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A priori estimates for the free boundary problem of incompressible inviscid Boussinesq and MHD-Boussinesq equations without heat diffusion

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Abstract: For all physical spatial dimensions $n = 2$ and 3, we establish a priori estimates of Sobolev norms for free boundary problem of inviscid Boussinesq and MHD-Boussinesq equations without heat diffusion under the Taylor-type sign condition on the initial free boundary. It is different from MHD equations because the energy of the system is not conserved.

Keywords: free boundary problem; Boussinesq; MHD-Boussinesq; incompressible flows; a priori estimates
Mathematics Subject Classification: 35K59, 76W05

1. Introduction

The Boussinesq equations for magnetohydrodynamic convection (Boussinesq-MHD) model are of relevance to study a number of models coming from atmospheric or oceanographic turbulence. Let us consider the following inviscid MHD-Boussinesq equations without heat diffusion in $\mathcal{D}$:

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = b \cdot \nabla b + h e_n, \\ \partial_t h + v \cdot \nabla h = 0, \\ \partial_t b + v \cdot \nabla b = b \cdot \nabla v, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \end{cases} \tag{1.1}$$

where v, b, p and h denote the velocity, the magnetic field, the fluid pressure and the temperature, respectively; $\mathcal{D} \subset \cup_{0 \leqslant t \leqslant T} \{t\} \times \mathbb{R}^n$ is an unknowns to be determined for $n \in \{2, 3\}$. We want to find a set $\mathcal{D}$, v, h and b solving (1.1) and satisfying the initial conditions:

$$\{x : (0, x) \in \mathcal{D}\} = \mathcal{D}_0, \quad (v, b, h)|_{t=0} = (v_0(x), b_0(x), h_0(x)) \text{ for } x \in \mathcal{D}_0. \tag{1.2}$$

Let $\mathcal{D}_t = \{x \in \mathbb{R}^n : (t, x) \in \mathcal{D}\}$, then the free boundary conditions should be

$$\begin{cases} v_N = \kappa, & \text{on } \partial\mathcal{D}_t, \\ p = 0, & \text{on } \partial\mathcal{D}_t, \\ b \cdot N = 0, & \text{on } \partial\mathcal{D}_t, \end{cases} \tag{1.3}$$

for each $t \in [0, T]$, where N is the exterior unit normal to $\partial\mathcal{D}_t$, $v_N = N^i v_i$, κ is the normal velocity of $\partial\mathcal{D}_t$. To get a priori bounds for (1.1) and (1.3) in Sobolev spaces, we need assume:

$$\nabla_N p \leqslant -\varepsilon < 0 \text{ on } \partial\mathcal{D}_t, \tag{1.4}$$

where ε is positive constant. On the other hand, the condition (1.4) can hold initially and it will hold true within some time. In fact, the pressure is larger in the interior than on the boundary. Moreover, (1.4) is called Taylor sign condition for the Euler Eq (1.4) also plays the same role, for the Boussinesq equations.

The free boundary problems of incompressible Euler equations and ideal incompressible MHD have been studied by many people in recent decades. Wu [20, 21] first made great contributions and proved well-posedness for incompressible and irrotational water wave problems. On the motion of the free surface of a liquid, Christodoulou-Lindblad [8] obtaind a priori energy estimate. In [9], Gu and Wang had benn proved local well-posedness of free-surface incompressible ideal magnetohydrodynamic equation. Hao and Luo obtain well-posedness for the linearized free boundary problem of incompressible ideal magnetohydrodynamics equations in [10]. More works for flows has been extensively studied in [13, 14, 16].

On the other hand, there have been some results for Boussinesq equations. Chae and Nam [6] proved local existence and blow-up criterion for the Boussinesq equations, and established [7] for initial data in Hlder spaces C^r with $r > 1$. Another local well-posedness was recently obtained [17] for the critical Besov spaces $B^{2/p+1}_{p,1}$. Numerical investigations on similar data for the 2-D inviscid Boussinesq equations appeared to indicate that there is no finite time singularity formation [19]. Miao and Zheng [18] obtained the global well-posedness for the Boussinesq system with horizontal dissipation. An analytical work on the inviscid Boussinesq equation can be found in [5]. The global well-posedness for weak or strong solutions was obtained by the stability and instability for a fully nonlinear 2-D MHD-Boussinesq equations in [1–3]. Liu, Bian and Pu [4] obtained global well-posedness of the 3D Boussinesq-MHD system without heat diffusion. There are few results about two-phase fluid motion in the Oberbeck-Boussinesq approximation. Hao and Zhang established the maximal $L^p - L^q$ regularity for the two-phase fluid motion of the linearized Oberbeck-Boussinesq approximation in [12]. In recently, they obtained Local well-posedness for two-phase fluid motion in the Oberbeck-Boussinesq approximation.

In this paper, we adopt a geometrical point of view used in [8, 11], and estimate quantities such as the second fundamental form. We denote the material derivative by $D_t = \partial_t + v^k \partial_k$, then the system (1.1) can be rewritten as:

$$\begin{aligned} D_t v_j + \partial_j p = b^k\partial_k b_j + \delta_{jn} h, &\quad \text{in } \mathcal{D},\\ D_t b_j = b^k\partial_k v_j, &\quad \text{in } \mathcal{D},\\ D_t h = 0, &\quad \text{in } \mathcal{D},\\ \partial_j v^j = 0, \quad \partial_j b^j = 0, &\quad \text{in } \mathcal{D},\\ v_N = \kappa, \quad b_j N^j = 0, &\quad \text{on } [0,T]\times\partial\mathcal{D}_t,\\ p = 0, &\quad \text{on } \partial\mathcal{D},\\ \nabla_N p < 0, &\quad \text{on } \{t=0\}\times\partial\mathcal{D}_0, \end{aligned} \tag{1.5}$$

where δ_{ij} is the Kronecker delta symbol such that $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$.

Remark 1.1. *Different from the fixed boundary problem, in this paper, we do not need to assume the condition of temperature on the boundary in (1.5). In the following proof, we can find that the higher order energy of temperature is actually controlled by its initial data , and is not affected by the boundary condition.*

Remark 1.2. *Owe to $D_t h = 0$, the first equation of Eq* (1.5) *can transform to $D_t v_j + \partial_j p = b^k\partial_k b_j + \delta_{jn} h_0$. Thus, let the zero order energy be*

$$E_0(t) = \frac{1}{2}\int_{\mathcal{D}_t}\left(|v(t,x)|^2 + |b(t,x)|^2\right)dx. \tag{1.6}$$

A direct computation yields

$$\frac{d}{dt}E_0(t) \neq 0.$$

Different from MHD equations, the energy of the system is not conserved, but it can be controlled by the initial data and time T.

Let's introduce some knowledge of Riemannian geometry such as second fundamental form of the free surface and tensor products by [8]. In order to get energies, we introduce orthogonal projection Π to the tangent space of the boundary of a (0,r) tensor α is defined to be the projection of each component along the normal:

$$(\Pi\alpha)_{i_1\cdots i_r} = \Pi_{i_1}^{j_1}\cdots\Pi_{i_r}^{j_r}\alpha_{j_1\cdots j_r}, \quad \text{where} \quad \Pi_i^j = \delta_i^j - N_i N^j.$$

Let $\bar\partial_i = \Pi_i^j\partial_j$ be a tangential derivative. In fact, we assume $p = 0$ on $\partial\mathcal{D}_t$, obviously, it follows that $\bar\partial_i p = 0$ and

$$\left(\Pi\partial^2 p\right)_{ij} = \theta_{ij}\nabla_N p, \tag{1.7}$$

where $\theta_{ij} = \bar\partial_i N_j$ is the second fundamental form of $\partial\mathcal{D}_t$. In fact,

$$\begin{aligned} 0 = \bar\partial_i\bar\partial_j p = \Pi_i^{i'}\partial_{i'}\Pi_j^{j'}\partial_{j'}p &= \Pi_i^{i'}\Pi_j^{j'}\partial_{i'}\partial_{j'}p - \left(\bar\partial_i N_j\right)N^k\partial_k p - N_j\left(\bar\partial_i N^k\right)\partial_k p\\ &= \left(\Pi\partial^2 p\right)_{ij} - \theta_{ij}\nabla_N p. \end{aligned}$$

Next, we define the quadratic form Q of the form:

$$Q(\alpha,\beta) = \langle\Pi\alpha,\Pi\beta\rangle = q^{i_1 j_1}\cdots q^{i_r j_r}\alpha_{i_1\cdots i_r}\beta_{j_1\cdots j_r},$$

where

$$q^{ij} = \delta^{ij} - \eta(d)^2 N^i N^j, \quad d(x) = \text{dist}\,(x, \partial\mathcal{D}_t)\,, \quad N^i = -\delta^{ij}\partial_j d,$$

where η is a smooth cutoff function satisfying $0 \leqslant \eta(d) \leqslant 1$, $\eta(d) = 1$ when $d < d_0/4$, and $\eta(d) = 0$ when $d > d_0/2$. d_0 is a fixed number that is smaller than the injectivity radius of the normal exponential map ς_0, defined to be the largest number ς_0 such that the map

$$\partial\mathcal{D}_t \times (-\varsigma_0, \varsigma_0) \to \{x \in \mathbb{R}^n : \text{dist}\,(x, \partial\mathcal{D}_t) < \varsigma_0\}\,,$$

given by

$$(\bar{x}, \varsigma) \to x = \bar{x} + \varsigma N(\bar{x}),$$

is an injection. Then higher energies for $r \geqslant 1$ can be denoted by

$$\begin{aligned} E_r(t) = & \int_{\mathcal{D}_t} \delta^{ij} \left(Q\left(\partial^r v_i, \partial^r v_j\right) + Q\left(\partial^r b_i, \partial^r b_j\right)\right) dx \\ & + \int_{\mathcal{D}_t} \left(\left|\partial^{r-1} \operatorname{curl} v\right|^2 + \left|\partial^{r-1} \operatorname{curl} b\right|^2 \right) dx \\ & + sgn(r-1) \int_{\partial\mathcal{D}_t} Q\left(\partial^r p, \partial^r p\right) \vartheta dS, \end{aligned} \tag{1.8}$$

where

$$\vartheta = (-\nabla_N p)^{-1}\,.$$

In this paper, we prove the following main theorem.

Theorem 1.1. *Let*

$$\begin{aligned} \mathcal{H}(0) &= \max\left(\|\theta(0,\cdot)\|_{L^\infty(\partial\mathcal{D}_0)}, 1/\varsigma_0(0)\right), \\ \mathcal{I}(0) &= \|1/\left(\nabla_N p(0,\cdot)\right)\|_{L^\infty(\partial\mathcal{D}_0)} = 1/\varepsilon(0) > 0. \end{aligned} \tag{1.9}$$

There exists a continuous function $\mathcal{K} > 0$ such that if

$$T \leqslant \mathcal{K}\Big(\mathcal{H}(0), \mathcal{I}(0), E_0(0), \|\Theta_0\|^2_{L^2(\Omega)} \cdots, E_{n+1}(0), \left\|\nabla^{n+1}\Theta_0\right\|^2_{L^2(\Omega)}, \text{Vol}\,\mathcal{D}_0\Big), \tag{1.10}$$

then any smooth solution of the free boundary problem for inviscid MHD-Boussinesq Eq (1.5) without heat diffusion satisfies

$$\sum_{s=0}^{n+1} E_s(t) \leqslant 2\sum_{s=0}^{n+1}\left(E_s(0) + \|\nabla^s\Theta_0\|^2_{L^2(\Omega)}\right), \quad 0 \leqslant t \leqslant T. \tag{1.11}$$

In order to prove Theorem 1.1, we consider the Boussinesq equations with zero viscosity and diffusivity in $\mathcal{D}$:

$$\begin{cases} \dfrac{\partial v}{\partial t} + (v\cdot\nabla)v = -\nabla p + he_n, \\ \dfrac{\partial h}{\partial t} + (v\cdot\nabla)h = 0, \\ \nabla\cdot v = 0, \end{cases} \tag{1.12}$$

with the following conditions on the free boundary

$$\begin{cases} v_N = \kappa, & \text{on } \partial\mathcal{D}_t, \\ p = 0, & \text{on } \partial\mathcal{D}_t. \end{cases} \tag{1.13}$$

We will prove a priori bounds for (1.12) and (1.13) in Sobolev spaces under the assumption (1.4).

Let us now outline the proof of Theorem 1.1. Firstly, for the Boussinesq equations, we transform the free boundary problem to a fixed initial boundary problem in the Lagrangian coordinates in Section 2. In Section 3, we prove the zero order and the first order energy estimates. Section 4 is devoted to the higher order energy estimates by using the identities derived in Section 2, then, for the MHD-Boussinesq equations, we can get a similar conclusion in Section 5. Finally, we justify the a priori assumptions in Section 6.

2. Reformulation in Lagrangian coordinates

As we all know, the method to deal with the free boundary problem is to transform the unknown region into a fixed region. We can introduce Lagrangian coordinates or co-moving coordinates to transform the free boundary problem to a fixed boundary problem. Let $f_0 : \Omega \to \mathcal{D}_0$ is a diffeomorphism, where Ω be bounded domain in $\mathbb{R}^n$. Then the Lagrangian coordinates in (t, y) where $x = x(t, y) = f_t(y)$ are given by solving

$$\frac{dx}{dt} = v(t, x(t, y)), \quad x(0, y) = f_0(y), \quad y \in \Omega. \tag{2.1}$$

The Euclidean metric δ_{ij} in $\mathcal{D}_t$ then induces a metric

$$g_{ab}(t, y) = \delta_{ij} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b}, \tag{2.2}$$

and its inverse

$$g^{cd}(t, y) = \delta^{kl} \frac{\partial y^c}{\partial x^k} \frac{\partial y^d}{\partial x^l}, \tag{2.3}$$

in Ω for each fixed t. Furthermore, expressed in the y-coordinates, we have

$$\partial_i = \frac{\partial}{\partial x^i} = \frac{\partial y^a}{\partial x^i} \frac{\partial}{\partial y^a}. \tag{2.4}$$

Let us introduce the notation for the material derivative

$$D_t = \frac{\partial}{\partial t}\Big|_{y=\text{ const}} = \frac{\partial}{\partial t}\Big|_{x=\text{ const}} + v^k \frac{\partial}{\partial x^k}.$$

Let $u(t, y)$, Θ(t,y), $P(t, y)$ represent the velocity, temperature, pressure in the Lagrangian coordinates, respectively, then from ([15], Lemma 2.1) and (1.12), we can obtain

$$\begin{aligned} D_t u_a =& \frac{\partial x^j}{\partial y^a} \left(-\partial_j p + \delta_{nj} h \right) + v_j \frac{\partial x^k}{\partial y^a} \frac{\partial v^j}{\partial x^k} \\ =& -\nabla_a P + \delta_{na} \Theta + u^c \nabla_a u_c. \end{aligned} \tag{2.5}$$

Obviously, because the temperature Θ is a scalar, so we directly obtain

$$D_t\Theta = 0. \tag{2.6}$$

Thus, the temperature can be directly expressed by the initial temperature in Lagrangian coordinates, then the system (1.12) and (1.13) can be rewritten in the Lagrangian coordinates as

$$\begin{aligned} D_t u_a + \nabla_a P = \delta_{na}\Theta_0 + u^c\nabla_a u_c, &\quad \text{in } [0,T]\times\Omega,\\ \nabla_a u^a = 0, &\quad \text{in } [0,T]\times\Omega,\\ P = 0, \quad u\cdot N = \kappa, &\quad \text{on } [0,T]\times\partial\Omega. \end{aligned} \tag{2.7}$$

where $\Theta|_{t=0} = \Theta_0$.

3. The zero order and the first order energy estimates

Firstly, we need to define the zero order energy as

$$E_0(t) = \frac{1}{2}\int_\Omega \left(|u(t,y)|^2\right) dy. \tag{3.1}$$

Owe to ([15], Lemma 2.1) and (2.7), Gauss' formula and $D_t d\mu_g = 0$, we can get

$$\begin{aligned} \frac{d}{dt}E_0(t) =& \frac{1}{2}\int_\Omega D_t\left(g^{ab}u_a u_b\right) d\mu_g\\ =& \int_\Omega \left(u^a D_t u_a\right) d\mu_g + \int_\Omega \frac{1}{2}\left(D_t g^{ab}\right)(u_a u_b)\, d\mu_g\\ =& \int_\Omega \left[-u^a\nabla_a P + u^a\delta_{na}\Theta_0 + u^a u^c\nabla_a u_c\right] d\mu_g - \int_\Omega h^{ab}(u_a u_b)\, d\mu_g\\ =& -\int_{\partial\Omega} N_a u^a P d\mu_\gamma + \int_\Omega u^a\delta_{na}\Theta_0 d\mu_g + \int_\Omega u^a u^c\nabla_a u_c d\mu_g\\ & -\frac{1}{2}\int_\Omega g^{ac}\left(\nabla_c u_d + \nabla_d u_c\right) g^{db} u_a u_b d\mu_g\\ =& \int_\Omega u^a\delta_{na}\Theta_0 d\mu_g, \end{aligned}$$

then, using the Hlder inequality

$$\frac{d}{dt}E_0(t) \leqslant C\|u\|_{L^2(\Omega)}\|\Theta_0\|_{L^2(\Omega)} \leqslant C\left(E_0(t) + \|\Theta_0\|^2_{L^2(\Omega)}\right). \tag{3.2}$$

when $t\in[0,T]$ with a constant $T>0$, by the Gronwall inequality, it follows that

$$E_0(t) \leqslant C(T)\left(E_0(0) + \|\Theta_0\|^2_{L^2(\Omega)}\right) \leqslant C\left(E_0(0) + \|\Theta_0\|^2_{L^2(\Omega)}\right). \tag{3.3}$$

Zero order energy is controlled by initial value and time T. By ([11], Lemma 2.3), (2.5) and (2.7), we can get

$$\begin{aligned} & D_t\left(\nabla_b u_a\right) + \nabla_b\nabla_a P\\ =& [D_t,\nabla_b]\, u_a + \nabla_b D_t u_a + \nabla_b\nabla_a P\\ =& -\left(\nabla_a\nabla_b u^d\right) u_d + \nabla_b\delta_{na}\Theta_0 + \nabla_b\left(u^c\nabla_a u_c\right)\\ =& -\left(\nabla_a\nabla_b u^d\right) u_d + \delta_{na}\nabla_b\Theta_0 + \nabla_b u^c\nabla_a u_c + u^c\nabla_b\nabla_a u_c\\ =& \nabla_b u^c\nabla_a u_c + \delta_{na}\nabla_b\Theta_0. \end{aligned} \tag{3.4}$$

Then by (2.6) and ([11] , Lemma 2.3), we have that

$$D_t\left(\nabla\Theta\right)=\left[D_t,\nabla\right]\Theta+\nabla D_t\Theta=0. \tag{3.5}$$

In fact, the estimates of Θ and $\nabla\Theta$ are only related to their initial data. The difficulty is that the velocity term will appear in the higher derivatives, but we can still control them.

Now, we can calculate the first order energy estimates. By (3.4), ([15], Lemma 2.1) and ([11], (A.13)), we can get the material derivative of $g^{bd}\gamma^{ae}\nabla_a u_b\nabla_e u_d$,

$$\begin{aligned}
&D_t\left(g^{bd}\gamma^{ae}\nabla_a u_b\nabla_e u_d\right)\\
=&\left(D_t g^{bd}\right)\gamma^{ae}\nabla_a u_b\nabla_e u_d+g^{bd}\left(D_t\gamma^{ae}\right)\nabla_a u_b\nabla_e u_d+2g^{bd}\gamma^{ae}\left(D_t\nabla_a u_b\right)\nabla_e u_d\\
=&-2g^{bc}h_{cf}g^{fd}\gamma^{ae}\nabla_a u_b\nabla_e u_d-2g^{bd}\gamma^{ac}h_{cf}\gamma^{fe}\nabla_a u_b\nabla_e u_d-2g^{bd}\gamma^{ae}\nabla_e u_d\nabla_a\nabla_b P\\
&+2g^{bd}\gamma^{ae}\nabla_e u_d\delta_{nb}\nabla_a\Theta_0+2g^{bd}\gamma^{ae}\nabla_e u_d\nabla_a u^c\nabla_b u_c\\
=&-\gamma^{ae}\left(\nabla_c u_f+\nabla_f u_c\right)\nabla_a u^c\nabla_e u^f-2\gamma^{ac}\gamma^{fe}\left(\nabla_c u_f+\nabla_f u_c\right)\nabla_a u^d\nabla_e u_d\\
&-2\gamma^{ae}\nabla_e u^b\nabla_a\nabla_b P+2\gamma^{ae}\nabla_e u^b\delta_{nb}\nabla_a\Theta_0+2\gamma^{ae}\nabla_e u^b\nabla_a u^c\nabla_b u_c\\
=&-2\gamma^{ae}\nabla_c u_f\nabla_a u^c\nabla_e u^f-4\gamma^{ae}\gamma^{fc}\nabla_e u_f\nabla_a u^d\nabla_c u_d+2\gamma^{ae}\nabla_e u^b\nabla_a u^c\nabla_b u_c\\
&-2\gamma^{ae}\nabla_e u^b\nabla_a\nabla_b P+2\gamma^{ae}\nabla_e u^b\delta_{nb}\nabla_a\Theta_0\\
=&-4\gamma^{ae}\gamma^{fc}\nabla_e u_f\nabla_a u^d\nabla_c u_d-2\gamma^{ae}\nabla_e u^b\nabla_a\nabla_b P+2\gamma^{ae}\nabla_e u^b\delta_{bn}\nabla_a\Theta_0\\
=&-4\gamma^{ae}\gamma^{fc}\nabla_e u_f\nabla_a u^d\nabla_c u_d+2\gamma^{ae}\nabla_e u^b\delta_{nb}\nabla_a\Theta_0-2\nabla_b\left(\gamma^{ae}\nabla_e u^b\nabla_a P\right)\\
&+2\left(\nabla_b\gamma^{ae}\right)\left(\nabla_e u^b\nabla_a P\right).
\end{aligned} \tag{3.6}$$

Similarly, we can easily calculate the material derivative of $|\operatorname{curl} u|^2$. It follows that

$$\begin{aligned}
D_t|\operatorname{curl} u|^2=&D_t\left(g^{ac}g^{bd}(\operatorname{curl} u)_{ab}(\operatorname{curl} u)_{cd}\right)\\
=&2\left(D_t g^{ac}\right)g^{bd}(\operatorname{curl} u)_{ab}(\operatorname{curl} u)_{cd}+4g^{ac}g^{bd}\left(D_t\nabla_a u_b\right)(\operatorname{curl} u)_{cd}\\
=&-2g^{ae}g^{fc}g^{bd}\left(\nabla_e u_f+\nabla_f u_e\right)(\operatorname{curl} u)_{ab}(\operatorname{curl} u)_{cd}\\
&+4g^{ac}g^{bd}(\operatorname{curl} u)_{cd}\nabla_a u^e\nabla_b u_e-4g^{ac}g^{bd}(\operatorname{curl} u)_{cd}\nabla_a\nabla_b P\\
&+4g^{ac}g^{bd}(\operatorname{curl} u)_{cd}\delta_{nb}\nabla_a\Theta_0\\
=&-4g^{ae}g^{bd}\nabla_e u^c(\operatorname{curl} u)_{ab}(\operatorname{curl} u)_{cd}+4g^{ac}g^{bd}(\operatorname{curl} u)_{cd}\delta_{nb}\nabla_a\Theta_0.
\end{aligned} \tag{3.7}$$

Now, we can define the first order energy as

$$E_1(t)=\int_\Omega g^{bd}\gamma^{ae}\nabla_a u_b\nabla_e u_d\,d\mu_g+\int_\Omega|\operatorname{curl} u|^2 d\mu_g. \tag{3.8}$$

Finally, we have the following theorem.

Theorem 3.1. *For any smooth solution of system (2.7) satisfying the following assumptions*

$$\begin{aligned}
&|\nabla P|\leqslant M,\quad |\nabla u|\leqslant M,\quad \text{in } [0,T]\times\Omega,\\
&|\theta|+|\nabla u|+\frac{1}{\varsigma_0}\leqslant K,\quad \text{on } [0,T]\times\partial\Omega,
\end{aligned}$$

when $t\in[0,T]$, we can get

$$E_1(t)\leqslant 2e^{CMt}\left(E_1(0)+\|\nabla\Theta_0\|^2_{L^2(\Omega)}\right)+CK^2\left(\operatorname{Vol}\Omega\right)\left(e^{CMt}-1\right). \tag{3.9}$$

Proof.

$$\frac{d}{dt}E_1(t) = \int_\Omega D_t\left(g^{bd}\gamma^{ae}\nabla_a u_b \nabla_e u_d\right) d\mu_g + \int_\Omega D_t\left(|\operatorname{curl} u|^2\right) d\mu_g.$$

From (3.6)–(3.8), ([15], Lemma 2.1) and Gauss' formula, it yield to

$$\begin{aligned}\frac{d}{dt}E_1(t) =& \int_\Omega \left(-4\gamma^{ae}\gamma^{fc}\nabla_e u_f \nabla_a u^d \nabla_c u_d + 2\gamma^{ae}\nabla_e u^b \delta_{nb}\nabla_a \Theta_0\right) d\mu_g \\ &+ 2\int_\Omega (\nabla_b \gamma^{ae})\left(\nabla_e u^b \nabla_a P\right) d\mu_g - 4\int_\Omega g^{ae}g^{bd}\nabla_e u^c (\operatorname{curl} u)_{ab}(\operatorname{curl} u)_{cd} d\mu_g \\ &+ 4\int_\Omega g^{ac}g^{bd}(\operatorname{curl} u)_{cd}\delta_{nb}\nabla_a \Theta_0 d\mu_g - 2\int_{\partial\Omega} N_b\left(\gamma^{ae}\nabla_e u^b \nabla_a P\right) d\mu_\gamma.\end{aligned}$$

In fact the integral on the boundary is zero. Since $P = 0$ on $\partial\Omega$, we have $\bar{\nabla}P = \gamma^{ae}\nabla_a P = 0$.

Then, by ([11], (A.3) and (A.5)) and the fact $\nabla_N N = 0$. we obtain that

$$\theta_{ab} = \left(\delta_a^c - N_a N^c\right)\nabla_c N_b = \nabla_a N_b - N_a \nabla_N N_b = \nabla_a N_b,$$

Thus, we have

$$\begin{aligned}\nabla_b \gamma^{ae} &= \nabla_b\left(g^{ae} - N^a N^e\right) = -\nabla_b\left(N^a N^e\right) = -\left(\nabla_b N^a\right)N^e - \left(\nabla_b N^e\right)N^a \\ &= -\theta_b^a N^e - \theta_b^e N^a.\end{aligned}$$

From the Hlder inequality and ([11], (A.5)), it follows that

$$\begin{aligned}\frac{d}{dt}E_1(t) \leqslant& CKM(\operatorname{Vol}\Omega)^{1/2}E_1^{1/2}(t) + CM(\operatorname{Vol}\Omega)^{1/2}\|\nabla\Theta_0\|_{L^2(\Omega)} \\ &+ C\|\nabla u\|_{L^\infty(\Omega)}\left(\|\nabla u\|_{L^2(\Omega)}^2 + \|\operatorname{curl} u\|_{L^2(\Omega)}^2\right) \\ \leqslant& CKM(\operatorname{Vol}\Omega)^{1/2}\left(E_1(t) + \|\nabla\Theta_0\|_{L^2(\Omega)}^2\right)^{1/2} + CME_1(t).\end{aligned}$$

By the Gronwall inequality, it yields the desired estimate.

Remark 3.1. *In fact, we can find that the integral involving P is zero, so pressure does not affect boundary integral in E_1. But for the higher order estimates, we have to introduce boundary integrals for P.*

4. The general r-th order energy estimates

In this section, we will get the higher order energy estimates. By ([11], Lemma 2.2) and (1.12), we obtain that

$$\begin{aligned}D_t\nabla^r u_a = D_t\nabla_{a_1}\cdots\nabla_{a_r}u_a &= D_t\left(\frac{\partial x^{i_1}}{\partial y^{a_1}}\cdots\frac{\partial x^{i_r}}{\partial y^{a_r}}\frac{\partial x^i}{\partial y^a}\partial_{i_1}\cdots\partial_{i_r}v_i\right) \\ &= -\nabla^r\nabla_a P - \sum_{s=1}^{r-1}\begin{pmatrix} r \\ s+1\end{pmatrix}\left(\nabla^{1+s}u\right)\cdot\nabla^{r-s}u_a + \delta_{na}\nabla^r\Theta_0 + \nabla_a u^c \nabla^r u_c,\end{aligned}$$

when $r \geqslant 2$, by moving terms, we have

$$D_t\nabla^r u_a + \nabla^r\nabla_a P = (\operatorname{curl} u)_{ac}\nabla^r u^c + \operatorname{sgn}(2-r)\sum_{s=1}^{r-2}\begin{pmatrix} r \\ s+1 \end{pmatrix}\left(\nabla^{1+s}u\right)\cdot\nabla^{r-s}u_a + \delta_{na}\nabla^r\Theta_0. \tag{4.1}$$

Define the r-th order energy for $r \geqslant 2$ as

$$\begin{aligned} E_r(t) = &\int_\Omega g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a u_b\nabla_F^{r-1}\nabla_f u_d d\mu_g + \int_\Omega \left|\nabla^{r-1}\operatorname{curl} u\right|^2 d\mu_g \\ &+ \int_{\partial\Omega}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a P\nabla_F^{r-1}\nabla_f P\vartheta d\mu_\gamma, \end{aligned} \tag{4.2}$$

where $\vartheta = 1/(-\nabla_N P)$, then we can get the following estimates.

Theorem 4.1. *For integer $2 \leqslant r \leqslant n+1$, there exists a constant $T > 0$ such that, for any smooth solution to system (2.7) for $0 \leqslant t \leqslant T$ satisfying*

$$\begin{aligned} |\nabla P| \leqslant M, \quad |\nabla u| \leqslant M, \quad |\nabla\Theta_0| \leqslant M, &\quad \text{in } [0,T]\times\Omega, \\ |\theta| + 1/\varsigma_0 \leqslant K, &\quad \text{on } [0,T]\times\partial\Omega, \\ -\nabla_N P \geqslant \varepsilon > 0, &\quad \text{on } [0,T]\times\partial\Omega, \\ \left|\nabla^2 P\right| + |\nabla_N D_t P| \leqslant L, &\quad \text{on } [0,T]\times\partial\Omega, \end{aligned} \tag{4.3}$$

when $t \in [0,T]$, we obtain

$$E_r(t) \leqslant e^{C_1 t}\left(E_r(0) + \|\nabla^r\Theta_0\|^2_{L^2(\Omega)}\right) + C_2\left(e^{C_1 t} - 1\right), \tag{4.4}$$

where the constants C_1 and C_2 depend on $K, M, L, 1/\varepsilon, \operatorname{Vol}\Omega, E_0(0), \|\Theta_0\|^2_{L^2(\Omega)}, E_1(0), \cdots, E_{r-1}(0)$ and $\left\|\nabla^{r-1}\Theta_0\right\|^2_{L^2(\Omega)}$.

Proof. Appling (4.2), the derivative of E_r with respect to t is

$$\begin{aligned} \frac{d}{dt}E_r(t) = &\int_\Omega D_t\left(g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a u_b\nabla_F^{r-1}\nabla_f u_d\right)d\mu_g \\ &+ \int_\Omega D_t\left|\nabla^{r-1}\operatorname{curl} u\right|^2 d\mu_g + \int_{\partial\Omega} D_t\left(\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a P\nabla_F^{r-1}\nabla_f P\right)\vartheta d\mu_\gamma \\ &+ \int_{\partial\Omega}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a P\nabla_F^{r-1}\nabla_f P\left(\frac{\vartheta_t}{\vartheta} - h_{NN}\right)\vartheta d\mu_\gamma. \end{aligned} \tag{4.5}$$

From ([15], Lemma 2.1) and (4.1), we directly have

$$\begin{aligned} &D_t\left(g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a u_b\nabla_F^{r-1}\nabla_f u_d\right) \\ = &-2\nabla_c u_e\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a u^c\nabla_F^{r-1}\nabla_f u^e - 4r\nabla_c u_e\gamma^{ac}\gamma^{ef}\gamma^{AF}\nabla_A^{r-1}\nabla_a u^d\nabla_F^{r-1}\nabla_f u_d \\ &-2\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f u^b\nabla_A^{r-1}\nabla_a\nabla_b P + 2\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f u^b(\operatorname{curl} u)_{bc}\nabla_A^{r-1}\nabla_a u^c \\ &+ 2\operatorname{sgn}(2-r)\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f u_d\sum_{s=1}^{r-2}\begin{pmatrix} r \\ s+1 \end{pmatrix}\left(\left(\nabla^{s+1}u\right)\cdot\nabla^{r-s}u^d\right)_{Aa} \\ &+ 2\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f u^b\delta_{nb}\nabla_A^{r-1}\nabla_a\Theta_0. \end{aligned}$$

and

$$D_t\left(\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a P\nabla_F^{r-1}\nabla_f P\right)$$
$$=-2r\nabla_c u_e\gamma^{ac}\gamma^{ef}\gamma^{AF}\nabla_A^{r-1}\nabla_a P\nabla_F^{r-1}\nabla_f P+2\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a PD_t\left(\nabla_F^{r-1}\nabla_f P\right).$$

Next we will deal with the integration of the higher derivatives of P on the boundary. It is the difficulty in this paper. By the Hlder inequality, we get

$$\begin{aligned}&\int_\Omega D_t\left(g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a u_b\nabla_F^{r-1}\nabla_f u_d\right)d\mu_g+\int_{\partial\Omega}D_t\left(\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a P\nabla_F^{r-1}\nabla_f P\right)\vartheta d\mu_\gamma\\ \leqslant&C\|\nabla u\|_{L^\infty(\Omega)}E_r(t)+CE_r(t)+\|\nabla^r\Theta_0\|_{L^2(\Omega)}^2+CE_r^{1/2}(t)\sum_{s=1}^{r-2}\left\|\nabla^{s+1}u\right\|_{L^4(\Omega)}\left\|\nabla^{r-s}u\right\|_{L^4(\Omega)}\\ &+2\int_{\partial\Omega}\gamma^{af}\gamma^{AF}\nabla_{Aa}^r P\left(D_t\nabla_{Ff}^r P-\frac{1}{\vartheta}N_b\nabla_{Ff}^r u^b\right)\vartheta d\mu_\gamma\\ &+2\int_\Omega\nabla_b\left(\gamma^{af}\gamma^{AF}\right)\nabla_F^{r-1}\nabla_f u^b\nabla_A^{r-1}\nabla_a Pd\mu_g.\end{aligned}\tag{4.6}$$

Then, using the Hlder inequality, it yield that

$$2\int_\Omega\nabla_b\left(\gamma^{af}\gamma^{AF}\right)\nabla_F^{r-1}\nabla_f u^b\nabla_A^{r-1}\nabla_a Pd\mu_g\leqslant CKE_r^{1/2}(t)\|\nabla^r P\|_{L^2(\Omega)}.\tag{4.7}$$

Obviously, we have to estimate $\|\nabla^r P\|_{L^2(\Omega)}$. Firstly, we need to take divergence on the first equation of (2.7), by ([11], Lemma 2.2), we nave

$$\Delta P=-\nabla_a u^b\nabla_b u^a+\partial_n\Theta_0.\tag{4.8}$$

For $r\geqslant 2$, we have

$$\nabla^{r-2}\Delta P=-\sum_{s=0}^{r-2}\begin{pmatrix}r-2\\ s\end{pmatrix}\nabla^s\nabla_a u^b\nabla^{r-2-s}\nabla_b u^a-\nabla^{r-2}\partial_n\Theta_0.$$

Then by ([11], (A.12)), for $\varsigma_1\geqslant 1/K_1$, it follows that

$$\|u\|_{L^\infty(\Omega)}\leqslant C\sum_{s=0}^{2}K_1^{n/2-s}\|\nabla^s u\|_{L^2(\Omega)}\leqslant C(K_1)\sum_{s=0}^{2}E_s^{1/2}(t).\tag{4.9}$$

In view of (4.9), for $s\geqslant 0$, similarly, we can directly get taht

$$\|\nabla^s u\|_{L^\infty(\Omega)}\leqslant C\sum_{\ell=0}^{2}K_1^{n/2-\ell}\left\|\nabla^{\ell+s}u\right\|_{L^2(\Omega)}\leqslant C(K_1)\sum_{\ell=0}^{2}E_{s+\ell}^{1/2}(t),\tag{4.10}$$

When $r = 3, 4$, By the Hlder inequality, (4.9) and (4.10), we have

$$\begin{aligned}\left\|\nabla^{r-2}\Delta P\right\|_{L^2(\Omega)} \leqslant & C\sum_{s=0}^{r-2}\left\|\nabla^s\nabla_a u^b\nabla^{r-2-s}\nabla_b u^a\right\|_{L^2(\Omega)} + C\left\|\nabla^{r-2}\partial_n\Theta_0\right\|_{L^2(\Omega)}\\ \leqslant & C\|\nabla u\|_{L^\infty(\Omega)}\left\|\nabla^{r-1}u\right\|_{L^2(\Omega)} + (r-3)C\left\|\nabla^2 u\right\|_{L^\infty(\Omega)}\left\|\nabla^2 u\right\|_{L^2(\Omega)}\\ & + C\left\|\nabla^{r-1}\Theta_0\right\|_{L^2(\Omega)}\\ \leqslant & C(K_1)\sum_{\ell=1}^{r-1}E_\ell(t) + C(K_1)E_2^{1/2}(t)E_r^{1/2}(t) + C(K_1)\|\nabla^{r-1}\Theta_0\|_{L^2(\Omega)}\\ \leqslant & C(K_1)\sum_{\ell=0}^{r-1}(E_\ell(t) + \|\nabla^\ell\Theta_0\|^2_{L^2(\Omega)}) + C(K_1)E_2^{1/2}(t)E_r^{1/2}(t).\end{aligned}\tag{4.11}$$

The last inequality is due to the inequality $\|\nabla^{r-1}\Theta_0\|_{L^2(\Omega)} \leqslant CE_0 + C\|\nabla^{r-1}\Theta_0\|^2_{L^2(\Omega)}$. For $r = 2$, we have the following estimate from the assumption of (4.3) and the Hlder inequality, i.e.,

$$\|\Delta P\|_{L^2(\Omega)} \leqslant CM\left(E_1(t) + \|\nabla\Theta_0\|^2_{L^2(\Omega)}\right)^{1/2},\tag{4.12}$$

which is a lower order energy term. Thus , from ([8], (A.17)), (4.11) and (4.12), we get for any $\delta_r > 0$

$$\begin{aligned}\|\nabla^r P\|_{L^2(\Omega)} \leqslant & \delta_r\|\Pi\nabla^r P\|_{L^2(\partial\Omega)} + C(1/\delta_r, K, \mathrm{Vol}\,\Omega)\sum_{s\leqslant r-2}\|\nabla^s\Delta P\|_{L^2(\Omega)}\\ \leqslant & \delta_r\|\Pi\nabla^r P\|_{L^2(\partial\Omega)} + C(1/\delta_r, K, K_1, M, \mathrm{Vol}\,\Omega)\sum_{\ell=0}^{r-1}\left(E_\ell(t) + \|\nabla^\ell\Theta_0\|^2_{L^2(\Omega)}\right)\\ & + (r-2)C(1/\delta_r, K, K_1, M, \mathrm{Vol}\,\Omega)(E_2^{1/2}(t)E_r^{1/2}(t)).\end{aligned}\tag{4.13}$$

Due to ([11], (A.18)) and $P = 0$ on $\partial\Omega$, we have for $r \geqslant 1$,

$$\|\Pi\nabla^r P\|_{L^2(\partial\Omega)} \leqslant C(K, K_1)\left(\|\theta\|_{L^\infty(\partial\Omega)} + (r-2)\sum_{k\leqslant r-3}\left\|\bar\nabla^k\theta\right\|_{L^2(\partial\Omega)}\right)\sum_{k\leqslant r-1}\left\|\nabla^k P\right\|_{L^2(\partial\Omega)}.\tag{4.14}$$

By ([11], (A.7)), we obtain that $\Pi\nabla^2 P = \theta\nabla_N P$, then from (4.3), we get

$$\|\theta\|_{L^2(\partial\Omega)} = \left\|\frac{\Pi\nabla^2 P}{\nabla_N P}\right\|_{L^2(\partial\Omega)} \leqslant \frac{1}{\varepsilon}\left\|\Pi\nabla^2 P\right\|_{L^2(\partial\Omega)}.\tag{4.15}$$

Then , for $r = 2, 3, 4$, we need to estimate $\|\Pi\nabla^r P\|_{L^2(\partial\Omega)}$ and $\|\nabla^r P\|_{L^2(\Omega)}$.

When r=2, by using (4.13) and (4.14), we have

$$\begin{aligned}\left\|\Pi\nabla^2 P\right\|_{L^2(\partial\Omega)} \leqslant & \|\theta\|_{L^\infty(\partial\Omega)}\|\nabla P\|_{L^2(\partial\Omega)}\\ \leqslant & C(K, \mathrm{Vol}\,\Omega)\left(\left\|\nabla^2 P\right\|_{L^2(\Omega)} + \|\nabla P\|_{L^2(\Omega)}\right)\\ \leqslant & C(K, \mathrm{Vol}\,\Omega)\delta_2\left\|\Pi\nabla^2 P\right\|_{L^2(\partial\Omega)} + C(K, \mathrm{Vol}\,\Omega)(\mathrm{Vol}\,\Omega)^{1/2}M\\ & + C(1/\delta_2, K, K_1, M, \mathrm{Vol}\,\Omega)\left(\sum_{\ell=0}^{1}(E_\ell(t) + \|\nabla^\ell\Theta_0\|^2_{L^2(\Omega)})\right).\end{aligned}$$

We can take δ_2 so small that the first term can be absorbed by the left-hand side. Thus

$$\left\|\Pi\nabla^2 P\right\|_{L^2(\partial\Omega)}, \left\|\nabla^2 P\right\|_{L^2(\Omega)} \leqslant C\,(K, K_1, M, \operatorname{Vol}\Omega)\left(1 + \sum_{\ell=0}^{1}(E_\ell(t) + \|\nabla^\ell\Theta_0\|^2_{L^2(\Omega)})\right), \tag{4.16}$$

$$\|\theta\|_{L^2(\partial\Omega)} \leqslant C\,(K, K_1, M, \operatorname{Vol}\Omega, 1/\varepsilon)\left(1 + \sum_{\ell=0}^{1}(E_\ell(t) + \|\nabla^\ell\Theta_0\|^2_{L^2(\Omega)})\right). \tag{4.17}$$

From Theorem 3.1 and zero order energy estimate, there exists a constant $T > 0$ such that $E_i(t) \leqslant CE_i(0) + \|\nabla^i\Theta_0\|^2_{L^2(\Omega)}$ for $t \in [0, T]$ and $i = 0, 1$.

When r=3, by (4.3), (4.14), (4.16) and (4.17), we have

$$\begin{aligned}\left\|\Pi\nabla^3 P\right\|_{L^2(\partial\Omega)} \leqslant& C\,(K, K_1)\left(K + \|\theta\|_{L^2(\partial\Omega)}\right)\sum_{k\leqslant 2}\left\|\nabla^k P\right\|_{L^2(\partial\Omega)}\\ \leqslant& C\left(K, K_1, M, \operatorname{Vol}\Omega, 1/\varepsilon, E_0(0), \|\Theta_0\|^2_{L^2(\Omega)}, E_1(0), \|\nabla\Theta_0\|^2_{L^2(\Omega)}\right)\left\|\nabla^3 P\right\|_{L^2(\Omega)}\\ &+ C\left(K, K_1, M, \operatorname{Vol}\Omega, 1/\varepsilon, E_0(0), \|\Theta_0\|^2_{L^2(\Omega)}, E_1(0), \|\nabla\Theta_0\|^2_{L^2(\Omega)}\right),\end{aligned} \tag{4.18}$$

and by (4.13), it follows that

$$\begin{aligned}\left\|\nabla^3 P\right\|_{L^2(\Omega)} \leqslant& \delta_3 C\left(K, K_1, M, \operatorname{Vol}\Omega, 1/\varepsilon, E_0(0), \|\Theta_0\|^2_{L^2(\Omega)}, E_1(0), \|\nabla\Theta_0\|^2_{L^2(\Omega)}\right)\left\|\nabla^3 P\right\|_{L^2(\Omega)}\\ &+ \delta_3 C\left(K, K_1, M, \operatorname{Vol}\Omega, 1/\varepsilon, E_0(0), \|\Theta_0\|^2_{L^2(\Omega)}, E_1(0), \|\nabla\Theta_0\|^2_{L^2(\Omega)}\right)\\ &+ C\,(1/\delta_3, K, K_1, M, \operatorname{Vol}\Omega)\left(\sum_{\ell=0}^{2}(E_\ell(t) + \|\nabla^\ell\Theta_0\|^2_{L^2(\Omega)})\right)\\ &+ C\,(1/\delta_3, K, K_1, M, \operatorname{Vol}\Omega)\,(E_2^{1/2}(t)E_3^{1/2}(t)).\end{aligned} \tag{4.19}$$

Then, we can choose a sufficiently small $\delta_3 > 0$, and by (4.18) and (4.19), it yields that

$$\begin{aligned}\left\|\nabla^3 P\right\|_{L^2(\Omega)}, \left\|\Pi\nabla^3 P\right\|_{L^2(\partial\Omega)} \leqslant& C\left(K, K_1, M, \operatorname{Vol}\Omega, 1/\varepsilon, E_0(0), \|\Theta_0\|^2_{L^2(\Omega)}, E_1(0), \|\nabla\Theta_0\|^2_{L^2(\Omega)}\right)\\ &\times\left(1 + \sum_{\ell=0}^{2}(E_\ell(t) + \|\nabla^\ell\Theta_0\|^2_{L^2(\Omega)}) + E_2^{1/2}(t)E_3^{1/2}(t)\right).\end{aligned} \tag{4.20}$$

When r=4, since

$$\begin{aligned}\bar\nabla_b\nabla_N P &= \gamma_b^d\nabla_d\,(N^a\nabla_a P) = \left(\delta_b^d - N_b N^d\right)((\nabla_d N^a)\,\nabla_a P + N^a\nabla_d\nabla_a P)\\ &= \theta_b^a\nabla_a P + N^a\nabla_b\nabla_a P - N_b N^d\,(\theta_d^a\nabla_a P + N^a\nabla_d\nabla_a P)\,,\end{aligned}$$

by [[11], (A.31) and (A.8)], (4.14), (4.16), (4.17) and (4.20), we have

$$\begin{aligned}\left\|\bar\nabla\nabla_N P\right\|_{L^2(\partial\Omega)} \leqslant& C\|\theta\|_{L^\infty(\partial\Omega)}\|\nabla P\|_{L^2(\partial\Omega)} + C\left\|\nabla^2 P\right\|_{L^2(\partial\Omega)}\\ \leqslant& C(K, \operatorname{Vol}\Omega)\left(\left\|\nabla^3 P\right\|_{L^2(\Omega)} + \left\|\nabla^2 P\right\|_{L^2(\Omega)} + \|\nabla P\|_{L^2(\Omega)}\right)\\ \leqslant& C\left(K, K_1, M, \operatorname{Vol}\Omega, 1/\varepsilon, E_0(0), \|\Theta_0\|^2_{L^2(\Omega)}, E_1(0), \|\nabla\Theta_0\|^2_{L^2(\Omega)}\right)\\ &\times\left(1 + \sum_{\ell=0}^{2}(E_\ell(t) + \|\nabla^\ell\Theta_0\|^2_{L^2(\Omega)}) + E_2^{1/2}(t)E_3^{1/2}(t)\right).\end{aligned}$$

Thus, from [[11], (A.8)], we can get $(\bar\nabla\theta)\nabla_N P = \Pi\nabla^3 P - 3\theta\tilde\otimes\bar\nabla\nabla_N P$, and

$$\begin{aligned}\|\bar\nabla\theta\|_{L^2(\partial\Omega)} &\leqslant \frac{1}{\varepsilon}\left(\left\|\Pi\nabla^3 P\right\|_{L^2(\partial\Omega)} + C\|\theta\|_{L^\infty(\partial\Omega)}\left\|\bar\nabla\nabla_N p\right\|_{L^2(\partial\Omega)}\right)\\ &\leqslant C\left(K, K_1, M, \operatorname{Vol}\Omega, 1/\varepsilon, E_0(0), \|\Theta_0\|^2_{L^2(\Omega)}, E_1(0), \|\nabla\Theta_0\|^2_{L^2(\Omega)}\right)\\ &\quad\times\left(1 + \sum_{\ell=0}^{2}(E_\ell(t) + \|\nabla^\ell\Theta_0\|^2_{L^2(\Omega)}) + E_2^{1/2}(t)E_3^{1/2}(t)\right).\end{aligned}$$

Hence, by (4.14), it follows that

$$\left\|\Pi\nabla^4 P\right\|_{L^2(\partial\Omega)} \leqslant C(K, K_1)\left(K + \|\theta\|_{L^2(\partial\Omega)} + \|\bar\nabla\theta\|_{L^2(\partial\Omega)}\right)\sum_{k\leqslant 4}\left\|\nabla^k P\right\|_{L^2(\Omega)}.$$

In the end, by (4.15), we choose a sufficiently small $\delta_4 > 0$ which can absorb the highest order term in the right-hand side, and get

$$\begin{aligned}&\left\|\nabla^4 P\right\|_{L^2(\Omega)}, \left\|\Pi\nabla^4 P\right\|_{L^2(\partial\Omega)}\\ \leqslant& C\left(K, K_1, M, \operatorname{Vol}\Omega, 1/\varepsilon, E_0(0), \|\Theta_0\|^2_{L^2(\Omega)}, E_1(0), \|\nabla\Theta_0\|^2_{L^2(\Omega)}\right)\\ &\times\left(1 + \sum_{\ell=0}^{3}((E_\ell(t) + \|\nabla^\ell\Theta_0\|^2_{L^2(\Omega)}) + E_2^{1/2}(t)E_4^{1/2}(t)\right).\end{aligned} \tag{4.21}$$

Therefore, by using (4.16), (4.20) and (4.21), we can get for $r \geqslant 2$

$$\begin{aligned}\|\nabla^r P\|_{L^2(\Omega)} \leqslant& C\left(K, K_1, M, \operatorname{Vol}\Omega, 1/\varepsilon, E_0(0), \|\Theta_0\|^2_{L^2(\Omega)}, E_1(0), \|\nabla\Theta_0\|^2_{L^2(\Omega)}\right)\\ &\times\left(1 + \sum_{\ell=0}^{r-1}(E_\ell(t) + \|\nabla^\ell\Theta_0\|^2_{L^2(\Omega)}) + (r-2)(E_2^{1/2}(t)E_r^{1/2}(t))\right),\end{aligned}$$

from which and (4.7), we have

$$\begin{aligned}&2\int_\Omega \nabla_b\left(\gamma^{af}\gamma^{AF}\right)\nabla_F^{r-1}\nabla_f u^b\nabla_A^{r-1}\nabla_a P d\mu_g\\ &\leqslant C\left(K, K_1, M, \operatorname{Vol}\Omega, 1/\varepsilon, E_0(0), \|\Theta_0\|^2_{L^2(\Omega)}, E_1(0), \|\nabla\Theta_0\|^2_{L^2(\Omega)}\right)E_r^{1/2}(t)\\ &\times\left(1 + \sum_{\ell=0}^{r-1}(E_\ell(t) + \|\nabla^\ell\Theta_0\|^2_{L^2(\Omega)}) + (r-2)E_2^{1/2}(t)E_r^{1/2}(t)\right).\end{aligned}$$

Now, since $P = 0$ on $\partial\Omega$, we have $\gamma_b^a\nabla_a P = 0$ on $\partial\Omega$. Thus we also get

$$-\vartheta^{-1}N_b = \nabla_b P. \tag{4.22}$$

By using the Hlder inequality and (4.22), we obtain

$$\begin{aligned}&\int_{\partial\Omega}\gamma^{af}\gamma^{AF}\nabla_{Aa}^r P\left(D_t\nabla_{Ff}^r P - \frac{1}{\vartheta}N_b\nabla_{Ff}^r u^b\right)\vartheta d\mu_\gamma\\ &\leqslant C\|\vartheta\|^{1/2}_{L^\infty(\partial\Omega)}E_r^{1/2}(t)\left\|\Pi\left(D_t(\nabla^r P) - \vartheta^{-1}N_b\nabla^r u^b\right)\right\|_{L^2(\partial\Omega)}\\ &= C\|\vartheta\|^{1/2}_{L^\infty(\partial\Omega)}E_r^{1/2}(t)\|\Pi(D_t(\nabla^r P) + \nabla^r u\cdot\nabla P)\|_{L^2(\partial\Omega)},\end{aligned} \tag{4.23}$$

hence, we will estimate $\Pi D_t(\nabla^r P)$ and $\nabla^r u \cdot \nabla P$, from ([11], Lemma 2.3), it yields that

$$\begin{aligned} D_t\nabla^r P + \nabla^r u \cdot \nabla P &= [D_t, \nabla^r] P + \nabla^r D_t P + \nabla^r u \cdot \nabla P \\ &= \operatorname{sgn}(2-r)\sum_{s=1}^{r-2}\begin{pmatrix} r \\ s+1 \end{pmatrix}\left(\nabla^{s+1}u\right)\cdot\nabla^{r-s}P + \nabla^r D_t P. \end{aligned} \tag{4.24}$$

For $2 \leqslant r \leqslant 4$, From ([11], (A.18), (A.31) and (A.17)), we can get

$$\begin{aligned} \|\Pi\nabla^r D_t P\|_{L^2(\partial\Omega)} \leqslant & C(K, K_1, \operatorname{Vol}\Omega)\left(\|\theta\|_{L^\infty(\partial\Omega)} + (r-2)\sum_{k\leqslant r-3}\left\|\bar{\nabla}^k\theta\right\|_{L^2(\partial\Omega)}\right) \\ & \times \sum_{k\leqslant r}\left\|\nabla^k D_t P\right\|_{L^2(\Omega)}, \end{aligned} \tag{4.25}$$

then since by ([11], (A.17)), we get that

$$\|\nabla^r D_t P\|_{L^2(\Omega)} \leqslant \delta\,\|\Pi\nabla^r D_t P\|_{L^2(\partial\Omega)} + C(1/\delta, K, \operatorname{Vol}\Omega)\sum_{s\leqslant r-2}\|\nabla^s \Delta D_t P\|_{L^2(\Omega)}. \tag{4.26}$$

Now, by ([11], Lemmas 2.1 and 2.3), (2.7), (3.4), (3.5) and (4.8), we have

$$\begin{aligned} \Delta D_t P =& 2h^{ab}\nabla_a\nabla_b P + (\Delta u^e)\nabla_e P - D_t(\triangle P) \\ =& 2h^{ab}\nabla_a\nabla_b P + (\Delta u^e)\nabla_e P - 2D_t\left(g^{bd}\right)\nabla_a u_d\nabla_b u^a - 2g^{bd}D_t(\nabla_a u_d)\nabla_b u^a + D_t(\partial_n\Theta_0) \\ =& 2h^{ab}\nabla_a\nabla_b P + (\Delta u^e)\nabla_e P + 4h^{bd}\nabla_a u_d\nabla_b u^a + 2g^{bd}\nabla_b u^a\nabla_a\nabla_d P \\ & - 2g^{bd}\nabla_b u^a\nabla_a u^c\nabla_d u_c - 2g^{bd}\nabla_d u^a\delta_{nb}\nabla_a\Theta_0 + D_t(\partial_n\Theta_0) \\ =& 4g^{ac}\nabla_c u^b\nabla_a\nabla_b P + (\Delta u^e)\nabla_e P + 2\nabla_e u^b\nabla_b u^a\nabla_a u^e - 2g^{bd}\nabla_b u^a\delta_{nb}\nabla_a\Theta_0. \end{aligned}$$

For $s = 2$ (similarly for $s = 0, 1$), From (4.12), (4.15) and ([11], Lemma A.12), it follows that,

$$\begin{aligned} \left\|\nabla^2\Delta D_t P\right\|_{L^2(\Omega)} \leqslant & C\left\|\nabla^3 u\nabla^2 P + \nabla^2 u\nabla^3 P + \nabla u\nabla^4 P + \nabla P\nabla^4 u\right\|_{L^2(\Omega)} \\ & + C\left\|\nabla^3 u\nabla u\nabla u + \nabla^2 u\nabla^2 u\nabla u\right\|_{L^2(\Omega)} \\ & + C\left\|\nabla^3 u\nabla\Theta_0 + \nabla^2 u\nabla^2\Theta_0 + \nabla u\nabla^3\Theta_0\right\|_{L^2(\Omega)} \\ \leqslant & C\|\nabla u\|_{L^\infty(\Omega)}\left\|\nabla^4 P\right\|_{L^2(\Omega)} + C\left\|\nabla^3 u\right\|_{L^2(\Omega)}\left\|\nabla^2 P\right\|_{L^\infty(\Omega)} \\ & + C\left\|\nabla^2 u\right\|_{L^4(\Omega)}\left\|\nabla^3 P\right\|_{L^4(\Omega)} + C\left\|\nabla^4 u\right\|_{L^2(\Omega)}\|\nabla P\|_{L^\infty(\Omega)} \\ & + C\|\nabla u\|_{L^\infty(\Omega)}\|\nabla u\|_{L^\infty(\Omega)}\left\|\nabla^3 u\right\|_{L^2(\Omega)} \\ & + C\|\nabla u\|_{L^\infty(\Omega)}\left\|\nabla^2 u\right\|_{L^4(\Omega)}\left\|\nabla^2 u\right\|_{L^4(\Omega)} \\ & + C\|\nabla u\|_{L^\infty(\Omega)}\left\|\nabla^3\Theta_0\right\|_{L^2(\Omega)} + C\left\|\nabla^3 u\right\|_{L^2(\Omega)}\|\nabla\Theta_0\|_{L^\infty(\Omega)} \\ & + C\left\|\nabla^2 u\right\|_{L^4(\Omega)}\left\|\nabla^2\Theta_0\right\|_{L^4(\Omega)}. \end{aligned}$$

By ([11], (A.11)) and (4.10), we have

$$\begin{aligned} \left\|\nabla^{s+1}u\right\|_{L^4(\Omega)} &\leqslant C\,\|\nabla^s u\|_{L^\infty(\Omega)}^{1/2}\left(\sum_{\ell=0}^{2}\left\|\nabla^{s+\ell}u\right\|_{L^2(\Omega)}K_1^{2-\ell}\right)^{1/2} \\ &\leqslant C(K_1)\sum_{\ell=0}^{2}E_{s+\ell}^{1/2}(t). \end{aligned} \tag{4.27}$$

Obviously, we can get

$$\begin{aligned}\left\|\nabla^{s+1}\Theta_0\right\|_{L^4(\Omega)} &\leqslant C\,\|\nabla^s\Theta_0\|_{L^\infty(\Omega)}^{1/2}\left(\sum_{\ell=0}^{2}\left\|\nabla^{s+\ell}\Theta_0\right\|_{L^2(\Omega)}K_1^{2-\ell}\right)^{1/2}\\ &\leqslant C\,(K_1)\sum_{\ell=0}^{2}\left\|\nabla^{s+\ell}\Theta_0\right\|_{L^2(\Omega)}.\end{aligned}\tag{4.28}$$

From (4.27) and (4.28), we will estimate all terms with $L^4(\Omega)$ norms and the similar estimate of P by the assumptions. Thus, we have

$$\begin{aligned}\|\nabla^s\Delta D_tP\|_{L^2(\Omega)} \leqslant & C\left(K, K_1, M, L, 1/\varepsilon, \mathrm{Vol}\,\Omega, E_0(0), \|\Theta_0\|_{L^2(\Omega)}\right)\\ &\times\left(1+\sum_{\ell=0}^{r-1}(E_\ell(t)+\left\|\nabla^\ell\Theta_0\right\|_{L^2(\Omega)}^2)\right)\left(1+(E_r(t)+\|\nabla^r\Theta_0\|_{L^2(\Omega)}^2)^{1/2}\right).\end{aligned}\tag{4.29}$$

By (4.25), (4.26) and (4.29), for small δ independent of $E_r(t)$, by induction argument for r, we have

$$\begin{aligned}\|\Pi\nabla^rD_tP\|_{L^2(\partial\Omega)} \leqslant & C(K, K_1, M, L, 1/\varepsilon, \mathrm{Vol}\,\Omega, E_0(0), \|\Theta_0\|_{L^2(\Omega)})\\ &\times\left(1+\sum_{\ell=0}^{r-1}(E_\ell(t)+\left\|\nabla^\ell\Theta_0\right\|_{L^2(\Omega)}^2)\right)\left(1+(E_r(t)+\|\nabla^r\Theta_0\|_{L^2(\Omega)}^2)^{1/2}\right).\end{aligned}\tag{4.30}$$

Finally, for the case $r = 3, 4$ and $s = r-2$, we need to estimate the remaining term $\Pi\left(\left(\nabla^{s+1}u\right)\cdot\nabla^{r-s}P\right)$, by ([11], Lemma A.14) and (4.3), we get

$$\begin{aligned}\left\|\Pi\left(\left(\nabla^{r-1}u\right)\cdot\nabla^2P\right)\right\|_{L^2(\partial\Omega)} &\leqslant \left\|\nabla^{r-1}u\right\|_{L^2(\partial\Omega)}\left\|\nabla^2P\right\|_{L^\infty(\partial\Omega)}\\ &\leqslant CL\left\|\nabla^2u\right\|_{L^{2(n-1)/(n-2)}(\partial\Omega)}\\ &\leqslant C(K,\mathrm{Vol}\,\Omega)L\left(\|\nabla^ru\|_{L^2(\Omega)}+\left\|\nabla^{r-1}u\right\|_{L^2(\Omega)}\right)\\ &\leqslant C(K,L,\mathrm{Vol}\,\Omega)\left(E_{r-1}^{1/2}(t)+E_r^{1/2}(t)\right).\end{aligned}$$

For $n = 3$, $r = 4$ and $s = 1$, we have similarly

$$\begin{aligned}&\left\|\Pi\left(\left(\nabla^2u\right)\cdot\nabla^3P\right)\right\|_{L^2(\partial\Omega)} = \left\|\Pi\nabla^2u\cdot\Pi\nabla^3p+\Pi\left(\nabla^2u\cdot N\right)\tilde{\otimes}\Pi\left(N\cdot\nabla^3P\right)\right\|_{L^2(\partial\Omega)}\\ \leqslant & C\left\|\Pi\nabla^2u\right\|_{L^4(\partial\Omega)}\left\|\Pi\nabla^3P\right\|_{L^4(\partial\Omega)}+C\left\|\Pi\left(N^a\nabla^2u_a\right)\right\|_{L^4(\partial\Omega)}\left\|\Pi\left(\nabla_N\nabla^2P\right)\right\|_{L^4(\partial\Omega)}\\ \leqslant & C\left\|\nabla^2u\right\|_{L^4(\partial\Omega)}\left\|\nabla^3P\right\|_{L^4(\partial\Omega)}\\ \leqslant & C(K,\mathrm{Vol}\,\Omega)\left(\left\|\nabla^3u\right\|_{L^2(\Omega)}+\left\|\nabla^2u\right\|_{L^2(\Omega)}\right)\left(\left\|\nabla^4P\right\|_{L^2(\Omega)}+\left\|\nabla^3P\right\|_{L^2(\Omega)}\right)\\ \leqslant & C\left(K,K_1,\mathrm{Vol}\,\Omega\right)\left(E_3^{1/2}(t)+E_2^{1/2}(t)\right)\left(\sum_{s=0}^{3}(E_s(t)+\|\nabla^s\Theta_0\|_{L^2(\Omega)}^2)+\left(\sum_{\ell=0}^{4}E_\ell^{1/2}(t)\right)E_2^{1/2}(t)\right)\\ \leqslant & C\left(K,K_1,\mathrm{Vol}\,\Omega\right)\sum_{s=0}^{3}(E_s(t)+\|\nabla^s\Theta_0\|_{L^2(\Omega)}^2)\sum_{\ell=0}^{4}E_\ell^{1/2}(t).\end{aligned}$$

Then, we can get

$$
\begin{aligned}
|(4.23)| \leqslant & C\left(K, K_1, M, L, 1/\varepsilon, \operatorname{Vol}\Omega, E_0(0), \|\Theta_0\|^2_{L^2(\Omega)}\right) \\
& \times\left(1+\sum_{s=0}^{r-1}(E_s(t)+\|\nabla^s\Theta_0\|^2_{L^2(\Omega)})\right)\left(1+E_r(t)+\|\nabla^r\Theta_0\|^2_{L^2(\Omega)}\right).
\end{aligned}
$$

By combining (4.27) with (4.28), it yields that

$$
\begin{aligned}
|(4.5)|+|(4.6)| \leqslant & C\left(K, K_1, M, L, 1/\varepsilon, \operatorname{Vol}\Omega, E_0(0), \|\Theta_0\|^2_{L^2(\Omega)}\right) \\
& \times\left(1+\sum_{s=0}^{r-1}(E_s(t)+\|\nabla^s\Theta_0\|^2_{L^2(\Omega)})\right)\left(1+E_r(t)+\|\nabla^r\Theta_0\|^2_{L^2(\Omega)}\right).
\end{aligned}
$$

Now, we calculate the material derivatives of $\left|\nabla^{r-1}\operatorname{curl} u\right|^2$. By ([15], Lemma 2.1) and (4.1), we can get

$$
\begin{aligned}
& D_t\left(\left|\nabla^{r-1}\operatorname{curl} u\right|^2\right) \\
=& D_t\left(g^{ac}g^{bd}g^{AF}\nabla_A^{r-1}(\operatorname{curl} u)_{ab}\nabla_F^{r-1}(\operatorname{curl} u)_{cd}\right) \\
=& (r+1)D_t\left(g^{ac}\right)g^{bd}g^{AF}\nabla_A^{r-1}(\operatorname{curl} u)_{ab}\nabla_F^{r-1}(\operatorname{curl} u)_{cd} \\
& + 4g^{ac}g^{bd}g^{AF}D_t\left(\nabla_A^{r-1}\nabla_a u_b\right)\nabla_F^{r-1}(\operatorname{curl} u)_{cd} \\
=& -2(r+1)g^{ae}\nabla_e u^c g^{bd}g^{AF}\nabla_A^{r-1}(\operatorname{curl} u)_{ab}\nabla_F^{r-1}(\operatorname{curl} u)_{cd} \\
& - 4g^{ac}g^{bd}g^{AF}\nabla_F^{r-1}(\operatorname{curl} u)_{cd}\nabla_{Aa}^r\nabla_b P \\
& + 4g^{ac}g^{bd}g^{AF}\nabla_F^{r-1}(\operatorname{curl} u)_{cd}\delta_{bn}\nabla_{Aa}^r\Theta_0 \\
& + 4g^{ac}g^{bd}g^{AF}\nabla_F^{r-1}(\operatorname{curl} u)_{cd}(\operatorname{curl} u)_{be}\nabla_{Aa}^r u^e \\
& + 4\operatorname{sgn}(2-r)g^{ac}g^{AF}\nabla_F^{r-1}(\operatorname{curl} u)_{cd}\sum_{s=1}^{r-2}\begin{pmatrix} r \\ s+1 \end{pmatrix}\left(\left(\nabla^{1+s}u\right)\cdot\nabla^{r-s}u^d\right)_{Aa}.
\end{aligned}
$$

The higher order term involving pressure P will vanish by symmetry. For other terms, we can apply the Hlder inequality and the Gauss formula to obtain that

$$
\begin{aligned}
\int_\Omega D_t\left(\left|\nabla^{r-1}\operatorname{curl} u\right|^2\right)d\mu_g \leqslant & C\left(K, K_1, M, L, 1/\varepsilon, \operatorname{Vol}\Omega, E_0(0), \|\Theta_0\|^2_{L^2(\Omega)}\right) \\
& \times\left(1+\sum_{s=0}^{r-1}(E_s(t)+\|\nabla^s\Theta_0\|^2_{L^2(\Omega)})\right)\left(1+E_r(t)+\|\nabla^r\Theta_0\|^2_{L^2(\Omega)}\right).
\end{aligned}
$$

Hence, we need to estimate the last term in (4.5). From ([11], (A.12)), we get

$$
D_t\left(\nabla_N p\right) = -2h_d^a N^d\nabla_a p + h_{NN}\nabla_N p + \nabla_N D_t p,
$$

and

$$
\frac{\vartheta_t}{\vartheta} = \frac{2h_d^a N^d\nabla_a p}{\nabla_N p} - h_{NN} + \frac{\nabla_N D_t p}{\nabla_N p}.
$$

Thus, the integrals can be controlled by $C(K, M, L, 1/\varepsilon)\left(E_r(t)+\|\nabla^r\Theta_0\|^2_{L^2(\Omega)}\right)$.

In general, we have

$$\begin{aligned}\frac{d}{dt}E_r(t) \leqslant & C\left(K, K_1, M, L, 1/\varepsilon, \operatorname{Vol}\Omega, E_0(0), \|\Theta_0\|^2_{L^2(\Omega)}\right)\\ & \times\left(1+\sum_{s=0}^{r-1}(E_s(t)+\|\nabla^s\Theta_0\|^2_{L^2(\Omega)})\right)\left(1+E_r(t)+\|\nabla^r\Theta_0\|^2_{L^2(\Omega)}\right),\end{aligned} \tag{4.31}$$

which implies the desired result by Gronwall's inequality and the induction argument for $r \in \{2, \cdots, n+1\}$.

5. A priori estimates for inviscid MHD-Boussinesq equations without heat diffusion

The system (1.1) can be written in the Lagrangian coordinates as

$$\begin{aligned} D_t u_a + \nabla_a P = u^c \nabla_a u_c + \beta^d \nabla_d \beta_a + \delta_{na}\Theta_0, &\quad \text{in } [0,T]\times\Omega,\\ D_t \beta_a = \beta^d \nabla_d u_a + \beta^c \nabla_a u_c, &\quad \text{in } [0,T]\times\Omega,\\ \nabla_a u^a = 0, \quad \nabla_a \beta^a = 0, &\quad \text{in } [0,T]\times\Omega,\\ \beta_a N^a = 0, \quad P = 0, &\quad \text{on } [0,T]\times\partial\Omega. \end{aligned} \tag{5.1}$$

Thus, by (5.1) and ([15], Lemma 2.1), we also get the zero order energy

$$E_0(t) = \frac{1}{2}\int_\Omega \left(|u(t,x)|^2 + |\beta(t,x)|^2\right) d\mu_g \leqslant CE_0(0) + \|\Theta_0\|^2_{L^2(\Omega)}.$$

Similarly, we can define the first order energy as

$$\begin{aligned}E_1(t) = & \int_\Omega \left(g^{bd}\gamma^{ae}\nabla_a u_b \nabla_e u_d + g^{bd}\gamma^{ae}\nabla_a\beta_b\nabla_e\beta_d\right) d\mu_g\\ & + \int_\Omega \left(|\operatorname{curl} u|^2 + |\operatorname{curl}\beta|^2\right) d\mu_g.\end{aligned}$$

Theorem 5.1. *For* $0 \leqslant t \leqslant T$ *for any smooth solution of system (5.1) satisfying*

$$|\nabla P| \leqslant M, \quad |\nabla u| \leqslant M, \quad \text{in } [0,T]\times\Omega, \tag{5.2}$$

$$|\theta| + |\nabla u| + \frac{1}{\varsigma_0} \leqslant K, \quad \text{on } [0,T]\times\partial\Omega, \tag{5.3}$$

when t ∈ [0,T], we can get

$$E_1(t) \leqslant 2e^{CMt}\left(E_1(0) + \|\nabla\Theta_0\|^2_{L^2(\Omega)}\right) + CK^2(\operatorname{Vol}\Omega)\left(e^{CMt}-1\right). \tag{5.4}$$

Define the r-th order energy for $r \geqslant 2$ as

$$\begin{aligned}E_r(t) = & \int_\Omega g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a u_b \nabla_F^{r-1}\nabla_f u_d d\mu_g + \int_\Omega \left|\nabla^{r-1}\operatorname{curl} u\right|^2 d\mu_g\\ & + \int_\Omega g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a \beta_b \nabla_F^{r-1}\nabla_f \beta_d d\mu_g + \int_\Omega \left|\nabla^{r-1}\operatorname{curl}\beta\right|^2 d\mu_g\\ & + \int_{\partial\Omega} \gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a P \nabla_F^{r-1}\nabla_f P \vartheta d\mu_\gamma.\end{aligned} \tag{5.5}$$

Theorem 5.2. *For the integer $2 \leqslant r \leqslant n+1$, there exists a constant $T > 0$ such that, for any smooth solution to system (5.1) for $0 \leqslant t \leqslant T$ satisfying*

$$\begin{aligned} |\beta| \leqslant M_1, \quad for \quad r=2, \quad & in\ [0,T]\times\Omega, \\ |\nabla P| \leqslant M, \quad |\nabla u| \leqslant M, \quad |\nabla\beta| \leqslant M, \quad |\nabla\Theta_0| \leqslant M, \quad & in\ [0,T]\times\Omega, \\ |\theta| + 1/\varsigma_0 \leqslant K, \quad & on\ [0,T]\times\partial\Omega, \\ -\nabla_N P \geqslant \varepsilon > 0, \quad & on\ [0,T]\times\partial\Omega, \\ \left|\nabla^2 P\right| + |\nabla_N D_t P| \leqslant L, \quad & on\ [0,T]\times\partial\Omega, \end{aligned} \tag{5.6}$$

when $t \in [0,T]$, we can get

$$E_r(t) \leqslant e^{C_1 t}\left(E_r(0) + \|\nabla^r\Theta_0\|^2_{L^2(\Omega)}\right) + C_2\left(e^{C_1 t} - 1\right), \tag{5.7}$$

where the constants C_1 and C_2 depend on $K, M, L, 1/\varepsilon, \mathrm{Vol}\,\Omega, E_0(0), \|\Theta_0\|^2_{L^2(\Omega)}, E_1(0), \cdots, E_{r-1}(0)$ and $\left\|\nabla^{r-1}\Theta_0\right\|^2_{L^2(\Omega)}$.

Theorems 5.1 and 5.2 can be proved similarly. For the case involving the magnetic field, one can refer to [11]. We omit the details of the proof.

6. Justification of a priori assumptions

Since some a priori assumptions are made, them will be justified in this section, at time t, we denote the following values

$$\begin{aligned} &\mathcal{H}(t) = \max\left(\|\theta(t,\cdot)\|_{L^\infty(\partial\Omega)}, 1/\varsigma_0(t)\right), \\ &\mathcal{I}(t) = \|1/\left(\nabla_N P(t,\cdot)\right)\|_{L^\infty(\partial\Omega)}, \quad \varepsilon(t) = \frac{1}{\mathcal{I}(t)}. \end{aligned} \tag{6.1}$$

Our judgment is very similar to those in [8, 11], so we only state the results and omit their proofs as follows.

Lemma 6.1. *Let $K_1 \geqslant 1/\varsigma_1(t)$, then there are continuous functions F_j, j=1,2,3,4, such that*

$$\begin{aligned} &\|\nabla u\|_{L^\infty(\Omega)} + \|\beta, \nabla\beta\|_{L^\infty(\Omega)} + \|\nabla\Theta_0\|_{L^\infty(\Omega)} \leqslant F_1\left(K_1, E_0, \|\Theta_0\|^2_{L^2(\Omega)}, \cdots, E_{n+1}, \left\|\nabla^{n+1}\Theta_0\right\|^2_{L^2(\Omega)}\right), \\ &\|\nabla P\|_{L^\infty(\Omega)} + \left\|\nabla^2 P\right\|_{L^\infty(\Omega)} \leqslant F_2\left(K_1, \mathcal{I}, E_0, \|\Theta_0\|^2_{L^2(\Omega)}, \cdots, E_{n+1}, \left\|\nabla^{n+1}\Theta_0\right\|^2_{L^2(\Omega)}, \mathrm{Vol}\,\Omega\right), \\ &\|\nabla D_t P\|_{L^\infty(\partial\Omega)} \leqslant F_4\left(K_1, \mathcal{I}, E_0, \|\Theta_0\|^2_{L^2(\Omega)}, \cdots, E_{n+1}, \left\|\nabla^{n+1}\Theta_0\right\|^2_{L^2(\Omega)}, \mathrm{Vol}\,\Omega\right), \\ &\|\theta\|_{L^\infty(\partial\Omega)} \leqslant F_3\left(K_1, \mathcal{I}, E_0, \|\Theta_0\|^2_{L^2(\Omega)}, \cdots, E_{n+1}, \left\|\nabla^{n+1}\Theta_0\right\|^2_{L^2(\Omega)} \mathrm{Vol}\,\Omega\right), \\ &\left|\frac{d}{dt}\mathcal{I}\right| \leqslant C_r\left(K_1, \mathcal{I}, E_0, \|\Theta_0\|^2_{L^2(\Omega)}, \cdots, E_{n+1}, \left\|\nabla^{n+1}\Theta_0\right\|^2_{L^2(\Omega)}, \mathrm{Vol}\,\Omega\right), \\ &\left|\frac{d}{dt}E_r\right| \leqslant C_r\left(K_1, \mathcal{I}, E_0, \|\Theta_0\|^2_{L^2(\Omega)}, \cdots, E_{n+1}, \left\|\nabla^{n+1}\Theta_0\right\|^2_{L^2(\Omega)}, \mathrm{Vol}\,\Omega\right)\sum_{s=0}^{r}(E_s + \|\nabla^s\Theta_0\|^2_{L^2(\Omega)}). \end{aligned}$$

Lemma 6.2. *There exists a continuous function* $\mathcal{K} > 0$ *depending on* K_1, $E_0(0)$, $\|\Theta_0\|^2_{L^2(\Omega)}$, $E_1(0), \cdots,$ $E_{n+1}(0)$, $\left\|\nabla^{n+1}\Theta_0\right\|^2_{L^2(\Omega)}$ *and* $\mathrm{Vol}\,\Omega$ *such that for*

$$0 \leqslant t \leqslant \mathcal{K}\left(K_1, E_0(0), \|\Theta_0\|^2_{L^2(\Omega)}, E_1(0), \cdots, E_{n+1}(0), \left\|\nabla^{n+1}\Theta_0\right\|^2_{L^2(\Omega)}, \mathrm{Vol}\,\Omega\right)$$

the following statements hold

$$E_s(t) \leqslant 2\left(E_s(0) + \|\nabla^s\Theta_0\|^2_{L^2(\Omega)}\right), \quad 0 \leqslant s \leqslant n+1, \quad \mathcal{I}(t) \leqslant 2\mathcal{I}(0).$$

Furthermore,

$$\frac{1}{2}g_{ab}(0,y)Y^aY^b \leqslant g_{ab}(t,y)Y^aY^b \leqslant 2g_{ab}(0,y)Y^aY^b,$$

and

$$\begin{aligned} |N(x(t,\bar{y})) - N(x(0,\bar{y}))| &\leqslant \frac{\varepsilon_1}{16}, \quad \bar{y} \in \partial\Omega, \\ |x(t,y) - x(t,y)| &\leqslant \frac{\varsigma_1}{16}, \quad y \in \Omega, \\ \left|\frac{\partial x(t,\bar{y})}{\partial y} - \frac{\partial(0,\bar{y})}{\partial y}\right| &\leqslant \frac{\varepsilon_1}{16}, \quad \bar{y} \in \partial\Omega. \end{aligned}$$

Lemma 6.3. *Let* $\mathcal{K}$ *be as in Lemma 6.2. There exists some* $\varepsilon_1 > 0$ *such that, if*

$$|N(x(0,y_1)) - N(x(0,y_2))| \leqslant \frac{\varepsilon_1}{2},$$

then for $t \leqslant \mathcal{K}$*, it holds*

$$|N(x(t,y_1)) - N(x(t,y_2))| \leqslant \varepsilon_1.$$

Consequently, Lemmas 6.2 and 6.3 yield immediately Theorem 1.1.

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Conflict of interest

The authors declare that there is no conflict of interest.

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