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Research article

Some Schläfli type modular equations of composite degrees

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Abstract: S. Ramanujan documented several modular equations of degrees in his notebooks. These identities are used to evaluate Weber's class in variants, continued fractions and many more. In the present work, we establish modular equations of composite degrees using the known identities.

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1. Introduction

Ramanujan [4, pp. 204–237], [7] recorded 23 interesting modular equations that we describe now. The q-shifted factorial is defined as

$$f(-q) := (q;q)_\infty = \prod_{n=1}^{\infty}(1-q^n), \qquad |q| < 1.$$

Ramanujan's modular equations contain quotients of the function $f(-q)$ at certain arguments. For example, [4, p.206], Let

$$P := \frac{f(-q)}{q^{1/3}f(-q^5)} \quad \text{and} \quad Q := \frac{f(-q^2)}{q^{1/3}f(-q^{10})}.$$

Then, we have

$$PQ + \frac{5}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3. \tag{1.1}$$

Many authors, after the publication of [4], discovered several new modular equations of nature (1.1). For the wonderful work, one may refer to [1, 2, 6, 9, 11–16]. Ramanujan's contributions in the area of modular equations are immense and he found several classical modular equations. An article by M.

Hanna [5] summarizes centuries of work on modular equations and presents in various forms. In fact, Ramanujan [8] devoted more space in his notebooks to modular equations than to any other topic. He documented more than 200 different modular equations. Most of his work on modular equations can be found in Chapters 18–21 and in 100 pages of unorganized material in his second notebook [7]. Inspired by the above work, in this paper we obtain some P-Q type modular equations. Before concluding this section, we define some basic definitions and modular equations. The purpose of this paper is to prove four new modular equations of the type (1.1). In Section 2, we list four preliminary results which we need in our main results. Section 3 is devoted to proving modular equations.

If $|ab| < 1$, Ramanujan's general theta function $f(a, b)$ is stated as follows

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}.$$

The above identity enjoys the Jacobi's triple product identity [3], we have

$$f(a,b) := (-a, -b, ab; ab)_\infty.$$

The two particular facts of $f(a, b)$ [3], are as follows

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.$$

It is fair that for any complex number τ, if $q = e^{2i\pi\tau}$ then $f(-q) = e^{-i\pi\tau/12}\eta(\tau)$, where $\eta(\tau)$ is the classical Dedekind η-function with $Im(\tau) > 0$ and defined as

$$\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) = e^{\pi i\tau/12} \prod_{n=1}^{\infty} (1 - e^{2n\pi i\tau}).$$

After Ramanujan, we define

$$\chi(q) = (-q; q^2)_\infty.$$

A modular equation of degree n is an equation relating α and β that is stated as

$$n \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)},$$

where

$$ {}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n \qquad |z| < 1,$$

represents an ordinary hypergeometric function with

$$(\alpha)_k = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)}$$

and $(\alpha)_m = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+m-1)$ is the Pochhammer notation. Then, we say that β is degree n over α and call the ratio

$$m := \frac{z_1}{z_n},$$

the multiplier, with $z_1 = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)$ and $z_n = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)$.

2. Preliminary results

Lemma 2.1. *[15] Let*

$$A_n = \frac{\varphi(q^n)}{q^{n/4}\psi(q^{2n})}. \tag{2.1}$$

Then, we have

$$A_1A_3 + \frac{16}{A_1A_3} = \left(\frac{A_1}{A_3}\right)^2 + \left(\frac{A_3}{A_1}\right)^2 + 6 \tag{2.2}$$

and

$$\left(\frac{A_1}{A_5}\right)^3 + \left(\frac{A_5}{A_1}\right)^3 + 10\left(\left(\frac{A_1}{A_5}\right)^2 + \left(\frac{A_5}{A_1}\right)^2\right) + 15\left(\frac{A_1}{A_5} + \frac{A_5}{A_1}\right) - (A_1A_5)^2 - \frac{256}{(A_1A_5)^2} - 20 = 0. \tag{2.3}$$

Lemma 2.2. *[4, Entry 69, p.236] Let*

$$B_n = \frac{f_1}{q^{1/12}f_3}. \tag{2.4}$$

Then, we have

$$(B_1B_7)^3 + \frac{27}{(B_1B_7)^3} = \left(\frac{B_7}{B_1}\right)^4 - 7\left(\frac{B_7}{B_1}\right)^2 + 7\left(\frac{B_1}{B_7}\right)^2 - \left(\frac{B_1}{B_7}\right)^4. \tag{2.5}$$

Lemma 2.3. *[4, Entry 72, p.237] Let*

$$B_n = \frac{f_1}{q^{1/2}f_{13}}. \tag{2.6}$$

Then, we have

$$C_1C_3 + \frac{13}{C_1C_3} = \left(\frac{C_3}{C_1}\right)^2 - 3\frac{C_3}{C_1} - 3\frac{C_1}{C_3} + \left(\frac{C_1}{C_3}\right)^2 - 3. \tag{2.7}$$

3. Main results

Throughout this section, we use

$$x_n = (PQ)^n + \frac{1}{(PQ)^n} \qquad \text{and} \qquad y_n = \left(\frac{P}{Q}\right)^n + \left(\frac{Q}{P}\right)^n.$$

Theorem 3.1. *Let*

$$P := q^{1/2}\frac{\varphi(q)\psi(q^6)}{\varphi(q^3)\psi(q^2)} \quad \textit{and} \quad Q := q^{3/2}\frac{\varphi(q^3)\psi(q^{18})}{\varphi(q^9)\psi(q^6)}.$$

Then, we have

$$x_3 - 16x_2 + 37x_1 - x_2y_2 + x_1y_2 - 6x_2y_1 - 12x_1y_1 + 12y_1 - 44 = 0.$$

Proof. On rewriting P and Q on making use of (2.1), we obtain

$$P = \frac{A_1}{A_3} \qquad \text{and} \qquad Q = \frac{A_3}{A_9}. \tag{3.1}$$

Also, from (2.2) and (3.1), we find that

$$\frac{PA_3^2}{4} + \frac{4}{PA_3^2} = \frac{1}{4}\left(P^2 + \frac{1}{P^2} + 6\right).$$

On solving the above for $PA_3^2/4$, we obtain

$$\frac{PA_3^2}{4} = \frac{a \pm \sqrt{a^2-4}}{2} \tag{3.2}$$

where

$$a = \frac{1}{4}\left(P^2 + \frac{1}{P^2} + 6\right).$$

Further, the identity (3.2) implies that

$$\frac{4}{PA_3^2} = \frac{a \mp \sqrt{a^2-4}}{2}. \tag{3.3}$$

Similarly, we can deduce

$$\frac{QA_9^2}{4} = \frac{b \pm \sqrt{b^2-4}}{2} \tag{3.4}$$

and

$$\frac{4}{QA_9^2} = \frac{b \mp \sqrt{b^2-4}}{2} \tag{3.5}$$

where

$$b = \frac{1}{4}\left(Q^2 + \frac{1}{Q^2} + 6\right).$$

Multiplying (3.2) with (3.5), (3.3) with (3.4) and employing (3.1) and after a little simplification, we obtain

$$4PQ = ab \pm b\sqrt{a^2-4} \mp a\sqrt{b^2-4} - \sqrt{a^2-4}\sqrt{b^2-4} \tag{3.6}$$

and

$$\frac{4}{PQ} = ab \mp b\sqrt{a^2-4} \pm a\sqrt{b^2-4} - \sqrt{a^2-4}\sqrt{b^2-4} \tag{3.7}$$

respectively. Adding (3.6) and (3.7) and then rearranging, we obtain

$$2\left(PQ + \frac{1}{PQ}\right) - ab = -\sqrt{a^2-4}\sqrt{b^2-4}.$$

Squaring on both sides, employing a and b in the resulting equation and then consolidating, we complete the proof. □

Theorem 3.2. *Let*

$$P := q\frac{\varphi(q)\psi(q^{10})}{\varphi(q^5)\psi(q^2)} \qquad \textit{and} \qquad Q := q^5\frac{\varphi(q^5)\psi(q^{50})}{\varphi(q^{25})\psi(q^{10})}.$$

Then, we have

$$\begin{aligned}&x_5 - x_4(156 - 15y_1) + x_3(225 - 200y_1 - y_3 + 15y_2) + x_2(225y_1 - 30y_2 + y_3) + x_1(226 - 55y_1 + 15y_2)\\&\quad + x_{1/2}(440y_{1/2} - 20y_{3/2}) - x_{3/2}(300y_{1/2} + 110y_{3/2} - 10y_{5/2}) - x_{5/2}(300y_{1/2} - 150y_{3/2} + 10y_{5/2})\\&\quad + x_{7/2}(150y_{1/2} - 20y_{3/2}) + x_{9/2}y_{1/2} + 30y_1 - 1392 = 0.\end{aligned}$$

Proof. On rewriting P and Q on making use of (2.3), we obtain

$$P = \frac{A_1}{A_5} \qquad \text{and} \qquad Q = \frac{A_5}{A_{25}}. \tag{3.8}$$

Also, from (2.6) and (3.8), we find that

$$\frac{PA_5^2}{16} + \frac{16}{PA_5^2} = \frac{1}{16}\left[\left(P^3 + \frac{1}{P^3}\right) + 10\left(P^2 + \frac{1}{P^2}\right) + 15\left(P + \frac{1}{P}\right) - 20\right].$$

On solving the above for $PA_5^2/16$, we obtain

$$\frac{PA_5^2}{16} = \frac{a \pm \sqrt{a^2 - 4}}{2} \tag{3.9}$$

where

$$a = \frac{1}{16}\left[\left(P^3 + \frac{1}{P^3}\right) + 10\left(P^2 + \frac{1}{P^2}\right) + 15\left(P + \frac{1}{P}\right) - 20\right].$$

Further, the identity (3.9) implies that

$$\frac{16}{PA_5^2} = \frac{a \mp \sqrt{a^2 - 4}}{2}. \tag{3.10}$$

Similarly, we can deduce

$$\frac{QA_{25}^2}{16} = \frac{b \pm \sqrt{b^2 - 4}}{2} \tag{3.11}$$

and

$$\frac{16}{QA_{25}^2} = \frac{b \mp \sqrt{b^2 - 4}}{2} \tag{3.12}$$

where

$$b = \frac{1}{16}\left[\left(Q^3 + \frac{1}{Q^3}\right) + 10\left(Q^2 + \frac{1}{Q^2}\right) + 15\left(Q + \frac{1}{Q}\right) - 20\right].$$

Multiplying (3.9) with (3.12), (3.10) with (3.11) and employing (3.8) and after a little simplification, we obtain

$$4(PQ)^2 = ab \pm b\sqrt{a^2 - 4} \mp a\sqrt{b^2 - 4} - \sqrt{a^2 - 4}\sqrt{b^2 - 4} \tag{3.13}$$

and

$$\frac{4}{P^2Q^2} = ab \mp b\sqrt{a^2-4} \pm a\sqrt{b^2-4} - \sqrt{a^2-4}\sqrt{b^2-4} \tag{3.14}$$

respectively. Adding (3.13) and (3.14) and then simplifying, we deduce

$$2\left((PQ)^2 + \frac{1}{(PQ)^2}\right) - ab = -\sqrt{a^2-4}\sqrt{b^2-4}.$$

Squaring on both sides, employing a and b in the resulting equation and then streamlining, we complete the proof. □

Theorem 3.3. *Let*

$$P := \frac{f_1}{q^{1/12}f_3} \qquad \textit{and} \qquad Q := \frac{f_7}{q^{7/12}f_{21}}.$$

Then, we have

$$\begin{aligned} x_7 - &x_6(7y_1+27) + 49x_5 + x_4(7y_3 - y_4) - x_3(y_4 - 14y_3 + 49y_2) \\ &+ 7x_2(y_3 - 7y_2) - x_1(14y_1 - 50)14y_1 + 254 = 0. \end{aligned}$$

Proof. From (2.4) and together with P and Q, we obtain d together with P and Q, we obtain

$$P = \frac{B_1}{B_3} \qquad \text{and} \qquad Q = \frac{B_7}{B_{21}}. \tag{3.15}$$

Also, from (2.5) and (3.15), we find that

$$\frac{P^3B_7^6}{3\sqrt{3}} + \frac{3\sqrt{3}}{P^3B_7^6} = \frac{1}{3\sqrt{3}}\left[\left(\frac{1}{P^4} - P^4\right) - 7\left(\frac{1}{P^2} - P^2\right)\right].$$

On solving the above for $P^3B_7^6/3\sqrt{3}$, we obtain

$$\frac{P^3B_7^6}{3\sqrt{3}} = \frac{a \pm \sqrt{a^2-4}}{2} \tag{3.16}$$

where

$$a = \frac{1}{3\sqrt{3}}\left[\left(\frac{1}{P^4} - P^4\right) - 7\left(\frac{1}{P^2} - P^2\right)\right].$$

Further, the identity (3.16) implies that

$$\frac{3\sqrt{3}}{P^3B_7^6} = \frac{a \mp \sqrt{a^2-4}}{2}. \tag{3.17}$$

Similarly, we can deduce

$$\frac{Q^3B_7^6}{3\sqrt{3}} = \frac{b \pm \sqrt{b^2-4}}{2} \tag{3.18}$$

and

$$\frac{3\sqrt{3}}{Q^3 B_7^6} = \frac{b \mp \sqrt{b^2-4}}{2} \tag{3.19}$$

where

$$b = \frac{1}{3\sqrt{3}}\left[\left(\frac{1}{Q^4} - Q^4\right) - 7\left(\frac{1}{Q^2} - Q^2\right)\right].$$

Multiplying (3.16) with (3.19), (3.17) with (3.18) and employing (3.22) and after a little simplification, we obtain

$$4PQ = ab \pm b\sqrt{a^2-4} \mp a\sqrt{b^2-4} - \sqrt{a^2-4}\sqrt{b^2-4} \tag{3.20}$$

and

$$\frac{4}{PQ} = ab \mp b\sqrt{a^2-4} \pm a\sqrt{b^2-4} - \sqrt{a^2-4}\sqrt{b^2-4} \tag{3.21}$$

respectively. Adding (3.20) and (3.21) and then reframing, we have

$$2\left(PQ + \frac{1}{PQ}\right) - ab = -\sqrt{a^2-4}\sqrt{b^2-4}.$$

Squaring on both sides and then rearranging, we deduce

$$\left((PQ)^3 + \frac{1}{(PQ)^3}\right)^2 - ab\left((PQ)^3 + \frac{1}{(PQ)^3}\right) + a^2 + b^2 - 4 = 0.$$

Finally, employing a and b in the above and then re-framing the terms, we obtain the required result. □

Theorem 3.4. *Let*

$$P := q\frac{f_1 f_{39}}{f_3 f_{13}} \qquad \textit{and} \qquad Q := q^3\frac{f_3 f_{117}}{f_9 f_{39}}.$$

Then, we have

$$x_3 - 4x_2 + 10x_1 - 3x_{1/2}(2y_{1/2} + y_{3/2}) + x_1(6y_1 + y_2) + 3x_{3/2}(3y_{1/2} + y_{3/2})$$
$$- x_2(3y_1 + y_2) - 3x_{5/2}y_{1/2} - 14 = 0.$$

Proof. From (2.6) and together with P and Q, we obtain

$$P = \frac{C_1}{C_3} \qquad \text{and} \qquad Q = \frac{C_3}{C_9}. \tag{3.22}$$

Also, from (2.7) and (3.22), we find that

$$\frac{PC_3^2}{\sqrt{13}} + \frac{\sqrt{13}}{PC_3^2} = \frac{1}{\sqrt{13}}\left(\frac{1}{P^2} - \frac{3}{P} - 3P + P^2 - 3\right).$$

On solving the above for $PC_3^2/\sqrt{13}$, we obtain

$$\frac{PC_3^2}{\sqrt{13}} = \frac{a \pm \sqrt{a^2-4}}{2} \tag{3.23}$$

where

$$a = \frac{1}{\sqrt{13}}\left(\frac{1}{P^2} - \frac{3}{P} - 3P + P^2 - 3\right).$$

Further, the identity (3.23) implies that

$$\frac{\sqrt{13}}{PC_3^2} = \frac{a \mp \sqrt{a^2-4}}{2}. \tag{3.24}$$

Similarly, we can deduce

$$\frac{QC_9^2}{2} = \frac{b \pm \sqrt{b^2-4}}{2} \tag{3.25}$$

and

$$\frac{2}{QC_9^2} = \frac{b \mp \sqrt{b^2-4}}{2} \tag{3.26}$$

where

$$b = \frac{1}{\sqrt{13}}\left(\frac{1}{Q^2} - \frac{3}{Q} - 3Q + Q^2 - 3\right).$$

Multiplying (3.23) with (3.25), (3.24) with (3.26) and employing (3.22) and after a little simplification, we obtain

$$4PQ = ab \pm b\sqrt{a^2-4} \mp a\sqrt{b^2-4} - \sqrt{a^2-4}\sqrt{b^2-4} \tag{3.27}$$

and

$$\frac{4}{PQ} = ab \mp b\sqrt{a^2-4} \pm a\sqrt{b^2-4} - \sqrt{a^2-4}\sqrt{b^2-4} \tag{3.28}$$

respectively. Adding (3.27) and (3.28) and then simplifying, we have

$$2\left(PQ + \frac{1}{PQ}\right) - ab = -\sqrt{a^2-4}\sqrt{b^2-4}.$$

Squaring on both sides and then rearranging, we obtain

$$(PQ)^2 + \frac{1}{(PQ)^2} - ab\left(PQ + \frac{1}{PQ}\right) + a^2 + b^2 - 2 = 0.$$

Finally, employing a and b in the above and then re-framing the terms, we deduce Theorem 3.4. □

4. Conclusions

Motivated by earlier works, which we have cited herein, in the present paper some modular equations that are analogous to Ramanujan's identities are established. It is believed that many of the recent works (see, for example, [1, 2, 10, 12, 13]), which we have chosen to cite in this paper, are potentially useful for indicating directions for further research on q-series, q-polynomials, and q-differences based upon the subject matter which is related to that of our present investigation.

Conflict of interest

The authors declare no conflict of interest.

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