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Research article

Sequential fractional order Neutral functional Integro differential equations on time scales with Caputo fractional operator over Banach spaces

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Abstract: In this work, we scrutinize the existence and uniqueness of the solution to the Integro differential equations for the Caputo fractional derivative on the time scale. We derive the solution of the neutral fractional differential equations along the finite delay conditions. The fixed point theory is demonstrated, and the solution depends upon the fixed point theorems: Banach contraction principle, nonlinear alternative for Leray-Schauder type, and Krasnoselskii fixed point theorem.

Keywords: Neutral differential equations; fixed point; Caputo fractional derivative; time scales; Semigroup theory; delay differential equations
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1. Introduction

The derivatives of arbitrary order, where the integer-order differentiation and n-fold integration are unified and generalized, the theory of integrals is known as Fractional Calculus. Abel in 1823, described the first application of derivative of order. In 1965, the fractional derivative (FD) was introduced by Lebinitz as a generalization of the Integral order derivative. Later it was reconsidered by Euler, Abel, Riemann Liouville, Grunwald and Letnikov. In several field of research, the topic of fractional calculus plays a major role in the real world problems. Among the diverse fields of science, the fractional calculus has the great application, say in physics, thermodynamics, viscoelasticity, biology, control theory, electrochemistry [1–3] and acts a controller model for population growth, practically etc. In recent years, fractional differentiation has been drawing increasing attention in the study of social and physical behaviors where scaling power law of fractional order appears universal as an empirical description of such complex phenomena [4]. Over the past few decades, numerous analyses, real-world problems, and numerical methods were resolved by fractional derivatives and integrals [5]. Fractional operators used to illustrate better the reality of real-world phenomena with the hereditary property [6]. The determination of curve's shape such that the time of descent of non-friction point mass sliding down through the curve under the action of gravity is not independent of the starting point is dealt with by the tautochrome problem in which it is the relation of application of fractional calculus with the solution of integral equations. Fractional derivative are amazing tool for illustrating the memory and hereditary properties of diverse materials and processes. On finding the approximation of the solution of the system, numerical analysis have played a major role [7].

The Caputo fractional differential operator is introduced by the Italian mathematician Caputo in 1967 [8]. Some problems of visco-elasticity are formulated and solved by M.Caputo [9] with his own definitions of fractional differentiation. The relationship between the Caputo Fractional Derivative(CFD) with Riemann-Liouville(RL) fractional derivative, Atangana-Baleanu(AB) fractional derivative has been very strong which describes the generalized Mittag-Leffler(ML) functions among their kernels, using certain mathematical model to obtain the results in betterment [10].

Neutral differential equations occur when $\max\{n_1, n_2, ..., n_k\} = n$. The past and present values of the function is dependent by the neutral differential equations, which is similar to retarded differential equations, but the neutral differential equations also depends on derivatives with delays. Neutral type differential equations [11–13] acts as a model for elastic network arise in high speed computers. That is, for the use of interconnection of switching circuit. Neutral differential equation occur in various branches of applied mathematics, as a result, seeking major heed in recent decades. The development of neutral differential system have been done by many researchers, mentioning the diverse fixed point approaches, mild solutions, and nonlocal conditions [14, 15]. The delay differential equations has the major application in the behaviour of real populations. The systems with impulses are utilized for studying the dynamics of processes subject to abrupt changes at discrete moments [16, 17].

A set with no empty closed subset of $\mathscr{R}$ is called Time scale. The time scale has been introduced to federate and enhance the theory of differential equation, and many other defined difference systems. The differential equations on time scales for the existence and uniqueness of IVP has been stated by Hilger with some applications. The union of disjoint closed real intervals on time scales acts as an excellent framework for the study of population. In the last few years, differential equations in time scale is majorly developed (see for example [18–32]).

In [33], by applying fixed point theorems the authors discussed the existence, uniqueness and stability for the non linear fractional differential equations with non linear integral boundary condition on time scales. In our work, we discuss the existence and uniqueness solution to the neutral functional sequential integro differential equations with Caputo fractional derivative on time scale's $\mathbb{T}$ Cauchy problem,

$$\begin{aligned} {}^{c}\Delta^{\rho}\Big[{}^{c}\Delta^{\varrho}p(\nu) - \Phi(\nu, p_\nu, \int_0^t k_1(t, s, p_s)ds)\Big] &= \psi(\nu, p_\nu, \int_0^t k_2(t, s, p_s)ds), \\ \nu \in \mathscr{J} &:= [0, T]_{\mathbb{T}} = [0, T] \cap \mathbb{T}, \\ p(\nu) = \varsigma(\nu), \nu \in [-\epsilon, 0]_{\mathbb{T}} &= [-\epsilon, 0] \cap \mathbb{T}, \\ {}^{c}\Delta^{\varrho}p(0) &= \phi \in \mathscr{R}. \end{aligned} \tag{1.1}$$

Here,${}^{c}\Delta^{\rho}, {}^{c}\Delta^{\varrho}$ are CFD. The given functions are $0 < \rho, \varrho < 1$, $\phi : \mathscr{J} \times C([-\epsilon, 0]_{\mathbb{T}}, \mathscr{R}) \to \mathscr{R}$ and

$\varsigma \in C([-\epsilon, 0]_{\mathbb{T}}, \mathscr{R})$. The function p in $[-\epsilon, T]_{\mathbb{T}}$ and for $\nu \in \mathscr{J}$ and $\epsilon > 0$, the element of $C_\epsilon := C([-\epsilon, 0]_{\mathbb{T}}, \mathscr{R})$ and we denote by p_ν as,

$$p(\eta) = p(\varrho + \eta), \quad \eta \in [-\epsilon, 0]_{\mathbb{T}}.$$

2. Preliminaries

Definition 2.1. [8] The CFD of order β is defined by, for $\beta > 0, t > 0, \beta, x, t \in \mathscr{R}$. The fractional operator is,

$$D_*^\beta u(t) = \begin{cases} \frac{1}{\Gamma(n-\beta)} \int_x^t \frac{u^{(n)}\tau}{(t-\tau)^{\beta+1-m}}, & m-1 < \beta < m \in N, \\ \frac{d^m}{dt^m} u(t), & \beta = m \in N. \end{cases} \tag{2.1}$$

Definition 2.2. The mapping Σ: $\mathbb{T} \to \mathbb{T}$, defined as $\Sigma(t) = inf\{s \in \mathbb{T} : s > t\}$ having inf $\phi = sup\mathbb{T}$. The forward jump operator is defined as a Time scale with an arbitrary non empty closed subset of $\mathscr{R}$ and is denoted by $\mathbb{T}$. (i.e., Σ(N) = N if $\mathbb{T}$ has a maximum N).

The mapping Ω: $\mathbb{T} \to \mathbb{T}$, defined by $\Omega(t) = sup\{s \in \mathbb{T} : s < t\}$ with sup $\phi = inf\mathbb{T}$. (i.e., Ω(N) = N if $\mathbb{T}$ has a maximum N) is called the backward jump operator.

Here, the symbol ϕ denotes the empty set.

Remark 2.3. In definition 2.2, if $\mathbb{T}$ contains $\bar{n}$ a maximum, then inf $\phi = sup\mathbb{T}$ (i.e., $\Sigma(\bar{n}) = \bar{n}$) and if $\mathbb{T}$ contains $\underline{n}$ a minimum, then sup $\phi = inf\mathbb{T}$ (i.e $\Omega(\underline{n}) = n$), ϕ is denoted as the empty set.

Definition 2.4. [30] A continuous function $u : \mathbb{T} \to \mathscr{R}$ at right dense points is called rd-continuous and in left-dense points left sided limit exists.

Definition 2.5. [33] A function U: $[x, y] \to \mathscr{R}$ is known as Δ anti-derivative of function u: $[x, y] \to \mathscr{R}$, where U is continuous on $[x, y]$, Δ differentiable on (x, y), and $U^\Delta(t) = u(t)\ \forall\ t \in (x, y)$, where $[x, y]$ is the closed bounded interval in $\mathbb{T}$.

The Δ-integral of u from x to y is,

$$\int_x^y u(t)\Delta t := U(y) - U(x). \tag{2.2}$$

Definition 2.6. [34] The fractional integral of order β of g is,

$${}_x^{\mathbb{T}}I_t^\beta h(t) := \int_x^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s)\Delta s, \tag{2.3}$$

where Γ is the Gamma function, $\mathbb{T}$ be a time scale with interval $[x, y]$ of $\mathbb{T}$, and g be an integrable function on $[x, y]$.

Definition 2.7. [35] Suppose that $\mathbb{T}$ is a time scale. The CFD of order β of g is defined by

$${}_x^c\Delta_t^\beta h(t) := \int_x^t \frac{(t-s)^{m-\beta-1}}{\Gamma(m-\beta)} g^{\Delta^m}(s)\Delta s, \tag{2.4}$$

where m=[β]+1 and [β] denotes the integer part of β.

Theorem 2.8. [36] Let β, β' and g be an integrable function on $[x, y]$, then,

$${}^{\mathbb{T}}I_x^\beta I^{\beta'} g(t) = {}^{\mathbb{T}}I^{\beta+\beta'} g(t).$$

Lemma 2.9. (Nonlinear alternative for Leray-Schauder type) [37–39]. Consider U a mapping from $\bar{F}$ to C, where C is a closed, convex subset of E, which is a Banach space and F, an open subset of C with 0∈F. Then either of the conditions hold:

(i) $\bar{F}$ has a fixed point,

(ii) $f = \Lambda F(f), f \in \partial F$ and $\Lambda \in (0, 1)$.

Lemma 2.10. (Krasnoselskii fixed point theorem) [37]. Let W be closed, bounded, convex and non-empty subset of a Banach space S. Let M, N be the operators such that (a) $Mu + Nv \in W$ whenever $u, v \in W$; (b) M is continuous and compact; (c) N is a contraction mapping. Then there exists $z \in W$ such that $z = Mz + Nz$.
Lemma 2.11. [20] Let $\mathbb{T}$ be a time scale and g be an increasing continuous function on the interval $[x, y]$with the time-scale . Let G be the extension of g to the real interval $[x, y]$, then

$$G(s) = \begin{cases} g(s) & if \quad s \in \mathbb{T}, \\ g(\tau) & if \quad s \in (t, \Sigma(\tau)) \notin \mathbb{T}, \end{cases}$$

then,

$$\int_x^y g(\tau)\Delta\tau \leq \int_x^y G(\tau)d\tau.$$

3. Main sequels

We need the following assumptions

(A1) There exists $\Lambda > 0$ such that, $|\psi(\nu, z) - \psi(\nu, \bar{z})| \leq \Lambda\|z - \bar{z}\|_c$ for $\nu \in \mathscr{J}$ and every $z, \bar{z} \in C_\epsilon$.

(A2) There exists a non '- ve'constant κ such that, $|\phi(\nu, z) - \phi(\nu, \bar{z})| \leq \kappa\|z - \bar{z}\|_c$ for $\nu \in \mathscr{J}$ and every $z, \bar{z} \in C_\epsilon$.

(A3) The function ψ, ϕ are continuous.

(A4) There exists a continuous non-decreasing function $\psi : [0, \infty]_{\mathbb{T}} \to (0, \infty)$ and a function $\Upsilon \in C(\mathscr{J}, \mathscr{J}^+)$ such that, $|\psi(\nu, z)| \leq \Upsilon(\nu)\psi(\|z\|_c)$ for each $(\nu, z) \in \mathscr{J} \times C_\epsilon$.

(A5) There exists a constant $\mathbf{L} > 0$ such that,

$$\frac{(\Gamma(\varrho + 1) - \kappa T^\varrho)\mathbf{L}}{\mathbf{H} + \psi(\mathbf{L})T^{\rho+\varrho}\|\Upsilon\|_\infty \frac{\Gamma(\varrho+1)}{\Gamma(\rho+\varrho+1)}} > 1$$

where, $\mathbf{H} = \Gamma(\varrho + 1)\|\varsigma\|_c + [|\phi| + \kappa\|\varsigma\|_c + 2\Phi_o]T^\varrho$

(A6) There exists a continuous function $\psi, \phi : \mathscr{J} \times C_\epsilon \to \mathscr{R}$, For each

$$p_s, q_s \in \mathscr{J}, |\phi(s, p_s, k_1 p_s(\nu)) - \phi(s, p_s, k_1 q_s(\nu))| = k_1\|p_s - q_s\|.$$

(A7) For each $p_s, q_s \in \mathscr{J}$ such that

$$p \text{ and } q \in C([0, T]_{\mathbb{T},\mathscr{R}}), |\psi(s, p_s, k_2 p_s(\nu) - \psi(s, p_s, k_2 q_s(\nu))| = k_2\|p_s - q_s\|.$$

Remark 3.1. By (A2) for each $(\nu, z) \in \mathscr{J}$,

$$\begin{aligned}|\Phi(\nu,z)| &= |\Phi(\nu,z) - \Phi(\nu,0) + \Phi(\nu,0)| \\ &\leq |\Phi(\nu,z) - \Phi(\nu,0)| + |\Phi(\nu,0)| \\ &\leq \kappa|z| + \Phi_o,\end{aligned}$$

where $\Phi_o = sup_{s\in[0,T]_{\mathbb{T}}}|\Phi(s,0)|$.

Remark 3.2. A function $p \in \tilde{\mathrm{E}}$ is said to be a solution of the problem

$$\begin{aligned}{}^{c}\Delta^{\rho}\Big[{}^{c}\Delta^{\varrho}p(\nu) - \Phi\Big(\nu, p_\nu, \int_0^t k_1(t,s,p_s)ds\Big)\Big] &= \psi\Big(\nu, p_\nu, \int_0^t k_2(t,s,p_s)ds\Big) \\ \nu \in \mathscr{J} &:= [0,T]_{\mathbb{T}} = [0,T]\cap\mathbb{T}, \\ p(\nu) = \varsigma(\nu), \nu \in [-\epsilon,0]_{\mathbb{T}} &= [-\epsilon,0]\cap\mathbb{T}, \\ {}^{c}\Delta^{\varrho}p(0) &= \phi \in \mathscr{R}.\end{aligned}$$

If p satisfies the equation ${}^{c}\Delta^{\rho}\Big[{}^{c}\Delta^{\varrho}p(\nu) - \Phi\Big(\nu, p_\nu, \int_0^t k_1(t,s,p_s)ds\Big)\Big] = \psi\Big(\nu, p_\nu, \int_0^t k_2(t,s,p_s)ds\Big)$ on $\mathscr{J}$, the condition $p(\nu) = \varsigma(\nu)$ on $[-\epsilon,0]_{\mathbb{T}}$ and ${}^{c}\Delta^{\varrho}p(0) = \phi$.

Theorem 3.3. The function $p \in \tilde{\mathrm{E}}$ is the solution of the problem.

$$\begin{aligned}{}^{c}\Delta^{\rho}\Big[{}^{c}\Delta^{\varrho}p(\nu) - \Phi\Big(\nu, p_\nu, \int_0^t k_1(t,s,p_s)ds\Big)\Big] &= \psi\Big(\nu, p_\nu, \int_0^t k_2(t,s,p_s)ds\Big) \\ \nu \in \mathscr{J} &:= [0,T]_{\mathbb{T}} = [0,T]\cap\mathbb{T}, \\ p(\nu) &= \varsigma(\nu), \nu \in [-\epsilon,0]_{\mathbb{T}} = [-\epsilon,0]\cap\mathbb{T}, \\ {}^{c}\Delta^{\varrho}p(0) &= \phi \in \mathscr{R}.\end{aligned}$$

If

$$p(\nu) = \begin{cases} \varsigma(\nu),\ if \nu \in [-\epsilon,0]_{\mathbb{T}} \\ \varsigma(0) + \frac{\phi - \Phi(0,\varsigma(0))}{\Gamma(\varrho)}\int_0^\nu (\nu - s)^{\varrho-1}\Delta s + \frac{1}{\Gamma(\varrho)}\int_0^\nu (\nu - s)^{\varrho-1}\Phi(\nu, p_\nu, k_1 p(\nu))\Delta s \\ + \frac{1}{\Gamma(\rho+\varrho)}\int_0^\nu (\nu - s)^{\rho+\varrho-1}\psi(\nu, p_\nu, k_2 p(\nu))\Delta s. \end{cases}$$

Proof. Using,

$${}^{c}\Delta^{\rho}\Big[{}^{c}\Delta^{\varrho}p(\nu) - \Phi\Big(\nu, p_\nu, \int_0^t k_1(t,s,p_s)ds\Big)\Big] = \psi\Big(\nu, p_\nu, \int_0^t k_2(t,s,p_s)ds\Big), \quad \nu \in \mathscr{J},$$

we get,

$${}^{c}\Delta^{\varrho}p(\nu) - \Phi[\nu, p_\nu, k_1 p(\nu)] = \beta + \frac{1}{\Gamma(\varrho)}\int_0^\nu (\nu - s)^{\rho-1}\psi(s, p_s, k_2 p(s))ds,$$

where $\beta \in \mathscr{R}$.

$D^{\varrho}p(0) = \phi, \beta = \phi - \Phi(0,\varsigma(0))$, is given as,

$${}^{c}\Delta^{\varrho} = \phi - \Phi(0,\varsigma(0)) + \Phi[\nu, p_\nu, k_1 p(\nu)] + \frac{1}{\Gamma(\varrho)}\int_0^\nu (\nu - s)^{\rho-1}\psi(s, p_s, k_2 p(s))ds.$$

Thus,

$$p(\nu) = \beta + \frac{\phi - \Phi(0, \varsigma(0))}{\Gamma(\varrho)} \int_0^\nu (\nu - s)^{\varrho - 1} \Delta s + \frac{1}{\Gamma(\varrho)} \int_0^\nu (\nu - s)^{\varrho - 1} \Phi(\nu, p_\nu, k_1 p(\nu)) \Delta s$$
$$+ \frac{1}{\Gamma(\rho + \varrho)} \int_0^\nu (\nu - s)^{\rho + \varrho - 1} \psi(\nu, p_\nu, k_2 p(\nu)) \Delta s.$$

we find $\beta = 0$ and hence the proof.

Theorem 3.4. Assume that (A1) and (A2) holds if,

$$T^\rho \Big(\frac{\kappa k_1}{\Gamma(\varrho + 1)} + \frac{\Lambda T^\varrho k_2}{\Gamma(\rho + \varrho + 1)} \Big) < 1,$$

then there exists a unique element for Initial value problem (1.1) in Ẽ. *Proof:*. Choose $E : C([-\epsilon, T]_{\mathbb{T}}, \mathscr{R}) \to C([-\epsilon, T]_{\mathbb{T}}, \mathscr{R})$ by,

$$E(p)(\nu) = \begin{cases} \varsigma(\nu), \\ \varsigma(0) + \frac{\phi - \Phi(0, \varsigma(0))}{\Gamma(\varrho)} \int_0^\nu (\nu - s)^{\varrho - 1} \Delta s + \frac{1}{\Gamma(\varrho)} \int_0^\nu (\nu - s)^{\varrho - 1} \Phi(s, p_s, k_1 p(\nu)) \Delta s \\ + \frac{1}{\Gamma(\rho + \varrho)} \int_0^\nu (\nu - s)^{\rho + \varrho - 1} \psi(s, p_s, k_2 p(\nu)) \Delta s. \end{cases}$$

Let $p, q \in C([0, T]_{\mathbb{T}}, \mathscr{R})$. Then by (A1) and (A2) we get,

$$|E(p)(\nu) - E(q)(\nu)| \leq \frac{1}{\Gamma(\varrho)} \int_0^\nu (\nu - s)^{\varrho - 1} |\Phi(s, p_s, k_1 p_s(\nu)) - \Phi(s, q_s, k_1 p_s(\nu))| \Delta s$$
$$+ \frac{1}{\Gamma(\rho + \varrho)} \int_0^\nu (\nu - s)^{\rho + \varrho - 1} |\psi(s, p_s, k_2 p_s(\nu)) - \Phi(s, q_s, k_2 p_s(\nu))| \Delta s$$
$$\leq \frac{\kappa}{\Gamma(\varrho)} \int_0^\nu (\nu - s)^{\varrho - 1} k_1 \|p_s - q_s\|_c \Delta s$$
$$+ \frac{\Lambda}{\Gamma(\rho + \varrho)} \int_0^\nu (\nu - s)^{\rho + \varrho - 1} k_2 \|p_s - q_s\|_c \Delta s.$$

By Lemma 2.11 we get,

$$|E(p)(\nu) - E(q)(\nu)| \leq \frac{1}{\Gamma(\varrho)} \int_0^\nu (\nu - s)^{\varrho - 1} k_1 \|p_s - q_s\|_c ds$$
$$+ \frac{\Lambda}{\Gamma(\rho + \varrho - 1)} \int_0^\nu (\nu - s)^{(\rho + \varrho - 1)} k_2 \|p_s - q_s\|_c ds$$
$$\leq \frac{\kappa \nu^\varrho}{\Gamma(\varrho + 1)} k_1 \|p - q\|_{[-\epsilon, T]_{\mathbb{T}}} + \frac{\Lambda \nu^{\rho + \varrho}}{\Gamma(\rho + \varrho + 1)} k_2 \|p - q\|_{[0, T]_{\mathbb{T}}}$$
$$\leq \frac{\kappa T^\varrho}{\Gamma(\varrho + 1)} k_1 \|p - q\|_{[-\epsilon, T]_{\mathbb{T}}} + \frac{\Lambda T^{\rho + \varrho}}{\Gamma(\rho + \varrho + 1)} k_2 \|p - q\|_{[0, T]_{\mathbb{T}}}.$$

Thus,

$$|E(p)(\nu) - E(q)(\nu)| \leq T^\rho \Big(\frac{\kappa k_1}{\Gamma(\varrho + 1)} + \frac{\Lambda T^\varrho k_2}{\Gamma(\rho + \varrho + 1)} \Big) \|p - q\|_{[-\epsilon, T]_{\mathbb{T}}}.$$

The operator is contraction E . Hence, in problem (1.1) By using Banach contraction principle E is a unique solution on $[-\epsilon, T]_{\mathbb{T}}$ and it has a unique fixed point.

Theorem 3.5. Assume the hypothesis (A2)–(A5) hold. If $\dfrac{\kappa T^{\varrho}}{\Gamma(\varrho+1)} < 1$, then the IVP (1.1) must contain atleast one solution on $\tilde{\mathrm{E}}$.

Proof. Let us prove the operator $E : C(-\epsilon, T)_{\mathbb{T},\mathscr{R}} \to C(-\epsilon, T)_{\mathbb{T},\mathscr{R}}$ is continuous and completely continuous.

Claim 1: Let $U_n \to y$ in $C(-\epsilon, T)_{\mathbb{T},\mathscr{R}}$ then E is continuous, where $\{U_n\}$ is the sequence. Then,

$$\begin{aligned}
|E(p)(\nu) - E(q)(\nu)| &\leq \frac{1}{\Gamma(\varrho)} \int_0^{\nu} (\nu - s)^{\varrho-1} |\Phi(s, p_{ns}, k_1 p_{ns}(\nu)) - \Phi(s, p_s, k_1 p_s(\nu))| \Delta s \\
&\quad + \frac{1}{\Gamma(\rho+\varrho)} \int_0^{\nu} (\nu - s)^{\rho+\varrho-1} |\psi(s, p_{ns}, k_2 p_{ns}(\nu)) - \psi(s, p_s, k_2 p_s(\nu))| \Delta s \\
&\leq \frac{1}{\Gamma(\varrho)} \int_0^{\nu} (\nu - s)^{\varrho-1} \sup_{s\in[0,T]} |\Phi(s, p_{ns}, k_1 p_{ns}(\nu))| - |\Phi(s, p_s, k_1 p_s(\nu)| \Delta s \\
&\quad + \frac{1}{\Gamma(\rho+\varrho)} \int_0^{\nu} (\nu - s)^{\rho+\varrho-1} \sup_{s\in[0,T]_{\mathbb{T}}} |\psi(s, p_{ns}, k_2 p_{ns}(\nu))| \\
&\quad - |\psi(s, p_s, k_2 p_s(\nu)| \Delta s \\
&\leq \frac{\|\Phi(\cdot, p_n, k_1 p_{n\cdot}) - \Phi(\cdot, p_\cdot, k_1 p_\cdot)\|_\infty}{\Gamma(\varrho)} \int_0^{T} (\nu - s)^{\rho-1} \Delta s \\
&\quad + \frac{\|\psi(\cdot, p_n, k_2 p_{n\cdot}) - \psi(\cdot, p_\cdot, k_2 p_\cdot)\|_\infty}{\Gamma(\rho+\varrho)} \int_0^{T} (\nu - s)^{\rho+\varrho-1} \Delta s.
\end{aligned}$$

By Lemma 2.11 we get,

$$\begin{aligned}
|E(p)(\nu) - E(q)(\nu)| &\leq \frac{\|\Phi(\cdot, p_n, k_1 p_{n\cdot}) - \Phi(\cdot, p_\cdot, k_1 p_\cdot)\|_\infty}{\Gamma(\varrho)} \int_0^{T} (\nu - s)^{\rho-1} ds \\
&\quad + \frac{\|\psi(\cdot, p_n, k_2 p_{n\cdot}) - \psi(\cdot, p_\cdot, k_2 p_\cdot)\|_\infty}{\Gamma(\rho+\varrho)} \int_0^{T} (\nu - s)^{\rho+\varrho-1} ds, \\
&\leq \frac{T^{\varrho}\|\Phi(\cdot, p_n, k_1 p_{n\cdot}) - \Phi(\cdot, p_\cdot, k_1 p_\cdot)\|_\infty}{\Gamma(\varrho+1)} \\
&\quad + \frac{T^{\rho+\varrho}\|\psi(\cdot, p_n, k_2 p_{n\cdot}) - \psi(\cdot, p_\cdot, k_2 p_\cdot)\|_\infty}{\Gamma(\rho+\varrho+1)}.
\end{aligned}$$

Since ψ and Φ are continuous functions,

$$\begin{aligned}
|E(p)(\nu) - E(q)(\nu)| &\leq \frac{T^{\varrho}\|\Phi(\cdot, p_n, k_1 p_{n\cdot}) - \Phi(\cdot, p_\cdot, k_1 p_\cdot)\|_\infty}{\Gamma(\varrho+1)} \\
&\quad + \frac{T^{\rho+\varrho}\|\psi(\cdot, p_n, k_2 p_{n\cdot}) - \psi(\cdot, p_\cdot, k_2 p_\cdot)\|_\infty}{\Gamma(\rho+\varrho+1)},
\end{aligned}$$

as $n \to \infty$.

Claim 2: E maps bounded sets into bounded sets in $C([-\epsilon, T]_{\mathbb{T}}, \mathscr{R})$. It is necessary to prove for that $k > 0$, and '+ ve'constant $\tilde{\Lambda}$ such that, for every $p \in B_k = p \in C([\epsilon, T]_{\mathbb{T}}, \mathscr{R}) : \|p\|_\infty \leq k$. we have, $\|E(p)\|_\infty \leq \tilde{\Lambda}$.

Using (A4) and (A5), for every $\nu \in \mathscr{J}$,we have,

$$\begin{aligned}|E(p)(\nu)| &\le \|\varsigma\|_c + \frac{|\phi + \kappa\|\varsigma\|_c + \Phi_o}{\Gamma(\varrho)}\int_s^{\nu}(\nu - s)^{\varrho-1}\Delta s + \frac{1}{\Gamma(\varrho)}\int_0^{\nu}(\nu - s)^{\varrho-1}|\Phi(s, p_s, k_1 p_s(\nu))|\Delta s\\ &+ \frac{1}{\Gamma(\rho+\varrho)}\int_0^{\nu}(\nu - s)^{\rho+\varrho-1}|\psi(s, p_s, k_2 p_s(\nu))|\Delta s\\ &\le \|\varsigma\|_c + \frac{|\phi + \kappa\|\varsigma\|_c + \Phi_o}{\Gamma(\varrho)}\int_s^{\nu}(\nu - s)^{\varrho-1}\Delta s + \frac{\kappa\|p\|_{[-\epsilon,T]_{\mathbb{T}}} + k_1\Phi_o}{\Gamma(\varrho)}\int_s^{\nu}(\nu - s)^{\varrho-1}\Delta s\\ &+ \frac{\psi(\|p\|_{[-\epsilon,T]_{\mathbb{T}}})\|\Upsilon\|_{\infty}k_2}{\Gamma(\rho+\varrho)}\int_0^{\nu}(\nu - s)^{\rho+\varrho-1}\Delta s.\end{aligned}$$

By Lemma 2.11 we get,

$$\begin{aligned}|E(p)(\nu)| &\le \|\varsigma\|_c + \frac{|\phi| + \kappa\|\varsigma\|_c + \Phi_o}{\Gamma(\varrho)}\int_s^{\nu}(\nu - s)^{\varrho-1}ds + \frac{\kappa\|p\|_{[-\epsilon,T]_{\mathbb{T}}} + k_1\Phi_o}{\Gamma(\varrho)}\int_s^{\nu}(\nu - s)^{\varrho-1}ds\\ &+ \frac{\psi(\|p\|_{[-\epsilon,T]_{\mathbb{T}}})\|\Upsilon\|_{\infty}k_2}{\Gamma(\rho+\varrho)}\int_0^{\nu}(\nu - s)^{\rho+\varrho-1}ds,\\ &\le \|\varsigma\|_c + \frac{[|\phi| + \kappa\|\varsigma\|_c + \Phi_o]T^{\varrho}}{\Gamma(\varrho+1)} + \frac{[\kappa\|p\|_{[-\epsilon,T]_{\mathbb{T}}} + k_1\Phi_o]T^{\varrho}}{\Gamma(\varrho+1)}\\ &+ \frac{[\psi(\|p\|_{[-\epsilon,T]_{\mathbb{T}}})\|\Upsilon\|_{\infty}k_2]T^{\rho+\varrho}}{\Gamma(\rho+\varrho+1)}.\end{aligned}$$

Thus,

$$\begin{aligned}\|E(p)\|_{\infty} &\le \|\varsigma\|_c + \frac{[|\phi| + \kappa\|\varsigma\|_c + \Phi_o]T^{\varrho}}{\Gamma(\varrho+1)} + \frac{[\kappa k + k_1\Phi_o]T^{\varrho}}{\Gamma(\varrho+1)} + \frac{\psi(k)\|\Upsilon\|_{\infty}k_2 T^{\rho+\varrho}}{\Gamma(\rho+\varrho+1)},\\ &= \tilde{\Lambda}.\end{aligned}$$

Claim 3: E is mapped from bounded to equicontinuous sets of $C([\epsilon, T]_{\mathbb{T}}, \mathscr{R})$. $\nu_1, \nu_2 \in \mathscr{J}, \nu_1 < \nu_2, B_k$ is the bounded set of $C([\epsilon, T]_{\mathbb{T}}, \mathscr{R})$.Let $p \in B_k$.

$$\begin{aligned}|E(p)(\nu_2) - E(p)(\nu_1)| \le{}& \frac{|(\phi - \Phi(0, \varsigma(0)))|}{\Gamma(\varrho)}\int_0^{\nu_1}((\nu_2 - s)^{\varrho-1} - (\nu_1 - s)^{\varrho-1})\Delta s\\ &+ \frac{|(\phi - \Phi(0, \varsigma(0)))|}{\Gamma(\varrho)}\int_{\nu_1}^{\nu_2}((\nu_2 - s)^{\varrho-1}\Delta s\\ &+ \frac{1}{\Gamma(\varrho)}\int_0^{\nu_1}[(\nu_2 - s)^{\varrho-1} - (\nu_1 - s)^{\varrho-1}]|\Phi(s, p_s, k_1 p_s(\nu))|\Delta s\\ &+ \frac{1}{\Gamma(\varrho)}\int_{\nu_1}^{\nu_2}(\nu_2 - s)^{\rho-1}|\Phi(s, p_s, k_1 p_s(\nu))|\Delta s\\ &+ \frac{1}{\Gamma(\rho+\varrho)}\int_0^{\nu_1}[(\nu_2 - s)^{\rho+\varrho-1} - (\nu_1 - s)^{\rho+\varrho-1}]|\psi(s, p_s, k_2 p_s(\nu))|\Delta s\\ &+ \frac{1}{\Gamma(\rho+\varrho)}\int_{\nu_1}^{\nu_2}(\nu_2 - s)^{\rho+\varrho-1}|\psi(s, p_s, k_2 p_s(\nu))|\Delta s.\end{aligned}$$

By Lemma 2.11 we get,

$$\begin{aligned}|E(p)(\nu_2)-E(p)(\nu_1)| \leq& \frac{|(\phi-\Phi(0,\varsigma(0)))|}{\Gamma(\varrho)}\int_0^{\nu_1}((\nu_2-s)^{\varrho-1}-(\nu_1-s)^{\varrho-1})ds\\ &+\frac{|(\phi-\Phi(0,\varsigma(0)))|}{\Gamma(\varrho)}\int_{\nu_1}^{\nu_2}((\nu_2-s)^{\varrho-1}ds\\ &+\frac{1}{\Gamma(\varrho)}\int_0^{\nu_1}[(\nu_2-s)^{\varrho-1}-(\nu_1-s)^{\varrho-1}]|\Phi(s,p_s,k_1p_s(\nu))|ds\\ &+\frac{1}{\Gamma(\varrho)}\int_{\nu_1}^{\nu_2}(\nu_2-s)^{\varrho-1}|\Phi(s,p_s,k_1p_s(\nu))|ds\\ &+\frac{1}{\Gamma(\rho+\varrho)}\int_0^{\nu_1}[(\nu_2-s)^{\rho+\varrho-1}-(\nu_1-s)^{\rho+\varrho-1}]|\psi(s,p_s,k_2p_s(\nu))|ds\\ &+\frac{1}{\Gamma(\rho+\varrho)}\int_{\nu_1}^{\nu_2}(\nu_2-s)^{\rho+\varrho-1}|\psi(s,p_s,k_2p_s(\nu))|ds.\end{aligned}$$

Thus,

$$\begin{aligned}|E(p)(\nu_2)-E(p)(\nu_1)| \leq& \frac{|\phi|+\kappa\|\varsigma\|_c+\Phi_o}{\Gamma(\varrho+1)}[\nu_2^{\varrho}-\nu_1^{\varrho}]+\frac{\kappa_k+\Phi_ok_1}{\Gamma(\varrho+1)}[|\nu_2^{\varrho}-\nu_1^{\varrho}|+|\nu_2-\nu_1|^{\varrho}]\\ &+\frac{\psi(k)\|\Upsilon\|_\infty k_2}{\Gamma(\rho+\varrho+1)}[|\nu_2^{\rho+\varrho}-\nu_1^{\rho+\varrho}+|\nu_2-\nu_1|^{\rho+\varrho}].\end{aligned}$$

As $\nu_1 \to \nu_2$, the R.H.S of the above inequality $\to 0$. The equicontinuity for the case $\nu_1 \leq \nu_2 \leq 0$ & $\nu_1 \leq 0 \leq \nu_2$ is obvious. By using Arzela Ascoli theorem, $C([-\epsilon,T]_{\mathbb{T}},\mathscr{R}) \to C([-\epsilon,T]_{\mathbb{T}},\mathscr{R})$ is continuous and completely continuous.

Claim 4: A set $U \subseteq C([-\epsilon,T]_{\mathbb{T}},\mathscr{R})$ with $p \neq m\,E(p)$ and for $m \in (0,1)$and $p \in \partial p$.

Let $U \in C([-\epsilon,T]_{\mathbb{T}},\mathscr{R})$ & $p = mE(p)$ for some $0 < m < 1$. Then, for each $\nu \in \mathscr{J}$, we have,

$$\begin{aligned}p(\nu) =& m(\varsigma(0))+(\phi-\Phi(0,\varsigma(0)))\int_0^{\nu}\frac{(\nu-s)^{\varrho-1}}{\Gamma(\varrho)}\Delta s+\frac{1}{\Gamma(\varrho)}\int_0^{\nu}(\nu-s)^{\varrho-1}\Phi(s,p_s,k_1p(s))\Delta s\\ &+\frac{1}{\Gamma(\rho+\varrho)}\int_0^{\nu}(\nu-s)^{\rho+\varrho-1}\psi(s,p_s,k_2p(s))\Delta s.\end{aligned}$$

Considering, for every $\nu \in \mathscr{J}$, we obtain,

$$\begin{aligned}p(\nu) \leq& \|\varsigma\|_c+\|\phi\|+\kappa\|\varsigma\|_c+\Phi_o\int_0^{\nu}\frac{(\nu-s)^{\varrho-1}}{\Gamma(\varrho)}\Delta s+\frac{\kappa\|p\|_{[-\epsilon,\mathbb{T}]_{\mathbb{T}}}+\Phi_ok_1}{\Gamma(\varrho)}\int_0^{\nu}(\nu-s)^{\varrho-1}\Delta s\\ &+\frac{1}{\Gamma(\rho+\varrho)}\int_0^{\nu}(\nu-s)^{\rho+\varrho-1}\Upsilon(s)\psi(\|p_s\|_c)\Delta s\\ \leq& \|\varsigma\|_c+\|\phi\|+\kappa\|\varsigma\|_c+\Phi_o\int_0^{\nu}\frac{(\nu-s)^{\varrho-1}}{\Gamma(\varrho)}ds+\frac{\kappa\|p\|_{[-\epsilon,\mathbb{T}]_{\mathbb{T}}}+\Phi_ok_1}{\Gamma(\varrho)}\int_0^{\nu}(\nu-s)^{\varrho-1}ds\\ &+\frac{1}{\Gamma(\rho+\varrho)}\int_0^{\nu}(\nu-s)^{\rho+\varrho-1}\Upsilon(s)\psi(\|p\|_s\|_c)ds\\ \leq& \|\varsigma\|_c+\frac{\|\phi\|+\kappa\|\varsigma\|_c+\Phi_oT^{\varrho}}{T(\varrho+1)}+\frac{\kappa\|p\|_{[-\epsilon,\mathbb{T}]_{\mathbb{T}}}+\Phi_ok_1T^{\varrho}}{\Gamma(\varrho+1)}+\frac{\|\Upsilon\|_\infty k_2\psi(\|p\|_{[-\epsilon,T]_{\mathbb{T}}})}{\Gamma(\rho+\varrho+1)}T^{\rho+\varrho}.\end{aligned}$$

Thus,

$$(\Gamma(\varrho+1)-\kappa T^{\varrho})\|p\|_{[-\epsilon,T]_{\mathbb{T}}} \leq \Gamma(\varrho+1)\|\varsigma\|_c + [|\phi+\kappa\|\varsigma\|_c + 2\Phi_o k_1]T^{\varrho} + T^{\rho+\varrho}\|\Upsilon\|_\infty k_2\psi(\|p\|_{[-\epsilon,T]_{\mathbb{T}}})\frac{\Gamma(\varrho+1)}{\rho+\varrho+1},$$

which can be expressed as

$$\frac{(\Gamma(\varrho+1)-\kappa T^{\varrho})\|p\|_{[-\epsilon,T]_{\mathbb{T}}}}{\mathbf{H}+T^{\rho+\varrho}\|\Upsilon\|_\infty k_2\psi(\|p\|_{[-\epsilon,T]_{\mathbb{T}}})\frac{\Gamma(\varrho+1)}{\rho+\varrho+1}} \leq 1.$$

There exists $\mathbf{L}$ and such that $\|p\|_{-\epsilon,T_{\mathbb{T}}} \neq L$, Setting for $p \in \{p \in C([-\epsilon,T]_{\mathbb{T}},\mathscr{R}) : \|p\|_{[-\epsilon,T]_{\mathbb{T}}} \leq \mathbf{L}\}$. From p, there is no $p \in \partial p$ such that $p = mEp$ there exists $m \in (0,1)$. E has a fixed point $u \in \bar{U}$ by using the nonlinear alternative for Leary-Schauder type as the solution of the problem (1.1).

Theorem 3.6. Consider that (A2)–(A3) and $\frac{\kappa T^{\varrho}}{\Gamma(\varrho+1)} < 1$ holds, and (A6) $|\psi(\nu,x)| \leq \mathcal{X}_1(\nu)$, $|\Phi(\nu,x)| \leq \mathcal{X}_2(\nu)$ for all $(p,x) \in \mathscr{J}\times\mathscr{R}$, where $\mathcal{X}_1,\mathcal{X}_2 \in (\mathscr{J},\mathscr{R}^+)$. On $\tilde{E}$ the problem (1.1) has atleast one solution defined.

Proof. The operator Q_1 and Q_2 :

$$Q_1p(\nu) = \begin{cases} 0, \ if \ \nu \in [-\epsilon,0]_{\mathbb{T}}, \\ (\phi-\Phi(0,\varsigma(0)))\frac{1}{\Gamma(\varrho)}\int_0^\nu (\nu-s)^{\varrho-1}\Delta s \\ +\frac{1}{\Gamma(\varrho)}\int_0^\nu (\nu-s)^{\varrho-1}\Phi(s,p_s,k_1p_s(\nu))\Delta s \quad if \ \nu\in[0,T]_{\mathbb{T}}, \end{cases} \tag{2.5}$$

$$Q_2p(\nu) = \begin{cases} \varsigma(p), \ if \ \nu\in[-\epsilon,0]_{\mathbb{T}}, \\ \varsigma(0)+\frac{1}{\Gamma(\rho+\varrho)}\int_0^\nu (\nu-s)^{\rho+\varrho-1}\psi(s,p_s,k_2p_s(\nu))\Delta s \quad if \ \nu\in[0,T]_{\mathbb{T}}. \end{cases} \tag{2.6}$$

Put

$$\sup_{\nu\in[0,T]_{\mathbb{T}}} \mathcal{X}_1(\nu) = \|\mathcal{X}_1\|_\infty, \qquad \sup_{\nu\in[0,T]_{\mathbb{T}}} \mathcal{X}_2(\nu) = \|\mathcal{X}_2\|_\infty,$$

and

$$\omega \geq \|\varsigma\|_c + T^{\varrho}\Big[\frac{[|\phi|+2\|\mathcal{X}_2\|_\infty]}{\Gamma(\varrho+1)} + \frac{T^{\rho}\|\mathcal{X}_1\|_\infty}{\Gamma(\rho+\varrho+1)}\Big],$$

and define $D_\omega = p \in C([\epsilon,T]_{\mathbb{T}},\mathscr{R}) = \|p\|_\infty \leq \omega$

Claim 1: Any sort of $p,q \in D_\omega$: $Q_1p+Q_2q \in D_\omega$: Any sort of $p,q\in D_\omega$. From (2.5), (2.6) and Lemma 2.11, we have,

$$|Q_1p(\nu)+Q_2q(\nu)| \leq \sup_{\nu\in[0,T]_{\mathbb{T}}}\Big(\frac{\phi-\Phi(0,\varsigma(0))}{\Gamma(\varrho)}\int_0^\nu(\nu-s)^{\varrho-1}\Delta s + \frac{1}{\Gamma(\varrho)}\int_0^\nu(\nu-s)^{\varrho-1}\Phi(s,p_s,k_1p_s(\nu))\Delta s + \varsigma(0) + \frac{1}{\Gamma(\rho+\varrho)}\int_0^\nu(\nu-s)^{\rho+\varrho-1}\psi(s,p_s,k_2p_s(\nu))\Delta s\Big)$$

$$
\begin{aligned}
&\leq \sup_{\nu\in[0,T]_{\mathbb{T}}}\Big(\frac{\phi-\Phi(0,\varsigma(0))}{\Gamma(\varrho)}\int_0^{\nu}(\nu-s)^{\varrho-1}ds+\frac{1}{\Gamma(\varrho)}\int_0^{\nu}(\nu-s)^{\varrho-1}\Phi(s,p_s,k_1p_s(\nu))ds \\
&\quad +\varsigma(0)+\frac{1}{\Gamma(\rho+\varrho)}\int_0^{\nu}(\nu-s)^{\rho+\varrho-1}\psi(s,p_s,k_2p_s(\nu))ds\Big) \\
&\leq \|\varsigma\|_c+T^{\varrho}\Big[\frac{k_1[|\phi|+2|\mathcal{X}_2|_\infty]}{\Gamma(\varrho+1)}+\frac{T^{\rho}k_2\|\mathcal{X}\|_\infty}{\Gamma(\rho+\varrho+1)}\Big] \\
&\leq \omega k_1k_2.
\end{aligned}
$$

This shows that $\mathcal{Q}_1y+\mathcal{Q}_2z=D_\omega$.

Claim 2: On D_ω, $\mathcal{Q}_1$ is a contraction mapping:
Let $p,q\in D_\omega$. By (2.5) and Lemma 2.11 we have,

$$
\begin{aligned}
|\mathcal{Q}_1p(\nu)+\mathcal{Q}_2q(\nu)| &\leq \frac{1}{\Gamma(\varrho)}\int_0^{\nu}(\nu-s)^{\varrho-1}|\Phi(s,p_s,k_1p_s(\nu))-\Phi(s,q_s,k_1p_s(\nu))|\Delta s \\
&\leq \frac{1}{\Gamma(\varrho)}\int_0^{\nu}(\nu-s)^{\varrho-1}k_1\|p_s-q_s\|_c\Delta s \\
&\leq \frac{\kappa}{\Gamma(\varrho)}\int_0^{\nu}(\nu-s)^{\varrho-1}k_1\|p_s-q_s\|_c ds \\
&\leq \frac{\kappa\nu^{\varrho}}{\Gamma(\varrho+1)}k_1\|p-q\|_{[-\epsilon,T]_{\mathbb{T}}} \\
&\leq \frac{\kappa T^{\varrho}}{\Gamma(\varrho+1)}k_1\|p-q\|_{[-\epsilon,T]_{\mathbb{T}}}.
\end{aligned}
$$

Thus,

$$
\|\mathcal{Q}_1p(\nu)+\mathcal{Q}_2q(\nu)\|_{[-\epsilon,T]_{\mathbb{T}}}\leq \frac{\kappa T^{\varrho}}{\Gamma(\varrho+1)}k_1\|p-q\|_{[-\epsilon,T]_{\mathbb{T}}}.
$$

Claim 3: $\mathcal{Q}_2$ and ψ is continuous so $\mathcal{Q}_2$ is continuous. $\mathcal{Q}_2$ is uniformly bounded on D_ω. i.e.,

$$
\|\mathcal{Q}_2p\|\leq\|\varsigma\|_c+\frac{T^{\rho+\varrho}k_2\|\mathcal{X}_1\|_\infty}{\Gamma(\rho+\varrho+1)}.
$$

Claim 4: $\mathcal{Q}_2$ is equicontinuous. Defining

$$
\psi_o=\sup_{(\nu,p)\in[0,T]_{\mathbb{T}}\times D_\omega}|\psi(\nu,p)|\leq\infty.
$$

For $\nu_1,\nu_2\in[0,T]_{\mathbb{T}}, \nu_1,\nu_2$ by (2.6) and Lemma 2.11 we have,

$$
\begin{aligned}
|\mathcal{Q}_2p(\nu_2)-\mathcal{Q}_2p(\nu_1)| &\leq \frac{\psi_ok_1}{\Gamma(\rho+\varrho)}\int_0^{\nu_1}|(\nu_2-s)^{\rho+\varrho-1}-(\nu_1-s)^{\rho+\varrho-1}|\Delta s \\
&\quad+\frac{\psi_ok_2}{\Gamma(\rho+\varrho)}\int_0^{\nu_1}(\nu_2-s)^{\rho+\varrho-1}\Delta s \\
&\leq \frac{\psi_ok_1}{\Gamma(\rho+\varrho)}\int_0^{\nu_1}|(\nu_2-s)^{\rho+\varrho-1}-(\nu_1-s)^{\rho+\varrho-1}|ds
\end{aligned}
$$

$$+\frac{\psi_o k_2}{\Gamma(\rho+\varrho)}\int_0^{\nu_1}(\nu_2-s)^{\rho+\varrho-1}ds$$
$$\le\frac{\psi_o k_1 k_2}{\Gamma(\rho+\varrho+1)}[|\nu_2^{\rho+\varrho}|+|\nu_2-\nu_1|^{\rho+\varrho}].$$

As $\nu_1\to\nu_2$ the R.H.S of the above inequality $\to 0$. Therefore Q_2 is equicontinuous, where on D_ω Q_2 is relatively compact . Therefore, by Arzela-Ascoli theorem Q_2 is compact on D_ω .

The problem (1.1) has atleast one solution $[-\epsilon, T]_{\mathbb{T}}$.

4. Example

Let us consider the fractional functional integro-differential equation on a Time scale T.

$$^{c}\Delta^{\frac{1}{3}}\Big[{}^{c}\Delta^{\frac{1}{2}}p(\nu)-\frac{1}{1000}(\nu cos\|p_\nu\|_c-\|p_\nu\|_c sin\nu)\int_0^t k_1(t,\nu cos\|p_\nu\|_c,\|p_\nu\|_c sin\nu)d\nu\Big]$$
$$=\frac{1}{100e^\nu}\frac{\|p_\nu\|_c}{5+\|p_\nu\|_c}\int_0^t k_1(t,\nu cos\|p_\nu\|_c,\|p_\nu\|_c sin\nu)d\nu,$$
$$\nu\in\mathscr{J}:[0,1]\cap\mathbb{T},$$
$$p(\nu)=\varsigma(\nu),\quad \nu\in[-\in,0]\cap\mathbb{T},$$
$$D^{\frac{1}{2}}p(0)=\frac{1}{3}.$$

Let

$$\psi(\nu,x)=\frac{1}{100e^\nu}\frac{x}{5+x}\int_0^t k_2(t,\nu cosx,xsin\nu)d\nu,$$
$$\phi(\nu,x)=\frac{1}{1000}(\nu cosx-xsin\nu)\int_0^t k_1(t,\nu cosx,xsin\nu)d\nu,$$
$$(\nu,x)\in[0,1]\cap T\times[0,\infty].$$

For $p,q\in[0,\infty]$ and $\nu\in\mathscr{J}$, we have

$$|\psi(\nu,p)-\psi(\nu,q)|=\frac{1}{100e^\nu}\left|\frac{p}{5+p}-\frac{q}{5+q}\right|\int_0^t k_2(t,\nu cospq,pqsin\nu)d\nu$$
$$=\frac{5|p-q|}{100e^\nu(5+p)(5+q)}\int_0^t k_2(t,\nu cospq,pqsin\nu)d\nu$$
$$\le\frac{1}{500}|p-q|\int_0^t k_2(t,\nu cospq,pqsin\nu)d\nu,$$

and

$$|\phi(\nu,p)-\phi(\nu,q)|\le\left[\frac{1}{1000}|\nu||cosp-cosq+\frac{1}{1000}|sin\nu||p-q|\right]\int_0^t k_2(t,\nu cospq,pqsin\nu)d\nu$$
$$\le\frac{1}{1000}|p-q|+\frac{1}{1000}|p-q|\int_0^t k_2(t,\nu cospq,pqsin\nu)d\nu$$

$$\leq \frac{1}{500}|p-q|\int_0^t k_2(t, vcospq, pqsinv)dv.$$

And put $k_1 = k_2$,

$$T^{\rho}\Big(\frac{\kappa k_1}{\Gamma(\varrho+1)} + \frac{\Lambda T^{\varrho}}{\Gamma(\rho+\varrho+1)}\Big) = \Big(\frac{\frac{1}{500}}{\Gamma(\frac{1}{2}+1)} + \frac{\frac{1}{500}}{\Gamma(\frac{1}{3}+\frac{1}{2}+1)}\Big)\int_0^t k_2(t, vcospq, pqsinv)dv$$
$$\approx 0.04385118\int_0^t k_2(t, vcospq, pqsinv)dv$$
$$< \int_0^t k_2(t, vcospq, pqsinv)dv.$$

Hence the conditions (A1) and (A2) hold with $\Lambda = \kappa = \frac{1}{500}$. Thus the problem has a unique solution on $[-\in, 1]_{\mathbb{T}}$.

5. Conclusions

In this work, we obtain the existence and uniqueness solution to the integro differential equations for the Caputo fractional derivative on Time scale. The solution of the neutral fractional differential equations along the finite delay condition is derived by using the fixed point theory.

In future we look forward more on circuit analysis, in particular by using Sequential fractional order Neutral functional Integro differential equations on time scales with Caputo fractional operator over Banach spaces.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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