Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(3): 5838–5846.
DOI: 10.3934/math.2023294
Received: 02 November 2022
Revised: 04 December 2022
Accepted: 09 December 2022
Published: 26 December 2022

Research article

Some identities involving the bi-periodic Fibonacci and Lucas polynomials

Tingting Du and Zhengang Wu*

Northwest University, Xian, Shaanxi 710000, China

* **Correspondence:** Email: 20144743@nwu.edu.cn.

Abstract: In this paper, by using generating functions for the Chebyshev polynomials, we have obtained the convolution formulas involving the bi-periodic Fibonacci and Lucas polynomials.

Keywords: bi-periodic Fibonacci polynomials; bi-periodic Lucas polynomials; convolution formula; Chebyshev polynomials
Mathematics Subject Classification: 11B37, 11B39

1. Introduction

For any real number x, the Fibonacci polynomials $\{F_n(x)\}$ and Lucas polynomials $\{L_n(x)\}$ are defined by the recurrence relations as follows:

$$F_0(x) = 0, \quad F_1(x) = 1, \quad F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \quad n \geq 2,$$

and

$$L_0(x) = 2, \quad L_1(x) = x, \quad L_n(x) = xL_{n-1}(x) + L_{n-2}(x), \quad n \geq 2.$$

For $x = 1$, the Fibonacci and Lucas polynomials are well known, respectively, Fibonacci sequences $\{F_n\}$ and Lucas sequences $\{L_n\}$. The various properties of $\{F_n(x)\}$ and $\{L_n(x)\}$ have been investigated by many authors; see [1–5]. In particular, in [6–8] the authors established a series of connection formulaes between Fibonacci polynomials, Lucas polynomials and Chebyshev polynomials.

In [9], Yi and Zhang considered the convolution involving the Fibonacci polynomials:

$$\sum_{a_1+a_2+\cdots+a_k=n} F_{a_1+1}(x) \cdot F_{a_2+1}(x) \cdots F_{a_k+1}(x),$$

where the summation is over all k-dimension nonnegative integer coordinates $(a_1, a_2, \cdots, a_k)$ such that $a_1 + a_2 + \cdots + a_k = n$, and k is any positive integer.

In [10], Zhang obtained a series of identities that consists of the Fibonacci and Lucas sequences, by using generating functions for the second kind Chebyshev polynomials $\{U_n(x)\}$ and their partial derivatives to prove the following:

$$\sum_{a_1+a_2+\cdots+a_{k+1}=n} F_{m(a_1+1)}\cdot F_{m(a_2+1)}\cdots F_{m(a_{k+1}+1)} = (-i)^{mn}\,\frac{F_m^{k+1}}{2^k\cdot k!}U_{n+k}^{(k)}\left(\frac{i^m}{2}L_m\right),$$

and

$$\sum_{a_1+a_2+\cdots+a_{k+1}=n+k+1} L_{ma_1}\cdot L_{ma_2}\cdots L_{ma_{k+1}}$$
$$= (-i)^{m(n+k+1)}\,\frac{2}{k!}\sum_{h=0}^{k+1}\left(\frac{i^{m+2}}{2}L_m\right)^h\binom{k+1}{h}U_{n+2k+1-h}^{(k)}\left(\frac{i^m}{2}L_m\right),$$

where k, m are any positive integers, $n, a_1, a_2, \cdots, a_{k+1}$ are nonnegative integers, i is the square root of -1, $U^{(k)}(x)$ denotes the k-order derivative of $U(x)$ for x, and $\binom{k+1}{h}=\frac{(k+1)!}{h!(k+1-h)!}$.

In addition, in [11], the author introduced the bi-periodic Fibonacci polynomials $\{f_n(x)\}$, defined by

$$f_0(x)=0,\quad f_1(x)=1,\quad f_n(x)=\begin{cases} axf_{n-1}(x)+f_{n-2}(x), & if\, n\, is\, even;\\ bxf_{n-1}(x)+f_{n-2}(x), & if\, n\, is\, odd,\end{cases}\quad n\geq 2, \tag{1.1}$$

where a, b are any nonzero real numbers, and x is any real numbers. The bi-periodic Lucas polynomials $\{l_n(x)\}$ are defined by

$$l_0(x)=2,\quad l_1(x)=ax,\quad l_n(x)=\begin{cases} bxl_{n-1}(x)+l_{n-2}(x), & if\, n\, is\, even;\\ axl_{n-1}(x)+l_{n-2}(x), & if\, n\, is\, odd,\end{cases}\quad n\geq 2, \tag{1.2}$$

where a, b are any nonzero real numbers, and x is any real numbers. For $x=1$, the bi-periodic Fibonacci and Lucas polynomials are well known, respectively, bi-periodic Fibonacci and Lucas sequences.

Recently, in [12], Komatsu and Ramirez considered the convolution identities of order 2, 3 and 4 for the bi-periodic Fibonacci sequences $\{f_n\}$ are given with binomial cofficients. Other works related to convolved sequences can be found in [13–20].

In this paper inspired by [10], we use the generating function of the first kind Chebyshev polynomials $\{T_n(x)\}$, the second kind Chebyshev polynomials $\{U_n(x)\}$ and their partial derivative to study the following two theorems:

Theorem 1. Let $\{f_n(x)\}$ be bi-periodic Fibonacci polynomials defined by (1.1), for any positive integer k, nonnegative integer $n, a_1, a_2, \cdots, a_{k+1}$. Then, we have the identity

$$\sum_{a_1+a_2+\cdots+a_{k+1}=n}\left(\frac{b}{a}\right)^{\frac{\xi(a_1)+\xi(a_2)+\cdots+\xi(a_{k+1})}{2}} f_{a_1+1}(x)\cdot f_{a_2+1}(x)\cdots f_{a_{k+1}+1}(x)$$
$$=\frac{1}{k!}\sum_{h=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^h\,\frac{(n+k-h)!\cdot i^{2h}}{h!\,(n-2h)!}\left(\sqrt{ab}x\right)^{n-2h},$$

where $\lfloor z \rfloor$ denotes the floor function, the greatest integer less than or equal to z, $\xi(n) = n - \left\lfloor \frac{n}{2} \right\rfloor$ is the parity function, and i is the square root of -1.

Theorem 2. Let $\{l_n(x)\}$ be bi-periodic Lucas polynomials defined by (1.2), for any positive integer r, nonnegative integer $m, a_1, a_2, \cdots, a_{r+1}$. Then, we have the identity

$$\sum_{a_1+a_2+\cdots+a_{r+1}=m} \left(\frac{b}{a}\right)^{\frac{\xi(a_1)+\xi(a_2)+\cdots+\xi(a_{r+1})}{2}} l_{a_1}(x) \cdot l_{a_2}(x) \cdots l_{a_{r+1}}(x)$$

$$= \frac{2^{r+1}(1-xt)^{r+1}(m+r)}{(1-t^2)\cdot r!} \sum_{h=0}^{\left\lfloor \frac{m}{2} \right\rfloor} (-1)^h \frac{(m+r-h-1)! \cdot i^{2h}}{h!\,(m-2h)!} \left(\sqrt{ab}x\right)^{m-2h},$$

where $\lfloor z \rfloor$ denotes the floor function, the greatest integer less than or equal to z, $\xi(n) = n - \left\lfloor \frac{n}{2} \right\rfloor$ is the parity function, and i is the square root of -1.

From these two theorems we may immediately deduce the following corollaries:

Corollary 1. Let $\{f_n\}$ be bi-periodic Fibonacci sequences and $\{l_n\}$ be bi-periodic Lucas sequences , for any positive integers k, r, and any nonnegative integers n, m. Then, we have the following:

$$\sum_{a_1+a_2+\cdots+a_{k+1}=n} \left(\frac{b}{a}\right)^{\frac{\xi(a_1)+\xi(a_2)+\cdots+\xi(a_{k+1})}{2}} f_{a_1+1} \cdot f_{a_2+1} \cdots f_{a_{k+1}+1}$$

$$= \frac{1}{k!} \sum_{h=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^h \frac{(n+k-h)! \cdot i^{2h}}{h!\,(n-2h)!} \left(\sqrt{ab}\right)^{n-2h},$$

and

$$\sum_{a_1+a_2+\cdots+a_{r+1}=m} \left(\frac{b}{a}\right)^{\frac{\xi(a_1)+\xi(a_2)+\cdots+\xi(a_{r+1})}{2}} l_{a_1} \cdot l_{a_2} \cdots l_{a_{r+1}}$$

$$= \frac{2^{r+1}(1-t)^{r+1}(m+r)}{(1-t^2)\cdot r!} \sum_{h=0}^{\left\lfloor \frac{m}{2} \right\rfloor} (-1)^h \frac{(m+r-h-1)! \cdot i^{2h}}{h!\,(m-2h)!} \left(\sqrt{ab}\right)^{m-2h},$$

where $\lfloor z \rfloor$ denotes the floor function, the greatest integer less than or equal to z, $\xi(n) = n - \left\lfloor \frac{n}{2} \right\rfloor$ is the parity function, and i is the square root of -1.

Corollary 2. Let $\{F_n(x)\}$ be Fibonacci polynomials and $\{L_n(x)\}$ be Lucas polynomials , for any positive integers k, r, and any nonnegative integers n, m. Then, we have the following:

$$\sum_{a_1+a_2+\cdots+a_{k+1}=n} F_{a_1+1}(x) \cdot F_{a_2+1}(x) \cdots F_{a_{k+1}+1}(x)$$

$$= \frac{1}{k!} \sum_{h=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^h \frac{(n+k-h)! \cdot i^{2h}}{h!\,(n-2h)!} x^{n-2h},$$

and

$$\sum_{a_1+a_2+\cdots+a_{r+1}=m} L_{a_1}(x)\cdot L_{a_2}(x)\cdots L_{a_{r+1}}(x)$$

$$=\frac{2^{r+1}(1-xt)^{r+1}(m+r)}{(1-t^2)\cdot r!}\sum_{h=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^h\frac{(m+r-h-1)!\cdot i^{2h}}{h!\,(m-2h)!}x^{m-2h},$$

where $\lfloor z\rfloor$ denotes the floor function, the greatest integer less than or equal to z, and i is the square root of -1.

Corollary 3. Let $\{F_n\}$ be Fibonacci sequences and $\{L_n\}$ be Lucas sequences, for any positive integers k, r and any nonnegative integers n, m. Then, we have the following:

$$\sum_{a_1+a_2+\cdots+a_{k+1}=n} F_{a_1+1}\cdot F_{a_2+1}\cdots F_{a_{k+1}+1}=\frac{1}{k!}\sum_{h=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^h\frac{(n+k-h)!\cdot i^{2h}}{h!\,(n-2h)!},$$

and

$$\sum_{a_1+a_2+\cdots+a_{r+1}=m} L_{a_1}\cdot L_{a_2}\cdots L_{a_{r+1}}$$

$$=\frac{2^{r+1}(1-t)^{r+1}(m+r)}{(1-t^2)\cdot r!}\sum_{h=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^h\frac{(m+r-h-1)!\cdot i^{2h}}{h!\,(m-2h)!},$$

where $\lfloor z\rfloor$ denotes the floor function, the greatest integer less than or equal to z, and i is the square root of -1.

2. Auxiliary results

In this section, we shall give several lemmas which are necessary in the proofs of the theorems. First, we introduce Chebyshev polynomials $\{T_n(x)\}$ and $\{U_n(x)\}$. For any positive integer n, the first kind Chebyshev polynomials $\{T_n(x)\}$ are defined by

$$T_0(x)=1,\quad T_1(x)=x,\quad T_n(x)=2xT_{n-1}(x)-T_{n-2}(x),\quad n\geq 2,$$

and the second kind Chebyshev polynomials $\{U_n(x)\}$ are defined by

$$U_0(x)=1,\quad U_1(x)=2x,\quad U_n(x)=2xU_{n-1}(x)-U_{n-2}(x),\quad n\geq 2.$$

The generating function $F(t,x)$ of the polynomials $\{T_n(x)\}$ is given by

$$F(t,x)=\sum_{n=0}^{\infty}T_n(x)t^n=\frac{1-xt}{1-2xt+t^2},\quad |x|<1,\quad |t|<1, \tag{2.1}$$

and the generating function $G(t,x)$ of the polynomials $\{U_n(x)\}$ is given by

$$G(t,x)=\sum_{n=0}^{\infty}U_n(x)t^n=\frac{1}{1-2xt+t^2},\quad |x|<1,\quad |t|<1. \tag{2.2}$$

Their general expressions are

$$T_n(x) = \frac{n}{2} \sum_{h=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^h (n-h-1)!}{h!\,(n-2h)!} (2x)^{n-2h}, \quad |x| < 1, \tag{2.3}$$

and

$$U_n(x) = \sum_{h=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^h (n-h)!}{h!\,(n-2h)!} (2x)^{n-2h}, \quad |x| < 1. \tag{2.4}$$

Lemma 1. [11] Let $\{f_n(x)\}$ be bi-periodic Fibonacci polynomials, $\{l_n(x)\}$ be bi-periodic Lucas polynomials, $\{T_n(x)\}$ be Chebyshev polynomials of the first kind and $\{U_n(x)\}$ be Chebyshev polynomials of the second kind. For any positive integer n, we have the following identities:

$$f_{n+1}(x) = \left(\frac{a}{b}\right)^{\frac{\xi(n)}{2}} i^n U_n\left(\frac{\sqrt{ab}}{2i} x\right), \tag{2.5}$$

and

$$l_n(x) = 2\left(\frac{a}{b}\right)^{\frac{\xi(n)}{2}} i^n T_n\left(\frac{\sqrt{ab}}{2i} x\right), \tag{2.6}$$

where i is the square root of -1, and $\xi(n) = n - \lfloor \frac{n}{2} \rfloor$ is the parity function.

Lemma 2. Let $\{U_n(x)\}$ be Chebyshev polynomials of the second kind, and for any positive integer k and nonnegative integer n, we have the identity:

$$\sum_{a_1+a_2+\cdots+a_{k+1}=n} U_{a_1}(x) \cdot U_{a_2}(x) \cdots U_{a_{k+1}}(x) = \frac{1}{k!} \sum_{h=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^h \frac{(n+k-h)!}{h!\,(n-2h)!} (2x)^{n-2h}.$$

Proof. Noting that the degree of $U_n(x)$ has degree n and taking the partial derivative $\left(\frac{\partial}{\partial x}\right)^k$ on both sides of (2.2), we have

$$\frac{\partial^k G(t,x)}{\partial x^k} = \frac{(2t)^k \cdot k!}{(1-2xt+t^2)^{k+1}} = \sum_{n=0}^{\infty} U_{n+k}^{(k)}(x)\, t^{n+k},$$

where $U^{(k)}(x)$ denotes the k-order derivative of $U_n(x)$ for x. Then, we obtain that

$$\begin{aligned} &\sum_{n=0}^{\infty} \left(\sum_{a_1+a_2+\cdots+a_{k+1}=n} U_{a_1}(x) \cdot U_{a_2}(x) \cdots U_{a_{k+1}}(x) \right) t^n = \left(\sum_{n=0}^{\infty} U_n(x)\, t^n \right)^{k+1} \\ &= \frac{1}{(1-2xt+t^2)^{k+1}} = \frac{1}{(2t)^k \cdot k!} \frac{\partial^k G(t,x)}{\partial x^k} = \frac{1}{2^k \cdot k!} \sum_{n=0}^{\infty} U_{n+k}^{(k)}(x)\, t^n. \end{aligned} \tag{2.7}$$

Comparing the coefficients of t^n on both sides of Eq (2.7), we obtain that

$$\sum_{a_1+a_2+\cdots+a_{k+1}=n} U_{a_1}(x) \cdot U_{a_2}(x) \cdots U_{a_{k+1}}(x) = \frac{1}{2^k \cdot k!} U_{n+k}^{(k)}(x). \tag{2.8}$$

From (2.4), we can deduce the k^{th} derivative of $U_{n+k}(x)$,

$$U_{n+k}^{(k)}(x) = \sum_{h=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^h \binom{n+k-h}{h} (n+k-2h)_k \, 2^{n+k-2h} x^{n-2h}, \tag{2.9}$$

where the falling factorial polynomials $(x)_n$ are given by

$$(x)_0 = 1, \quad (x)_n = x(x-1)\cdots(x-n+1), \quad n \geq 1. \tag{2.10}$$

Then, combining (2.8) and (2.9), we obtain

$$\sum_{a_1+a_2+\cdots+a_{k+1}=n} U_{a_1}(x) \cdot U_{a_2}(x) \cdots U_{a_{k+1}}(x) = \frac{1}{k!} \sum_{h=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^h \frac{(n+k-h)!}{h!\,(n-2i)!} (2x)^{n-2h}.$$

This completes the proof of the Lemma. □

Lemma 3. Let $\{T_n(x)\}$ be Chebyshev polynomials of the first kind. For any positive integer r and nonnegative integer m, we have the identity

$$\sum_{a_1+a_2+\cdots+a_{r+1}=m} T_{a_1}(x) \cdot T_{a_2}(x) \cdots T_{a_{r+1}}(x)$$
$$= \frac{(1-xt)^{r+1}(m+r)}{(1-t^2) \cdot r!} \sum_{h=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^h \frac{(m+r-h-1)!}{h!\,(m-2h)!} (2x)^{m-2h}.$$

Proof. Noting that the degree of $T_m(x)$ has degree m and taking the partial derivative $\left(\frac{\partial}{\partial x}\right)^r$ on both sides of (2.1), we have

$$\frac{\partial^r F(t,x)}{\partial x^r} = \frac{\left(t-t^3\right)(2t)^{r-1} \cdot r!}{\left(1-2xt+t^2\right)^{r+1}} = \sum_{m=0}^{\infty} T_{m+r}^{(r)}(x)\, t^{m+r},$$

where $T^{(r)}(x)$ denotes the r-order derivative of $T_m(x)$ for x. Then, we obtain that

$$\begin{aligned}
&\sum_{m=0}^{\infty} \left(\sum_{a_1+a_2+\cdots+a_{r+1}=m} T_{a_1}(x) \cdot T_{a_2}(x) \cdots T_{a_{r+1}}(x) \right) t^m \\
&= \left(\sum_{m=0}^{\infty} T_m(x)\, t^m \right)^{r+1} = \frac{(1-xt)^{r+1}}{\left(1-2xt+t^2\right)^{r+1}} \\
&= \frac{(1-xt)^{r+1}}{(t-t^3)(2t)^{r-1} \cdot r!} \cdot \frac{\partial^r F(t,x)}{\partial x^r} = \frac{(1-xt)^{r+1}}{(1-t^2)\, 2^{r-1} \cdot r!} \sum_{m=0}^{\infty} T_{m+r}^{(r)}(x)\, t^m.
\end{aligned} \tag{2.11}$$

Comparing the coefficients of t^m on both sides of equation (2.11), we obtain that

$$\sum_{a_1+a_1+\cdots+a_{r+1}=m} T_{a_1}(x) \cdot T_{a_2}(x) \cdots T_{a_{r+1}}(x) = \frac{(1-xt)^{r+1}}{(1-t^2)\, 2^{r-1} \cdot r!} T_{m+r}^{(r)}(x). \tag{2.12}$$

From (2.3), we can deduce the r^{th} derivative of $T_{m+r}(x)$,

$$T_{m+r}^{(r)}(x) = \frac{m+r}{2} \sum_{h=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^h \frac{1}{m+r-h} \binom{m+r-h}{h} 2^{m+r-2h} (m+r-2h)_r \, x^{m-2h}. \tag{2.13}$$

Then, combining (2.12) and (2.13), where the condition $(x)_n$ is defined by (2.10), we obtain

$$\sum_{a_1+a_2+\cdots+a_{r+1}=m} T_{a_1}(x) \cdot T_{a_2}(x) \cdots T_{a_{r+1}}(x)$$
$$= \frac{(1-xt)^{r+1}(m+r)}{(1-t^2)\, r!} \sum_{h=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^h \frac{(m+r-h-1)!}{h!\,(m-2h)!} (2x)^{m-2h}.$$

This completes the proof of the Lemma. □

3. Proof of the theorems

Proof of Theorem 1. By Lemma 2, we obtain that

$$\sum_{a_1+a_2+\cdots+a_{k+1}=n} U_{a_1}\left(\frac{\sqrt{ab}}{2i}x\right) \cdot U_{a_2}\left(\frac{\sqrt{ab}}{2i}x\right) \cdots U_{a_{k+1}}\left(\frac{\sqrt{ab}}{2i}x\right)$$
$$= \frac{1}{k!} \sum_{h=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^h \frac{(n+k-h)!}{h!\,(n-2h)!} \left(\frac{\sqrt{ab}}{i}x\right)^{n-2h}.$$

By (2.5) of Lemma 1, we obtain that

$$\sum_{a_1+a_2+\cdots+a_{k+1}=n} \frac{f_{a_1+1}(x) \cdot f_{a_2+1}(x) \cdots f_{a_{k+1}+1}(x)}{\left(\frac{a}{b}\right)^{\frac{\xi(a_1)+\xi(a_2)+\cdots+\xi(a_{k+1})}{2}} \cdot i^{a_1+a_2+\cdots+a_{k+1}}}$$
$$= \frac{1}{k!} \sum_{h=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^h \frac{(n+k-h)!}{h!\,(n-2h)!} \left(\frac{\sqrt{ab}}{i}x\right)^{n-2h}.$$

Therefore, we obtain

$$\sum_{a_1+a_2+\cdots+a_{k+1}=n} \left(\frac{b}{a}\right)^{\frac{\xi(a_1)+\xi(a_2)+\cdots+\xi(a_{k+1})}{2}} f_{a_1+1}(x) \cdot f_{a_2+1}(x) \cdots f_{a_{k+1}+1}(x)$$
$$= \frac{1}{k!} \sum_{h=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^h \frac{(n+k-h)! \cdot i^{2h}}{h!\,(n-2h)!} \left(\sqrt{ab}x\right)^{n-2h}.$$

This completes the proof of the Theorem. □

Proof of Theorem 2. By Lemma 3, we obtain that

$$\sum_{a_1+a_2+\cdots+a_{r+1}=m} T_{a_1}\left(\frac{\sqrt{ab}}{2i}x\right) \cdot T_{a_2}\left(\frac{\sqrt{ab}}{2i}x\right) \cdots T_{a_{r+1}}\left(\frac{\sqrt{ab}}{2i}x\right)$$
$$= \frac{(1-xt)^{r+1}(m+r)}{(1-t^2)\, r!} \sum_{h=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^h \frac{(m+r-h-1)!}{h!\,(m-2h)!} \left(\frac{\sqrt{ab}}{i}x\right)^{m-2h}.$$

By (2.6) of Lemma 1, we obtain that

$$\sum_{a_1+a_2+\cdots+a_{k+1}=m} \frac{l_{a_1}(x)\cdot l_{a_2}(x)\cdots l_{a_{r+1}}(x)}{2^{r+1}\left(\frac{a}{b}\right)^{\frac{\xi(a_1)+\xi(a_2)+\cdots+\xi(a_{r+1})}{2}}\cdot i^{a_1+a_2+\cdots+a_{r+1}}}$$
$$=\frac{(1-xt)^{r+1}(m+r)}{(1-t^2)\cdot r!}\sum_{h=0}^{\left\lfloor\frac{m}{2}\right\rfloor}(-1)^h\frac{(m+r-h-1)!}{h!\,(m-2h)!}\left(\frac{\sqrt{ab}}{i}x\right)^{m-2h}.$$

Therefore, we obtain

$$\sum_{a_1+a_2+\cdots+a_{r+1}=m}\left(\frac{b}{a}\right)^{\frac{\xi(a_1)+\xi(a_2)+\cdots+\xi(a_{r+1})}{2}} l_{a_1}(x)\cdot l_{a_2}(x)\cdots l_{a_{r+1}}(x)$$
$$=\frac{2^{r+1}(1-xt)^{r+1}(m+r)}{(1-t^2)\cdot r!}\sum_{h=0}^{\left\lfloor\frac{m}{2}\right\rfloor}(-1)^h\frac{(m+r-h-1)!i^{2h}}{h!\,(m-2h)!}\left(\sqrt{ab}x\right)^{m-2h}.$$

This completes the proof of the Theorem. □

4. Conclusions

In this paper, by using generating functions for the Chebyshev polynomials, we have obtained the convolution formulas involving the bi-periodic Fibonacci and Lucas polynomials. In the past, scholars considered the convolution of linear recursive polynomials. In this paper, we extend the previous research to non-linear recursive polynomials. Specifically, we consider the convolution formula of bi-periodic recursive polynomials. Furthermore, we hope to consider extending the convolution formula of t-periodic recursive polynomials.

Acknowledgments

The authors express their gratitude to the referee for very helpful and detailed comments. Supported by the National Natural Science Foundation of China (Grant No. 11701448)

Conflicts of interest

All authors declare no conflicts of interest in this paper.

References

1. V. E. Hoggatt, M. Bicknell, Roots of Fibonacci polynomials, *Fibonacci Q.*, **11** (1973), 271–274.
2. Z. Wu, W. Zhang, The sums of the reciprocals of Fibonacci polynomials and Lucas polynomials, *J. Inequal. Appl.*, **2012** (2012), 134. https://doi.org/10.1186/1029-242X-2012-134
3. U. Dutta, P. Ray, On the finite reciprocal sums of Fibonacci and Lucas polynomials, *AIMS Math.*, **4** (2019), 1569–1581. https://doi.org/10.3934/math.2019.6.1569

4. T. Du, Z. Wu, On the reciprocal products of generalized Fibonacci sequences, *J. Inequal. Appl.*, **2022** (2022), 154. https://doi.org/10.1186/s13660-022-02889-8
5. P. Relhan, V. Verma, On the sum of reciprocals of Jacobsthal polynomials, *J. Phys. Conf. Ser.*, **1531** (2020), 012070. https://doi.org/10.1088/1742-6596/1531/1/012070
6. W. M. Abd-Elhameed, A. N. Philippou, N. A. Zeyada, Novel results for two generalized classes of Fibonacci and Lucas polynomials and their uses in the reduction of some radicals, *Mathematics*, **10** (2022), 2342. https://doi.org/10.3390/math10132342
7. W. M. Abd-Elhameed, N. A. Zeyada, New identities involving generalized Fibonacci and generalized Lucas numbers, *Indian J. Pure. Appl. Math.*, **49** (2018), 527–537. https://doi.org/10.1007/s13226-018-0282-7
8. W. M. Abd-Elhameed, Y. H. Youssri, N. El-Sissi, M. Sadek, New hypergeometric connection formulae between Fibonacci and Chebyshev polynomials, *Ramanujan J.*, **42** (2017), 347–361. https://doi.org/10.1007/s11139-015-9712-x
9. Y. Yuan, W. Zhang, Some identities involving the Fibonacci polynomials, *Fibonacci Quart.*, **40** (2002), 314–318.
10. W. Zhang, Some identities involving the Fibonacci numbers and Lucas numbers, *Fibonacci Quart.*, **42** (2004), 149–154.
11. N. Yilmaz, A. Coskun, N. Taskara, On properties of bi-periodic Fibonacci and Lucas polynomials, *AIP Conf. P.*, **1863** (2017), 310002. https://doi.org/10.1063/1.4992478
12. T. Komatsu, J. Ramírez, Convolutions of the bi-periodic Fibonacci numbers, *Hacet. J. Math. Stat.*, **49** (2020), 565–577. https://doi.org/10.15672/hujms.568340
13. T. Kim, D. Dolgy, D. Kim, J. Seo, Convolved fibonacci numbers and their applications, *Ars Comb.*, **135** (2016), 119–131. https://doi.org/10.48550/arXiv.1607.06380
14. Z. Chen, L. Qi, Some convolution formulae related to the second-order linear recurrence sequence, *Symmetry*, **11** (2019), 788. https://doi.org/10.3390/sym11060788
15. E. Kılıç, Tribonacci sequences with certain indices and their sums, *Ars Comb.*, **86** (2008), 13–22.
16. T. Agoh, K. Dilcher, Higher-order convolutions for Bernoulli and Euler polynomials, *J. Math. Anal. Appl.*, **419** (2014), 1235–1247. https://doi.org/10.1016/j.jmaa.2014.05.050
17. Y. He, T. Kim, A higher-order convolution for Bernoulli polynomials of the second kind, *Appl. Math. Comput.*, 324 (2018), 51–58. https://doi.org/10.1016/j.amc.2017.12.014
18. S. Falcon, A. Plaza, On k-Fibonacci numbers of arithmetic indexes, *Appl. Math. Comput.*, **208** (2009), 180–185 . https://doi.org/10.1016/j.amc.2008.11.031
19. T. Kim, D. Kim, D. Dolgy, J. Kwon, Representing sums of finite products of Chebyshev polynomials of the first kind and Lucas polynomials by Chebyshev polynomials, *Mathematics*, **7** (2019), 26. https://doi.org/10.3390/math7010026
20. W. M. Abd-Elhameed, N. A. Zeyada, New formulas including convolution, connection and radicals formulas of k-Fibonacci and k-Lucas polynomials, *Indian J. Pure. Appl. Math.*, **53** (2022), 1006–1016. https://doi.org/10.1007/s13226-021-00214-5

AIMS Press