*, w) - g(v^*, v^*) \rangle &\geq 0, \quad \forall w \in \Omega, \\ \langle \nabla_w \Phi(v^*, v^*), w - v^* \rangle + \langle \lambda^*, \nabla_w^T g(v^*, v^*)(w - v^*) \rangle &\geq 0, \quad \forall w \in \Omega, \\ \langle \nabla_w \Phi(v^*, v^*) + \nabla_w g(v^*, v^*) \lambda^*, w - v^* \rangle &\geq 0, \quad \forall w \in \Omega. \end{aligned}$$

Through the above discussion, we have

$$\begin{cases} \langle \nabla_w \Phi(v^*, v^*) + \nabla_w g(v^*, v^*) \lambda^*, w - v^* \rangle \geq 0, & \forall w \in \Omega, \\ \langle \lambda - \lambda^*, -g(v^*, v^*) \rangle \geq 0, & \forall \lambda \in \mathcal{K}. \end{cases} \quad (3.13)$$

As we all know, by the equivalent relationship between the projection operator and the variational inequality, inequality (3.13) can be transformed into

$$\begin{cases} v^* = \Pi_{\Omega}(v^* - \alpha(\nabla_w \Phi(v^*, v^*) + \nabla_w g(v^*, v^*)\lambda^*)), \\ \lambda^* = \Pi_{\mathcal{K}}(\lambda^* + \alpha g(v^*, v^*)), \end{cases} \quad (3.14)$$

where  $\alpha > 0$  is a parameter, and  $\Pi_{\Omega}(\cdot)$  and  $\Pi_{\mathcal{K}}(\cdot)$  are projection operators onto the set  $\Omega$  and second-order cone  $\mathcal{K}$ , respectively.

After the above discussion, it is easy to find that when the objective function  $\Phi(v, w)$  is a skew symmetric function, and the constraint  $g(v, w)$  is a symmetric function,  $v^*$  is the solution of the fixed point problem of set-valued mappings with second-order cone double constraints if and only if  $v^*$  satisfies inequality (3.13) or (3.14).

#### 4. Alternating direction method and its convergence

The classic methods for solving the variational inequality problem (3.13) are the alternating direction method, the projection shrinkage method and the external gradient method. The alternating direction method is more suitable for dealing with large-scale problems that solve the original problem by solving a series of sub-variational inequalities in each iteration.

Solving the variational inequality problem (3.13) is equivalent to solving the projection equation (3.14). Let the residual function be

$$e(u) = \begin{pmatrix} e_1(u) \\ e_2(u) \end{pmatrix} = \begin{pmatrix} v - \Pi_{\Omega}(v - \alpha(\nabla_w \Phi(v, v) + \nabla_w g(v, v)\lambda)) \\ \lambda - \Pi_{\mathcal{K}}(\lambda + \alpha g(v, v)) \end{pmatrix}, \quad (4.1)$$

where  $u = \begin{pmatrix} v \\ \lambda \end{pmatrix}$ , and  $\alpha > 0$  is a parameter. Hence, solving the problem (3.13) is equivalent to finding the zero point of the residual function (4.1). [21] proposes an alternating direction method for structural variational inequalities with equality constraints. Referring to its ideas, we propose Algorithm 1 for the fixed point problem of set-valued mappings with the second-order cone double constraints.

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##### Algorithm 1. Alternating direction method

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Step 0. Given  $\epsilon_1 > 0$ ,  $\alpha > 0$  and  $\lambda^0 \in \mathcal{K}$ , set  $k = 0$ .

Step 1. Find  $v^{k+1} \in \Omega$  such that

$$(w - v^{k+1})^T (\nabla_w \Phi(v^{k+1}, v^{k+1}) + \nabla_w g(v^{k+1}, v^{k+1})\Pi_{\mathcal{K}}(\lambda^k + \alpha g(v^{k+1}, v^{k+1}))) \geq 0, \quad \forall w \in \Omega.$$

Step 2. Update the multiplier

$$\lambda^{k+1} = \Pi_{\mathcal{K}}(\lambda^k + \alpha g(v^{k+1}, v^{k+1})).$$

Step 3. Convergence verification: If  $r_k = \|e(u^{k+1})\|_{\infty} \leq \epsilon_1$ , the iteration is terminated. Otherwise, proceed to the next step.

Step 4. Set  $k = k + 1$ , and return to Step 1.

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On the one hand, it can be obtained in Algorithm 1:

$$\begin{cases} v^{k+1} = \Pi_{\Omega}(v^{k+1} - \alpha(\nabla_w \Phi(v^{k+1}, v^{k+1}) + \nabla_w g(v^{k+1}, v^{k+1})\Pi_{\mathcal{K}}(\lambda^k + \alpha g(v^{k+1}, v^{k+1}))), \\ \lambda^{k+1} = \Pi_{\mathcal{K}}(\lambda^k + \alpha g(v^{k+1}, v^{k+1})). \end{cases} \quad (4.2)$$

On the other hand, according to the residual function  $e(u^{k+1})$ ,

$$e(u^{k+1}) = \begin{pmatrix} v^{k+1} - \Pi_{\Omega}(v^{k+1} - \alpha(\nabla_w \Phi(v^{k+1}, v^{k+1}) + \nabla_w g(v^{k+1}, v^{k+1})\lambda^{k+1})) \\ \lambda^{k+1} - \Pi_{\mathcal{K}}(\lambda^{k+1} + \alpha g(v^{k+1}, v^{k+1})) \end{pmatrix}. \quad (4.3)$$

Substituting the iteration points in Eq (4.2) into Eq (4.3), the following equation holds by using the non-expansive property of projection.

$$\begin{aligned} \|e_1(u^{k+1})\| &= \|v^{k+1} - \Pi_{\Omega}(v^{k+1} - \alpha(\nabla_w \Phi(v^{k+1}, v^{k+1}) + \nabla_w g(v^{k+1}, v^{k+1})\Pi_{\mathcal{K}}(\lambda^{k+1} + \alpha g(v^{k+1}, v^{k+1})))\| \\ &= \|\Pi_{\Omega}(v^{k+1} - \alpha(\nabla_w \Phi(v^{k+1}, v^{k+1}) + \nabla_w g(v^{k+1}, v^{k+1})\Pi_{\mathcal{K}}(\lambda^k + \alpha g(v^{k+1}, v^{k+1}))) \\ &\quad - \Pi_{\Omega}(v^{k+1} - \alpha(\nabla_w \Phi(v^{k+1}, v^{k+1}) + \nabla_w g(v^{k+1}, v^{k+1})\Pi_{\mathcal{K}}(\lambda^{k+1} + \alpha g(v^{k+1}, v^{k+1})))\| \\ &\leq \|\alpha \nabla_w g(v^{k+1}, v^{k+1})(\Pi_{\mathcal{K}}(\lambda^{k+1} + \alpha g(v^{k+1}, v^{k+1})) - \Pi_{\mathcal{K}}(\lambda^k + \alpha g(v^{k+1}, v^{k+1})))\| \\ &\leq \alpha \|\nabla_w g(v^{k+1}, v^{k+1})\| \|\lambda^k - \lambda^{k+1}\|. \end{aligned}$$

$$\begin{aligned} \|e_2(u^{k+1})\| &= \|\lambda^{k+1} - \Pi_{\mathcal{K}}(\lambda^{k+1} + \alpha g(v^{k+1}, v^{k+1}))\| \\ &= \|\Pi_{\mathcal{K}}(\lambda^k + \alpha g(v^{k+1}, v^{k+1})) - \Pi_{\mathcal{K}}(\lambda^{k+1} + \alpha g(v^{k+1}, v^{k+1}))\| \\ &\leq \|\lambda^k - \lambda^{k+1}\|. \end{aligned}$$

With the help of  $\|e(u)\|^2 \leq \|e_1(u)\|^2 + \|e_2(u)\|^2$ , we have

$$\|e(u^{k+1})\|^2 \leq (1 + \alpha^2 \|\nabla_w g(v^{k+1}, v^{k+1})\|^2) \|\lambda^k - \lambda^{k+1}\|^2. \quad (4.4)$$

Therefore, in order to prove that the alternating direction method is convergent, it is necessary to prove  $\lim_{k \rightarrow \infty} \|\lambda^{k+1} - \lambda^k\|^2 = 0$ .

**Theorem 4.1.** *Let  $u^* = (v^*, w^*, \lambda^*)$  be the solution of problem (3.1), and let  $\{u^k\} = \{v^k, w^k, \lambda^k\}$  be the sequence generated by Algorithm 1. If  $\lim_{k \rightarrow \infty} \|\lambda^{k+1} - \lambda^k\|^2 = 0$  holds, then the algorithm is convergent.*

*Proof.* The iterative system in Algorithm 1 can be equivalently expressed as the following variational inequalities:

$$\langle \nabla_w \Phi(v^{k+1}, v^{k+1}) + \nabla_w g(v^{k+1}, v^{k+1})\lambda^{k+1}, v^* - v^{k+1} \rangle \geq 0, \quad (4.5)$$

$$\langle \lambda^{k+1} - \lambda^k - \alpha g(v^{k+1}, v^{k+1}), \lambda^* - \lambda^{k+1} \rangle \geq 0. \quad (4.6)$$

Inequalities (4.5) and (4.6) can be expressed in an equivalent manner as

$$\langle \nabla_w \Phi(v^{k+1}, v^{k+1}), v^* - v^{k+1} \rangle + \langle \nabla_w g(v^{k+1}, v^{k+1})\lambda^{k+1}, v^* - v^{k+1} \rangle \geq 0, \quad (4.7)$$

$$\langle \lambda^{k+1} - \lambda^k, \lambda^* - \lambda^{k+1} \rangle - \alpha \langle g(v^{k+1}, v^{k+1}), \lambda^* - \lambda^{k+1} \rangle \geq 0. \quad (4.8)$$

Let  $w = v^{k+1}$  in the first inequality of (3.13), and adding it to (4.7), we get

$$\langle \nabla_w \Phi(v^{k+1}, v^{k+1}) - \nabla_w \Phi(v^*, v^*), v^* - v^{k+1} \rangle + \langle \nabla_w g(v^{k+1}, v^{k+1})\lambda^{k+1} - \nabla_w g(v^*, v^*)\lambda^*, v^* - v^{k+1} \rangle \geq 0. \quad (4.9)$$

According to the monotonicity of  $\nabla_w \Phi(v, w)|_{v=w}$ , we have

$$\langle \nabla_w g(v^{k+1}, v^{k+1})\lambda^{k+1} - \nabla_w g(v^*, v^*)\lambda^*, v^* - v^{k+1} \rangle \geq 0. \quad (4.10)$$

In view of the convexity of  $g$  and Property 2.2, we obtain

$$\langle g(v^{k+1}, v^{k+1}) - g(v^*, v^*), \lambda^* - \lambda^{k+1} \rangle \geq 0. \quad (4.11)$$

Since  $\lambda \in \mathcal{K}$ ,  $-g(v^*, w) \in \mathcal{K}$ . Therefore,  $\langle \lambda^*, -g(v^*, v^*) \rangle = 0$ ,  $\langle \lambda^{k+1}, g(v^*, v^*) \rangle \leq 0$ , and

$$\langle \lambda^{k+1} - \lambda^k, \lambda^* - \lambda^{k+1} \rangle - \alpha \langle g(v^{k+1}, v^{k+1}) - g(v^*, v^*), \lambda^* - \lambda^{k+1} \rangle \geq 0. \quad (4.12)$$

Combining (4.11) and (4.12), we have  $\langle \lambda^{k+1} - \lambda^k, \lambda^* - \lambda^{k+1} \rangle \geq 0$ . So, the following formula holds:

$$\|\lambda^* - \lambda^k\|^2 = \|\lambda^* - \lambda^{k+1} + \lambda^{k+1} - \lambda^k\|^2 \geq \|\lambda^* - \lambda^{k+1}\|^2 + \|\lambda^{k+1} - \lambda^k\|^2. \quad (4.13)$$

Summing (4.13) from  $k = 0$  to  $k = n$  yields

$$\sum_{k=0}^n \|\lambda^* - \lambda^k\|^2 \geq \sum_{k=0}^n \|\lambda^* - \lambda^{k+1}\|^2 + \sum_{k=0}^n \|\lambda^{k+1} - \lambda^k\|^2, \quad (4.14)$$

which is

$$\|\lambda^0 - \lambda^*\|^2 \geq \|\lambda^* - \lambda^{n+1}\|^2 + \sum_{k=0}^n \|\lambda^{k+1} - \lambda^k\|^2. \quad (4.15)$$

Therefore, the series  $\sum_{k=0}^{\infty} \|\lambda^{k+1} - \lambda^k\|^2$  is bounded. According to the convergence of the series,  $\lim_{k \rightarrow \infty} \|\lambda^{k+1} - \lambda^k\|^2 = 0$  can be obtained.

In summary,  $\lim_{k \rightarrow \infty} \|e(u^{k+1})\|^2 = 0$ , and the convergence of Algorithm 1 is proved.  $\square$

Algorithm 1 is a theoretical algorithm. From a practical point of view, we will give an inexact alternating direction method to solve our problem, which is what we will discuss in the next section.

## 5. Inexact alternating direction method based on projection and its convergence

Although the alternating direction method can decompose the large-scale original problem into a series of sub-problems for solution, it still needs to solve the sub-variational inequalities during each iteration. Due to the difficulty of solving the variational inequalities, this section proposes the new alternating direction method of projection. It can be approximately solved according to the properties of orthogonal projection and second-order cone projection in each iteration. Compared with the precise solution of sub-problems in the previous section, inexact solution is more feasible and faster.

Using Definition 2.8 of orthogonal projection, the residual function can be expressed as

$$\begin{aligned} e(u) &= \begin{pmatrix} e_1(u) \\ e_2(u) \end{pmatrix} = \begin{pmatrix} v - \Pi_{\Omega}(v - \alpha(\nabla_w \Phi(v, v) + \nabla_w g(v, v)\lambda)) \\ \lambda - \Pi_{\mathcal{K}}(\lambda + \alpha g(v, v)) \end{pmatrix} \\ &= \begin{pmatrix} \alpha(\nabla_w \Phi(v, v) + \nabla_w g(v, v)\lambda) \\ \lambda - \Pi_{\mathcal{K}}(\lambda + \alpha g(v, v)) \end{pmatrix}. \end{aligned}$$

Then, solving the problem (3.13) is equivalent to finding the zero point of the above residual function. We propose an inexact alternating direction method, namely, Algorithm 2, which improves Algorithm 1 and is feasible.



**Algorithm 2.** Inexact alternating direction method

Step 0. Given  $\epsilon_1 > 0, \alpha > 0$  and  $\lambda^0 \in \mathcal{K}$ , set  $k = 0$ .

Step 1. Find  $v^{k+1} \in \Omega$  such that

$$v^{k+1} - v^k + \alpha(\nabla_w \Phi(v^{k+1}, v^{k+1}) + \nabla_w g(v^{k+1}, v^{k+1})\Pi_{\mathcal{K}}(\lambda^k + \alpha g(v^{k+1}, v^{k+1}))) = 0.$$

Step 2. Update multiplier

$$\lambda^{k+1} = \Pi_{\mathcal{K}}(\lambda^k + \alpha g(v^{k+1}, v^{k+1})).$$

Step 3. Convergence verification: If  $r_k = \|e(u^{k+1})\|_{\infty} \leq \epsilon_1$ , the iteration is terminated. Otherwise, proceed to the next step.

Step 4. Set  $k = k + 1$ , return to Step 1.

This article selects  $\epsilon_1 = 10^{-6}$ .

Let  $G^k(v) = v - v^k + \alpha(\nabla_w \Phi(v, v) + \nabla_w g(v, v)\Pi_{\mathcal{K}}(\lambda^k + \alpha g(v, v)))$ . Because the projection operator  $\Pi_{\mathcal{K}}(\cdot)$  is semi-smooth (see [23,24]), the equation  $G^k(\cdot)$  in the inexact alternating direction method is also semi-smooth. If any element in  $\partial G^k(v^{k+1})$  is nonsingular, and  $v^k$  is sufficiently close to  $v^{k+1}$  when  $k$  is sufficiently large, then the first iterative formulation in Algorithm 2 can be solved by applying the semi-smooth Newton method of Qi and Sun [23].

**Newton method**

Step 0. Given  $\epsilon_0 > 0$  and  $\xi^0 = v^k$ , set  $j = 0$ .

Step 1. Determine whether  $\|G^k(\xi^j)\| \leq \epsilon_0$  is satisfied. If it is satisfied, the iteration is terminated, and let  $v^{k+1} = \xi^j$ . Otherwise, proceed to the next step.

Step 2. Calculate  $d^j = -[\partial G^k(\xi^j)]^{-1}G^k(\xi^j)$ .

Step 3. Set  $\xi^{j+1} = \xi^j + d^j$ ,  $j = j + 1$ , return to Step 1.

This article selects  $\epsilon_0 = 10^{-6}$ . On the one hand, it can be obtained with Algorithm 2:

$$\begin{cases} v^{k+1} - v^k + \alpha(\nabla_w \Phi(v^{k+1}, v^{k+1}) + \nabla_w g(v^{k+1}, v^{k+1})\Pi_{\mathcal{K}}(\lambda^k + \alpha g(v^{k+1}, v^{k+1}))) = 0, \\ \lambda^{k+1} = \Pi_{\mathcal{K}}(\lambda^k + \alpha g(v^{k+1}, v^{k+1})). \end{cases} \quad (5.1)$$

On the other hand, according to the residual function  $e(u^{k+1})$ ,

$$e(u^{k+1}) = \begin{pmatrix} \alpha(\nabla_w \Phi(v^{k+1}, v^{k+1}) + \nabla_w g(v^{k+1}, v^{k+1})\lambda^{k+1}) \\ \lambda^{k+1} - \Pi_{\mathcal{K}}(\lambda^{k+1} + \alpha g(v^{k+1}, v^{k+1})) \end{pmatrix}. \quad (5.2)$$

Combining Eqs (5.1) and (5.2), the following equations are established:

$$\begin{aligned} \|e_1(u^{k+1})\| &= \|\alpha(\nabla_w \Phi(v^{k+1}, v^{k+1}) + \nabla_w g(v^{k+1}, v^{k+1})\Pi_{\mathcal{K}}(\lambda^{k+1} + \alpha g(v^{k+1}, v^{k+1})) - 0\| \\ &= \|\alpha(\nabla_w \Phi(v^{k+1}, v^{k+1}) + \nabla_w g(v^{k+1}, v^{k+1})\Pi_{\mathcal{K}}(\lambda^{k+1} + \alpha g(v^{k+1}, v^{k+1})) \\ &\quad - v^{k+1} + v^k - \alpha(\nabla_w \Phi(v^{k+1}, v^{k+1}) + \nabla_w g(v^{k+1}, v^{k+1})\Pi_{\mathcal{K}}(\lambda^{k+1} + \alpha g(v^{k+1}, v^{k+1})))\| \\ &\leq \|\alpha \nabla_w g(v^{k+1}, v^{k+1})(\Pi_{\mathcal{K}}(\lambda^{k+1} + \alpha g(v^{k+1}, v^{k+1})) - \Pi_{\mathcal{K}}(\lambda^k + \alpha g(v^{k+1}, v^{k+1})))\| + \|v^{k+1} - v^k\| \\ &\leq \alpha \|\nabla_w g(v^{k+1}, v^{k+1})\| \|\lambda^k - \lambda^{k+1}\| + \|v^{k+1} - v^k\|. \end{aligned}$$

$$\begin{aligned}
\|e_2(u^{k+1})\| &= \|\lambda^{k+1} - \Pi_{\mathcal{K}}(\lambda^{k+1} + \alpha g(v^{k+1}, v^{k+1}))\| \\
&= \|\Pi_{\mathcal{K}}(\lambda^k + \alpha g(v^{k+1}, v^{k+1})) - \Pi_{\mathcal{K}}(\lambda^{k+1} + \alpha g(v^{k+1}, v^{k+1}))\| \\
&\leq \|\lambda^{k+1} - \lambda^k\|.
\end{aligned}$$

With the help of  $\|e(u)\|^2 \leq \|e_1(u)\|^2 + \|e_2(u)\|^2$ , we have

$$\|e(u^{k+1})\|^2 \leq (1 + 2\alpha^2 \|\nabla_w g(v^{k+1}, v^{k+1})\|^2) \|\lambda^k - \lambda^{k+1}\|^2 + 2\|v^{k+1} - v^k\|^2.$$

As in Section 4, in order to illustrate that Algorithm 2 is convergent, it is necessary to prove  $\lim_{k \rightarrow \infty} \|\lambda^{k+1} - \lambda^k\|^2 = 0$  and  $\lim_{k \rightarrow \infty} \|v^{k+1} - v^k\|^2 = 0$ .

**Theorem 5.1.** *Let  $u^* = (v^*, w^*, \lambda^*)$  be the solution of problem (3.1), and let  $\{u^k\} = \{v^k, w^k, \lambda^k\}$  be the sequence generated by Algorithm 2. If  $\lim_{k \rightarrow \infty} \|\lambda^{k+1} - \lambda^k\|^2 = 0$  and  $\lim_{k \rightarrow \infty} \|v^{k+1} - v^k\|^2 = 0$  hold, then the algorithm is convergent.*

*Proof.* The iterative system in the inexact alternating direction method can be equivalently expressed as the following variational inequality:

$$\langle v^{k+1} - v^k + \alpha(\nabla_w \Phi(v^{k+1}, v^{k+1}) + \nabla_w g(v^{k+1}, v^{k+1}))\lambda^{k+1}, v^* - v^{k+1} \rangle = 0, \quad (5.3)$$

$$\langle \lambda^{k+1} - \lambda^k - \alpha g(v^{k+1}, v^{k+1}), \lambda^* - \lambda^{k+1} \rangle \geq 0. \quad (5.4)$$

(5.3) and (5.4) can be expressed in an equivalent manner as

$$\langle v^{k+1} - v^k, v^* - v^{k+1} \rangle + \alpha \langle \nabla_w \Phi(v^{k+1}, v^{k+1}), v^* - v^{k+1} \rangle + \alpha \langle \nabla_w g(v^{k+1}, v^{k+1})\lambda^{k+1}, v^* - v^{k+1} \rangle = 0, \quad (5.5)$$

$$\langle \lambda^{k+1} - \lambda^k, \lambda^* - \lambda^{k+1} \rangle - \alpha \langle g(v^{k+1}, v^{k+1}), \lambda^* - \lambda^{k+1} \rangle \geq 0. \quad (5.6)$$

Let  $w = v^{k+1}$  in the first inequality of (3.13). Adding it to the (5.5), we get

$$\begin{aligned}
&\langle v^{k+1} - v^k, v^* - v^{k+1} \rangle + \alpha \langle \nabla_w \Phi(v^{k+1}, v^{k+1}) \\
&- \nabla_w \Phi(v^*, v^*), v^* - v^{k+1} \rangle + \alpha \langle \nabla_w g(v^{k+1}, v^{k+1})\lambda^{k+1} - \nabla_w g(v^*, v^*)\lambda^*, v^* - v^{k+1} \rangle \geq 0.
\end{aligned}$$

According to the monotonicity of  $\nabla_w \Phi(v, w)|_{v=w}$ , we have

$$\langle v^{k+1} - v^k, v^* - v^{k+1} \rangle + \alpha \langle \nabla_w g(v^{k+1}, v^{k+1})\lambda^{k+1} - \nabla_w g(v^*, v^*)\lambda^*, v^* - v^{k+1} \rangle \geq 0. \quad (5.7)$$

In view of the convexity of  $g$  and Property 2.2, we obtain

$$\langle v^{k+1} - v^k, v^* - v^{k+1} \rangle + \frac{\alpha}{2} \langle g(v^{k+1}, v^{k+1}) - g(v^*, v^*), \lambda^* - \lambda^{k+1} \rangle \geq 0. \quad (5.8)$$

Since  $\lambda \in \mathcal{K}$ ,  $-g(v^*, w) \in \mathcal{K}$ . Therefore,  $\langle \lambda^*, -g(v^*, v^*) \rangle = 0$ ,  $\langle \lambda^{k+1}, g(v^*, v^*) \rangle \leq 0$ , and

$$\langle \lambda^{k+1} - \lambda^k, \lambda^* - \lambda^{k+1} \rangle - \alpha \langle g(v^{k+1}, v^{k+1}) - g(v^*, v^*), \lambda^* - \lambda^{k+1} \rangle \geq 0. \quad (5.9)$$

Combining (5.8) and (5.9), we have  $\langle v^{k+1} - v^k, v^* - v^{k+1} \rangle + \frac{1}{2} \langle \lambda^{k+1} - \lambda^k, \lambda^* - \lambda^{k+1} \rangle \geq 0$ . So, the following inequation holds:

$$\frac{1}{2} \|\lambda^* - \lambda^k\|^2 + \|v^* - v^k\|^2 \geq \frac{1}{2} \|\lambda^* - \lambda^{k+1}\|^2 + \frac{1}{2} \|\lambda^{k+1} - \lambda^k\|^2 + \|v^* - v^{k+1}\|^2 + \|v^{k+1} - v^k\|^2. \quad (5.10)$$

Summing (5.10) from  $k = 0$  to  $k = n$  yields

$$\frac{1}{2} \|\lambda^0 - \lambda^*\|^2 + \|v^0 - v^*\|^2 \geq \frac{1}{2} \|\lambda^* - \lambda^{n+1}\|^2 + \|v^* - v^{n+1}\|^2 + \frac{1}{2} \sum_{k=0}^n \|\lambda^{k+1} - \lambda^k\|^2 + \sum_{k=0}^n \|v^{k+1} - v^k\|^2. \quad (5.11)$$

Therefore, the series  $\sum_{k=0}^{\infty} \|\lambda^{k+1} - \lambda^k\|^2$  and  $\sum_{k=0}^{\infty} \|v^{k+1} - v^k\|^2$  are bounded. According to the convergence of the series,  $\lim_{k \rightarrow \infty} \|\lambda^{k+1} - \lambda^k\|^2 = 0$  and  $\lim_{k \rightarrow \infty} \|v^{k+1} - v^k\|^2 = 0$  can be obtained.

In summary,  $\lim_{k \rightarrow \infty} \|e(u^{k+1})\|^2 = 0$ , and the convergence of the inexact alternating direction method is proved.  $\square$

## 6. Numerical experiment

### 6.1. Mathematical programming problem with equilibrium constraints

The mathematical programming problem with equilibrium constraints is a constraint optimization problem with variational inequalities or complementary systems in the constraints. The function  $f$  is the overall objective function of the minimization problem,  $F$  is the internal equilibrium function, and  $Z$  is the joint upper-level feasible region. For each given  $x \in X$ ,  $X = \{x \in \mathbb{R}^n | (x, y) \in Z\}$ , and the range  $C(x)$  of  $y$  is defined. Assuming that  $X \subseteq \text{dom}(C)$ , the mathematical programming problem with equilibrium constraints can be expressed as

$$\begin{aligned} & \min f(x, y) \\ & \text{s.t. } (x, y) \in Z \\ & \quad y \in \Omega^*, \end{aligned} \quad (6.1)$$

where, for each  $x \in X$ ,  $\Omega^*$  is the solution set of the variational inequality defined by  $VI(F(x, \cdot), C(x))$ . That is,  $y \in \Omega^*$  is equivalent to  $y \in C(x)$  and satisfies  $\langle F(x, y), v - y \rangle \geq 0, \forall v \in C(x)$ .

If there is a real-valued function  $\Phi(x, y)$  that satisfies  $F(x, y) = \nabla_y \Phi(x, y)$ , in this special case, problem (6.1) is usually regarded as a bi-level optimization problem. Then, the bottom optimization problem of the bi-level optimization problem can be expressed as

$$\begin{aligned} & \min \Phi(x, y) \\ & \quad y \in \Omega^*. \end{aligned} \quad (6.2)$$

Therefore, for any given vector  $x$ , there is  $y \in C(x)$ ,  $C = \{x \in \Omega | -g(x, y^*) \in \mathcal{K}\}$ ,

$$y^* \in \text{argmin}\{\Phi(x, y^*) | -g(x, y^*) \in \mathcal{K}, x \in \Omega\}.$$

It can represent the optimal solution mapping  $S(x)$  of the above problem (6.2), and the solution mapping  $S$  is a set-valued mapping from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ . Next, we will solve Examples 1–5 with Algorithm 2.

**Example 1.** Economic equilibrium model

$$\begin{aligned} & \min f(v, w) \\ & \text{s.t. } (v, w) \in Z \\ & \langle Nw + Mv^* + m, w - v^* \rangle \geq 0, \quad \forall w \in \Omega \\ & v \in C(w), \end{aligned}$$

where  $C(w) = \{w \in \Omega \mid -g(v, w) \in \mathcal{K}\}$ ,  $g(v, w) = v + w$ . The bottom optimization problem can be processed as

$$v^* \in \operatorname{argmin}\{\langle \frac{N}{2}w + Mv^* + m, w \rangle \mid -g(v^*, w) \in \mathcal{K}, w \in \Omega\}. \quad (6.3)$$

Here,  $\Phi(v, w) = \langle \frac{N}{2}w + Mv + m, w \rangle$  is a skew symmetric function, and  $g(v, w) = v + w$  is a symmetric function.

In this example,  $\alpha = 0.5$  is used, and then the expression of  $H^j$  in step  $j$  in Newton method is as follows:

$$H^j = I_n + \alpha(N + M) + 2\alpha^2 \partial \Pi_{\mathcal{K}}(\lambda^k + 2\alpha \xi^j).$$

Table 1 shows the numerical results of Example 1, where *Iter* is the number of iterations of the solution process, and *Time* is the CPU calculation time when the program terminates.

**Table 1.** Numerical calculation results of problem (6.3).

$n$	$\mathcal{K}$	<i>Iter</i>	<i>Time</i>	$r_k$	$\epsilon_0$	$\epsilon_1$
50	$\mathcal{K}^{50}$	21	5.468750e-01	1.797038e-07	$10^{-6}$	$10^{-6}$
100	$\mathcal{K}^{50} \times \mathcal{K}^{50}$	28	8.437500e-01	6.575534e-07	$10^{-6}$	$10^{-6}$
200	$\mathcal{K}^{50} \times \mathcal{K}^{50} \times \mathcal{K}^{100}$	28	1.281250e+00	8.227579e-07	$10^{-6}$	$10^{-6}$

## 6.2. Nash equilibrium $n$ -person game problem

The more general situation of Nash equilibrium  $n$ -person game: Let  $f_i(x_i, x_{-i}^*)$  be the economic cost function of the  $i$ -th player,  $i \in I$ . This function not only depends on the strategy  $x_i \in X_i$  of this player, where  $X_i = (x_i)_{i \in I}$  is a convex set, but it also depends on the strategies  $x_{-i} \in X_{-i}$ ,  $X_{-i} = (x_j)_{j \in I \setminus i}$  of other players. Then, the equilibrium point of the  $n$ -person game is the solution contained in the following set:

$$x_i^* \in \operatorname{argmin}\{f_i(x_i, x_{-i}^*) \mid x_i \in X_i\}.$$

Let function  $\Phi(v, w) = \sum_{i=1}^n f_i(x_i, x_{-i})$ , where  $v = (x_{-i})$ ,  $w = (x_i)$ ,  $g(v, w) = (g_i(x_i, x_{-i}))$ ,  $i = 1, 2, \dots, n$ .

Here,  $(v, w) = (x_i, x_{-i}) \in \Omega \times \Omega$ ,  $\Omega = X_1 \times X_2 \times \dots \times X_n$ , and then

$$v^* \in \operatorname{argmin}\{\Phi(v^*, w) \mid -g(v^*, w) \in \mathcal{K}, w \in \Omega\}.$$

**Example 2.** Cournot duopoly model

$$v^* \in \operatorname{argmin}\{\Phi(v^*, w) \mid -g(v^*, w) \in \mathcal{K}, w \in \Omega\}, \quad (6.4)$$

where  $\Phi(v, w) = w^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} w + v^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} w - w^T u$ , ( $u > 0$ ), is a skew symmetric function, and  $g(v, w) = \begin{pmatrix} -(w_1 - 1)^2 - (v_1 - 1)^2 \\ -(w_2 - 1)^2 - (v_2 - 1)^2 \end{pmatrix}$  is a symmetric function.

Let  $N = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ ,  $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . In this example,  $\alpha = 0.5$  is used, and then the expression of  $H^j$  in step  $j$  in Newton method is as follows:

$$H^j = I_n + \alpha(N + M) - 2\alpha \operatorname{diag}_{1 \leq i \leq n} \{\Pi_{\mathcal{K}}(\lambda_i^k - \alpha(\xi_i^j - 1)^2) - 4\alpha(\xi_i^j - 1)^2 \partial \Pi_{\mathcal{K}}(\lambda_i^k - 2\alpha(\xi_i^j - 1)^2)\}.$$

Table 2 shows the numerical results of Example 2.

**Table 2.** Numerical calculation results of problem (6.4).

$n$	$\mathcal{K}$	$Iter$	$Time$	$r_k$	$\epsilon_0$	$\epsilon_1$
2	$\mathcal{K}^2$	26	6.093750e-01	9.330831e-07	$10^{-6}$	$10^{-6}$

### 6.3. Bi-level optimization problem

Consider the following bi-level programming problem:

$$\begin{aligned} & \min_{(v,w)} f(v, w) \\ & \text{s.t. } h(v, w) \leq 0 \\ & v \in \Omega^*, \end{aligned} \quad (6.5)$$

where  $\Omega^*$  represents the solution set of the underlying problem, which is

$$\begin{aligned} & \min_v \Phi(v, w) \\ & \text{s.t. } -g(v, w) \in \mathcal{K}. \end{aligned} \quad (6.6)$$

Then, the set of reasonable reflections of the underlying problem is

$$\Omega^* = \{v^* \mid v^* \in \operatorname{argmin}\{\Phi(v^*, w) \mid -g(v^*, w) \in \mathcal{K}, w \in \Omega\}\}.$$

**Example 3.** Consider the following bi-level programming problem in  $\mathbb{R}^2$ :

$$\begin{aligned} & \min w - v \\ & \text{s.t. } (v, w) \geq 0 \\ & v \in \operatorname{argmin}\left\{\frac{v^2}{2} \mid v + w \in \mathcal{K}, w \in \Omega\right\}. \end{aligned} \quad (6.7)$$

Obviously, problem (6.7) has the only optimal solution  $(0, 0)$ .

**Example 4.** Consider the bottom optimization problem in the bi-level programming problem in  $\mathbb{R}^n$ :

$$v^* \in \operatorname{argmin}\left\{\frac{w^2}{2} \mid -g(v^*, w) \in \mathcal{K}, w \in \Omega\right\}, \quad (6.8)$$

where  $\Phi(v, w) = \frac{w^2}{2}$  is a skew symmetric function, and  $g(v, w) = \begin{pmatrix} w_1 \sin(v_1) + v_1 \sin(w_1) \\ w_2 \sin(v_2) + v_2 \sin(w_2) \\ \vdots \\ w_n \sin(v_n) + v_n \sin(w_n) \end{pmatrix}$  is a symmetric function.

In this example,  $\alpha = 0.5$  is used, and then the expression of  $H^j$  in step  $j$  in Newton method is as follows:

$$H^j = (1 + \alpha)I_n + \alpha \operatorname{diag}_{1 \leq i \leq n} \{ [2 \cos(\xi_i^j) - \xi_i^j \sin(\xi_i^j)] \Pi_{\mathcal{K}}(\lambda_i^k + 2\alpha \xi_i^j \sin(\xi_i^j)) \} \\ + 2\alpha^2 \operatorname{diag}_{1 \leq i \leq n} \{ [\sin(\xi_i^j) + \xi_i^j \cos(\xi_i^j)]^2 \partial \Pi_{\mathcal{K}}(\lambda_i^k + 2\alpha \xi_i^j \sin(\xi_i^j)) \}.$$

Table 3 shows the numerical results of Example 3.

**Table 3.** Numerical calculation results of problem (6.8).

$n$	$\mathcal{K}$	<i>Iter</i>	<i>Time</i>	$r_k$	$\epsilon_0$	$\epsilon_1$
50	$\mathcal{K}^{50}$	11	5.312500e-01	5.085837e-07	$10^{-6}$	$10^{-6}$
100	$\mathcal{K}^{50} \times \mathcal{K}^{50}$	17	1.218750e+00	5.810239e-07	$10^{-6}$	$10^{-6}$
200	$\mathcal{K}^{50} \times \mathcal{K}^{50} \times \mathcal{K}^{100}$	17	3.843750e+00	8.325520e-07	$10^{-6}$	$10^{-6}$

To illustrate the effectiveness and feasibility of the algorithm, we compared Algorithm 2 with the algorithm in [19] under the condition of achieving the same accuracy, and the numerical results are shown below. Here, ALM denotes the augmented Lagrange method in [19], and IADM denotes Algorithm 2 in this paper.

**Table 4.** Comparison of numerical calculation results of Algorithm 2 and algorithm in [19].

<i>Method</i>	Ex1(n=50)		Ex4(n=50)		Ex1(n=100)		Ex4(n=100)	
	<i>Time</i>	<i>Iter</i>	<i>Time</i>	<i>Iter</i>	<i>Time</i>	<i>Iter</i>	<i>Time</i>	<i>Iter</i>
ALM	7.187500e-01	23	6.562500e-01	11	9.843750e-01	23	1.187500e+00	12
IADM	5.468750e-01	21	5.312500e-01	11	8.437500e-01	28	1.218750e+00	17
<i>Method</i>	Ex1(n=200)		Ex4(n=200)		Ex2(n=2)			
	<i>Time</i>	<i>Iter</i>	<i>Time</i>	<i>Iter</i>	<i>Time</i>	<i>Iter</i>		
ALM	1.859375e+00	23	4.218750e+00	18	6.250000e-01	42		
IADM	1.281250e+00	28	3.843750e+00	17	6.093750e-01	26		

As can be seen from the above table, Algorithm 2 is better than the algorithm in [19].

#### 6.4. Equilibrium problem

Let  $K$  be a nonempty closed and convex subset of a Hilbert space  $\mathbb{E}$  and  $f : \mathbb{E} \times \mathbb{E} \rightarrow \mathbb{R}$  be a bi-function satisfying  $f(x, x) = 0$  for any  $x \in K$ . The equilibrium problem for the bi-function  $f$  on  $K$  is defined as follows:

$$\text{Find } x \in K \text{ such that } f(x^*, y) > 0, \forall y \in K. \quad (6.9)$$

Problem (6.9) is equivalent to the following optimization problem with non-negative optimal value:

$$\begin{aligned} \min f(x^*, y) \\ \text{s.t. } y \in K. \end{aligned} \quad (6.10)$$

That is,

$$x^* \in \operatorname{argmin}\{f(x^*, y) \mid y \in K\}. \quad (6.11)$$

**Example 5.** Consider that  $f : \mathbb{E} \times \mathbb{E} \rightarrow \mathbb{R}$  is defined by

$$f(x, y) = \langle Ax + By + c, y - x \rangle, \forall y \in K, \quad (6.12)$$

$$\text{where } A = \begin{pmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, B = \begin{pmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, c = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \\ -1 \end{pmatrix}, \text{ and } K = \{x \in \mathbb{R}^5 \mid$$

$-5 \leq x_i \leq 5\}$ . It is easy to verify that  $f(x, y)$  is a skew symmetric function. Let  $a = (5, 5, 5, 5, 5)^T$ ,  $g(x) = -x - a$ , and  $h(x) = x - a$ . We have

$$x^* \in \operatorname{argmin}\{\langle Ax^* + By + c, y - x^* \rangle \mid y \in K\}, \quad (6.13)$$

where  $K = \{x \in \mathbb{R}^5 \mid g(x) \leq 0, h(x) \leq 0\}$ .

In this example,  $\alpha = 0.5$  is used, and then the expression of  $H^j$  in step  $j$  in Newton method is as follows:

$$H^j = I_5 + \alpha(A + B) - \alpha I_5(\beta - \gamma),$$

where  $\beta = \begin{cases} 0, & \text{if } p - \alpha(\xi + c) < 0, \\ -\alpha, & \text{if } p - \alpha(\xi + c) > 0, \\ 0 \text{ or } -\alpha, & \text{if } p - \alpha(\xi + c) = 0, \end{cases}$  and  $\gamma = \begin{cases} 0, & \text{if } \mu + \alpha(\xi - c) < 0, \\ \alpha, & \text{if } \mu + \alpha(\xi - c) > 0, \\ 0 \text{ or } \alpha, & \text{if } \mu + \alpha(\xi - c) = 0. \end{cases}$  Note that  $p \in \mathbb{R}_+^5$  and  $\mu \in \mathbb{R}_+^5$  are Lagrange multipliers of problem (6.13).

Table 5 shows the numerical results of Example 5.

**Table 5.** Numerical calculation results of problem (6.13).

$n$	Iter	Time	$r_k$	$\epsilon_0$	$\epsilon_1$
5	23	0.468750	6.202562e-07	$10^{-6}$	$10^{-6}$

Table 6 shows the numerical results of [26].

**Table 6.** Numerical calculation results of [26].

$n$	$\lambda_n$	$Iter$	$Time$	$\epsilon_1$
5	$\frac{1}{n+1}$	102	0.9632	$10^{-6}$
5	$\frac{1}{\sqrt{n+1}}$	18	0.1919	$10^{-6}$
5	$\frac{1}{\log(n+2)}$	13	0.1279	$10^{-6}$
5	$\frac{1}{(n+1)\log(n+3)}$	123	1.1854	$10^{-6}$
5	$\frac{1}{\log(n+3)}$	77	0.7494	$10^{-6}$
5	$\frac{1}{n+1}$	18	0.1797	$10^{-6}$
	$\frac{1}{\log \log(n+20)}$			

By comparing with [26], we can see that in the cases of  $\frac{1}{n+1}$ ,  $\frac{1}{(n+1)\log(n+3)}$  and  $\frac{\log(n+3)}{n+1}$ , the algorithm of this paper has obvious advantages.

## 7. Conclusions

This paper mainly studies the fixed point problem of set-valued mappings with second-order cone double constraints by using the alternating direction method. With the help of generalized saddle point theory, the variational inequality problems of the fixed point problem of set-valued mappings with second-order cone double constraints are obtained. By using the projection idea, a new alternating direction method is proposed. The new alternating direction method requires only simple projection calculations for each iteration, avoiding the problem of solving sub-variational inequalities. Finally, five complex problems with second-order cone double constraints in practical applications are given. An inexact alternating direction method is used to solve them, which further illustrates the practicability and effectiveness of the algorithm. Furthermore, the results for fixed-point problems can also be applied to [27–29].

Although the research in this paper has achieved some results, there are still many questions worthy of our continued research. For example, the convergence rate of the new alternating direction method still needs to be estimated. In practical applications, the number of variables is often more than two, and the problem of which alternating direction method is applicable to multiple separable variables is also the focus of future research.

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