Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(3): 6356–6374.
DOI: 10.3934/math.2023321
Received: 28 October 2022
Revised: 16 December 2022
Accepted: 20 December 2022
Published: 03 January 2023

Research article

Stochastic dynamic analysis of a chemostat model of intestinal microbes with migratory effect

Yue Dong and Xinzhu Meng*

College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, China

* **Correspondence:** Email: mxz721106@sdust.edu.cn.

Abstract: This paper proposes a stochastic intestinal chemostat model considering microbial migration, intraspecific competition and stochastic perturbation. First, the extinction and persistence in mean of the intestinal microbe of the chemostat model are investigated by constructing the appropriate Lyapunov functions. Second, we explore and obtain sufficient conditions for the existence and uniqueness of an ergodic stationary distribution of the model by using ergodic theory. The results show stochastic interference has a critical impact on the extinction and sustainable survival of the intestinal microbe. Eventually, numerical simulations are carried out to verify the theoretical results.

Keywords: stochastic chemostat model; microbial migration; extinction; persistence in mean; stationary distribution
Mathematics Subject Classification: 37A50, 37H05, 37N25, 60G10

1. Introduction

We are not individuals but a collection of living things [1]. Up to 10 million microbes inhabit every square centimeter of skin [2]. Surface-dwelling microbes even play a crucial role in maintaining the homeostasis of the host microbial community [3]. At the same time, our teeth, throat, and esophagus are even more infested with microbes, which accumulate thousands of times more than the surface of the skin. The human body has a large number of microorganisms living in the gut, and the vast majority of them are beneficial to us. A correct view of their value and continuous research and discovery is the proper attitude of human beings towards the microbes in the body. Researchers have long applied various methods to study the role of the microbiome in living organisms and achieved significant results. In addition to the application of biological methods [4], the application of mathematical methods is undoubtedly a great help for this research direction [5, 6].

The flow characteristics of chemostat model are very suitable for simulating the flow ecological environment. And the culture of microorganisms in the chemostat can be regulated by controlling the concentration of nutrients. Combined with the features of intestinal circulation, the chemostat model can be used to further investigate the microbial system in the intestinal tract. Presently, research on chemostat models has been further improved [9–11], and researchers have studied ecosystems with different types of functional responses through chemostat models, such as Monod-Haldane [12,13,23], Lotka-Volterra [14], Holling II [16], etc. In spite of this, few papers have applied the chemostat model to specific ecological environment. This paper proposes that, it is an innovation to apply the chemostat model to the characterization of intestinal microorganisms. In addition, based on the study of reaction function in the above literature and the characteristics of intestinal microorganisms, we adopted Holling II functional response function. In the gut, microbes circulate with material and may attach to the folds in the intestinal wall as they flow, or they may fall off the wall. It is an innovative point of this paper to consider and study the impact of the migration between the flowing microorganisms and the microorganisms on the intestinal wall as the microorganisms on the two plates. This phenomenon can be described by adding a migration term to the model. In [7], the authors considered the variable flow rate, microbial decomposition, and other factors to construct a detailed intestinal chemostat model.

Inevitably, microbial survival in the gut is affected by uncertainties such as body temperature, water supply and pH-value.

In stochastic biological models, scholars use Brownian motion to describe the perturbation of the system caused by uncertain factors and have made important achievements [8, 17, 18, 21, 22]. On the basis of deterministic models, many authors consider the linear disturbance of mortality, which further enrich the content of the models [15,25,27,29]. In addition, bilinear perturbation and nonlinear perturbation have been considered in [24, 28], respectively. In [26], stochastic models perturbed by Lévy noise are concerned. From what has been discussed above, we assume that uncertain factors can cause an impact on microbial growth, that is, environmental disturbance, mainly interferes with the consumption function. Then the following chemostat model with nonlinear perturbation, intraspecific competition and migration is established.

$$\begin{cases} dS(t) = \left[D(S_0 - S(t)) - \dfrac{aS(t)}{m+S(t)}(x_1(t)+x_2(t))\right]dt - \dfrac{\sigma a S(t)}{m+S(t)}(x_1(t)+x_2(t))dB(t), \\ dx_1(t) = \left[x_1(t)\left(\dfrac{bS(t)}{m+S(t)} - \alpha x_1(t) - \beta x_2(t)\right) - v x_1(t) - D x_1(t) + r_2 x_2(t) - r_1 x_1(t)\right]dt \\ \qquad + \dfrac{\sigma b S(t) x_1(t)}{m+S(t)}dB(t), \\ dx_2(t) = \left[x_2(t)\left(\dfrac{bS(t)}{m+S(t)} - \alpha x_1(t) - \beta x_2(t)\right) - v x_2(t) - r_2 x_2(t) + r_1 x_1(t)\right]dt \\ \qquad + \dfrac{\sigma b S(t) x_2(t)}{m+S(t)}dB(t). \end{cases} \tag{1.1}$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). At the same time, there are the corrective coefficients of Wong-Zakai in the model, and the same calculation results can be obtained by using Itô integral and Stratonovich integral [30]. The function $B(t)$ is independent standard Brownian motion defined on this complete probability space with $B(0) = 0$ and $\sigma > 0$ represents for the intensity of the white noise. And other parameters are defined in the Table 1.

Table 1. Parameter definition.

Parameters	Paraphrase
$S(t) \geq 0$	the concentration of nutrients in the intestine at time t
$x_1(t) \geq 0$	the amount of bacteria in the gut at time t
$x_2(t) \geq 0$	the amount of bacteria on the wall at time t
$D \geq 0$	input and output flow rate
$S_0 \geq 0$	the initial concentration of nutrients
$a \geq 0$	the maximum rate of nutrient uptake by microorganisms
$b \geq 0$	the effective rate of nutrient uptake by microorganisms
$m \geq 0$	the half-saturation constant
$r_1 \geq 0$	the rate at which microbe attaches to the wall
$r_2 \geq 0$	the rate at which microbe detaches from wall
$\alpha \geq 0$	intra-specific competition rate of the gut population of microbe
$\beta \geq 0$	intra-specific competition rate of wall population of microbe
$v \geq 0$	the mortality of the microbial

The characteristics of the proposed model are the application of the chemostat model to simulate the intestinal environment, and the introduction of microbial migration, interspecific competition and nonlinear disturbance on the basis of the classical chemostat model. Compared with [7], we consider the effect of environmental disturbance on the microbial response function based on the corresponding autonomous model.

The organizational structure of the paper is as follows. In Section 2, we provide the lemmas and partial proofs used in the argument of this paper. In Section 3, we verify the existence and uniqueness of global positive solution and the boundedness of solution of the system (1.1). Section 4 presents all the research results for the proposed system. In the first subsection, sufficient conditions for microbial extinction are proved. Then, the condition of persistence in mean of the system is given. Next, the existence of ergodic stationary distribution is established. Finally, the behavior of system (1.1) is simulated numerically in Section 5, and the conclusions are given.

2. Preliminaries

We let $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$. If $f(t)$ is an intergrable function on $[0, +\infty)$, let $\langle f(t) \rangle = \frac{1}{t}\int_0^t f(\theta)\mathrm{d}\theta$. Here we give the common lemmas of stochastic differential equations which will be used in the proof below.

Lemma 2.1 (Stationary distribution [20]). *Let $X(t)$ be a homogenous Markov process in E_l (E_l denotes l-dimensional Euclidean space), and it is described by the following stochastic differential equation*

$$\mathrm{d}X(t) = b(X)\mathrm{d}t + \sum_{r=1}^{k} g_r(X)\mathrm{d}B_r(t).$$

The diffusion matrix is defined as follows

$$A(x) = (a_{ij}(x)),\ a_{ij}(x) = \sum_{r=1}^{k} g_r^i(x)g_r^j(x).$$

If there is a bounded open set $U \subset R^n$ with regular that satisfies the following conditions.
(i) For any $x \in U, \varepsilon \in R^n$, there is a constant $\varsigma > 0$, satisfying that:

$$\textstyle\sum_{i,j=1}^{n} a_{ij}(x)\varepsilon_i\varepsilon_j \geq \varsigma\,|\varepsilon|^2.$$

(ii) For any $x \in R^n \setminus U$, there is a $C^{1,2}$ function V, such that $LV < 0$.
Then, the Markov process $X(t)$ exists a stationary distribution $\psi(\cdot)$.

Lemma 2.2 (Existence and uniqueness of global positive solutions)**.** *For any initial value $(S(0), x_1(0), x_2(0)) \in \mathbb{R}^3_+$, there is a unique positive solution $(S(t), x_1(t), x_2(t))$ of system* (1.1) *on $t \geq 0$ and the solution will remain in $\mathbb{R}^3_+$ with probability one.*

Proof. Since the coefficients of the system (1.1) satisfy the local Lipschitz conditions, then for any given initial value $(S(0), x_1(0), x_2(0)) \in \mathbb{R}^3_+$, there exists a unique local solution $(S(t), x_1(t), x_2(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time. Now we just need to prove $\tau_e = \infty$ a.s..

Let $k_0 > 0$ be so sufficiently large that any initial value $S(0)$, $x_1(0)$ and $x_2(0)$ lying within the interval $\left[\frac{1}{k_0}, k_0\right]$. For any integer k satisfying $k \geq k_0$, define the stopping time as follows:

$$\tau_k = \inf\left\{t \in [0, \tau_e) : \min\{S(t), x_1(t), x_2(t)\} \leq \frac{1}{k} \text{ or } \max\{S(t), x_1(t), x_2(t)\} \geq k\right\},$$

with $\inf \emptyset = \infty$ (where $\emptyset$ denotes the empty set). It is easy to get that τ_k is increasing as $k \to \infty$. Let $\tau_\infty = \lim_{k\to\infty} \tau_k$, hence $\tau_\infty \leq \tau_k$ a.s.. Next, we only need to verify $\tau_\infty = \infty$ a.s.. If this statement is false, then there exist two constants $T > 0$ and $\epsilon \in (0, 1)$ such that

$$\mathbb{P}\{\tau_\infty \leq T\} > \epsilon.$$

Thus there is an integer $k_1 \geq k_0$ such that

$$\mathbb{P}\{\tau_k \leq T\} \geq \epsilon,\ k \geq k_1.$$

Define a C^2-function I: $\mathbb{R}^3_+ \to \mathbb{R}_+$ as follows,

$$I(S(t), x_1(t), x_2(t)) = (S - 1 - \ln S) + (x_1 - 1 - \ln x_1) + (x_2 - 1 - \ln x_2) + \frac{x_1^2}{2} + \frac{x_2^2}{2}. \tag{2.1}$$

Clearly,

$$I > 0.$$

Applying Itô's formula yields

$$\begin{aligned} LI = &\left(1 - \frac{1}{S}\right)\left[DS_0 - DS - \frac{aS}{m+S}(x_1 + x_2)\right] + \frac{\sigma^2 a^2}{2(m+S)^2}(x_1 + x_2)^2 \\ &+ \left(1 - \frac{1}{x_1} + x_1\right)\left[\frac{bSx_1}{m+S} - (v + D)x_1 - \alpha x_1^2 - \beta x_1 x_2 + r_2 x_2 - r_1 x_1\right] + \frac{\sigma^2 b^2 S^2(1 + x_1^2)}{2(m+S)^2} \\ &+ \left(1 - \frac{1}{x_2} + x_2\right)\left[\frac{bSx_2}{m+S} - vx_2 - \alpha x_1 x_2 - \beta x_2^2 - r_2 x_2 + r_1 x_1\right] + \frac{\sigma^2 b^2 S^2(1 + x_2^2)}{2(m+S)^2} \end{aligned}$$

$$
\begin{aligned}
&= DS_0 - DS - \frac{aS}{m+S}(x_1+x_2) - \frac{DS_0}{S} + D + \frac{a}{m+S}(x_1+x_2) + \frac{\sigma^2 a^2}{2(m+S)^2}(x_1+x_2)^2 \\
&+ \frac{bSx_1}{m+S} - (v+D)x_1 - \alpha x_1^2 - \beta x_1x_2 + r_2x_2 - r_1x_1 - \frac{bS}{m+S} + (v+D+r_1) \\
&+ \alpha x_1 + \beta x_2 - \frac{r_2x_2}{x_1} + \frac{bSx_1^2}{m+S} - (v+D)x_1^2 - \alpha x_1^3 - \beta x_1^2x_2 + r_2x_1x_2 - r_1x_1^2 \\
&+ \frac{bSx_2}{m+S} - vx_2 - \alpha x_1x_2 - \beta x_2^2 - r_2x_2 + r_1x_1 - \frac{bS}{m+S} + (v+r_2) + \alpha x_1 + \beta x_2 - \frac{r_1x_1}{x_2} \\
&+ \frac{bSx_2^2}{m+S} - vx_2^2 - \alpha x_1x_2^2 - \beta x_2^3 - r_2x_2^2 + r_1x_1x_2 + \frac{\sigma^2a^2}{2(m+S)^2}(x_1+x_2)^2 \\
&+ \frac{\sigma^2b^2S^2}{2(m+S)^2}\left(2 + x_1^2 + x_2^2\right) \\
&\le DS_0 + 2D + 2v + r_1 + r_2 + \left(\frac{a}{m} + b + 2\alpha\right)x_1 + \left(\frac{a}{m} + 2\beta + b\right)x_2 + (r_1+r_2)x_1x_2 \\
&+ bx_1^2 + bx_2^2 - \alpha x_1^3 - \beta x_2^3 - (v+D+r_1)x_1^2 - \beta x_1^2x_2 - (v+r_2)x_2^2 - \alpha x_1x_2^2 \\
&+ \frac{\sigma^2a^2}{2m^2}(x_1+x_2)^2 + \frac{\sigma^2b^2}{2}\left(2 + x_1^2 + x_2^2\right) \\
&\le DS_0 + 2v + 2D + r_1 + r_2 + \sigma^2b^2 + h_1 := \kappa,
\end{aligned}
$$

where

$$
\begin{aligned}
h_1 = \max\Bigg\{&\left(\frac{a}{m} + b + 2\alpha\right)x_1 + \left(\frac{a}{m} + b + 2\beta\right)x_2 + (r_1+r_2)x_1x_2 + bx_1^2 + bx_2^2 - \alpha x_1^3 \\
&-\beta x_2^3 - (v+D+r_1)x_1^2 - (v+r_2)x_2^2 - \beta x_1^2x_2 - \alpha x_1x_2^2 + \frac{\sigma^2a^2}{2m^2}(x_1+x_2)^2 \\
&+\frac{\sigma^2b^2}{2}\left(x_1^2+x_2^2\right)\Bigg\},
\end{aligned}
$$

then

$$
LI \le DS_0 + 2v + 2D + r_1 + r_2 + \sigma^2b^2 + h_1 := \kappa, \tag{2.2}
$$

where κ is a positive constant.

The proof is completed. □

Lemma 2.3 (Boundedness)**.** *For any initial value* $(S(0), x_1(0), x_2(0)) \in \mathbb{R}_+^3$*, the solution* $(S(t), x_1(t), x_2(t))$ *of the stochastic system* (1.1) *is bounded. And it satisfies*

$$
\limsup_{t\to\infty}\left[S(t) + \frac{a}{b}x_1(t) + \frac{a}{b}x_2(t)\right] \le \omega, \quad a.s., \tag{2.3}
$$

where $\omega = \max\left\{\frac{DS_0}{\mu}, S(0) + \frac{a}{b}x_1(0) + \frac{a}{b}x_2(0)\right\}$.

Proof. Denote

$$
Y(t) = S(t) + \frac{a}{b}x_1(t) + \frac{a}{b}x_2(t),
$$

then from the system (1.1), we obtain

$$\begin{aligned}
\mathrm{d}Y(t) &= \Big[DS_0 - DS - \frac{a}{b}vx_1 - \frac{a}{b}Dx_1 - \frac{a}{b}\alpha x_1^2 - \frac{a}{b}\beta x_1x_2 - \frac{a}{b}vx_2 - \frac{a}{b}\alpha x_1x_2 - \frac{a}{b}\beta x_2^2\Big]\mathrm{d}t \\
&\leq \Big[DS_0 - DS - \frac{a}{b}vx_1 - \frac{a}{b}Dx_1 - \frac{a}{b}vx_2\Big]\mathrm{d}t \\
&\leq \Big[DS_0 - \mu\Big(S + \frac{a}{b}x_1 + \frac{a}{b}x_2\Big)\Big]\mathrm{d}t, \qquad (2.4)
\end{aligned}$$

where $\mu = \min\{D, v\}$, $Y(t)$ is the solution of (2.4). And it has the following form

$$Y(t) \leq \omega.$$

Furthermore,

$$\limsup_{t\to+\infty} x_1(t) \leqslant \frac{b\omega}{a}, \quad \limsup_{t\to+\infty} x_2(t) \leqslant \frac{b\omega}{a}.$$

The proof is completed. □

3. Main results

3.1. Extinction

In this subsection, we establish threshold condition for intestinal microbial extinction.

Define a parameter

$$R_1^* = \frac{bS_0}{m+S_0} - \frac{\sigma^2 b^2 S_0^2}{4(m+S_0)^2} - v, \quad R_2^* = \frac{1}{\sigma^2} - v.$$

Theorem 3.1. *(1) If $\frac{bS_0}{m+S_0} < \frac{2}{\sigma^2}$ and $R_1^* < 0$, then microbials tend to extinction almost surely.*
(2) If $\frac{bS_0}{m+S_0} \geq \frac{2}{\sigma^2}$ and $R_2^* < 0$, then microbials tend to extinction almost surely.

Proof. Through the system, we can get,

$$\begin{aligned}
\mathrm{d}(x_1 + x_2) = &\Big[-vx_1 - Dx_1 + x_1\Big(\frac{bS}{m+S} - \alpha x_1 - \beta x_2\Big) - vx_2 + x_2\Big(\frac{bS}{m+S} - \alpha x_1 - \beta x_2\Big)\Big]\mathrm{d}t \\
&+ \frac{\sigma bS}{m+S}(x_1 + x_2)\,\mathrm{d}B(t).
\end{aligned}$$

Applying Itô's formula gives

$$\begin{aligned}
\mathrm{d}\ln(x_1(t) + x_2(t)) = &\Big\{\frac{1}{x_1+x_2}\Big[-v(x_1+x_2) - Dx_1 + (x_1+x_2)\Big(\frac{bS}{m+S} - \alpha x_1 - \beta x_2\Big)\Big] \\
&- \frac{\sigma^2 b^2 S^2}{2(x_1+x_2)^2(m+S)^2}(x_1^2 + x_2^2)\Big\}\mathrm{d}t + \frac{\sigma bS}{m+S}\mathrm{d}B(t) \\
\leq &\Big[-v - \frac{Dx_1}{x_1+x_2} + \Big(\frac{bS}{m+S} - \alpha x_1 - \beta x_2\Big) - \frac{\sigma^2 b^2 S^2}{4(m+S)^2}\Big]\mathrm{d}t + \frac{\sigma bS}{m+S}\mathrm{d}B(t) \\
\leq &\Big[-v + \frac{bS}{m+S} - \frac{\sigma^2 b^2 S^2}{4(m+S)^2}\Big]\mathrm{d}t + \frac{\sigma bS}{m+S}\mathrm{d}B(t). \qquad (3.1)
\end{aligned}$$

Consider the function

$$g(S)=\frac{bS}{m+S}-\frac{\sigma^2b^2S^2}{4(m+S)^2}.$$

Obviously, when $\frac{bS}{m+S}=\frac{2}{\sigma^2}$, $g(S)$ gets the maximum. Otherwise, $S\le S_0$. Therefore, the maximum value of $g(S)$ is discussed in the following two cases.
Case 1. If $\frac{bS_0}{m+S_0}<\frac{2}{\sigma^2}$,

$$g(S)\le\frac{bS_0}{m+S_0}-\frac{\sigma^2b^2S_0^2}{4(m+S_0)^2}.$$

Combining (3.1), we get

$$\mathrm{d}\ln(x_1(t)+x_2(t))\le\left[\frac{bS_0}{m+S_0}-\frac{\sigma^2b^2S_0^2}{4(m+S_0)^2}-v\right]\mathrm{d}t+\frac{\sigma bS}{m+S}\mathrm{d}B(t). \tag{3.2}$$

Integrating from 0 to t and dividing by t on both sides of (3.2) give

$$\frac{\ln(x_1(t)+x_2(t))}{t}\le\frac{bS_0}{m+S_0}-\frac{\sigma^2b^2S_0^2}{4(m+S_0)^2}-v+\frac{1}{t}\int_0^t\frac{\sigma bS(\tau)}{m+S(\tau)}\mathrm{d}B(\tau)+\frac{\ln(x_1(0)+x_2(0))}{t}.$$

According to the strong law of large numbers [19], we obtain

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\frac{\sigma bS(\tau)}{m+S(\tau)}\mathrm{d}B(\tau)=0.$$

So we can get

$$\limsup_{t\to\infty}\frac{\ln(x_1(t)+x_2(t))}{t}\le\frac{bS_0}{m+S_0}-\frac{\sigma^2b^2S_0^2}{4(m+S_0)^2}-v<0,$$

which means

$$\lim_{t\to\infty}(x_1(t)+x_2(t))=0.$$

Case 2. If $\frac{bS_0}{m+S_0}\ge\frac{2}{\sigma^2}$,

$$g(S)\le\frac{1}{\sigma^2}.$$

Combining (3.1), we get

$$\mathrm{d}\ln(x_1(t)+x_2(t))\le\left[\frac{1}{\sigma^2}-v\right]\mathrm{d}t+\frac{\sigma bS}{m+S}\mathrm{d}B(t). \tag{3.3}$$

Similar to Case 1, we can obtain

$$\limsup_{t\to\infty}\frac{\ln(x_1(t)+x_2(t))}{t}\le\frac{1}{\sigma^2}-v<0,$$

which means

$$\lim_{t\to\infty}(x_1(t)+x_2(t))=0.$$

□

3.2. Persistence in mean

In this subsection, we examine the conditions sufficient for microbes to persist over time.

Define
$$R^* = \frac{2bS_0}{m+S_0} - 2v - D - r_1 - r_2 - \sigma^2 b^2 - \frac{4b^2\omega^2(\alpha+\beta)}{(m+S_0)Da}.$$

Theorem 3.2. *If $R^* > 0$, then for any initial value $(S(t), x_1(t), x_2(t)) \in R^3_+$, the system (1.1) is persistent in mean.*

Proof. Integrating the equations of the system (1.1) from 0 to t and dividing t on both sides yield
$$\begin{aligned}\varepsilon(t) &\triangleq \frac{b(S(t)-S(0))}{at} + \frac{x_1(t)-x_1(0)}{t} + \frac{x_2(t)-x_2(0)}{t} \\ &\geq \frac{bDS_0}{a} - \frac{2b^2\omega^2(\alpha+\beta)}{a^2} - \frac{bD}{a}\langle S(t)\rangle - (v+D)\langle x_1(t)\rangle - v\langle x_2(t)\rangle .\end{aligned}$$

Then we can get
$$\langle S(t)\rangle \geq S_0 - \frac{2b\omega^2(\alpha+\beta)}{Da} - \frac{a(v+D)}{Db}\langle x_1(t)\rangle - \frac{av}{Db}\langle x_2(t)\rangle - \frac{a}{Db}\varepsilon(t). \tag{3.4}$$

Applying Itô's formula gives
$$\begin{aligned}\mathrm{d}(\ln x_1 + \ln x_2) = &\left[-v - D + \frac{bS}{m+S} - \alpha x_1 - \beta x_2 + \frac{r_2x_2}{x_1} - r_1 - \frac{\sigma^2b^2S^2}{2(m+S)^2} - v + \frac{bS}{m+S}\right. \\ &\left. -\alpha x_1 - \beta x_2 - r_2 + \frac{r_1x_1}{x_2} - \frac{\sigma^2b^2S^2}{2(m+S)^2}\right]\mathrm{d}t + \frac{2\sigma bS}{m+S}\mathrm{d}B(t) \\ \geq &\left[-2v - D - r_1 - r_2 - 2\alpha x_1 - 2\beta x_2 + \frac{2bS}{m+S} - \frac{\sigma^2b^2S^2}{(m+S)^2}\right]\mathrm{d}t + \frac{2\sigma bS}{m+S}\mathrm{d}B(t) \\ \geq &\left[-2v - D - r_1 - r_2 - 2\alpha x_1 - 2\beta x_2 + \frac{2bS}{m+S_0} - \sigma^2b^2\right]\mathrm{d}t + \frac{2\sigma bS}{m+S}\mathrm{d}B(t),\end{aligned}$$

from which we can obtain
$$\begin{aligned}&\frac{\ln x_1(t) - \ln x_1(0)}{t} + \frac{\ln x_2(t) - \ln x_2(0)}{t} \\ &\geq -2v - D - r_1 - r_2 - 2\alpha\langle x_1\rangle - 2\beta\langle x_2\rangle + \frac{2b}{m+S_0}\langle S\rangle - \sigma^2b^2 + \frac{1}{t}\int_0^t \frac{2\sigma bS(\tau)}{m+S(\tau)}\mathrm{d}B(\tau) \\ &\geq -2v - D - r_1 - r_2 - \sigma^2b^2 - 2\alpha\langle x_1\rangle - 2\beta\langle x_2\rangle + \frac{1}{t}\int_0^t \frac{2\sigma bS(\tau)}{m+S(\tau)}\mathrm{d}B(\tau) + \frac{2bS_0}{m+S_0} \\ &\quad - \frac{4b^2\omega^2(\alpha+\beta)}{(m+S_0)Da} - \frac{2a(v+D)}{D(m+S_0)}\langle x_1\rangle - \frac{2av}{D(m+S_0)}\langle x_2\rangle - \frac{2a}{(m+S_0)D}\varepsilon(t) \\ &\geq \frac{2bS_0}{m+S_0} - 2v - D - r_1 - r_2 - \sigma^2b^2 - \left(2\alpha + \frac{2a(v+D)}{D(m+S_0)}\right)\langle x_1\rangle - \left(2\beta + \frac{2av}{D(m+S_0)}\right)\langle x_2\rangle \\ &\quad - \frac{4b^2\omega^2(\alpha+\beta)}{(m+S_0)Da} + \frac{1}{t}\int_0^t \frac{2\sigma bS(\tau)}{m+S(\tau)}\mathrm{d}B(\tau) - \frac{2a}{(m+S_0)D}\varepsilon(t).\end{aligned}$$

From the strong law of large number of martingales, we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\frac{2\sigma bS(\tau)}{m+S(\tau)}\mathrm{d}B(\tau)=0.$$

Since

$$\lim_{t\to\infty}\frac{\ln x_1(t)+\ln x_2(t)}{t}=0,\lim_{t\to\infty}\varepsilon(t)=0.$$

When $R^*>0$ we can get

$$\liminf_{t\to\infty}(\langle x_1\rangle+\langle x_2\rangle)\geq\frac{1}{Q}\left[\frac{2bS_0}{m+S_0}-2v-D-r_1-r_2-\sigma^2b^2-\frac{4b^2\omega^2(\alpha+\beta)}{(m+S_0)Da}\right]>0,$$

where

$$Q=\max\left\{2\alpha+\frac{2a(v+D)}{D(m+S_0)},2\beta+\frac{2av}{D(m+S_0)}\right\}.$$

This completes the proof. □

3.3. Existence of ergodic stationary distribution of system (1.1)

In this subsection, we explore sufficient conditions for the existence of a unique stationary ergodic distribution.

Define

$$R=\frac{DbS_0}{2(m+S_0)\left(D+v+r_1+\frac{\sigma^2b^2}{2}\right)\left(D+v+r_2+\frac{\sigma^2b^2}{2}\right)}.$$

Theorem 3.3. *Assume $R>1$, then system* (1.1) *exists a unique ergodic stationary distribution.*

Proof. In order to prove this theorem, it is necessary to prove that system (1.1) satisfies the conditions in Lemma 2.1. Define

$$\begin{aligned}V_1(S,x_1,x_2)&=M\left[-c_1\ln S-c_2\ln x_1-c_3\ln x_2\right]+\frac{x_1^2}{2}+\frac{x_2^2}{2}\\&=MV_2+V_3.\end{aligned}$$

$V_1(S,x_1,x_2)$ is a continuous function, so it exists a minimum V_{1min}. We can get a C^2-function $V:\mathbb{R}_+^3\to\mathbb{R}_+$:

$$V(S,x_1,x_2)=V_1(S,x_1,x_2)-V_{1min}.$$

Applying Itô's formula gives

$$\begin{aligned}LV_2=&-\frac{c_1}{S}\left[DS_0-DS-\frac{aS}{m+S}(x_1+x_2)\right]+\frac{c_1\sigma^2a^2}{2(m+S)^2}(x_1+x_2)^2-\frac{c_2}{x_1}\left[-vx_1-Dx_1\right.\\&\left.+x_1\left(\frac{bS}{m+S}-\alpha x_1-\beta x_2\right)+r_2x_2-r_1x_1\right]+\frac{c_2\sigma^2b^2S^2}{2(m+S)^2}-\frac{c_3}{x_2}\left[-vx_2-r_2x_2\right.\\&\left.+x_2\left(\frac{bS}{m+S}-\alpha x_1-\beta x_2\right)+r_1x_1\right]+\frac{c_3\sigma^2b^2S^2}{2(m+S)^2}\end{aligned}$$

$$
\begin{aligned}
&= -\frac{c_1 D S_0}{S} + c_1 D + \frac{c_1 a}{m+S}(x_1+x_2) + \frac{c_1\sigma^2 a^2}{2(m+S)^2}(x_1+x_2)^2 + c_2 v + c_2 D - \frac{c_2 b S}{m+S} \\
&+ c_2\alpha x_1 + c_2\beta x_2 - \frac{c_2 r_2 x_2}{x_1} + c_2 r_1 + \frac{c_2\sigma^2 b^2 S^2}{2(m+S)^2} + c_3 v - \frac{c_3 b S}{m+S} + c_3\alpha x_1 + c_3\beta x_2 \\
&+ c_3 r_2 - \frac{c_3 r_1 x_1}{x_2} + \frac{c_3\sigma^2 b^2 S^2}{2(m+S)^2} \\
&\le -\frac{c_1 D S_0}{2S} - \frac{c_2 b S}{m+S_0} - c_3 D + c_1 D + c_2\left(D + v + r_1 + \frac{\sigma^2 b^2}{2}\right) + c_3\left(D + v + r_2 + \frac{\sigma^2 b^2}{2}\right) \\
&+ \frac{c_1\sigma^2 a^2}{2m^2}(x_1+x_2)^2 + \left(\frac{c_1 a}{m} + c_2\alpha + c_3\alpha\right)x_1 + \left(\frac{c_1 a}{m} + c_2\beta + c_3\beta\right)x_2 - \frac{c_3 b S}{m+S_0} - \frac{c_1 D S_0}{2S} \\
&\le -3\sqrt[3]{\frac{c_1 c_2 c_3 D^2 b S_0}{2(m+S_0)}} + c_1 D + c_2\left(D + v + r_1 + \frac{\sigma^2 b^2}{2}\right) + c_3\left(D + v + r_2 + \frac{\sigma^2 b^2}{2}\right) \\
&+ \frac{c_1\sigma^2 a^2}{2m^2}(x_1+x_2)^2 + \left(\frac{c_1 a}{m} + c_2\alpha + c_3\alpha\right)x_1 + \left(\frac{c_1 a}{m} + c_2\beta + c_3\beta\right)x_2 - \frac{c_3 b S}{m+S_0} - \frac{c_1 D S_0}{2S} \\
&= -3DS_0\left(\sqrt[3]{R} - 1\right) + \left(\frac{c_1 a}{m} + c_2\alpha + c_3\alpha\right)x_1 + \left(\frac{c_1 a}{m} + c_2\beta + c_3\beta\right)x_2 - \frac{c_3 b S}{m+S_0} \\
&- \frac{c_1 D S_0}{2S} + \frac{c_1\sigma^2 a^2}{2m^2}(x_1+x_2)^2,
\end{aligned}
$$

where $c_1 = S_0$, $c_2 = \frac{DS_0}{D+v+r_1+\frac{\sigma^2 b^2}{2}}$ and $c_3 = \frac{DS_0}{D+v+r_2+\frac{\sigma^2 b^2}{2}}$, so that

$$
c_1 D = c_2\left(D + v + r_1 + \frac{\sigma^2 b^2}{2}\right) = c_3\left(D + v + r_2 + \frac{\sigma^2 b^2}{2}\right) = DS_0.
$$

In the same way, we can get

$$
\begin{aligned}
LV_3 &= \frac{bSx_1^2}{m+S} - \alpha x_1^3 - \beta x_2 x_1^2 - v x_1^2 - D x_1^2 + r_2 x_1 x_2 - r_1 x_1^2 + \frac{\sigma^2 b^2 S^2 x_1^2}{2(m+S)^2} + \frac{bSx_2^2}{m+S} \\
&- \alpha x_1 x_2^2 - \beta x_2^3 - v x_2^2 - r_2 x_2^2 + r_1 x_1 x_2 + \frac{\sigma^2 b^2 S^2 x_2^2}{2(m+S)^2} \\
&\le b x_1^2 - \alpha x_1^3 - \beta x_2 x_1^2 + r_2 x_1 x_2 + \frac{\sigma^2 b^2}{2}\left(x_1^2 + x_2^2\right) + b x_2^2 - \alpha x_1 x_2^2 - \beta x_2^3 + r_1 x_1 x_2.
\end{aligned}
$$

Then,

$$
\begin{aligned}
LV &= MLV_2 + LV_3 \\
&\le -3MDS_0\left(\sqrt[3]{R} - 1\right) + \left(\frac{c_1 a}{m} + c_2\alpha + c_3\alpha\right)Mx_1 + \left(\frac{c_1 a}{m} + c_2\beta + c_3\beta\right)Mx_2 - \frac{Mc_3 b S}{m+S_0} \\
&- \frac{Mc_1 D S_0}{2S} + \frac{Mc_1\sigma^2 a^2}{2m^2}(x_1+x_2)^2 + \left(b + \frac{\sigma^2 b^2}{2}\right)x_1^2 + \left(b + \frac{\sigma^2 b^2}{2}\right)x_2^2 + (r_1 + r_2)x_1 x_2 \\
&- \alpha x_1^3 - \beta x_2^3 - \alpha x_1 x_2^2 - \beta x_1^2 x_2,
\end{aligned}
$$

where M is a sufficiently large constant such that

$$
-3MDS_0\left(\sqrt[3]{R} - 1\right) + \max\{A, B, C, E, F\} \le -2, \tag{3.5}
$$

where A, B, C, E, F are defined in the following.

Define a compact bounded subset U,

$$U = \left\{(S, x_1, x_2) \in \mathbb{R}_+^3 : \epsilon \le S \le \frac{1}{\epsilon}, \epsilon \le x_1 \le \frac{1}{\epsilon}, \epsilon \le x_2 \le \frac{1}{\epsilon}\right\},$$

and in the set $\mathbb{R}_+^3 \setminus U$, choosing ϵ small enough such that

$$-3MDS_0\left(\sqrt[3]{R} - 1\right) - \frac{c_1MDS_0}{2\epsilon} + A \le -1, \tag{3.6}$$

$$-3MDS_0\left(\sqrt[3]{R} - 1\right) + \left(\frac{c_1a}{m} + c_2\alpha + c_3\alpha\right)M\epsilon + \left(b + \frac{\sigma^2b^2}{2}\right)\epsilon^2 + B \le -1, \tag{3.7}$$

$$-3MDS_0\left(\sqrt[3]{R} - 1\right) + \left(\frac{c_1a}{m} + c_2\beta + c_3\beta\right)M\epsilon + \left(b + \frac{\sigma^2b^2}{2}\right)\epsilon^2 + C \le -1, \tag{3.8}$$

$$-3MDS_0\left(\sqrt[3]{R} - 1\right) - \frac{c_3Mb}{(m + S_0)\epsilon} + A \le -1, \tag{3.9}$$

$$-3MDS_0\left(\sqrt[3]{R} - 1\right) - \frac{\alpha}{2\epsilon^3} + E \le -1, \tag{3.10}$$

$$-3MDS_0\left(\sqrt[3]{R} - 1\right) - \frac{\beta}{2\epsilon^3} + F \le -1. \tag{3.11}$$

Next, six domains are given in the following,

$$U_1 = \left\{(S, x_1, x_2) \in \mathbb{R}_+^3 : 0 < S < \epsilon\right\}, U_2 = \left\{(S, x_1, x_2) \in \mathbb{R}_+^3 : 0 < x_1 < \epsilon\right\},$$
$$U_3 = \left\{(S, x_1, x_2) \in \mathbb{R}_+^3 : 0 < x_2 < \epsilon\right\}, U_4 = \left\{(S, x_1, x_2) \in \mathbb{R}_+^3 : S > \frac{1}{\epsilon}\right\},$$
$$U_5 = \left\{(S, x_1, x_2) \in \mathbb{R}_+^3 : x_1 > \frac{1}{\epsilon}\right\}, U_6 = \left\{(S, x_1, x_2) \in \mathbb{R}_+^3 : x_2 > \frac{1}{\epsilon}\right\}.$$

Case 1. If $(S, x_1, x_2) \in U_1$,

$$\begin{aligned} LV \le & -3MDS_0\left(\sqrt[3]{R} - 1\right) + \left(\frac{c_1a}{m} + c_2\alpha + c_3\alpha\right)Mx_1 + \left(\frac{c_1a}{m} + c_2\beta + c_3\beta\right)Mx_2 - \frac{Mc_1DS_0}{2S} \\ & + \frac{Mc_1\sigma^2a^2}{2m^2}(x_1 + x_2)^2 + \left(b + \frac{\sigma^2b^2}{2}\right)x_1^2 + \left(b + \frac{\sigma^2b^2}{2}\right)x_2^2 + (r_1 + r_2)x_1x_2 - \alpha x_1^3 - \beta x_2^3 \\ & - \alpha x_1x_2^2 - \beta x_1^2x_2 \\ \le & -3MDS_0\left(\sqrt[3]{R} - 1\right) - \frac{c_1MDS_0}{2\epsilon} + A, \end{aligned}$$

where

$$A=\sup_{(S,x_1,x_2)\in\mathbb{R}^3_+}\left\{\left(\frac{c_1a}{m}+c_2\alpha+c_3\alpha\right)Mx_1+\left(\frac{c_1a}{m}+c_2\beta+c_3\beta\right)Mx_2-\alpha x_1x_2^2-\beta x_1^2x_2-\beta x_2^3\right.$$
$$\left.+\frac{Mc_1\sigma^2a^2}{2m^2}(x_1+x_2)^2+\left(b+\frac{\sigma^2b^2}{2}\right)x_1^2+\left(b+\frac{\sigma^2b^2}{2}\right)x_2^2+(r_1+r_2)x_1x_2-\alpha x_1^3\right\}. \tag{3.12}$$

Together with (3.5) and (3.6), we get $LV\leq -1$.

Case 2. If $(S,x_1,x_2)\in U_2$,

$$\begin{aligned}LV\leq& -3MDS_0\left(\sqrt[3]{R}-1\right)+\left(\frac{c_1a}{m}+c_2\alpha+c_3\alpha\right)Mx_1+\left(\frac{c_1a}{m}+c_2\beta+c_3\beta\right)Mx_2-\beta x_2^3\\&+\frac{Mc_1\sigma^2a^2}{2m^2}(x_1+x_2)^2+\left(b+\frac{\sigma^2b^2}{2}\right)x_1^2+\left(b+\frac{\sigma^2b^2}{2}\right)x_2^2+(r_1+r_2)x_1x_2-\alpha x_1^3\\&-\alpha x_1x_2^2-\beta x_1^2x_2\\ \leq& -3MDS_0\left(\sqrt[3]{R}-1\right)+\left(\frac{c_1a}{m}+c_2\alpha+c_3\alpha\right)M\epsilon+\left(b+\frac{\sigma^2b^2}{2}\right)\epsilon^2+B,\end{aligned}$$

where

$$B=\sup_{(S,x_1,x_2)\in\mathbb{R}^3_+}\left\{\left(\frac{c_1a}{m}+c_2\beta+c_3\beta\right)Mx_2+\frac{Mc_1\sigma^2a^2}{2m^2}(x_1+x_2)^2+\left(b+\frac{\sigma^2b^2}{2}\right)x_2^2\right.$$
$$\left.+(r_1+r_2)x_1x_2-\alpha x_1^3-\alpha x_1x_2^2-\beta x_1^2x_2-\beta x_2^3\right\}. \tag{3.13}$$

On account of (3.5) and (3.7), we get $LV\leq -1$.

Case 3. If $(S,x_1,x_2)\in U_3$,

$$\begin{aligned}LV\leq& -3MDS_0\left(\sqrt[3]{R}-1\right)+\left(\frac{c_1a}{m}+c_2\alpha+c_3\alpha\right)Mx_1+\left(\frac{c_1a}{m}+c_2\beta+c_3\beta\right)Mx_2-\beta x_2^3\\&+\frac{Mc_1\sigma^2a^2}{2m^2}(x_1+x_2)^2+\left(b+\frac{\sigma^2b^2}{2}\right)x_1^2+\left(b+\frac{\sigma^2b^2}{2}\right)x_2^2+(r_1+r_2)x_1x_2-\alpha x_1^3\\&-\alpha x_1x_2^2-\beta x_1^2x_2\\ \leq& -3MDS_0\left(\sqrt[3]{R}-1\right)+\left(\frac{c_1a}{m}+c_2\beta+c_3\beta\right)M\epsilon+\left(b+\frac{\sigma^2b^2}{2}\right)\epsilon^2+C,\end{aligned}$$

where

$$C=\sup_{(S,x_1,x_2)\in\mathbb{R}^3_+}\left\{\left(\frac{c_1a}{m}+c_2\alpha+c_3\alpha\right)Mx_1+\frac{Mc_1\sigma^2a^2}{2m^2}(x_1+x_2)^2+\left(b+\frac{\sigma^2b^2}{2}\right)x_1^2\right.$$
$$\left.+(r_1+r_2)x_1x_2-\alpha x_1^3-\alpha x_1x_2^2-\beta x_1^2x_2-\beta x_2^3\right\}. \tag{3.14}$$

In view of (3.5) and (3.8), we get $LV\leq -1$.

Case 4. If $(S,x_1,x_2)\in U_4$,

$$LV\leq -3MDS_0\left(\sqrt[3]{R}-1\right)+\left(\frac{c_1a}{m}+c_2\alpha+c_3\alpha\right)Mx_1+\left(\frac{c_1a}{m}+c_2\beta+c_3\beta\right)Mx_2-\frac{Mc_3bS}{m+S_0}$$

$$
\begin{aligned}
&+\frac{Mc_1\sigma^2a^2}{2m^2}(x_1+x_2)^2+\left(b+\frac{\sigma^2b^2}{2}\right)x_1^2+\left(b+\frac{\sigma^2b^2}{2}\right)x_2^2+(r_1+r_2)x_1x_2-\alpha x_1^3\\
&-\beta x_2^3-\alpha x_1x_2^2-\beta x_1^2x_2\\
&\leq -3MDS_0\left(\sqrt[3]{R}-1\right)-\frac{c_3Mb}{(m+S_0)\epsilon}+A.
\end{aligned}
$$

Considering (3.5) and (3.9), we get $LV \leq -1$.

Case 5. If $(S, x_1, x_2) \in U_5$,

$$
\begin{aligned}
LV \leq& -3MDS_0\left(\sqrt[3]{R}-1\right)+\left(\frac{c_1a}{m}+c_2\alpha+c_3\alpha\right)Mx_1+\left(\frac{c_1a}{m}+c_2\beta+c_3\beta\right)Mx_2\\
&+\frac{Mc_1\sigma^2a^2}{2m^2}(x_1+x_2)^2+\left(b+\frac{\sigma^2b^2}{2}\right)x_1^2+\left(b+\frac{\sigma^2b^2}{2}\right)x_2^2+(r_1+r_2)x_1x_2-\alpha x_1^3\\
&-\beta x_2^3-\alpha x_1x_2^2-\beta x_1^2x_2\\
&\leq -3MDS_0\left(\sqrt[3]{R}-1\right)-\frac{\alpha}{2\epsilon^3}+E,
\end{aligned}
$$

where

$$
\begin{aligned}
E=\sup_{(S,x_1,x_2)\in\mathbb{R}_+^3}&\left\{\left(\frac{c_1a}{m}+c_2\alpha+c_3\alpha\right)Mx_1+\left(\frac{c_1a}{m}+c_2\beta+c_3\beta\right)Mx_2+\frac{Mc_1\sigma^2a^2}{2m^2}(x_1+x_2)^2\right.\\
&\left.+\left(b+\frac{\sigma^2b^2}{2}\right)x_1^2+\left(b+\frac{\sigma^2b^2}{2}\right)x_2^2+(r_1+r_2)x_1x_2-\frac{\alpha}{2}x_1^3-\beta x_2^3-\alpha x_1x_2^2-\beta x_1^2x_2\right\}. \qquad (3.15)
\end{aligned}
$$

Combining with (3.5) and (3.10), we get $LV \leq -1$.

Case 6. If $(S, x_1, x_2) \in U_6$,

$$
\begin{aligned}
LV \leq& -3MDS_0\left(\sqrt[3]{R}-1\right)+\left(\frac{c_1a}{m}+c_2\alpha+c_3\alpha\right)Mx_1+\left(\frac{c_1a}{m}+c_2\beta+c_3\beta\right)Mx_2\\
&+\frac{Mc_1\sigma^2a^2}{2m^2}(x_1+x_2)^2+\left(b+\frac{\sigma^2b^2}{2}\right)x_1^2+\left(b+\frac{\sigma^2b^2}{2}\right)x_2^2+(r_1+r_2)x_1x_2-\alpha x_1^3\\
&-\beta x_2^3-\alpha x_1x_2^2-\beta x_1^2x_2\\
&\leq -3MDS_0\left(\sqrt[3]{R}-1\right)-\frac{\beta}{2\epsilon^3}+F,
\end{aligned}
$$

where

$$
\begin{aligned}
F=\sup_{(S,x_1,x_2)\in\mathbb{R}_+^3}&\left\{\left(\frac{c_1a}{m}+c_2\alpha+c_3\alpha\right)Mx_1+\left(\frac{c_1a}{m}+c_2\beta+c_3\beta\right)Mx_2+\frac{Mc_1\sigma^2a^2}{2m^2}(x_1+x_2)^2\right.\\
&\left.+\left(b+\frac{\sigma^2b^2}{2}\right)x_1^2+\left(b+\frac{\sigma^2b^2}{2}\right)x_2^2+(r_1+r_2)x_1x_2-\frac{\beta}{2}x_2^3-\alpha x_1^3-\alpha x_1x_2^2-\beta x_1^2x_2\right\}. \qquad (3.16)
\end{aligned}
$$

Together with (3.5) and (3.11), we get $LV \leq -1$.

Evidently, there exists a sufficiently small ε such that $LV \leq -1$, for any $(S, x_1, x_2) \in R_+^3 \setminus U$.

In addition, the diffusion matrix of the system (1.1) is

$$\begin{pmatrix} \frac{\sigma^2 a^2 S^2}{(m+S)^2}(x_1+x_2)^2 & 0 & 0 \\ 0 & \frac{\sigma^2 b^2 S^2 x_1^2}{(m+S)^2} & 0 \\ 0 & 0 & \frac{\sigma^2 b^2 S^2 x_2^2}{(m+S)^2} \end{pmatrix}.$$

Then there exists a positive $N = \min_{(S,x_1,x_2)\in U_\varepsilon} \left\{ \frac{\sigma^2 a^2 S^2}{(m+S)^2}(x_1+x_2)^2, \frac{\sigma^2 b^2 S^2 x_1^2}{(m+S)^2}, \frac{\sigma^2 b^2 S^2 x_2^2}{(m+S)^2} \right\}$ such that

$$\sum_{i,j=1}^{3} a_{i,j}\xi_i\xi_j = \frac{\sigma^2 a^2 S^2}{(m+S)^2}(x_1+x_2)^2\xi_1^2 + \frac{\sigma^2 b^2 S^2 x_1^2}{(m+S)^2}\xi_2^2 + \frac{\sigma^2 b^2 S^2 x_2^2}{(m+S)^2}\xi_3^2 \geq N|\xi|^2,$$

where $(S, x_1, x_2) \in U$, $\xi = (\xi_1, \xi_2, \xi_3) \in R_+^3$. Obviously this satisfies the conditions in the Lemma 2.1.

Hence, the system (1.1) exists a unique stationary distribution. □

4. Numerical simulations

For the theoretical results obtained above, we will conduct further verification through numerical simulation.

Example 1. We use numerical simulations to verify the conditions under which microbes would go extinct. For details about parameter values, see Tables 2 and 3.

Table 2. Parameter settings for Figure 1(a).

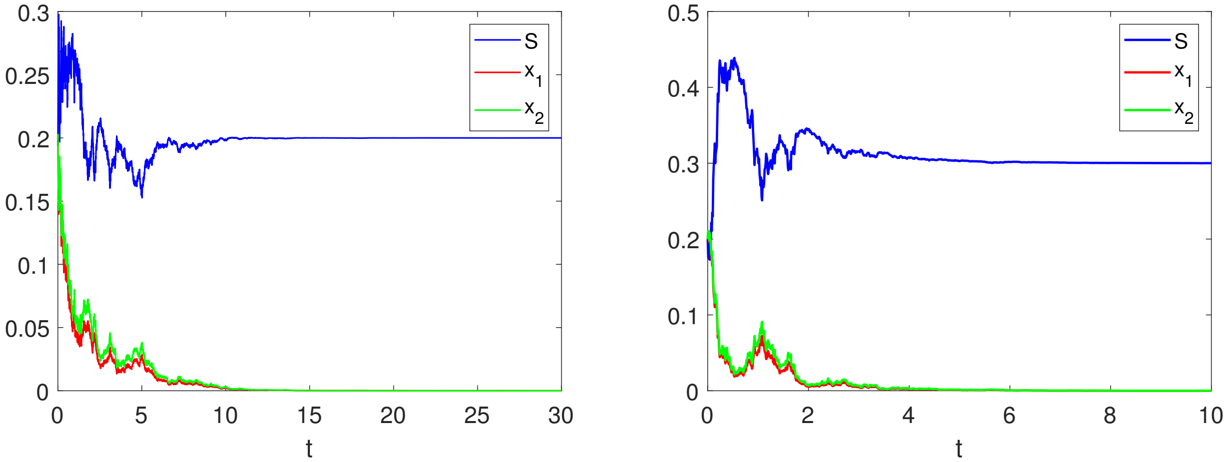
Parameters	$S(0)$	$x_1(0)$	$x_2(0)$	S_0	D	a	b	m	v	α	β	r_1	r_2	σ
value	0.2	0.2	0.2	0.2	1	2	4	1	0.5	0.1	0.2	0.5	0.8	1.4

Table 3. Parameter settings for Figure 1(b).

Parameters	$S(0)$	$x_1(0)$	$x_2(0)$	S_0	D	a	b	m	v	α	β	r_1	r_2	σ
value	0.2	0.2	0.2	0.3	1	2	4	1	0.5	0.1	0.2	0.5	0.8	2

And the results are as follows.

The changes in the density of microorganisms and nutrient are shown in Figure 1, the red and green tracks gradually approach zero. Under those parameter settings, $R_1^* = -0.0511 < 0$ and $R_2^* = -0.25 < 0$, the threshold conditions of microbial extinction are met. Therefore, the theoretical results are consistent with the numerical simulation results.



(a) $\frac{bS_0}{m+S_0} < \frac{2}{\sigma^2}$ and $R_1^* < 0$ (b) $\frac{bS_0}{m+S_0} \geq \frac{2}{\sigma^2}$ and $R_2^* < 0$

Figure 1. The trajectories of the solution of the stochastic model.

From the point of view of extinction threshold, it is easy to analyze that large noise leads to microbial extinction. Next, we further analyze the effect of noise on the survival of gut microbes. In Example 2, we demonstrate that the enteric microorganism could still survive for a long time under the influence of certain noise.

Example 2. The parameters of composite number are selected to meet the conditions of microbial persistence, that is, $R^* > 0$. The correctness of the conclusion is verified by the following numerical simulation. Parameter settings are shown in Table 4, at this point, $R^* = 0.1096 > 0$, and simulation results are shown in Figure 2.

Table 4. Parameter Settings for Example 2.

Parameters	$S(0)$	$x_1(0)$	$x_2(0)$	S_0	D	a	b	m	ν	α	β	r_1	r_2	σ
value	0.03	0.03	0.03	0.4	0.03	2	0.4	0.25	0.03	0.05	0.05	0.01	0.02	0.03

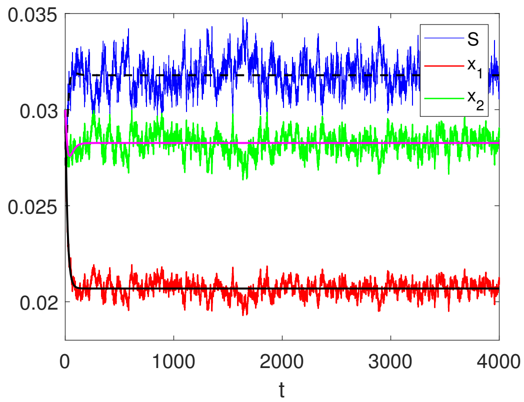


Figure 2. The trajectories of the solutions of the stochastic model and the deterministic model.

In addition to the trajectories indicated by the legend in Figure 2, the black dashed line, the black solid line and the purple solid line represent the trajectories of S, x_1 and x_2 of the deterministic model, respectively.

As we can see from the figure, when it is not affected by noise, that is, $\sigma = 0$, the microorganism will survive for a long time. When affected by a certain noise, the trajectories fluctuate near the trajectories of the corresponding deterministic model, and all of them are above the X-axis. In other words, microbes are still persistent.

Example 3. On the basis of Example 2, with other parameters unchanged and $\sigma = 0.7$, the numerical simulation results are shown in Figure 3.

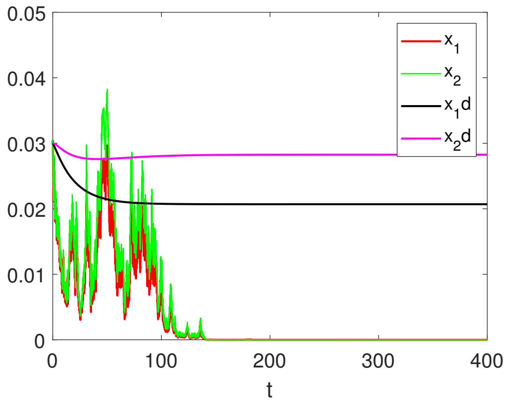


Figure 3. The trajectories of x_1, x_2 of the stochastic model and the deterministic model.

Under the influence of high noise, the originally persistent microorganisms tend to become extinct. Therefore, it can be concluded that the strong noise has adverse effects on the survival of microorganisms. This conclusion is reflected in the threshold values of microbial persistence and extinction.

Example 4. Finally, we verify the existence of stationary ergodic distribution through numerical simulation. The density distributions of microorganisms and nutrient are shown in Figure 4. Set the parameters as Table 5.

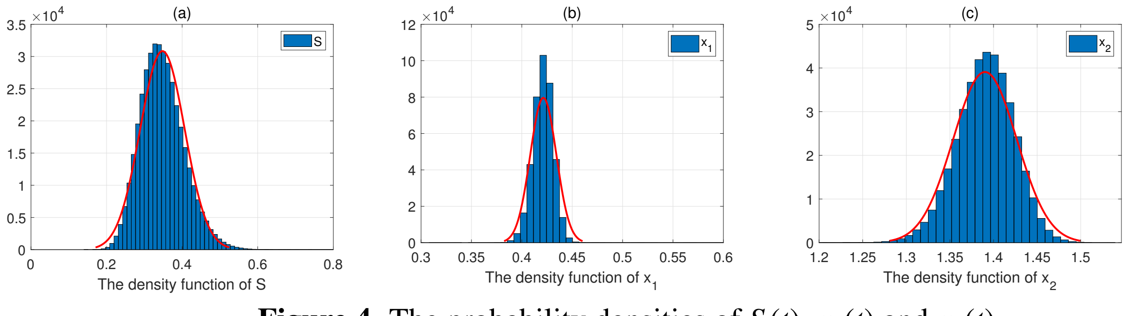


Figure 4. The probability densities of $S(t)$, $x_1(t)$ and $x_2(t)$.

Table 5. Parameter Settings for Example 4.

Parameters	$S(0)$	$x_1(0)$	$x_2(0)$	S_0	D	a	b	m	v	α	β	r_1	r_2	σ
value	0.2	0.2	0.2	10	0.3	3	3	0.3	0.05	0.5	0.9	0.1	0.1	0.05

In this case, $R = 2.0535 > 1$, and the theoretical results are consistent with the numerical simulation.

5. Conclusions

This paper proposes a stochastic chemostat model for intestinal microorganisms that considers migration and intraspecific competition. The dynamic behavior of the model is analyzed by ordinary differential theory. By constructing the Lyapunov functions, the threshold conditions of intestinal microbial extinction and persistence in mean are obtained, and the condition for the existence of stationary ergodic distribution is analyzed. Analysis shows that high noise increases the risk of extinction of gut microbes. At the same time, too fast intestinal velocity will also affect the survival of microorganisms. Now, we sum up the main results as follows.

(1) Microorganisms tend to become extinct when the following conditions are met.
 (i) If $\frac{bS_0}{m+S_0} < \frac{2}{\sigma^2}$ and $R_1^* < 0$, then microbials tend to extinction almost surely.
 (ii) If $\frac{bS_0}{m+S_0} \geq \frac{2}{\sigma^2}$ and $R_2^* < 0$, then microbials tend to extinction almost surely.

(2) If $R^* > 0$, then the microorganism is persistent in mean. In addtion, we get

$$\liminf_{t\to\infty} (\langle x_1\rangle + \langle x_2\rangle) \geq \frac{R^*}{Q} > 0.$$

(3) Assume $R > 1$, then system (1.1) exists a unique ergodic stationary distribution.

By overcoming the difficulties in Lyapunov function construction and inequality reduction, the sufficient conditions for microbial extinction, persistence and stationary ergodic distribution were obtained. The threshold of microbial persistence or extinction is influenced by environmental disturbance (noise intensity). The results of numerical simulation also show that a large degree of noise will interfere with the survival of microorganisms, so that the microorganisms that can live for a long time tend to become extinct.

Acknowledgments

This work is supported by the Shandong Provincial Natural Science Foundation of China (No.ZR2019MA003), the Research Fund for the Taishan Scholar Project of Shandong Province of China and the SDUST Innovation Fund for Graduate Students (No.YC20210291).

Conflict of interest

The authors declare no conflicts of interest.

References

1. J. A. Gilbert, R. A. Quinn, J. Debelius, Z. Z. Xu, J. Morton, N. Garg, et al., Microbiome-wide association studies link dynamic microbial consortia to disease, *Nature*, **535** (2016), 94–103. https://doi.org/10.1038/nature18850

2. G. W. Tannock, *Normal microflora: an introduction to microbes inhabiting the human body*, Springer Science Business Media, 1995.
3. L. V. Hooper, D. R. Littman, A. J. Macpherson, Interactions between the microbiota and the immune system, *Science*, **336** (2012), 1268–1273. https://science.org/doi/abs/10.1126/science.1223490
4. J. Halfvarson, C. J. Brislawn, R. Lamendella, Y. Vázquez-Baeza, W. A. Walters, L. M. Bramer, et al., Dynamics of the human gut microbiome in inflammatory bowel disease, *Nat. Microbiol.*, **2** (2017), 1–7. https://doi.org/10.1038/nmicrobiol.2017.4
5. B. Tang, Mathematical investigations of growth of microorganisms in the gradostat, *J. Math. Biol.*, **23** (1986), 319–339. https://doi.org/10.1007/BF00275252
6. A. Mitchell, Y. Pilpel, A mathematical model for adaptive prediction of environmental changes by microorganisms, *Proc. Natl. Acad. Sci. U S A*, **108** (2011), 7271–7276. https://doi.org/10.1073/pnas.1019754108
7. B. Hall, X. Han, P. E. Kloeden, H. W. Van Wyk, A nonautonomous chemostat model for the growth of gut microbiome with varying nutrient, *Discrete Cont. Dyn. Syst.*, **15** (2022), 2889–2908. https://doi.org/10.3934/dcdss.2022075
8. H. Qi, X. Leng, X. Meng, T. Zhang, Periodic solution and ergodic stationary distribution of SEIS dynamical systems with active and latent patients, *Qual. Theory Dyn. Syst.*, **18** (2019), 347–369. https://doi.org/10.1007/s12346-018-0289-9
9. C. Fritsch, J. Harmand, F. Campillo, A modeling approach of the chemostat, *Ecol. Modell.*, **299** (2015), 1–13. https://doi.org/10.1016/j.ecolmodel.2014.11.021
10. J. Monod, Technique, theory and applications of continuous culture, *Ann. Inst. Pasteur*, **79** (1950), 390–410.
11. A. Novick, L. Szilard, Description of the chemostat, *Science*, **112** (1950), 715–716. https:/10.1126/science.112.2920.715
12. Z. Liu, R. Tan, Impulsive harvesting and stocking in a Monod-Haldane functional response predator-prey system, *Chaos Solitons Fract.*, **34** (2007), 454–464. https://doi.org/10.1016/j.chaos.2006.03.054
13. X. Zhang, R. Yuan, A stochastic chemostat model with mean-reverting Ornstein-Uhlenbeck process and Monod-Haldane response function, *Appl. Math. Comput.*, **394** (2021), 125833. https://doi.org/10.1016/j.amc.2020.125833
14. M. Gao, D. Jiang, Stationary distribution of a chemostat model with distributed delay and stochastic perturbations, *Appl. Math. Lett.*, **123** (2020), 107585. https://doi.org/10.1016/j.aml.2021.107585
15. Y. Cai, Y. Kang, W. Wang, A stochastic SIRS epidemic model with nonlinear incidence rate, *Appl. Math. Comput.*, **305** (2017), 221–240. https://doi.org/10.1016/j.amc.2017.02.003
16. R. Liu, W. Ma, Noise-induced stochastic transition: a stochastic chemostat model with two complementary nutrients and flocculation effect, *Chaos Solitons Fract.*, **147** (2021), 110951. https://doi.org/10.1016/j.chaos.2021.110951
17. G. Liu, H. Qi, Z. Chang, X. Meng, Asymptotic stability of a stochastic may mutualism system, *Comput. Math. Appl.*, **79** (2020), 735–745. https://doi.org/10.1016/j.camwa.2019.07.022

18. F. Li, S. Zhang, X. Meng, Dynamics analysis and numerical simulations of a delayed stochastic epidemic model subject to a general response function, *Comput. Appl. Math.*, **38** (2019), 95. https://doi.org/10.1007/s40314-019-0857-x

19. R. S. Liptser, A strong law of large numbers for local martingales, *Stochastics*, **3** (1980), 217–228. https://doi.org/10.1080/17442508008833146

20. R. Khasminskii, *Stochastic stability of differential equations*, Springer Science and Business Media, 2011.

21. H. Qi, X. Meng, T. Hayat, A. Hobiny, Stationary distribution of a stochastic predator-prey model with hunting cooperation, *Appl. Math. Lett.*, **124** (2022), 107662. https://doi.org/10.1016/j.aml.2021.107662

22. H. Qi, X. Meng, Mathematical modeling, analysis and numerical simulation of HIV: the influence of stochastic environmental fluctuations on dynamics, *Math. Comput. Simulat.*, **187** (2021), 700–719. https://doi.org/10.1016/j.matcom.2021.03.027

23. C. Zeng, B. Liao, J. Huang, Dynamics of the stochastic chemostat model with Monod-Haldane response function, *J. Nonlinear. Mod. Anal.*, **1** (2019), 335–354. https://doi.org/10.12150/jnma.2019.335

24. B. Cao, M. Shan, Q. Zhang, W. Wang, A stochastic SIS epidemic model with vaccination, *Phys. A*, **486** (2017), 127–143. https://doi.org/10.1016/j.physa.2017.05.083

25. Q. Liu, D. Jiang, T. Hayat, A. Alsaedi, Stationary distribution of a stochastic delayed SVEIR epidemic model with vaccination and saturation incidence, *Phys. A*, **512** (2018), 849–863. https://doi.org/10.1016/j.physa.2018.08.054

26. M. Liu, K. Wang, Stochastic Lotka-Volterra systems with Lévy noise, *J. Math. Anal. Appl.*, **410** (2014), 750–763. https://doi.org/10.1016/j.jmaa.2013.07.078

27. S. Zhang, S. Yuan, T. Zhang, A predator-prey model with different response functions to juvenile and adult prey in deterministic and stochastic environments, *Appl. Math. Comput.*, **413** (2022), 126598. https://doi.org/10.1016/j.amc.2021.126598

28. T. Feng, X. Meng, T. Zhang, Z. Qiu, Analysis of the predator-prey interactions: a stochastic model incorporating disease invasion, *Qual. Theory Dyn. Syst.*, **19** (2020), 55. https://doi.org/10.1007/s12346-020-00391-4

29. Y. Zhou, W. Zhang, S. Yuan, Survival and stationary distribution of a SIR epidemic model with stochastic perturbations, *Appl. Math. Comput.*, **244** (2014), 118–131. https://doi.org/10.1016/j.amc.2014.06.100

30. G. Falsone, Stochastic differential calculus for Gaussian and non-Gaussian noises: a critical review, *Commun. Nonlinear Sci.*, **56** (2018), 198–216. http://dx.doi.org/10.1016/j.cnsns.2017.08.001

AIMS Press