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Research article

N_b-fuzzy metric spaces with topological properties and applications

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Abstract: Our aim is to introduce the notion of N_b-fuzzy metric space (FMS). We also define quasi N-FMS, and pseudo N_b-FMS with examples and counterexamples and prove a decomposition theorem for pseudo N_b-FMS. We prove various theorems related to the convergence of sequences and analyze topology of symmetric N_b-FMS. At last, we provide an application of q-contraction mapping as a Banach contraction principle (BCP) in the structure of symmetric N_b-FMS and applied it in the solution of integral equations and linear equations.

Keywords: N_b-fuzzy metric space; pseudo N_b-fuzzy metric space; quasi N-fuzzy metric space; q-contraction
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1. Introduction and preliminaries

In the literature of mathematical analysis there exist many generalizations of metric space (MS). The famous generalization of MS is 2-MS introduced by Gähler [1] but notice that a 2-metric is not a continuous function of its variables whereas an ordinary metric is. This led Dhage [2] to introduce the notion of D-MS. After that Mustafa and Sims [3] showed that most of the topological properties of D-MS were not correct. Then they introduced the G-MS unfortunately Jleli and Samet [4] showed that most of the obtained fixed point theorems on G-MS can be deduced immediately from fixed point theorems on MS or quasi MS. After that Sedghi et al. [5] defined S-MS. On the other hand after replacing set of positive real number by ordered Banach space Huang and Zhang [6] introduced cone MS and many types of cone MSs are defined in [7–10]. The more generalized idea of the triangle inequality was introduced so that the concept of MS was extended to b-MS in 1989 by Bakhtin [11]. After that a S_b-MS [12–14] is also defined.

One of the most famous generalization of MS is FMS introduced by Kramosil and Michalek [15] in 1975. In 1994, George and Veeramani [16] modified the concept of FMS. It has wide applications in computer science and engineering like (see [17, 18]) as in colour image filtering, neural network, data mining, etc. Some of its generalizations are fuzzy metric type space (or fuzzy b-MS [19–21]), Q-FMS [22], dislocated G-FMS [23] and N-FMS [24] and also defined same space in [25] in 2020 etc.

Inspired by the work of Nadaban [20] we are introducing N_b-FMS which is the generalization of N-FMS.

In this work, after introduction and preliminaries in Section 2, we define N_b-FMS which is a generalization of N-FMS and S_b-MS. In Section 3, we define various types of N-FMS and N_b-FMS and for the base we have also define different S-MS and S_b-MS such as quasi S-MS, with examples and counterexamples. We also prove a decomposition theorem on pseudo N-FMS. In Section 4, we define convergent sequence, Cauchy sequence and topology of symmetric N_b-FMS and prove various theorems related to convergence and topology. In Section 5, we apply q-contraction mapping in symmetric N_b-FMS and prove celebrated BCP with example and integral analogue. We also provide application of BCP in solution of integral equations and linear equations.

We recall the following definitions which will be needed in the sequel.

Definition 1.1. [26] A map $o : [0, 1]^3 \to [0, 1]$ is called continuous t-norm if it satisfies the following conditions:

(1) $o(a, 1, 1) = a, o(0, 0, 0) = 0$;

(2) $o(a, b, c) = o(a, c, b) = o(b, c, a)$;

(3) $o(a_1, b_1, c_1) \geq o(a_2, b_2, c_2)$ for $a_1 \geq a_2, b_1 \geq b_2, c_1 \geq c_2$.

Examples of t-norm are
(1) $a \circ b \circ c = abc$ product t-norm;
(2) $a \circ b \circ c = \min\{a, b, c\}$ minimum t-norm.

Definition 1.2. [18] A triplet (X, N, o) is an N-FMS, if X is an arbitrary (nonempty) set, o is continuous t-norm and N is a fuzzy set on $X^3 \times (0, \infty)$ satisfying the following conditions for all $\zeta, \eta, \upsilon \in X$ and $r, s, t > 0$.

(N_1) $N(\zeta, \eta, \upsilon, t) > 0$,

(N_2) $N(\zeta, \eta, \upsilon, t) = 1$ iff $\zeta = \eta = \upsilon$,

(N_3) $N(\zeta, \eta, \upsilon, r + s + t) \geq N(\zeta, \zeta, a, r) \circ N(\eta, \eta, a, s) \circ N(\upsilon, \upsilon, a, t)$,

(N_4) $N(\zeta, \eta, \upsilon, \cdot) : (0, \infty) \to (0, 1]$ is a continuous function.

Definition 1.3. [13] Let X be a nonempty set and $k \geq 1$ be a given real number. A function $S_b : X^3 \to [0, \infty)$ is said to be S_b-metric iff for all $\zeta, \eta, \upsilon, a \in X$ the following conditions are satisfied:

(S_{b1}) $S_b(\zeta, \eta, \upsilon) = 0$ iff $\zeta = \eta = \upsilon$,

(S_{b2}) $S_b(\zeta, \eta, \upsilon) \leq k\Big[S_b(\zeta, \zeta, a) + S_b(\eta, \eta, a) + S_b(\upsilon, \upsilon, a)\Big]$.

The pair (X, S_b) is called an S_b-MS.

2. A new type of FMSs

In this section, we will define N_b-FMS with various examples and prove that N_b-FMS generalizes the N-FMS with the help of counterexamples.

Definition 2.1. A quadruple (X, N_b, o, k) is an N_b-FMS if X is an arbitrary (nonempty) set, o is continuous t-norm, $k \geq 1$ is a real number and N_b is a fuzzy set on $X^3 \times (0, \infty)$ satisfying the following conditions for all $\zeta, \eta, \upsilon, a \in X$ and $r, s, t > 0$:

(N_{b1}) $N_b(\zeta, \eta, \upsilon, t) > 0$,

(N_{b2}) $N_b(\zeta, \eta, \upsilon, t) = 1$ iff $\zeta = \eta = \upsilon$,

(N_{b3}) $N_b(\zeta, \eta, \upsilon, k(r + s + t)) \geq N_b(\zeta, \zeta, a, r) \circ N_b(\eta, \eta, a, s) \circ N_b(\upsilon, \upsilon, a, t)$,

(N_{b4}) $N_b(\zeta, \eta, \upsilon, \cdot)(0, \infty) \to (0, 1]$ is a continuous function.

$N_b(\zeta, \eta, \upsilon, t)$ is considered as the degree of nearness of ζ, η and υ with respect to t.

Remark 2.2. We note that N_b-FMS is the generalization of N-FMS and b-MS or we can say that it is the fuzzy version of S_b-MS. Indeed each N-FMS is a N_b-FMS for $k = 1$. However each N_b-FMS is need not be N-FMS.

Example 2.3. Let $X = R$ be real line and S_b be an S_b-metric on X defined by

$$S_b(\zeta, \eta, \upsilon) = \Big[\,|\,\zeta - \upsilon\,| + |\,\eta - \upsilon\,|\,\Big]^2,$$

or

$$S_b(\zeta, \eta, \upsilon) = \Big[\,|\,\eta + \upsilon - 2\zeta\,| + |\,\eta - \upsilon\,|\,\Big]^2,$$

or

$$S_b(\zeta, \eta, \upsilon) = k\Big(\,|\,\zeta - \upsilon\,| + |\,\zeta + \upsilon - 2\eta\,|\,\Big) \quad \forall\ \zeta, \eta, \upsilon \in X \text{ and } k \geq 1.$$

Let N_b be the function on $X^3 \times (0, \infty)$ defined by

$$N_b(\zeta, \eta, \upsilon, t) = \frac{t}{t + S_b(\zeta, \eta, \upsilon)},$$

for all $\zeta, \eta, \upsilon \in X$ and $t > 0$. Then (X, N_b, k, o) is an N_b-FMS with constant k and product t-norm.

Proof. N_{b1}, N_{b2} and N_{b4} are obvious we check only N_{b3}. Now,

$$N_b(\zeta, \zeta, a, r) \circ N_b(\eta, \eta, a, s) \circ N_b(\upsilon, \upsilon, a, t)$$
$$= \frac{r}{r + S_b(\zeta, \zeta, a)} \cdot \frac{s}{s + S_b(\eta, \eta, a)} \cdot \frac{t}{t + S_b(\upsilon, \upsilon, a)}$$

$$
\begin{aligned}
&= \frac{1}{1+\frac{S_b(\zeta,\zeta,a)}{r}} \cdot \frac{1}{1+\frac{S_b(\eta,\eta,a)}{s}} \cdot \frac{1}{1+\frac{S_b(\upsilon,\upsilon,a)}{t}} \\
&\leq \frac{1}{1+\frac{S_b(\zeta,\zeta,a)}{(r+s+t)}} \cdot \frac{1}{1+\frac{S_b(\eta,\eta,a)}{(r+s+t)}} \cdot \frac{1}{1+\frac{S_b(\upsilon,\upsilon,a)}{(r+s+t)}} \\
&\leq \frac{1}{1+\frac{S_b(\zeta,\zeta,a)+S_b(\eta,\eta,a)+S_b(\upsilon,\upsilon,a)}{(r+s+t)}} \\
&\leq \frac{1}{1+\frac{S_b(\zeta,\eta,\upsilon)}{k(r+s+t)}} \\
&= \frac{k(r+s+t)}{k(r+s+t)+S_b(\zeta,\eta,\upsilon)} \\
&= N_b(\zeta,\eta,\upsilon,k(r+s+t)).
\end{aligned}
$$

Hence N_{b3} satisfied. □

Example 2.4. Let $X = R$ be a real line and S_b be an S_b-MS as defined in Example 2.3. Let N_b is a function on $X^3 \times (0,\infty)$ defined by

$$N_b(\zeta,\eta,\upsilon,t) = e^{\frac{-S_b(\zeta,\eta,\upsilon)}{t}},$$

for all $\zeta,\eta,\upsilon \in X$ and t is product t-norm.Then $(X, N_b, \cdot, k)$ be an N_b-FMS with constant k.

Proof. N_{b1}, N_{b2} and N_{b4} conditions are standard. So we verify only N_{b3}, for this consider

$$
\begin{aligned}
N_b(\zeta,\zeta,a,r)\cdot N_b(\eta,\eta,a,s)\cdot N_b(\upsilon,\upsilon,a,t) &= e^{\frac{-S_b(\zeta,\zeta,a)}{r}} \cdot e^{\frac{-S_b(\eta,\eta,a)}{s}} \cdot e^{\frac{-S_b(\upsilon,\upsilon,a)}{t}} \\
&\leq e^{\frac{-S_b(\zeta,\zeta,a)}{r+s+t}} \cdot e^{\frac{-S_b(\eta,\eta,a)}{r+s+t}} \cdot e^{\frac{-S_b(\upsilon,\upsilon,a)}{r+s+t}} \\
&= e^{-\frac{S_b(\zeta,\zeta,a)+S_b(\eta,\eta,a)+S_b(\upsilon,\upsilon,a)}{r+s+t}} \\
&\leq e^{-\frac{S_b(\zeta,\eta,\upsilon)}{k(r+s+t)}} \\
&= N_b(\zeta,\eta,\upsilon,k(r+s+t)).
\end{aligned}
$$

Hence N_{b3} satisfied. □

Definition 2.5. Let (X, N_b, o, k) be an N_b-FMS and $k \geq 1$. An N_b-fuzzy metric N_b is called symmetric if

$$N_b(\zeta,\zeta,\eta,t) = N_b(\eta,\eta,\zeta,t), \tag{2.1}$$

for all $\zeta,\eta \in X$ and $t > 0$.

Remark 2.6. From Definition 2.1 for $k = 1$ N_b-FMS induced to an N-FMS and N-fuzzy metric satisfies symmetry property (2.1) automatically (see Proposition 3.5 of [24]). However N_b-fuzzy metric is not symmetric everywhere. So N_b-FMS is more general structure to N-FMS.

Example 2.7. Let $X = R$ and N_b be a function on $X^3 \times (0, \infty)$ defined by

$$N_b(\zeta, \eta, \upsilon, t) = \frac{t}{t + \Big[\mid \zeta - \eta \mid + \mid \eta - \upsilon \mid + \mid \upsilon - \zeta \mid \Big]^p},$$

for all $\zeta, \eta, \upsilon \in X$, $t > 0$ and $p > 1$. Noting that (X, N_b, o, k) be symmetric N_b-FMS with $k = 2^{2(p-1)}$ and product t norm o.

Example 2.8. Let $X = R^+ \cup \{0\}$, Define $S_b : X^3 \to [0, \infty)$ by

$$S_b(\zeta, \eta, \upsilon) = \begin{cases} 0, & \text{if } \zeta = \eta = \upsilon, \\ \Big[\left|\zeta - \frac{\upsilon}{2}\right| + \left|\eta - \frac{\upsilon}{2}\right| \Big]^2, & \text{otherwise.} \end{cases}$$

Then (X, S_b, k) is a S_b-MS where $k \geq 1$. Now we define N_b-fuzzy metric on $X^3 \times (0, \infty)$ by

$$N_b(\zeta, \eta, \upsilon, t) = \frac{t}{t + S_b(\zeta, \eta, \upsilon)}.$$

Then (X, N_b, o, k) be a N_b-FMS. But it is not N-fuzzy metric for some $\zeta, \eta, \upsilon \in X$. To prove it take $\zeta = 4, \eta = 6, \upsilon = 5$, and $a = 8$ and $r = 0.1, s = 0.7, t = 0.2$ and o by minimum t-norm. Then,

$$\begin{aligned} N_b(\zeta, \eta, \upsilon, (r + s + t)) &= N_b(4, 6, 5, 1) = \frac{1}{26}, \\ N_b(\zeta, \zeta, a, r) &= N_b(4, 4, 8, 0.1) = 1, \\ N_b(\eta, \eta, a, s) &= N_b(6, 6, 8, 0.7) = \frac{1}{23}, \\ N_b(\upsilon, \upsilon, a, t) &= N_b(5, 5, 8, 0.2) = \frac{1}{21}. \end{aligned}$$

It is clear that

$$\frac{1}{26} \ngeq \min\left\{1, \frac{1}{23}, \frac{1}{21}\right\}.$$

So that the usual triangle inequality of N-fuzzy metric is not satisfied. However if $k > 2$ then triangle inequality of N_b-fuzzy metric is satisfied. So above example satisfied all properties of N_b-FMS but not N-FMS. Moreover N-FMS satisfied symmetric property Proposition 3.5 of [24], i.e.,

$$N(\zeta, \zeta, \eta, t) = N(\eta, \eta, \zeta, t).$$

But in above example it is clear that

$$N_b(\zeta, \zeta, \eta, t) \neq N_b(\eta, \eta, \zeta, t).$$

Hence N_b-FMS does not satisfy symmetric property in general. So we conclude that N_b-FMS generalize the N-FMS.

3. Types of generalized FMSs

In this section, we will define quasi N-FMS, pseudo N_b-FMS for this we build up a base and define quasi-S-MS and pseudo-S_b-MS with counterexamples. We prove decomposition theorem on pseudo N_b-FMS.

Definition 3.1. Let X be a nonempty set. A function $S : X^3 \to [0, \infty)$ is said to be quasi-S-metric iff for all $\zeta, \eta, \upsilon, a \in X$ the following conditions are satisfied:

(S_{q1}) $S(\zeta, \eta, \upsilon) \geq 0$,

(S_{q2}) $S(\zeta, \eta, \upsilon) = S(P\{\zeta, \eta, \upsilon\}) = 0$ iff $\zeta = \eta = \upsilon$, where P is permutation,

(S_{q3}) $S(\zeta, \eta, \upsilon) \leq \left[S(\zeta, \zeta, a) + S(\eta, \eta, a) + S(\upsilon, \upsilon, a)\right]$.

The pair (X, S) is called a quasi S-MS.

Example 3.2. Let $X = R^+ \bigcup\{0\}$. Define $S : X^3 \to [0, \infty)$ by

$$S(\zeta, \eta, \upsilon) = \begin{cases} 0, & \text{if } \zeta = \eta = \upsilon, \\ \left|\zeta - \frac{\upsilon}{2}\right| + \left|\eta - \frac{\upsilon}{2}\right|, & \text{otherwise.} \end{cases}$$

Then (X, S) is a quasi S-MS.

Proof. Note that S_{q1} is obvious and for S_{q2}:

$$S(\zeta, \eta, \upsilon) = S\left(P\{\zeta, \eta, \upsilon\}\right) = 0 \text{ iff } \zeta = \eta = \upsilon,$$

that is if $\zeta \neq \eta \neq \upsilon$ then by definition of S in above example $S(\zeta, \eta, \upsilon) \neq S(P\{\zeta, \eta, \upsilon\})$. Moreover $S(\zeta, \zeta, \eta) \neq S(\eta, \eta, \upsilon)$. Hence quasi S-metric is not symmetric in general. However, S-MS is satisfied symmetric property automatically (see Lemma 2.5 of [5]). Hence quasi S-MS generalizes S-MS.

Now, for S_{q3}

$$\begin{aligned} S(\zeta, \eta, \upsilon) &= \left|\zeta - \frac{\upsilon}{2}\right| + \left|\eta - \frac{\upsilon}{2}\right| \\ &= \left|\zeta - \frac{a}{2} + \frac{a}{2} - \frac{\upsilon}{2}\right| + \left|\eta - \frac{a}{2} + \frac{a}{2} - \frac{\upsilon}{2}\right| \\ &\leq \left|\zeta - \frac{a}{2}\right| + \left|\frac{\upsilon}{2} - \frac{a}{2}\right| + \left|\eta - \frac{a}{2}\right| + \left|\frac{\upsilon}{2} - \frac{a}{2}\right| \\ &= \left|\zeta - \frac{a}{2}\right| + \left|\eta - \frac{a}{2}\right| + 2\left|\frac{\upsilon}{2} - \frac{a}{2}\right| \\ &\leq \left|\zeta - \frac{a}{2}\right| + \left|\eta - \frac{a}{2}\right| + \left|\upsilon - \frac{a}{2}\right| \\ &\leq 2\left|\zeta - \frac{a}{2}\right| + 2\left|\eta - \frac{a}{2}\right| + 2\left|\upsilon - \frac{a}{2}\right| \\ &= S(\zeta, \zeta, a) + S(\eta, \eta, a) + S(\upsilon, \upsilon, a). \end{aligned}$$

Hence S_{q3} also satisfied. □

Remark 3.3. Pseudo S-MS is already defined in [27].

Definition 3.4. A triplet (X, S_{bp}, k) is said to be pseudo S_b-MS if X is arbitrary (nonempty) set, $k \geq 1$ is given real number satisfying the following conditions:

(S_{bp1}) $\forall\ \zeta, \eta, \upsilon \in X,\ \ S_{bp}(\zeta, \eta, \upsilon) \geq 0$,

(S_{bp2}) $\forall\ \zeta, \eta, \upsilon \in X$ and $S_{bp}(\zeta, \eta, \upsilon) = 0$ if $\zeta = \eta = \upsilon$,

(S_{bp3}) $\forall\ \zeta, \eta, \upsilon, a \in X$ and $k \geq 1$,

$$S_{bp}(\zeta, \eta, \upsilon) \leq k\Big[S_{bp}(\zeta, \zeta, a) + S_{bp}(\eta, \eta, a) + S_{bp}(\upsilon, \upsilon, a)\Big].$$

Example 3.5. Let $X = R$, define a function S_{bp} on X^3 to $[0, \infty)$ by

$$S_{bp} = \Big[\ |\ \zeta^2 - \upsilon^2\ | + |\ \eta^2 - \upsilon^2\ |\ \Big]^2,$$

for all $\zeta, \eta, \upsilon \in X$, then (X, S_{bp}, k) is pseudo S_b-MS.

Proof. It is clear that S_{p1}, S_{p2} and S_{p3} satisfied by above example, so it is pseudo S_b-MS but it is not S_b-MS since if

$$\begin{aligned} &S_{bp}(\zeta, \eta, \upsilon) = 0 \\ &\Rightarrow \Big[\ |\ \zeta^2 - \upsilon^2\ | + |\ \eta^2 - \upsilon^2\ |\ \Big]^2 = 0 \\ &\Rightarrow \zeta = \pm\upsilon,\ \eta = \pm\upsilon \Rightarrow \zeta \neq \eta \neq \upsilon. \end{aligned}$$

Hence it is not satisfied S_{b_1} property of Definition 1.3. □

Remark 3.6. Every S_b-MS is pseudo S_b-MS but converse is not true in general, see above example.

Definition 3.7. A quasi N-FMS is a triplet (X, N_q, o) where X is a nonempty set, o is continuous t-norm and N_q is a fuzzy set on $X^3 \times (0, \infty)$ satisfying the following conditions:

(N_{q1}) $N_q(\zeta, \eta, \upsilon, t) > 0$,

(N_{q2}) $N_q(\zeta, \eta, \upsilon, t) = N_q\Big(P(\zeta, \eta, \upsilon), t\Big) = 1$ iff $\zeta = \eta = \upsilon$ where P is permutation,

(N_{q3}) $N_q\Big(\zeta, \eta, \upsilon, (r + s + t)\Big) \geq N_q(\zeta, \zeta, a, r) \circ N_q(\eta, \eta, a, s) \circ N_q(\upsilon, \upsilon, a, t)$,

(N_{q4}) $N_q(\zeta, \eta, \upsilon, \cdot) : (0, \infty) \to (0, 1]$ is a continuous function.

Example 3.8. Let (X, S) be a quasi S-MS (as defined in Example 3.2). Define a function N_q on $X^3 \times (0, \infty)$ such that

$$N_q(\zeta, \eta, \upsilon, t) = \frac{t}{t + S(\zeta, \eta, \upsilon)} \text{ for } t > 0.$$

Then (X, N_q, o) is a quasi-N-FMS.

Proof. Let t be a product t-norm then N_{q1}, N_{q2}, N_{q3} and N_{q4} are obvious. But in above Example $N_q(1, 1, 2, t) \neq N_q(2, 2, 1, t)$. □

Remark 3.9. N-FMS satisfies symmetric property automatically (see Proposition 3.5 of [24]), but quasi N-FMS is not symmetric, i.e., in quasi N-FMS for some $\zeta, \eta \in X$

$$N_q(\zeta, \zeta, \eta, t) \neq N_q(\eta, \eta, \zeta, t).$$

Hence quasi N-FMS are different FMS.

Definition 3.10. A quadruple (X, N_{bp}, o, k) is said to be pseudo N_b-FMS if X is an arbitrary (nonempty) set, o is a continuous t-norm, $k \geq 1$ is a given real number and N_{bp} is a fuzzy set on $X^3 \times (0, \infty)$ satisfying the following conditions:

(N_{bp1}) $\forall\ \zeta, \eta, \upsilon \in X$ and $\forall\ t > 0, N_{bp}(\zeta, \eta, \upsilon, t) > 0$,

(N_{bp2}) $\forall\ \zeta, \eta, \upsilon \in X$ and $\forall\ t > 0, N_{bp}(\zeta, \eta, \upsilon, t) = 1$ if $\zeta = \eta = \upsilon$,

(N_{bp3}) $\forall\ \zeta, \eta, \upsilon, a \in X$ and $\forall\ r, s, t > 0$

$$N_{bp}(\zeta, \eta, \upsilon, k(r + s + t)) \geq N_{bp}(\zeta, \zeta, a, r) \circ N_{bp}(\eta, \eta, a, s) \circ N_{bp}(\upsilon, \upsilon, a, t),$$

(N_{bp4}) $\forall\ \zeta, \eta, \upsilon \in X,\ N_{bp}(\zeta, \eta, \upsilon, \cdot) : (0, \infty) \to (0, 1]$ is continuous function.

Example 3.11. Consider R with the usual metric. Let $X = \{\{\zeta_n\} : \{\zeta_n\} \text{ is convergent in } R\}$. Define $a \circ b \circ c = abc$ for all $a, b, c \in [0, 1]$ and

$$N_{bp}(\zeta_n, \eta_n, \upsilon_n, t) = \left[\exp \lim \frac{(\mid \zeta_n - \upsilon_n \mid + \mid \eta_n - \upsilon_n \mid)^2}{t}\right]^{-1}.$$

Noting that (X, N_{bp}, o, k) is a pseudo N_b-FMS but not N_b-FMS.
To see this, let $\{\zeta_n\} = \{\frac{2}{n}\}, \{\eta_n\} = \{\frac{3}{n}\}, \{\upsilon_n\} = \{\frac{5}{n}\}$.
Then $\{\zeta_n\} \neq \{\eta_n\} \neq \{\upsilon_n\}$ for $\{\zeta_n\}, \{\eta_n\}$ and $\{\upsilon_n\} \in X$ but $N_{bp}(\zeta_n, \eta_n, \upsilon_n, t) = 1$.

Remark 3.12. Every N_b-FMS is a pseudo N_b-FMS denoted by (X, N_{bp}, o, k) but converse is not true, see above example.

Theorem 3.13. *Let (X, N_{bp}, o, k) be a pseudo N_b-FMS and*

$$S_{bp(x)}(\zeta, \eta, \upsilon) = \inf\{t > 0 : N_{bp}(\zeta, \eta, \upsilon, t) > x,\ x \in (0, 1)\}.$$

Then $S = \{S_{bp}(x)\}_{x \in (0,1)}$ is an ascending family of pseudo S_b-metric on X.

Proof. $S_{bp(x)}(\zeta, \zeta, \zeta) = \inf\{t > 0; N_{bp}(\zeta, \zeta, \zeta, t) > x\} = 0$.
Now,

$$\begin{aligned}
&k\Big[S_{bp(x)}(\zeta, \zeta, a) + S_{bp(x)}(\eta, \eta, a) + S_{bp(x)}(\upsilon, \upsilon, a)\Big] \\
&= k\Big[\inf\{r > 0 : N_{bp}(\zeta, \zeta, a, r) > x\} + \inf\{s > 0 : N_{bp}(\eta, \eta, a, s) > x\} \\
&\quad + \inf\{t > 0 : N_{bp}(\upsilon, \upsilon, a, t) > x\}\Big] \\
&= k\Big[\inf\{r + s + t > 0 : N_{bp}(\zeta, \zeta, a, r) > x, N_{bp}(\eta, \eta, a, s) > x, N_{bp}(\upsilon, \upsilon, a, t) > x\}\Big]
\end{aligned}$$

$$= \inf\left\{k(r+s+t) > 0 : N_{bp}(\zeta,\zeta,a,r) \circ N_{bp}(\eta,\eta,a,s) \circ N_{bp}(\upsilon,\upsilon,a,t) > x\right\}$$
$$\geq \inf\left\{k(r+s+t) > 0 : N_{bp}(\zeta,\eta,\upsilon,k(r+s+t)) > x\right\}$$
$$= S_{bp(x)}(\zeta,\eta,\upsilon).$$

It remain to prove that

$$\delta = \{S_{bp(x)}\}_{x\in(0,1)} \text{ is an ascending family.}$$

Let $x_1 \leq x_2$, then

$$\{t > 0 : N_{bp}(\zeta,\eta,\upsilon,t) > x_2\} \subseteq \{t > 0 : N_{bp}(\zeta,\eta,\upsilon,t) > x_1\}.$$

Thus,

$$\inf\{t > 0 : N_{bp}(\zeta,\eta,\upsilon,t) > x_2\} \geq \inf\{t > 0 : N_{bp}(\zeta,\eta,\upsilon,t) > x_1\},$$

namely

$$S_{bp(x_2)}(\zeta,\eta,\upsilon) \geq S_{bp(x_1)}(\zeta,\eta,\upsilon), \quad \forall\ (\zeta,\eta,\upsilon) \in X^3.$$

□

Example 3.14. Let $X = [-1,1]$, define $N_{bp}(\zeta,\eta,\upsilon,t)$ by

$$N_{bp} = \frac{t}{t + \left[\ |\ \zeta^2 - \upsilon^2\ | + |\ \eta^2 - \upsilon^2\ |\ \right]^2},$$

for all $\zeta,\eta,\upsilon \in X$, o is continuous t-norm, then (X, N_{bp}, o, k) is pseudo N_b-FMS. Define

$$S_{bp(x)}(\zeta,\eta,v) = \inf\{t > 0 : N_{bp}(\zeta,\eta,v,t) > x\}.$$

Take $x_1 = \frac{1}{4}$ and $x_2 = \frac{1}{2}$ in (0,1), then it is quiet natural

$$\{t > 0 : N_{bp}(\zeta,\eta,v,t) > \frac{1}{2}\} \subseteq \{t > 0 : N_{bp}(\zeta,\eta,v,t) > \frac{1}{4}\}.$$

Now, by property of infimum

$$\inf\{t > 0 : N_{bp}(\zeta,\eta,v,t) > \frac{1}{2}\} \geq \inf\{t > 0 : N_{bp}(\zeta,\eta,v,t) > \frac{1}{4}\}.$$

Hence,

$$S_{bp(\frac{1}{2})}(\zeta,\eta,v) \geq S_{bp(\frac{1}{4})}(\zeta,\eta,v), \quad \forall\ (\zeta,\eta,v) \in X^3.$$

Similarly, we can find for other distinct members in (0,1). Then $S = \{S_{bp}(x)\}_{x\in(0,1)}$ is an ascending family of pseudo S_b-metric on X.

We now define convergent sequence, Cauchy sequence, F-bounded set, q-contraction, continuous function and proved proposition and lemmas related to convergence and q-contraction.

Definition 3.15. Let (X, N_b, o, k) be a symmetric N_b-FMS. A sequence $\{\zeta_n\}$ in (X, N_b, o, k) converges to $\zeta \in X$, if $N_b(\zeta_n,\zeta_n,\zeta,t) \to 1$ or $N_b(\zeta,\zeta,\zeta_n,t) \to 1$ as $n \to \infty$ for each $t > 0$. That is, for $r > 0$ and $t > 0$ there exists $n_0 \in N$ such that for all $n \geq n_0$, $N_b(\zeta_n,\zeta_n,\zeta,t) > 1 - r$ or $N_b(\zeta,\zeta,\zeta_n,t) > 1 - r$.

Lemma 3.16. *Let (X, N_b, o, k) be a symmetric N_b-FMS, where o is product t norm. Let $\{\zeta_n\}$ be a sequence in X. If $\{\zeta_n\}$ converges to ζ and $\{\zeta_n\}$ also converges to η then $\zeta = \eta$. That is, if the limit of $\{\zeta_n\}$ exists, it is unique.*

Proof. Let $\{\zeta_n\}$ converge to ζ andη. Then $N_b(\zeta, \zeta, \zeta_n, s) \to 1$ as $n \to \infty$ for each $s > 0$ and $N_b(\eta, \eta, \zeta_n, t - 2s) \to 1$ as $n \to \infty$ for each $t - 2s > 0$.

$$\begin{aligned} N_b(\zeta, \zeta, \eta, t) &\geq N_b(\zeta, \zeta, \zeta_n, s) \circ N_b(\zeta, \zeta, \zeta_n, s) \circ N_b\Big(\eta, \eta, \zeta_n, \frac{t}{k} - 2s\Big) \\ &\to 1o1o1 = 1 \quad [\text{as } n \to \infty \ \& \ a \circ b \circ c = abc]. \end{aligned}$$

□

Example 3.17. Let (X, N_b, o, k) be a symmetric N_b-FMS as defined in Example 2.3. Define a sequence $\{x_n\} = \{\frac{1}{n}\}$, $\forall n \in N$. Then $\{x_n\}$ converges to unique limit 0.

Definition 3.18. Let (X, N_b, o, k) be a symmetric N_b-FMS and $\{\zeta_n\}$ be a sequence in X is called Cauchy sequence, if for each $r > 0$ and $t > 0$, there exists $n_0 \in N$ such that

$$N_b(\zeta_n, \zeta_n, \zeta_m, t) > 1 - r,$$
or
$$N_b(\zeta_m, \zeta_m, \zeta_n, t) > 1 - r,$$

for all $n, m \geq n_0$.

Definition 3.19. Let (X, N_b, o, k) be a symmetric N_b-FMS. If every Cauchy sequence in X is convergent in X, then X is called a complete symmetric N_b-FMS.

Definition 3.20. Let (X, N_b, o, k) be a symmetric N_b-FMS. A subset A of X is said to be F-bounded if there exist $t > 0$ and $0 < r < 1$ such that

$$N_b(\zeta, \zeta, \eta, t) > 1 - r, \quad \text{for all } \ \zeta, \eta \in A.$$

Example 3.21. Let $X = R$ and N_b be a function on $X^3 \times (0, \infty)$ defined by

$$N_b(\zeta, \eta, v, t) = \frac{t}{t + \Big[\,|\,\zeta - v\,| + |\,\eta - v\,|\,\Big]^2},$$

for all $\zeta, \eta, v \in X$ and $t > 0$. Then (X, N_b, o, k) be a symmetric N_b-FMS. Let A be a subset of R defined by $A = \{x : 0 < x < 1\}$, then A is F-bounded set. Particularly for $\zeta = 0$ and $\eta = 1$ the value of r is $\frac{4}{t+4}$.

Definition 3.22. Let (X, N_b, o, k) be a N_b-FMS. A self map $f : X \to X$ is a fuzzy q-contraction if for all $\zeta, \eta \in X$ and for some $q \in (0, 1)$, we have

$$N_b\Big(f(\zeta), f(\zeta), f(\eta), qt\Big) \geq N_b(\zeta, \zeta, \eta, t).$$

Lemma 3.23. *Let (X, N_b, o, k) be a symmetric N_b-FMS, where o is product t norm and $\{\zeta_n\}$ be a sequence in X. If $\{\zeta_n\}$ converges to ζ, then $\{\zeta_n\}$ is a Cauchy sequence.*

Proof. For each $s,t>0$ there is $p\in N$ such that

$$N_b(\zeta_n,\zeta_n,\zeta,s)\to 1 \quad \text{as } n\to\infty,$$

and

$$N_b(\zeta_{n+p},\zeta_{n+p},\zeta,\frac{t}{k}-2s)\to 1 \quad \text{as } n\to\infty,\ \text{for each } \frac{t}{k}-2s>0.$$

$$N_b(\zeta_n,\zeta_n,\zeta_{n+p},t)\geq N_b(\zeta_n,\zeta_n,\zeta,s)\circ N_b(\zeta_n,\zeta_n,\zeta,s)\circ N_b(\zeta_{n+p},\zeta_{n+p},\zeta,\frac{t}{k}-2s)$$
$$\to 1o1o1=1 \quad [\text{as } n\to\infty\ \&\ a\circ b\circ c=abc].$$

Hence, $\{\zeta_n\}$ is a Cauchy sequence. □

Example 3.24. Let (X,N_b,o,k) be a symmetric N_b-FMS as defined in Example 2.3. Define a sequence $\{x_n\}\in X$, by $\{1-\frac{1}{2^n}:n\in N\}$. Then $\{x_n\}$ is a convergence sequence and it converges to $1\in X$ and it is also a Cauchy sequence.

Remark 3.25. The converse of Lemma 3.23 is not always true, i.e., every Cauchy sequence need not be convergent. For example if we take $X=[0,1)$ in place of R in the above example $\{x_n\}$ is not convergent in X since 1 does not lie in X.

Remark 3.26. Symmetric N_b-FMS (X,N_b,o,k) is complete iff the symmetric S_b-MS (X,S_b) is complete where

$$N_b(\zeta,\eta,\upsilon,t)=\frac{t}{t+S_b(\zeta,\eta,\upsilon)},$$

for all $\zeta,\eta,\upsilon\in X$ and $t\in(0,\infty)$.

Definition 3.27. Let (X,N_b,o,k) and (X',N_b',o',k') be symmetric N_b-FMS. Then a function $f:X\to X'$ is said to be continuous at a point $\zeta\in X$ iff it is sequentially continuous at ζ, that is whenever $\{\zeta_n\}$ is convergent to ζ we have $\{f\zeta_n\}$ is convergent to $f(\zeta)$.

Proposition 3.28. *Let (X,N_b,o,k) be symmetric N_b-FMS and f be a fuzzy q-contraction. If any fixed point ζ of f satisfies*

$$N_b(\zeta,\zeta,\zeta,t)>0,$$

then

$$N_b(\zeta,\zeta,\zeta,t)=1.$$

Proof. Let $\zeta\in X$ be a fixed point of f, as f is a fuzzy q-contraction, so

$$\begin{aligned}N_b(\zeta,\zeta,\zeta,t)&=N_b(f(\zeta),f(\zeta),f(\zeta),t)\\&\geq N_b\Big(\zeta,\zeta,\zeta,\frac{t}{q}\Big)\\&\geq N_b\Big(\zeta,\zeta,\zeta,\frac{t}{q^2}\Big)\geq ...\\&\geq N_b\Big(\zeta,\zeta,\zeta,\frac{t}{q^n}\Big)\to 1 \quad \text{as } n\to\infty,\end{aligned}$$

and so

$$N_b(\zeta,\zeta,\zeta,t)=1.$$

□

Lemma 3.29. *Let (X, N_b, o, k) be a symmetric N_b-FMS. Let $\{\zeta_n\}$ and $\{\eta_n\}$ be two sequences in X and suppose $\zeta_n \to \zeta$, $\eta_n \to \eta$, as $n \to \infty$ and $N_b(\zeta, \zeta, \eta, t_n) \to N_b(\zeta, \zeta, \eta, t)$ as $n \to \infty$. Then $N_b(\zeta_n, \zeta_n, \eta_n, t_n) \to N_b(\zeta, \zeta, \zeta, t)$ as $n \to \infty$.*

Proof. Since $\lim_{n\to\infty} \zeta_n = \zeta$, $\lim_{n\to\infty} \eta_n = \eta$ and $\lim_{n\to\infty} N_b(\zeta, \zeta, \eta, t_n) = N_b(\zeta, \zeta, \zeta, t)$ there is $n_0 \in N$ such that $|t - t_n| < \delta$ for $n \geq n_0$ and $\delta < \frac{t}{2}$. We know that $N_b(\zeta, \zeta, \zeta, t)$ is nondecreasing with respect to t, so we have

$$
\begin{aligned}
N_b(\zeta_n, \zeta_n, \eta_n, t_n) &\geq N_b(\zeta_n, \zeta_n, \eta_n, t - \delta) \\
&\geq N_b\Big(\zeta_n, \zeta_n, \zeta, \frac{\delta}{3k}\Big) \circ N_b\Big(\zeta_n, \zeta_n, \zeta, \frac{\delta}{3k}\Big) \circ N_b\Big(\eta_n, \eta_n, \zeta, \frac{t}{k} - \frac{5\delta}{3k}\Big) \\
&\geq N_b\Big(\zeta_n, \zeta_n, \zeta, \frac{\delta}{3k}\Big) \circ N_b\Big(\zeta_n, \zeta_n, \zeta, \frac{\delta}{3k}\Big) \circ N_b\Big(\eta_n, \eta_n, \eta, \frac{\delta}{6k^2}\Big) \\
&\quad \circ N_b\Big(\eta_n, \eta_n, \eta, \frac{\delta}{6k^2}\Big) \circ N_b\Big(\eta, \eta, \zeta, \frac{t}{k^2} - \frac{7\delta}{6k^2}\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
N_b(\zeta, \zeta, \eta, t + 2\delta) &\geq N_b(\zeta, \zeta, \eta, t_n + \delta) \\
&\geq N_b\Big(\zeta, \zeta, \zeta_n, \frac{\delta}{3k}\Big) \circ N_b\Big(\zeta, \zeta, \zeta_n, \frac{\delta}{3k}\Big) \circ N_b\Big(\eta, \eta, \zeta_n, \frac{t_n}{k} + \frac{\delta}{3k}\Big) \\
&\geq N_b\Big(\zeta, \zeta, \zeta_n, \frac{\delta}{3k}\Big) \circ N_b\Big(\zeta, \zeta, \zeta_n, \frac{\delta}{3k}\Big) \circ N_b\Big(\eta, \eta, \eta_n, \frac{\delta}{6k^2}\Big) \\
&\quad \circ N_b\Big(\eta, \eta, \eta_n, \frac{\delta}{6k^2}\Big) \circ N_b\Big(\zeta_n, \zeta_n, \upsilon_n, \frac{t_n}{k^2}\Big).
\end{aligned}
$$

In view of Definition 3.15 and combining the arbitrariness of δ and the continuity for $N_b(\zeta, \zeta, \eta, \cdot)$ with respect to t. For large enough n, we have

$$
\begin{aligned}
N_b(\zeta, \zeta, \eta, t) &\geq N_b(\zeta_n, \zeta_n, \eta_n, t_n) \geq N_b(\eta, \eta, \zeta, t) \\
N_b(\zeta, \zeta, \eta, t) &\geq N_b(\zeta_n, \zeta_n, \upsilon_n, t_n) \\
&\geq N_b(\zeta, \zeta, \eta, t) \qquad \text{[by Definition 2.5]}.
\end{aligned}
$$

Consequently,

$$\lim_{n\to\infty} N_b(\zeta_n, \zeta_n, \eta_n, t_n) = N_b(\zeta, \zeta, \eta, t).$$

□

Lemma 3.30. *Let (X, N_b, o, k) be a symmetric N_b-FMS. If there exists $q \in (0, 1)$ such that $N_b(\zeta, \zeta, \eta, t) \geq N_b(\zeta, \zeta, \eta, \frac{t}{q})$ for all $\zeta, \eta \in X$, $t > 0$ and*

$$\lim_{t\to\infty} N_b(\zeta, \eta, \upsilon, t) = 1.$$

Then $\zeta = \eta$.

Proof. Suppose that there exists $q \in (0, 1)$ such that $N_b(\zeta, \zeta, \eta, t) \geq N_b\Big(\zeta, \zeta, \eta, \frac{t}{q}\Big)$ for all $\zeta, \eta \in X$ and $t > 0$. Then,

$$N_b(\zeta, \zeta, \eta, t) \geq N_b\Big(\zeta, \zeta, \eta, \frac{t}{q}\Big) \geq N_b\Big(\zeta, \zeta, \eta, \frac{t}{q^2}\Big),$$

and so

$$N_b(\zeta,\zeta,\eta,t) \geq N_b\Big(\zeta,\zeta,\eta,\frac{t}{q^n}\Big),$$

for positive integer n. Taking limit as $n \to \infty$, $N_b(\zeta,\zeta,\eta,t) \geq 1$ and hence $\zeta = \eta$. □

4. Application in fixed point theory

Fixed point theory is one of the most significant field of nonlinear functional analysis because of its wide applications in many disciplines such as studying the existence of solutions for algebraic equations, differential equations and integral equations, system of linear equations and convergence of many computational methods in economics, sports, medical science, computer science etc.

In this section as an application of fuzzy q-contraction in symmetric N_b-FMS, we establish an analogous of BCP in this space.

Theorem 4.1. *Let (X, N_b, o, k) be a complete symmetric N_b-FMS with*

$$\lim_{t\to\infty} N_b(x,y,z,t) = 1, \tag{4.1}$$

and f be a fuzzy q-contraction. Then f has a unique fixed point.

Proof. Let $\zeta_0 \in X$ and generate a sequence $\{\zeta_n\}$ by the iterative process $\zeta_n = f^n(\zeta_0), n \in N$. Since $n, t > 0$. So by definition of fuzzy q-contraction, we get

$$\begin{aligned} N_b\Big(\zeta_n,\zeta_n,\zeta_{n+1},qt\Big) &= N_b\Big(f\zeta_{n-1},f\zeta_{n-1},f\zeta_n,qt\Big) \\ &\geqslant N_b\Big(\zeta_{n-1},\zeta_{n-1},\zeta_n,t\Big) \\ &\geqslant N_b\Big(\zeta_{n-2},\zeta_{n-2},\zeta_{n-1},\frac{t}{q}\Big) \\ &\vdots \\ &\geqslant N_b\Big(\zeta_0,\zeta_0,\zeta_1,\frac{t}{q^{n-1}}\Big). \end{aligned}$$

Hence,

$$\begin{aligned} N_b\Big(\zeta_n,\zeta_n,\zeta_{n+1},qt\Big) &\geqslant N_b\Big(\zeta_0,\zeta_0,\zeta_1,\frac{t}{q^{n-1}}\Big) \\ N_b\Big(\zeta_n,\zeta_n,\zeta_{n+p},t\Big) &\geqslant N_b\Big(\zeta_n,\zeta_n,\zeta_{n+1},\frac{t}{3k}\Big) \circ N_b\Big(\zeta_n,\zeta_n,\zeta_{n+1},\frac{t}{3k}\Big) \\ &\quad \circ N_b\Big(\zeta_{n+p},\zeta_{n+p},\zeta_{n+1},\frac{t}{3k}\Big) \quad \text{[by } N_{b3} \text{ of Definition 2.1]} \\ &= N_b\Big(\zeta_n,\zeta_n,\zeta_{n+1},\frac{t}{3k}\Big) \circ N_b\Big(\zeta_n,\zeta_n,\zeta_{n+1},\frac{t}{3k}\Big) \\ &\quad \circ N_b\Big(\zeta_{n+1},\zeta_{n+1},\zeta_{n+p},\frac{t}{3k}\Big) \quad \text{[by symmetric property]} \\ &\geq N_b\Big(\zeta_n,\zeta_n,\zeta_{n+1},\frac{t}{3k}\Big) \circ N_b\Big(\zeta_n,\zeta_n,\zeta_{n+1},\frac{t}{3k}\Big) \\ &\quad \circ N_b\Big(\zeta_{n+1},\zeta_{n+1},\zeta_{n+2},\frac{t}{(3k)^2}\Big) \circ N_b\Big(\zeta_{n+1},\zeta_{n+1},\zeta_{n+2},\frac{t}{(3k)^2}\Big) \end{aligned}$$

$$
\begin{aligned}
&\circ N_b\Big(\zeta_{n+p}, \zeta_{n+p}, \zeta_{n+2}, \frac{t}{(3k)^2}\Big)\\
&= N_b\Big(\zeta_n, \zeta_n, \zeta_{n+1}, \frac{t}{3k}\Big) \circ N_b\Big(\zeta_n, \zeta_n, \zeta_{n+1}, \frac{t}{3k}\Big)\\
&\quad \circ N_b\Big(\zeta_{n+1}, \zeta_{n+1}, \zeta_{n+2}, \frac{t}{(3k)^2}\Big) \circ N_b\Big(\zeta_{n+1}, \zeta_{n+1}, \zeta_{n+2}, \frac{t}{(3k)^2}\Big)\\
&\quad \circ N_b\Big(\zeta_{n+2}, \zeta_{n+2}, \zeta_{n+p}, \frac{t}{(3k)^2}\Big) \quad \text{[by symmetric property]}\\
&\geq N_b\Big(\zeta_0, \zeta_0, \zeta_1, \frac{t}{q^n(3k)}\Big) \circ N_b\Big(\zeta_0, \zeta_0, \zeta_1, \frac{t}{q^n(3k)}\Big)\\
&\quad \circ N_b\Big(\zeta_0, \zeta_0, \zeta_1, \frac{t}{q^{n+1}(3k)^2}\Big) \circ N_b\Big(\zeta_0, \zeta_0, \zeta_1, \frac{t}{q^{n+1}(3k)^2}\Big).
\end{aligned}
$$

By the definition of fuzzy q-contraction (i.e., $q < 1$) together with condition (4.1) and letting $n \to \infty$, we get

$$\lim_{n\to\infty} N_b(\zeta_n, \zeta_n, \zeta_{n+1}, t) = 1o1o1o1 \cdots o1o1 = 1.$$

Hence, $\{\zeta_n\}$ is Cauchy sequence. Since (X, N_b, o, k) is a symmetric complete N_b-FMS, there exists $\zeta \in X$ such that

$$\lim_{n\to\infty} \zeta_n = \zeta.$$

Now, we will show that ζ is a fixed point of f.

$$
\begin{aligned}
N_b\Big(f(\zeta), f(\zeta), \zeta, t\Big) &\geq N_b\Big(f(\zeta), f(\zeta), f(\zeta_n), \frac{t}{3k}\Big) \circ N_b\Big(f(\zeta), f(\zeta), f(\zeta_n), \frac{t}{3k}\Big)\\
&\quad \circ N_b\Big(\zeta, \zeta, f(\zeta_n), \frac{t}{3k}\Big)\\
&\geq N_b\Big(\zeta, \zeta, \zeta_n, \frac{t}{3kq}\Big) \circ N_b\Big(\zeta, \zeta, \zeta_n, \frac{t}{3kq}\Big) \circ N_b\Big(\zeta, \zeta, \zeta_{n+1}, \frac{t}{3k}\Big)\\
&\quad \text{[since } f \text{ is } q \text{ contraction and } f(\zeta_n) = \zeta_{n+1}]\\
&\to 1o1o1 = 1 \text{ as } n \to \infty.
\end{aligned}
$$

This shows that $f(\zeta) = \zeta$, that is, ζ is fixed point of f. To prove the uniqueness, let $f(\eta) = \eta$ for some $\eta \in X$, then

$$
\begin{aligned}
N_b(\eta, \eta, \zeta, t) &= N_b(f(\eta), f(\eta), f(\zeta), t)\\
&\geq N_b(\eta, \eta, \zeta, \frac{t}{q})\\
&= N_b(f(\eta), f(\eta), f(\zeta), \frac{t}{q})\\
&\geq N_b(\eta, \eta, \zeta, \frac{t}{q^2}) \geq \cdots \geq N_b(\eta, \eta, \zeta, \frac{t}{q^n})\\
&\to 1 \text{ as } n \to \infty.
\end{aligned}
$$

Thus $\zeta = \eta$ and this completes the proof. □

Example 4.2. Let $X = [0,1]$ and (X, N_b, o, k) be the complete symmetric N_b-FMS where N_b is defined by

$$N_b(\zeta,\eta,\upsilon,t) = e^{-\dfrac{[|\zeta-\upsilon| + |\eta-\upsilon|]^2}{t}}, \quad \text{for all } \zeta,\eta,\upsilon \in X,\ t > 0.$$

Let $f(\zeta) = \lambda\zeta, \lambda < \frac{\sqrt{2}}{2}, \zeta \in X, t > 0$. Then, for $\frac{1}{2} > q$

$$\begin{aligned}
N_b(f(\zeta), f(\zeta), f(\eta), t) &= e^{-\dfrac{[|f(\zeta)-f(\eta)| + |f(\zeta)-f(\eta)|]^2}{t}} \\
&= e^{-\dfrac{[2\,|f(\zeta)-f(\eta)|\,]^2}{t}} \\
&= e^{-\dfrac{[2\,|\lambda\zeta-\lambda\eta|]^2}{t}} \\
&= e^{-\dfrac{4\lambda^2\,|\zeta-\eta|^2}{t}} \\
&= e^{-\dfrac{[|\zeta-\eta| + |\zeta-\eta|]^2}{t/\lambda^2}} \\
&= e^{-\dfrac{[|\zeta-\eta| + |\zeta-\eta|]^2}{t/q}} \\
&= N_b(\zeta,\zeta,\eta,\frac{t}{q}),
\end{aligned}$$

where $\lambda^2 = q$. Hence, all conditions of Theorem 4.1 are satisfied and so f has a unique fixed point 0 in X.

The first integral type of BCP was proved by Branciari [28] and see [19]. Let $\theta : (0,\infty) \to (0,\infty)$ as

$$\theta(t) = \int_0^t \phi(t)dt, \quad \forall\ t > 0,$$

be a nondecreasing and continuous function. Moreover for each $r > 0$, $\phi(r) > 0$. It also implies that $\phi(t) = 0$ iff $t = 0$.

In the following, we prove integral analogue of BCP in N_b-FMS.

Theorem 4.3. *Let (X, N_b, o, k) be a complete symmetric N_b-FMS and $f : X \to X$ be a map satisfying*

$$\int_0^{N_b(f(\zeta),f(\zeta),f(\eta),qt)} \phi(t)dt \geqslant \int_0^{N_b(\zeta,\zeta,\eta,t)} \phi(t)dt,$$

for all $\zeta,\eta \in X$, $\phi \in \theta$, and $q \in (0,1)$. Then f has a unique fixed point.

Proof. By taking $\phi(1) = 1$ and applying Theorem 4.1, we obtain the result. □

5. Application of BCP in integral equations

Solution of any kind of equations is the most important and interesting tasks in mathematics. There are many techniques for solving many class of equations. Facing the problem of finding solutions and studying whether such solutions are unique or multiple . Fixed point theory is one of the important methodology that has great success in the field of integral equations, since its a iterative procedures has great variety of contexts.

In the study of existence of solution of differential and integral equation, fixed point theory plays a very important role, see [29, 30].

In this section, we prove an application of Theorem 4.1 in particular nonlinear integral equation. The following theorem gives us an answer to the question "The solution of particular nonlinear equation (5.2) exists or not".

Consider $X = C[0, I]$ the class of all real valued continuous functions defined on $[0, I]$. Define a complete symmetric N_b-fuzzy metric $N_b : X^3 \times (0, \infty) \to [0, 1]$ by

$$N_b(\zeta, \eta, \upsilon, t) = e^{-\frac{\sup_{s\in[0,I]} [\, |\, \zeta(s) - \upsilon(s)\,| + |\, \eta(s) - \upsilon(s)\,|\,]^2}{t}}, \tag{5.1}$$

for $t > 0$ and for all $\zeta, \eta, \upsilon \in X$ and consider the integral equation.

$$\zeta(t) = g(t) + \int_0^I A(t, s)M(t, s, \zeta(s))ds, \tag{5.2}$$

where $I > 0$ and $g : [0, I] \to R$, $A : [0, I]^2 \to R$ and $M : [0, I]^2 \times R \to R$ are continuous functions.

Theorem 5.1. *Let (X, N_b, o, k) be a symmetric complete N_b-FMS defined by* (5.1). *Let $f : X \to X$ be the integral operator defined by*

$$f(\zeta(t)) = g(t) + \int_0^I A(t, s)M(t, s, \zeta(s))ds, \tag{5.3}$$

for all $\zeta \in X$ and $t, s \in [0, I]$. Suppose that the following conditions are satisfied:

(i) For all $t, s \in [0, 1]$ and $\zeta, \eta \in X$,

$$\left|M(t, s, \zeta(s)) - M(t, s, \eta(s))\right| \leq |\, \zeta(s) - \eta(s)\,| . \tag{5.4}$$

(ii) For all $t, s \in [0, I]$,

$$\sup_{s\in[0,I]} \left| \int_0^I (A(t, s))^2 ds \right| \leq q < 1. \tag{5.5}$$

Then the integral equation (5.2) *has a unique solution $\zeta^\star \in X$.*

Proof. For all $\zeta, \eta \in X$, we have

$$N_b(f(\zeta), f(\zeta), f(\eta), qt) = e^{-\frac{\sup_{s\in[0,I]} [\, |f(\zeta(t)) - f(\eta(t))| + |f(\zeta(t)) - f(\eta(t))|\,]^2}{qt}}$$

$$
\begin{aligned}
&= e^{-\frac{\sup_{s\in[0,I]}\left[2\left|f(\zeta(t))-f(\eta(t))\right|\right]^2}{qt}}\\
&= e^{-\frac{\sup_{s\in[0,I]} 4\left|\int_0^I (A(t,s)M(t,s,\zeta(s))-A(t,s)M(t,s,\eta(s)))ds\right|^2}{qt}}\\
&\geq e^{-\frac{\sup_{s\in[0,I]} 4\left|\int_0^I (A(t,s))^2 ds\right| \int_0^I \left|[M(t,s,\zeta(s))-M(t,s,\eta(s))]ds\right|^2}{qt}}\\
&\geq e^{-\frac{4q\int_0^I |(\zeta(s)-\eta(s))ds|^2}{qt}}\\
&\geq e^{-\frac{\sup_{s\in[0,I]} 4\left|\zeta(s)-\eta(s)\right|^2}{t}}\\
&= e^{-\frac{\sup_{s\in[0,I]}\left[\left|\zeta(s)-\eta(s)\right|+\left|\zeta(s)-\eta(s)\right|\right]^2}{t}}\\
&= N_b(\zeta,\zeta,\eta,t).
\end{aligned}
$$

Since all conditions of Theorem 4.1 are satisfied in complete symmetric N_b-FMS. Hence, the integral equation (5.2) has a unique solution. □

6. Application of BCP in solving linear equation

In this section, we study a very general class of linear equation and apply BCP to find existence of solution in the setting of symmetric N_b-FMS. The main advantage of BCP is that it is a general contractive condition can be generalized in the wide range of ways depending on many parameters. Further more such a contractive condition involves many distinct terms that can be either adding or multiplying between terms. We demonstrate a application of BCP that guarantees the existence and in some cases, the uniqueness of fixed points that can be interpreted as solution of the mentioned linear equations.

In this section, we give an application of Theorem 4.1 for solving system of linear equations.

Let $X = R^n$ and define complete symmetric N_b-fuzzy metric on $X^3 \times (0, \infty)$ by

$$N_b(\zeta,\eta,\upsilon,t) = \frac{t}{t+\left[\sum_{i=1}^n \mid \zeta_i-\upsilon_i \mid + \sum_{i=1}^n \mid \eta_i-\upsilon_i \mid\right]^2}, \tag{6.1}$$

for all $\zeta,\eta,\upsilon \in R^n$ and $k = 2$

$$\text{if} \quad \left[\max_{1\leq j\leq n}\sum_{i=1}^n \left|c_{ij}\right|\right]^2 \leq q < 1. \tag{6.2}$$

Then the following system of linear equations has a unique solution.

$$\begin{cases} c_{11}\zeta_1 + c_{12}\zeta_2 + \cdots + c_{1n}\zeta_n = d_1, \\ c_{21}\zeta_1 + c_{22}\zeta_2 + \cdots + c_{2n}\zeta_n = d_2, \\ \vdots \\ c_{n1}\zeta_1 + c_{n2}\zeta_2 + \cdots + c_{nn}\zeta_n = d_n. \end{cases} \tag{6.3}$$

Proof. Let $f : X \to X$ be defined by $f(\zeta) = c\zeta + d$ where $\zeta, d \in R^n$ and

$$c = \begin{pmatrix} c_{11} & c_{12} \cdots c_{1n} \\ c_{21} & c_{22} \cdots c_{2n} \\ \vdots & \vdots \\ c_{n1} & c_{n2} \cdots c_{nn} \end{pmatrix}.$$

For $\zeta, \eta \in R^n$, we get

$$\begin{aligned} N_b\Big(f(\zeta), f(\zeta), f(\eta), qt\Big) &= \frac{qt}{qt + 4\Big[\sum_{i=1}^n \Big|\sum_{j=1}^n c_{ij}(\zeta_j - \eta_j)\Big|\Big]^2} \\ &\geq \frac{qt}{qt + 4\Big[\sum_{j=1}^n \sum_{i=1}^n \mid c_{ij} \mid\mid (\zeta_j - \eta_j) \mid\Big]^2} \\ &= \frac{qt}{qt + \Big[\sum_{j=1}^n 2 \mid \zeta_j - \eta_j \mid \sum_{i=1}^n \mid c_{ij} \mid\Big]^2} \\ &\geq \frac{qt}{qt + \Big[\max_{1\leq j\leq n} \sum_{i=1}^n \mid c_{ij} \mid\Big]^2\Big[\sum_{j=1}^n 2 \mid \zeta_j - \eta_j \mid\Big]^2} \\ &\geq \frac{qt}{qt + q\Big[\sum_{j=1}^n 2 \mid \zeta_j - \eta_j \mid\Big]^2} \qquad \text{[by (6.2)]} \\ &= \frac{t}{t + \Big[\sum_{j=1}^n \mid \zeta_j - \eta_j \mid + \mid \zeta_j - \eta_j \mid\Big]^2} \\ &= N_b(\zeta, \zeta, \eta, t). \end{aligned}$$

Hence, f is a fuzzy q-contraction and by Theorem 4.1, f has a unique fixed point in complete symmetric N_b-FMS, that is, the system of linear equations (6.3) has a unique solution in X. □

7. Conclusions

In this article, we introduce the notions of N_b-FMS, quasi N-FMS, quasi N_b-FMS, pseudo N_b-FMS and proved decomposition theorem and BCP in the new setting with examples, counterexamples and applications.

We have built a fertile ground to study in further spaces like as extended N_b-FMS, partial N_b-FMS, extended partial N_b-FMS, intuitionistic N_b-FMS, partial intuitionistic N_b-FMS and many more generalized FMSs with fixed point theorems and their application in solution of different types of equations.

Conflict of interest

The authors declare to have no competing interests.

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