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Research article

About coincidence points theorems on 2-step Carnot groups with 1-dimensional centre equipped with Box-quasimetrics

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Abstract: For some class of 2-step Carnot groups D_n with 1-dimensional centre we find the exact values of the constants in $(1, q_2)$-generalized triangle inequality for their Box-quasimetrics $\rho_{\mathrm{Box}_{D_n}}$. Using this result we get the best version of the Coincidence Points Theorem of α-covering and β-Lipschitz mappings defined on $(D_n, \rho_{\mathrm{Box}_{D_n}})$.

Keywords: (q_1, q_2)-quasimetric spase; Carnot group; exact value; Box-quasimetric; coincidence points; estimates of divergence
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1. Introduction

Consider a set X consisting of at least two points. Function $\rho_X : X\times X \to \mathbb{R}^+$, $\rho_X(x, y) = 0 \Leftrightarrow x = 0$, is called (q_1, q_2)-quasimetric if the following (q_1, q_2)-*generalized triangle inequality* takes place:

$$\rho_X(x, z) \leq q_1\rho_X(x, y) + q_2\rho_X(y, z) \quad \forall x, y, z \in X,$$

where q_1, q_2 are some positive numbers. Pair (X, ρ_X) is called (q_1, q_2)-*quasimetric space* [1–9]. The expression $\rho_X(x, y)$ denotes a (q_1, q_2)-quasi-distance exactly from the point x to the point y. If $q_1 = q_2 = 1$, then (X, ρ_X) is a *quasimetric space* [11].

If for a (q_1, q_2)-quasimetric ρ_X the following condition holds

$$\rho_X(x, y) \leq q_0\rho_X(y, x) \quad \forall x, y \in X$$

for some $q_0 > 0$ then we refer to a (q_1, q_2)-quasimetric space (X, ρ_X) as a q_0-*symmetric* one; for the case when $q_0 = 1$, we use the notion of *symmetric* (q_1, q_2)-*quasimetric space*. The (q_2, q_1)-quasimetric $\overline{\rho}_X(x, y) = \rho_X(y, x)$ is said to be conjugate to $\rho_X(x, y)$. If ρ_X is symmetric then $\overline{\rho}_X$ is symmetric too.

The class of (q_1, q_2)-quasimetric spaces is suffciently wide; it includes quasimetric spaces, b-metric spaces introduced by Bakhtin in 1989, Carnot-Carathéodory spaces with Box-quasimetrics, L_p-spaces with $p \in (0, 1)$, etc. (see [10]).

Definition 1.1. *For a* (q_1, q_2)*-quasimetric space* (X, ρ_X) *we denote by* $R = R(\rho_X)$ *the set of points* $(q_1', q_2') \in \mathbb{R}^2$*, such that for* ρ_X (q_1', q_2')*-generalized triangle inequality holds.*

The next Property 1.2 follows directly from the Definition 1.1.

Property 1.2 ([1,2]). 1^0 *The set* $R = R(d)$ *is convex and closed, and, moreover,* $R \subseteq \{(x, y) \in \mathbb{R}^2 \mid x \geq 1,\ y \geq 1\}$*;*

2^0 *The condition* $(1, 1) \in R$ *is equivalent to the fact that* ρ_X *is a quasimetric;*

3^0 *If* (q_1, q_2)*-quasimetric is symmetric, then the set* R *is symmetric with respect to the bisector of the right upper coordinate angle of the Euclidean plane.*

If $q' \in R$ and $\tilde{q} \geq q'$ (in the sense that $\tilde{q}_1 \geq q_1'$, $\tilde{q}_2 \geq q_2'$), then $\tilde{q} \in R$. By considering the support lines at the boundary points of the closed convex set R we obtain that R has extreme points. (Recall that a point $x_0 \in A$ is called an *extreme point* of a set A, if there are no points $x_1, x_2 \in A$, such that $x_0 \in (x_1, x_2)$, that is, $x_0 = tx_1 + (1 - t)x_2$ for some $0 < t < 1$.) We easily see that each extreme point of R is a Pareto optimal point of R (in the sense of minimization of components), but not conversely. Extreme points of R are said to be extreme for ρ_X. A point $q \in R$ is said to be *best* for ρ_X if $q \leq q'$ for all $q' \in R$. See the examples of (q_1, q_2)-quasimetric spaces with the best points $q = (q_1^0, q_2^0)$ such that $q_1^0 + q_2^0 > 2$, in [1,4–6].

An important special case of symmetric (q_1, q_2)-quasimetric spaces are the symmetric $(1, q_2)$-quasimetric spaces [1]; these include Carnot groups and more general equiregular Carnot-Carathéodory spaces $(M, \rho_{\mathrm{Box}_M})$, equipped by Box-quasimetrics ρ_{Box_M} [6–15]. Moreover, in the general case, the constant q_2 does not equal 1 [16]. Box-quasimetrics were introduced in [17]. $(1, q_2)$-generalized triangle inequality plays a crucial role in obtaining the divergence estimates of the equiregular Carnot-Carathéodory space $(M, \rho_{\mathrm{Box}_M})$ from its nilpotent tangent cone, see [18, 19].

Define the sets

$$B_X^o(x, r) = \{y \in X \mid \rho_X(x, y) < r\}, \quad B_X(x, r) = \{y \in X \mid \rho_X(x, y) \leq r\}.$$

A set $U \subset X$ is said to be *open* if, for every point $u \in U$ there is a number $r_u > 0$ such that $B_X^o(u, r_u) \subset U$. A set is said to be *closed* if its complement is open. The open sets defined in this way determine a topology on X.

A sequence of points $\{x_i\} \subset (X, \rho_X)$ *converges* to a point $x_0 \in X$ (we write $x_i \to x_0$) if, for every $\varepsilon > 0$ ball $B_X^o(x_0, \varepsilon)$ contains all points x_i, starting with some of them. The point x_0 is called the *limit of the sequence* $\{x_i\}$. Clearly, this definition may equivalently be restated in the following form: A sequence $\{x_i\}$ *converges* to x_0, if $\lim\limits_{i\to\infty} \rho_X(x_0, x_i) = 0$.

A sequence $\{x_n\}$ in a (q_1, q_2)-quasimetric space (X, ρ_X) is called a *fundamental sequence* or a *Cauchy sequence*, if for every $\varepsilon > 0$ there is an N such that for all $n > m > N$ we have $\rho_X(x_m, x_n) < \varepsilon$. A (q_1, q_2)-quasimetric space (X, ρ_X) is said to be *complete* if each of its fundamental sequences has a limit (possibly non-unique).

Consider a (q_1, q_2)-quasimetric space (X, ρ_X) and a (q_1', q_2')-quasimetric space (Y, ρ_Y). Let $\Psi, \Phi : X \to Y$ be mappings and $\alpha > \beta \geq 0$ be numbers.

Definition 1.3 ([1–3]). *A point $x \in X$ is called a coincidence point of the mappings Ψ, Φ if*

$$\Psi(x) = \Phi(x).$$

Definition 1.4 ([1–3]). *A mapping Ψ is said to be α-covering if*

$$B_Y(\Psi(x), \alpha r) \subseteq \Psi(B_X(x, r)) \ \forall r \geq 0 \ \forall x \in X.$$

Definition 1.5. *A mapping Φ is said to be β-Lipschitz if*

$$\rho_Y(\Phi(x_1), \Phi(x_2)) \leq \beta \rho_X(x_1, x_2) \ \forall x_1, x_2 \in X.$$

α-covering of Ψ means that for every $x_0 \in X$, $y_1 \in Y$ there is $x_1 \in X$ such that $y_1 = \Psi(x_1)$, $\rho_X(x_0, x_1) \leq \frac{\rho_Y(\Psi(x_0), y_1)}{\alpha}$; hence the mapping Ψ is surjective.

The Banach open mapping theorem provides a classical example of a covering mapping. Recall that the theorem states that if X is a Banach space, Y is a normed space, and ψ is a linear, continuous, and surjective operator, then for some $\alpha > 0$ the operator ψ is α-covering. Covering mappings and their properties have been studied in detail since the middle of the 20th Century. One of the first papers devoted to this issue was the paper [20] by L. M. Graves. In this paper, the covering property of linear mappings in Banach spaces were used to derive conditions for smooth mappings to be locally covering. Subsequently, Milyutin [21] obtained a theorem that provides sufficient covering conditions.

Theorem 1.6 ([21]). *Let X be a complete metric space, Y be a linear metric space with a translation-invariant metric ρ_Y, $\psi : X \to Y$ be continuous and α-covering, $\phi : X \to Y$ be β-Lipschitz, and $\beta < \alpha$. Then the mapping $\psi + \phi$ is $(\alpha - \beta)$-covering.*

This result is commonly called the *Milyutin theorem on Lipschitz perturbations of covering mappings*. Subsequently, the covering property and its stability under perturbations was a subject of numerous studies (see, for instance, [22–25]). Another problem to which the covering mappings theory is applicable is the coincidence points problem. Sufficient conditions for the existence of coincidence points stated in terms of covering mappings were proved by Arutyunov in [26] on metric spaces. In [26] there were also obtained conditions for existence of coincidence points of set-valued mappings. In [27, 28] the stability of coincidence points was investigated. The discussed above and some other results of covering mappings theory has applications in investigations of control systems [29], implicit differential equations (see [30, 31]), Volterra equations (see [32]). The theory of coincidence points of both single-valued and set-valued mappings of metric spaces plays an important role in analysis (see [33]). This theory is applied to the study of inclusions (see [34]). We note the following recent interesting works on the theory of coincidence points [35–37].

In their recent papers [1–3, 10], Arutyunov and Greshnov introduced (q_1, q_2)-quasimetric spaces and studied their properties; they studied covering mappings from one (q_1, q_2)-quasimetric space to another and obtained sufficient conditions for the existence of coincidence points of two mappings between such spaces provided that one of them is covering and the other satisfies the Lipschitz condition. These results were extended to multi-valued mappings. Also it was proved that the coincidence points are stable under small perturbations of the mappings. The development of the theory of coincidence points of mappings on (q_1, q_2)-quasimetric spaces initiated interest in the study of more general f-quasimetric spaces [4] and in generalizing Banach's fixed point theorem to such spaces [38].

Let's formulate the results from [1, 2], we will deal with further.

As usual, $gph(F) = \{(x, y) \in X \times Y \mid y \in F(x)\}$ is the graph of the mapping $F : (X, \rho_X) \to (Y, \rho_Y)$. We say that a mapping F *closed* if, for all sequences $\{x_i\} \subset X$ and $\{y_i\} \subset Y$ converging to points x_0 and y_0 respectively, such that $(x_i, y_i) \in gph(F)$ for all i, it holds that $(x_0, y_0) \in gph(F)$.

Given a function $f : X \times X \to \mathbb{R}^+$ of two variables and a point $(x_1, x_2) \in X \times X$, we write $\varliminf\limits_{y \to x_1} f(y, x_1)$ for its lower limit in the first variable at the point (x_1, x_2). This limit is defined as the infimum of the lower limits $\inf \varliminf\limits_{y_i \to x_1} f(y_i, x_2)$, where the infimum is taken over all sequences $\{y_i\}$ that converge to x_1. The lower limit $\varliminf\limits_{y \to x_2} f(x_1, y)$ in the second variable is defined in a similar way.

Given any $u \in [0, 1)$ and any positive integer n, we write $S(u, n)$ for the sum of n terms of the geometric progression $\sum\limits_{i=0}^{n-1} u^i$ and, therefore, $S(u, n) = \frac{1-u^n}{1-u}$. We shall assume that $S(u, 0) = 0$ and $\beta^0 = 1$ for $\beta = 0$. For all $q_0, q_1, q_2 \geq 1$ we put

$$m_0 = \min\{j \in \mathbb{N} \mid q_2 \beta^j < \alpha^j\}$$

and under the assumption that $q_0^2 \beta < \alpha$, we put

$$n_0 = \min\{j \in \mathbb{N} \mid q_1 (q_0^2 \beta)^j < \alpha^j\}.$$

Theorem 1.7 (On the existence of coincidence points [1, 2])**.** *Assume that the (q_1, q_2)-quasimetric space (X, ρ_X) is complete. Let the mapping Ψ be α-covering and closed, while the mapping Φ be β-Lipschitz. Fix an arbitrary point $x_0 \in X$. Then the mappings Ψ and Φ have a coincidence point ξ, such that*

$$\varliminf_{\eta \to \xi} \rho_X(x_0, \eta) \leq \frac{q_1^2 \alpha^{m_0 - 1} S(q_2 \frac{\beta}{\alpha}, m_0 - 1) + q_1 (q_2 \beta)^{m_0 - 1}}{\alpha^{m_0} - q_2 \beta^{m_0}} \rho_Y(\Psi(x_0), \Phi(x_0)). \tag{1.1}$$

If the space (X, ρ_X) is q_0-symmetric, then ξ satisfies the estimate

$$\rho_X(x_0, \xi) \leq \frac{q_1^3 \alpha^{m_0 - 1} S(q_2 \frac{\beta}{\alpha}, m_0 - 1) + q_1^2 (q_2 \beta)^{m_0 - 1}}{\alpha^{m_0} - q_2 \beta^{m_0}} \rho_Y(\Psi(x_0), \Phi(x_0)), \tag{1.2}$$

and if, additionally, $q_0^2 \beta < \alpha$, then ξ also satisfies the estimates

$$\overline{\rho}_X(x_0, \xi) \leq q_0 q_2^2 \frac{q_2 \alpha^{n_0 - 1} S(q_1 q_0^2 \frac{\beta}{\alpha}, n_0 - 1) + (q_1 q_0^2 \beta)^{n_0 - 1}}{\alpha^{n_0} - q_1 (q_0^2 \beta)^{n_0}} \rho_Y(\Psi(x_0), \Phi(x_0)), \tag{1.3}$$

$$\varliminf_{\eta \to \xi} \overline{\rho}_X(x_0, \eta) \leq q_0 q_2 \frac{q_2 \alpha^{n_0 - 1} S(q_1 q_0^2 \frac{\beta}{\alpha}, n_0 - 1) + (q_1 q_0^2 \beta)^{n_0 - 1}}{\alpha^{n_0} - q_1 (q_0^2 \beta)^{n_0}} \rho_Y(\Psi(x_0), \Phi(x_0)). \tag{1.4}$$

Let $X = Y$ and Ψ be the identity mapping, i.e., $\Psi(x) \equiv x$. Then $\alpha = 1$, the condition $\beta < 1$ means that Φ is a contraction mapping, and the coincidence point becomes a fixed point.

Corollary 1.8 (Fixed-point theorem for a contraction mapping)**.** *A closed contraction mapping of a complete (q_1, q_2)-quasimetric space to itself has a fixed point, and this point is unique.*

An extended version of Theorem 1.7 is proved in [1, Theorem 4.5].

Theorem 1.9 ([1,2]). *Let the space (X,ρ_X) be complete, the mapping Ψ be α-covering and closed, and the mapping Φ be β-Lipschitz. Fix an arbitrary point $x_0 \in X$.*

1^0 *Let $q_1 = 1$. Then the mappings Ψ and Φ have a coincidence point ξ such that*

$$\varliminf_{\eta\to\xi}\rho_X(x_0,\eta) \leq \frac{\alpha-\beta+q_2\beta}{\alpha(\alpha-\beta)}\rho_Y(\Psi(x_0),\Phi(x_0)). \tag{1.5}$$

2^0 *Let the space (X,ρ_X) be q_0-symmetric, $q_0^2\beta < \alpha$, $q_2 = 1$. Then there exists a coincidence point ξ, such that*

$$\rho_X(\xi,x_0) \leq q_0\frac{q_1q_0^2\beta+\alpha-q_0^2\beta}{\alpha(\alpha-q_0^2\beta)}\rho_Y(\Psi(x_0),\Phi(x_0)). \tag{1.6}$$

The bounds (1.1)–(1.6) from the Theorem 1.7 and Theorem 1.9 are the estimates of divergence of coincidence point of α-covering and closed and β-Lipschitz mappings from an arbitrary point $x_0 \in X$.

Examples showing that the bounds (1.3)–(1.6) are unimprovable have been found in [1]. So the problem of finding the optimal bounds in (1.3)–(1.6) is directly related to finding optimal values of the constants q_1, q_2. Let us consider $(1,q_2)$-quasimetric spaces, in particular, Carnot groups G equipped with symmetric Box-quasimetric ρ_{Box_G}; from this point of view the problem of finding of minimal values of q_2 becomes relevant, see (1.5) and (1.6). Further, we will use the term *exact value* that implies such value of the constant q_2 that for every number q_2', $q_2' < q_2$, the $(1,q_2')$-generalized triangle inequality does not hold for ρ_{Box_G}. Note that the exact values of the constant q_2 for the $(1,q_2)$-generalized triangle inequality of Box-quasimetrics were obtained: on the canonical Heisenberg groups $\mathbb{H}^n_\alpha$, $n \in \mathbb{N}$, and the canonical Engel group $\mathbb{E}_{\alpha,\beta}$ in [16], on some low-dimensional 2-step canonical Carnot groups in [39]. (See the definition of canonical finite-dimensional Lie group in [39,40].)

The aim of this paper is to find the exact value of the constant q_2 on some $(n+1)$-dimensional 2-step canonical Carnot group D_n with the 1-dimensional centre of a special kind (see the Definition 2.2). The main result of our work is Theorem 3.7 where the exact values for q_2 on D_n are obtained. Using Theorem 3.7 we prove Theorem 3.9 which is the best version of the Theorem 1.9. The exact estimates obtained in the Theorem 3.9 can be used in optimal control problems, in particular, to obtain theorems Milyutin type theorems in sub-riemannian geometry. The proof of the Theorem 3.7 is based on some special combinatorial theorems, with which we find the maximum of some special bilinear function $f(A,B)$, where A, B belong to the set of all vertices of a standard unit n-dimensional cube (Theorem 3.1, Property 3.5); these results and methods of their proofs can be used in optimization problems of arbitrary functions over vertices of polytopes (see [41]).

2. Preliminaries

In this section, we recall some basic definitions and notations which will be required in proving our main results.

A Lie algebra is called graduated [42], if it decomposes into a direct sum of vector subspaces $V = \bigoplus_{i=1}^{r} V_i$, and, moreover, $[V_i,V_k] \subset V_{i+k}$, if $i+k \leq r$, and $[V_i,V_k] = 0$, if $i+k > r$. Note that a graduated algebra is always nilpotent of degree r. r-step stratifed Lie algebra V [43] is a Lie algebra nilpotent of degree r, that has a stratifcation, i. e.

$$V = \bigoplus_{i=1}^{r} V_i, \quad V_{i+1} = [V_1, V_i], \quad [V_1, V_r] = \{0\}.$$

An r-step Carnot algebra [43] is a graduated Lie algebra V, which has a stratifcation; a simply connected Lie group G, corresponding to an r-step Carnot algebra V, is called an r-step Carnot group. Let $N = \sum\limits_{i=1}^{r} n_i$, $n_i = \dim V_i$, and the basis of the left-invariant vector fields $\{X_1, \ldots, X_N\}$ of the Carnot group G is ordered such that the values of the first n_1 of them form at every point $u \in G$ the basis of the subspace $V_1(u)$, the values of the next n_2 of them form at every point $u \in G$ the basis of the subspace $V_2(u)$, and so on. We assign to every vector field X_k a natural number $j = \deg X_k$, defined by the inclusion $X_k \in V_j$.

Definition 2.1 ([12–15, 17]). *A Box-quasimetric ρ_{Box_G} is defined as*

$$\rho_{Box_G}(u, w) = \max\{|a_i|^{\frac{1}{\deg X_i}} \mid i = 1, \ldots, N\}, \; w = \exp\Big(\sum_{i=1}^{N} a_i X_i\Big)(u) \; \forall u, w \in G, \tag{2.1}$$

where exp is standard exponential mapping.

Note that standard exponential mapping is identical on canonical Lie group. The Definition 2.1 implies that ρ_{Box_G} satisfies the identity and symmetry axioms.

A canonical 2-step group $\mathbb{D}_n$ with the 1-dimensional centre is defined in the standard Euclidean space $\mathbb{R}^{n+1}$ with the coordinate system $(x_1, \ldots, x_n, t)$ and the coordinate frame $(O, e_1, \ldots, e_n, e)$ with the help of the following commutator table

$$[e_i, e_j] = \alpha_{ij} e, \quad \sum_{i,j=1}^{n} \alpha_{ij}^2 \neq 0, \tag{2.2}$$

the rest of possible commutators of $e_1, \ldots, e_{n+1}$ equal 0. Suppose that $x = (x_1, \ldots, x_n, t)$, $x' = (x'_1, \ldots, x'_n, t')$. Using the Campbell-Hausdorff formula [44], with the help of (2.2) we obtain

$$L_x^{\mathbb{D}_n} x' = x \cdot x' = \Big(x_1 + x'_1, \ldots, x_n + x'_n, \; t + t' + \sum_{i,j=1,\ldots,n,\, i<j} \frac{\alpha_{ij}}{2}(x_i x'_j - x_j x'_i)\Big). \tag{2.3}$$

The values of basis left-invariant vector fields $X_1, \ldots, X_n, T$ of the group $\mathbb{D}_n$ at every point $x = (x_1, \ldots, x_n, t)$ are defined as

$$(X_1, \ldots, X_n, T)(x) = \frac{\partial L_u^{\mathbb{D}_n}(x'_1, \ldots, x'_n, t')}{\partial(x'_1, \ldots, x'_n, t')}\Big|_{(x'_1,\ldots,x'_n,t')=(0,\ldots,0)}.$$

If in (2.2) we put $n = 2m$, $m \in \mathbb{N}$, $\sum\limits_{i=1}^{m-1} \alpha_{2i,2i+1}^2 = 0$ and $\alpha_{2j-1,2j} = \alpha \neq 0$, $j = 1, \ldots, m$, then we obtain a commutator table that defines the canonical Heisenberg group $\mathbb{H}_\alpha^m$ [16]. In particular, $\mathbb{D}_2 = \mathbb{H}_\alpha^1$.

According to (2.1), $(1, q_2)$-quasimetric $\rho_{\mathrm{Box}_{\mathbb{D}_n}}$ is defined by the rule

$$\rho_{\mathrm{Box}_{\mathbb{D}_n}}(u, w) = \max\{|a_1|, \ldots, |a_n|, |a_{n+1}|^{\frac{1}{2}}\}.$$

Definition 2.2. *Carnot group D_n is such a group $\mathbb{D}_n$ for which the relations $\alpha_{ij} = 1\ \forall i < j$ are fulfilled in (2.2).*

Let us consider some basis $E_0 = \{e_1, \dots, e_n\}$ in a n-dimensional vector space Vec_n.

Definition 2.3. *We say that a basis $E' = \{e'_1, \dots, e'_n\}$ is affine equivalent to basis E_0 on Vec_n if*

$$e'_i = \sum_{j=1} c_{ij} e_j, \quad i = 1, \dots, n,$$

where

$$\det \begin{pmatrix} c_{11} & \dots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \dots & c_{nn} \end{pmatrix} \neq 0.$$

Let us consider skew-symmetric bilinear bracket function $[x, y] : \mathrm{Vec}_n \times \mathrm{Vec}_n \to \mathbb{R}$. Let

$$[e_i, e_j] = a_{ij}, \quad \sum_{i,j=1}^{n} a_{ij}^2 \neq 0.$$

Since $[x, y]$ is skew-symmetric then $a_{ij} = -a_{ji}$, $a_{ii} = 0$ for all i.

Lemma 2.4. *Basis $E_0 = \{e_1, \dots, e_n\}$ is affine equivalent to some basis $E' = \{e'_1, \dots, e'_n\}$ such that*

$$[e'_i, e'_j] = b_{ij} > 0, \quad i < j.$$

Proof. Without loss of generality, we put $[e_1, e_2] = a_{12}$, where $a_{12} > 0$. Build the basis $\{e'_1, \dots, e'_n\}$ step by step.

1^0 Consider all brackets $[e_1, e_i]$, $i > 2$. Suppose that there are numbers $a_{1i} \leq 0$. Then, instead the basis $E_0 = \{e_1, \dots, e_n\}$ let us consider the basis

$$E_1 = \{e_1, A_1 e_2, e_3 + A_1 e_2, \dots, e_n + A_1 e_2\}, \quad A_1 > 0.$$

And we have

$$[e_1, e_i + A_1 e_2] = a_{1i} + a_{12} A_1;$$

if A_1 is a large enough number then $a_{12} A_1 + a_{1i} > 0$.

2^0 Next, we consider the basis E_1, but in order to avoid inconvenience, we will use the notation $\{e_1, \dots, e_n\}$ for E_1 and the symbols a_{ij}. So we have

$$[e_1, e_i] = a_{1i}, \quad a_{1i} > 0, \quad i = 2, \dots, n.$$

Consider all brackets $[e_2, e_i]$, $i > 3$. Suppose that there are numbers $a_{2i} \leq 0$. Then, instead the basis $E_1 = \{e_1, \dots, e_n\}$ let us consider the basis

$$E_2 = \{e_1, e_2, e_3 + A_2 e_1, \dots, e_n + A_2 e_1\}, \quad A_2 < 0.$$

We have

$$[e_1, e_i + A_2 e_1] = [e_1, e_i] = a_{1i}, \quad i > 3,$$

$$[e_2, e_i + A_2e_1] = a_{2i} - A_2a_{12}, \quad i > 3;$$

if $|A_2|$ is a large enough number then $a_{2i} - A_2a_{12} > 0$.

3^0 Next, we consider the basis E_2, but in order to avoid inconvenience, we will use the notation $\{e_1, \ldots, e_n\}$ for E_2 and the symbols a_{ij}. We have

$$[e_j, e_i] = a_{ji}, \quad a_{ji} > 0, \quad j = 1, 2, \ j < i, \quad i = 2, \ldots, n.$$

Consider all brackets $[e_3, e_i]$, $i > 3$. Suppose that there are numbers $a_{3i} \leq 0$. Then instead the basis $E_2 = \{e_1, \ldots, e_n\}$ let us consider the basis

$$E_3 = \{e_1, e_2, e_3 + A_3e_2, e_4, \ldots, e_n\}, \quad A_3 > 0.$$

We have

$$[e_1, e_3 + A_3e_2] = a_{13} + a_{12}A_3, \quad a_{13} + a_{12}A_3 > 0,$$

$$[e_2, e_3 + A_3e_2] = a_{23}, \quad a_{23} > 0,$$

$$[e_3 + A_3e_2, e_i] = a_{3i} + a_{2i}A_3, \ i > 3, \quad a_{3i} + a_{2i}A_3 > 0,$$

if A_3 is a large enough number.

4^0 Next, we consider the basis E_3, but in order to avoid inconvenience, we will use the notation $\{e_1, \ldots, e_n\}$ for E_3 and the symbols a_{ij}. We have

$$[e_j, e_i] = a_{ji}, \quad a_{ji} > 0, \ j < i, \quad i = 2, \ldots, n, \ j = 1, 2, 3.$$

Consider all brackets $[e_4, e_i]$, $i > 4$. Suppose that there are numbers $a_{4i} \leq 0$. Then, instead the basis $E_3 = \{e_1, \ldots, e_n\}$ let us consider the basis

$$E_4 = \{e_1, e_2, e_3, e_4 + A_4e_3, \ldots, e_n\}, \quad A_4 > 0.$$

We have

$$[e_i, e_4 + A_4e_3] = a_{i4} + A_4a_{i3} > 0, \quad i = 1, 2, 3,$$

$$[e_4 + A_4e_3, a_i] = a_{4i} + A_4a_{3i} > 0,$$

if A_4 is a large enough number.

The next steps are obvious. □

In some cases the basis $E_0 = \{e_1, \ldots, e_n\}$ is affine equivalent to such a basis $E' = \{e'_1, \ldots, e'_n\}$ that

$$[e'_i, e'_j] = 1, \quad i < j.$$

Consider, for example, the 3-dimensional case

$$[e_i, e_j] = a_{ij}, \quad i < j, \quad i, j = 1, 2, 3.$$

Taking into account Lemma 2.4, we can assume that $a_{ij} > 0$. Let

$$\begin{cases} x_1x_2 = a_{12}, \\ x_2x_3 = a_{23}, \\ x_1x_3 = a_{13}, \end{cases}$$

then

$$x_2^2 = \frac{a_{12}a_{23}}{a_{13}} \Leftrightarrow x_2 = \sqrt{\frac{a_{12}a_{23}}{a_{13}}},$$

so

$$x_1 = \sqrt{\frac{a_{12}a_{13}}{a_{23}}}, \quad x_3 = \sqrt{\frac{a_{13}a_{23}}{a_{12}}}.$$

It is not difficult to see that vectors $e'_i = \frac{e_i}{x_i}$, $i = 1,2,3$, satisfy the identities $[e'_i, e'_j] = 1$, $i < j$, $i, j = 1,2,3$.

3. Main results

Next, we consider the basis $\{e_1, \dots, e_n\}$ of Vec_n satisfying the table

$$[e_i, e_j] = 1, \quad \forall i < j. \tag{3.1}$$

Let $x = \sum\limits_{i=1}^{n} x_i e_i$, $y = \sum\limits_{i=1}^{n} y_i e_i$; then, using (3.1), we get

$$f(x,y) = [x,y] = \sum_{i<j}(x_i y_j - x_j y_i).$$

Let's find $\max\limits_{x,y\in \mathrm{Vert}(n)} f(x,y)$, where Vert($n$) is the set of all vertices of unit n-dimensional cube in vector space Vec_n, i. e. all possible points whose coordinates consist only of numbers ±1.

Let $A[n](x,y)$ is a $(n \times n)$-matrix consisting of elements

$$(A[n])_{ij} = \begin{cases} x_i y_j, & i > j, \\ -x_i y_j, & i < j, \\ 0, & i = j, \end{cases}$$

where $x, y \in \mathrm{Vert}(n)$. Denote by $L(A)$ the number of -1 in $A[n](x,y)$.

Theorem 3.1. 1) $\min\limits_{x,y\in Vert(2k)} L(A[2k](x,y)) \geq k^2 - k$, 2) $\min\limits_{x,y\in Vert(2k+1)} L(A[2k+1](x,y)) \geq k^2$.

Denote by A'_i the $(n-1)\times(n-1)$-matrix that is obtained from the matrix $A[n](x,y)$ by deleting i-line and i-column. Denote by A''_{ij} $(n-1)\times(n-1)$-matrix, that is obtained from the matrix $A[n](x,y)$ by deleting i-line and j-column.

Lemma 3.2. *Let us consider the matrix $A = A[2k+2](x,y)$. Then*

$$\frac{1}{C^2_{2k+2}} \sum_{i>j} L(A''_{ij}) = \frac{L(A)(4k^2 - 2k)}{4(k+1)^2 - 2(k+1)}.$$

Proof. We have

$$\sum_{i>j} L(A''_{ij}) = \sum_{i\neq j} l(i', j'),$$

where $l(i', j') = 0$ in the case when $(A'')_{i'j'} \neq -1$, and $l(i', j')$ is equal to the number of all matrices A''_{ij} containing the element $(A'')_{i'j'}$ in the case when $(A'')_{i'j'} = -1$. It is not difficult to see that $l = l(i', j')$ does not depend on the choice of a pair i', j'. We have

$$l \cdot N_{2k+2} = C^2_{2k+2} \cdot N_{2k},$$

where N_k is the number of non-diagonal elements of a $(k \times k)$-matrix, and C^2_{2k+2} is the number of ways to choose two pairs of i and j lines and columns in a $(2k + 2) \times (2k + 2)$-matrix. Then

$$\sum_{i>j} L(A''_{ij}) = \sum_{i \neq j} l(i', j') = l \cdot N_{2k+2},$$

hence Lemma 3.2 follows. □

Lemma 3.3. *Let us consider the matrix* $A = A[2k + 1](x, y)$. *Then*

$$\frac{1}{k+1} \sum_i L(A'_i) = \frac{L(A)(4k^2 - 2k)}{(2k + 1)^2 - (2k + 1)}.$$

Proof. The proof of Lemma 3.3 is similar to the proof of Lemma 3.2.

□

Proof of Theorem 3.1. 1) The proof is carried out using the method of mathematical induction. The cases $k = 1, 2, 3$ are clear. Suppose that for $k + 1$ the Theorem 3.1 does not hold, i. e. there is a matrix $A = A[2k + 2](x, y)$ such that

$$L(A) \leq (k + 1)^2 - (k + 1) - 1.$$

But then for matrix A there will be a matrix A''_{ij} for which the Theorem 3.1 does not hold too. Let A''_{ij} be a matrix for which $L(A''_{ij})$ is minimal. Then using Lemma 3.2 we have

$$\begin{aligned} L(A''_{ij}) &\leq \frac{L(A)(4k^2 - 2k)}{4(k + 1)^2 - 2(k + 1)} \leq \frac{((k + 1)^2 - (k + 1) - 1)(4k^2 - 2k)}{4(k + 1)^2 - 2(k + 1)} \\ &= \frac{((k + 1)^2 - (k + 1))(4k^2 - 2k)}{4(k + 1)^2 - 2(k + 1)} - \frac{4k^2 - 2k}{4(k + 1)^2 - 2(k + 1)} \\ &= \frac{k^2(2k - 1)}{2k + 1} - \frac{2k^2 - k}{2(k + 1)^2 - (k + 1)}. \end{aligned}$$

We have

$$\frac{k^2(2k - 1)}{2k + 1} = k^2\Big(1 - \frac{1}{k} + \frac{1}{k(2k + 1)}\Big).$$

Inequality

$$\frac{k^2}{k(2k + 1)} - \frac{2k^2 - k}{2(k + 1)^2 - (k + 1)} < 0$$

is equivalent to inequality

$$2k^2 > 3k + 2,$$

that is right for $k \geq 3$. Then $L(A''_{ij}) < k^2 - k$ but this is contradiction.

The proof of the point 2) is similar using Lemma 3.3. □

Corollary 3.4.

$$\max_{x,y\in Vert(2k)} f(x,y) \leq 2k^2, \qquad \max_{x,y\in Vert(2k+1)} f(x,y) \leq 2k^2+2k.$$

Property 3.5.

$$M_{2k} = \max_{x,y\in Vert(2k)} f(x,y) = 2k^2, \quad M_{2k+1} = \max_{x,y\in Vert(2k+1)} f(x,y) = 2k^2+2k.$$

Proof. Let $x,y \in \mathrm{Vert}(2k)$. You can see that if $x_i = 1,\ i = 1,\dots,2k,\ \ y_j = 1,\ j = 1,\dots,k,\ \ y_l = -1$, $l = k+1,\dots,2k$, then

$$\max_{x,y\in \mathrm{Vert}(2k)} f(x,y) = 2k^2.$$

Let $x,y \in \mathrm{Vert}(2k+1)$. You can see that if $x_i = 1,\ i = 1,\dots,2k,\ \ y_j = 1,\ j = 1,\dots,k,\ \ y_l = -1$, $l = k+1,\dots,2k+1$, then

$$\max_{x,y\in \mathrm{Vert}(2k+1)} f(x,y) = 2k^2+2k.$$

□

Using some results from work [39], we find the exact value of the constant in the $(1,q_2)$-generalized triangle inequality for the canonical Carnot group D_n. Let

$$M_{\mathbb{D}_n} = \sup_{x,x'\in \mathrm{Vert}(n)} \Big| \sum_{i,j=1,\dots,n,\, i<j} \frac{\alpha_{ij}}{2}(x_i x'_j - x_j x'_i) \Big|,$$

compare with (2.3).

Theorem 3.6 ([39]). *On canonical Carnot group $\mathbb{D}_n$ the following formula gives the exact value of the constant q_2 in the $(1,q_2)$-generalized triangle inequality*

$$q_2 = \begin{cases} 1, & M_{\mathbb{D}_n} \leq 2, \\ \frac{M_{\mathbb{D}_n}}{2}, & M_{\mathbb{D}_n} > 2. \end{cases}$$

Using Property 3.5 and Theorem 3.6 we get the following:

Theorem 3.7. *The exact value of the constant in the $(1,q_2)$-generalized triangle inequality for canonical Carnot group D_n is defined by the formula $q_2 = \frac{M_n}{2}$, where*

$$M_n = \begin{cases} 2k^2, & n = 2k, \\ 2k^2+2k, & n = 2k+1. \end{cases}$$

Lemma 3.8. *Let (X,ρ_X) be a symmetric $(1,q)$-quasimetric space. Then (X,ρ_X) is $(q,1)$-quasimetric space.*

Proof. Obviously. □

Consider a (q'_1,q'_2)-quasimetric space (Y,ρ_Y). The following Theorem 3.9 follows from Lemma 3.8, Theorem 1.9 and Theorem 3.7.

Theorem 3.9. *Let mapping $\Psi : (D_n, \rho_{Box_{D_n}}) \to (Y,\rho_Y)$ be α-covering and closed, and the mapping $\Phi : (D_n, \rho_{Box_{D_n}}) \to (Y,\rho_Y)$ be β-Lipschitz. Fix an arbitrary point $x_0 \in D_n$. The mappings Ψ and Φ have a coincidence point ξ such that*

$$\rho_X(x_0,\xi) \leq \frac{\alpha-\beta+\frac{M_n}{2}\beta}{\alpha(\alpha-\beta)}\rho_Y(\Psi(x_0),\Phi(x_0)).$$

4. Conclusions

In this paper, on some class of 2-step Carnot groups D_n with 1-dimensional centre we found the exact values of the constants in $(1, q_2)$-generalized triangle inequality for their Box-quasimetrics $\rho_{\mathrm{Box}_{D_n}}$. As a consequence, we obtained the best version of the Coincidence Points Theorem of α-covering and β-Lipschitz mappings defined on $(D_n, \rho_{\mathrm{Box}_{D_n}})$. The results obtained and the methods of their proofs can be applied in fixed point theory, optimal control theory, optimization problems, quasimetric analysis, sub-riemannian geometry.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. A. V. Arutyunov, A. V. Greshnov, (q_1, q_2)-quasimetric spaces. Covering mappings and coincidence points, *Izvestiya: Math.*, **82** (2018), 245–272. https://doi.org/10.4213/im8546 https://doi.org/10.1070/IM8546
2. A. V. Arutyunov, A. V. Greshnov, Theory of (q_1, q_2)-quasimetric spaces and coincidence points, *Dokl. Math.*, **94** (2016), 434–437. https://doi.org/10.1134/S1064562416040232
3. A. V. Arutyunov, A. V. Greshnov, Coincidence points of multi-valued mappings in (q_1, q_2)-quasimetric spaces, *Dokl. Math.*, **96** (2017), 438–441. https://doi.org/10.1134/S1064562417050064
4. A. V. Arutyunov, A. V. Greshnov, L. V. Lokoutsievskii, K. V. Storozhuk, Topological and geometrical properties of spaces with symmetric and nonsymmetric f-quasimetrics, *Topol. Appl.*, **221** (2017), 178–194. https://doi.org/10.1016/j.topol.2017.02.035
5. R. Sengupta, About fixed points of contraction mappings acting in (q_1, q_2)-quasi-metric spaces, *Eurasian Math. J.*, **8** (2017), 70–76.
6. A. V. Greshnov, (q_1, q_2)-quasimetrics bi-Lipschitz equivalent to 1-quasimetrics, *Sib. Adv. Math.*, **27** (2017), 253–262. https://doi.org/10.3103/S1055134417040034

7. A. V. Greshnov, Regularization of distance functions and separation axioms on (q_1, q_2)-quasimetric spaces, *Sib. Electron. Math. Re.*, **14** (2017), 765–773. https://doi.org/10.17377/semi.2017.14.065
8. A. V. Greshnov, Some problems of regularity of f-quasimetrics, *Sib. Electron. Math. Re.*, **15** (2018), 355–361. https://doi.org/10.17377/semi.2018.15.032
9. A. V. Greshnov, R. I. Zhukov, Completeness theorem in (q_1, q_2)-quasimetric spaces, *Sib. Electron. Math. Re.*, **16** (2018), 2090–2097. https://doi.org/10.33048/semi.2019.16.148
10. A. V. Arutyunov, A. V. Greshnov, (q_1, q_2)-quasimetric spaces. Covering mappings and coincidence points. A review of the results, *Fixed Point Theory*, **23** (2022), 473–486.
11. W. A. Wilson, On quasi-metric spaces, *Amer. J. Math.*, **53** (1931), 675–684. https://doi.org/10.2307/2371174
12. S. K. Vodopyanov, Geometry of Carnot-Carathéodory spaces and differentiability of mappings, *Contemp. Math.*, **424** (2007), 247–301.
13. S. G. Basalaev, S. K.Vodopyanov, Approximate differentiability of mappings of Carnot-Carathéodory spaces, *Eurasian Math. J.*, **4** (2013), 10–48.
14. A. V. Greshnov, On the generalized triangle inequality for quasimetrics induced by noncommuting vector fields, *Sib. Adv. Math.*, **22** (2012), 95–114. https://doi.org/10.3103/S1055134412020034
15. A. V. Greshnov, Proof of Gromov's theorem on homogeneous nilpotent approximation for vector fields of class C^1, *Sib. Adv. Math.*, **23** (2013), 180–191. https://doi.org/10.3103/S1055134413030036
16. A. V. Greshnov, M. V. Tryamkin, Exact values of constants in the generalized triangle inequality for some $(1, q_2)$-quasimetrics on canonical Carnot groups, *Math. Notes*, **98** (2015), 694–698. https://doi.org/10.1134/S0001434615090369
17. A. Nagel, E. M. Stein, S. Wainger, Balls and metrics defined by vector fields. I. Basic properties, *Acta Math.*, **155** (1985), 103–147. https://doi.org/10.1007/BF02392539
18. A. V. Greshnov, Local approximation of uniformly regular Carnot-Carathéodory quasispaces by their tangent cones, *Sib. Math. J.*, **48** (2007), 229–248. https://doi.org/10.1007/s11202-007-0024-2
19. A. V. Greshnov, Metrics and tangent cones of uniformly regular Carnot-Carathéodory spaces, *Sib. Math. J.*, **47** (2006), 209–238. https://doi.org/10.1007/s11202-006-0036-3
20. L. M. Graves, Some mapping theorems, *Duke Math. J.*, **17** (1950), 111–114. https://doi.org/10.1215/S0012-7094-50-01713-3
21. A. V. Dmitruk, A. A. Milyutin, N. P. Osmolovskii, Lyusternik's theorem and the theory of extrema, *Russ. Math. Surv.*, **35** (1980), 11–51. https://doi.org/10.1070/RM1980v035n06ABEH001973
22. A. L. Dontchev, R. T. Rockafellar, *Implicit functions and solution mappings*, Berlin: Springer, 2009. https://doi.org/10.1007/978-0-387-87821-8
23. B. S. Mordukhovich, Complete characterization of openness, metric regularity, and Lipschitzian properties of multifunctions, *Trans. Amer. Math. Soc.*, **340** (1993), 1–35. https://doi.org/10.2307/2154544
24. B. S. Mordukhovich, *Variational analysis and generalized differentiation I*, Basic Theory, Berlin: Springer, 2006. https://doi.org/10.1007/3-540-31247-1

25. A. Uderzo, A metric version of Milyutin theorem, *Set-Valued Var. Anal.*, **20** (2012), 279–306. https://doi.org/10.1007/s11228-011-0193-9

26. A. V. Arutyunov, Covering mappings in metric spaces and fixed points, *Dokl. Math.*, **76** (2007), 665–668. https://doi.org/10.1134/S1064562407050079

27. A. V. Arutyunov, Stability of coincidence points and properties of covering mappings, *Math. Notes*, **86** (2009), 153–158. https://doi.org/10.1134/S0001434609070177

28. A. V. Arutyunov, The coincidence point problem for set-valued mappings and Ulam–Hyers stability, *Dokl. Math.*, **89** (2014), 188–191. https://doi.org/10.1134/S1064562414020197

29. A. V. Arutyunov, S. E. Zhukovskiy, Local solvability of control systems with mixed constraints, *Differ. Equ.*, **46** (2010), 1561–1570. https://doi.org/10.1134/S0012266110110042

30. A. V. Arutyunov, E. S. Zhukovskii, S. E. Zhukovskiy, On the well-posedness of differential equations unsolved for the derivative, *Differ. Equ.*, **47** (2011), 1541–1555. https://doi.org/10.1134/S0012266111110012

31. E. R. Avakov, A. V. Arutyunov, E. S. Zhukovskii, Covering mappings and their applications to differential equations unsolved for the derivative, *Differ. Equ.*, **45** (2009), 627–649. https://doi.org/10.1134/S0012266109050024

32. A. V. Arutyunov, E. S. Zhukovskiy, S. E. Zhukovskiy, Covering mappings and well-posedness of nonlinear Volterra equations, *Nonlinear Anal.*, **75** (2012), 1026–1044. https://doi.org/10.1016/j.na.2011.03.038

33. B. S. Mordukhovich, B. Wang, Restrictive metric regularity and generalized differential calculus in Banach spaces, *Int. J. Math. Math. Sci.*, **2004** (2004), 683907. https://doi.org/10.1155/S0161171204405183

34. A. Arutyunov, V. A. de Oliveira, F. L. Pereira, E. Zhukovskiy, S. Zhukovskiy, On the solvability of implicit differential inclusions, *Appl. Anal.*, **94** (2015), 129–143. https://doi.org/10.1080/00036811.2014.891732

35. F. Sk, A. Hossain, Q. H. Khan, Relation-theoretic metrical coincidence theorems under weak C-contractions and K-contractions, *AIMS Math.*, **6** (2021), 13072–13091. https://doi.org/10.3934/math.2021756

36. F. A. Khan, F. Sk, M. G. Alshehri, Q. H. Khan, Aftab Alam, Relational Meir-Keeler cntractions and common fixed point theorems, *J. Funct. Spaces*, **2022** (2022), 3550923. https://doi.org/10.1155/2022/3550923

37. F. Sk, M. A. O. Tom, Q. H. Khan, F. A. Khan, On Prešić-Ćirić-type α-ψ contractions with an application, *Symmetry*, **14** (2022), 1166. https://doi.org/10.3390/sym14061166

38. E. S. Zhukovskiy, The fixed points of contractions of f-quasimetric spaces, *Sib. Math. J.*, **59** (2018), 1063–1072. https://doi.org/10.1134/S0037446618060095

39. A. V. Greshnov, On finding the exact values of the constant in a $(1, q_2)$-generalized triangle inequality for Box-quasimetrics on 2-step Carnot groups with 1-dimensional center, *Sib. Electron. Math. Re.*, **18** (2021), 1251–1260. https://doi.org/10.33048/semi.2021.18.095

40. L. V. Ovsyannikov, *Group analysis of differential equations*, New York: Academic Press, 1982. https://doi.org/10.1016/C2013-0-07470-1

41. S. Chubanov, A scaling algorith optimizing arbitrary functions over vertices of polytopes, *Math. Program.*, **190** (2021), 89–102. https://doi.org/10.1007/s10107-020-01522-0

42. L. P. Rothchild, E. S. Stein, Hypoelliptic differential operators and nilpotent groups, *Acta Math.*, **137** (1976), 247–320. https://doi.org/10.1007/BF02392419

43. A. Bonfiglioli, E. Lanconelli, F. Uguzzoni, *Stratified Lie groups and potential theory for their sub-Laplacian*, Springer Berlin, Heidelberg, 2007. http://doi.org/10.1007/978-3-540-71897-0

44. M. M. Postnikov, *Lie groups and Lie algebras*, Lectures in Geometry, Moscow: Mir, 1986.

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