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Research article

An improved reachable set estimation for time-delay linear systems with peak-bounded inputs and polytopic uncertainties via augmented zero equality approach

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Abstract: This paper proposes an improved estimation of the reachable set (RS) analysis in linear systems with polytopic uncertainties, peak-bounded inputs and time-varying delay. Inspired by past literature, Lyapunov-Krasovskii's functionals are dealt for treating the time-delay and bounding analysis effectively. So, the proposed method focuses on Lyapunov-Krasovskii's functionals via various time-delay conditions for linear systems. Based on the Lyapunov method, some integral inequalities, useful zero equalities, and the augmented zero equality approach are introduced. The results are expressed in terms of linear matrix inequalities, which are easy to get optimized solutions for obtaining guaranteed minimum RS of system dynamics. Finally, two numerical examples are shown to judge that the proposed estimation method can lead to less conservative results.

Keywords: reachable set; time delay systems; peak-input; linear systems; Lyapunov method
Mathematics Subject Classification: 34D20, 34K20, 34K25

1. Introduction

Dynamical systems have been recognized as one of the most important studies in the engineering field. Developments of the industrial machine, which has various dynamics from simple mechanical equipment to complex algorithm robots, are good examples for showing an importance of dynamical systems [1, 2]. Nowadays, as the need for heavy industrial equipment increases, the system engineers are interested in designing systems that have peak value inputs [3]. As discussed in [4,5], the reachable set estimation (RSE) has been selected to solve the minimization problems of peak gain and disturbance problems. In other words, the result of studying RSE in dynamic systems shows the ellipsoid bounded set of system trajectories with peak value inputs.

There are two well-known unavoidable phenomena when designing or operating dynamic hardware systems. The first is the systematic uncertainty phenomenon and the other is a time delay [6, 7]. The occurrence of them can derive low performance and system fault [8]. So, the bounding analysis with Lyapunov-Krasovskii's Functionals (LKFs) has been selected to design feedback controllers and find the stable region of the system with various conditions [9, 10]. By constructing appropriate LKFs, past works [6, 11] could investigate the RSE with bounding analysis. Furthermore, time-delay systems can be investigated in RSE problems with the uncertainties and disturbances phenomena by the LKFs methods [4, 6, 12].

The RSE of time-delay systems with bounded peak inputs gives essential meanings to dynamic differential systems [8, 13]. With the aforementioned concerns, studies of RSE have been developed in various systems. For adopting the practical dynamic systems, discrete-time systems [14–16], sampled-data systems [17], switched systems [18] have been considered. Furthermore, fuzzy systems [19, 20] that can provide effective solutions for nonlinear systems, Markov jump systems [21] for controlling complex uncertainties, and neural networks [22–24], which applied the mentioned techniques, are received attention in RSE research. As the above kinds of literature pointed out, some mathematical techniques affect improved estimation. The expression of polytopic uncertainties was used frequently in [6, 7, 25]. Jensen's inequality [26] and Wirtinger-based integral inequality (WBII) [27], which are represented as integral inequalities, are used for treating single or multi integral terms in the derivative of LKFs. Many works showed that utilizing these inequalities can expand the feasible region of stability criteria in time-delay systems [28–30]. The reciprocally convex approach (RCA) derives a tighter bound with an appropriate matrix [31–33]. It is well-known that expanding augmented vectors in integral terms of the Lyapunov function leads to less conservative results [34]. However, existing RSE methods are limited to expanding augmented vectors in integral terms of Lyapunov functions by containing $e^{\alpha(s-t)}$. In order to compensate for this limitation, an integral term of the Lyapunov function candidate without $e^{\alpha(s-t)}$ can be considered for RSE [35]. In [35], the absence of the $e^{\alpha(s-t)}$ method focused on calculating time-varying delay. This paper concentrates on the expanding augmented vectors for improved RSE with the mentioned method.

To get more optimized reachable bounding set, this paper introduces augmented integral terms of LKFs. And an advanced generalized integral inequality [36] for a derivative term of double integral Lyapunov function is utilized. And the convex parameters are treated by RCA [37]. Finally, inspired by [38,39], the appropriate augmented vectors and free-weighting matrices are used for the Augmented zero equality approach (AZEA) [40, 41]. With the mentioned approaches, the reachable bounding set for linear time-delay systems with polytopic uncertainties and bounded peak inputs is proposed. For the proving proposed suggestions, next section introduces some inequalities and mathematical facts. In Section 3, Theorems and Corollary are organized by the conditions of time-varying delay $\eta(t)$ and its time-derivative $\dot{\eta}(t)$. Section 4 introduces our investigation results and compares them with the past studies. Finally, two numerical examples are included to prove the superiority of proposed approaches with listed ellipsoidal bounds sizes and state trajectories.

Notation. $\mathbf{R}^n$ is the n-dimensional Euclidean space, $\mathbf{R}^{m\times n}$ denotes the set of $m\times n$ real matrix. $\mathcal{M} \in \mathbf{S}^n_+$ denotes the sets of positive definite $n \times n$ matrix. $\mathcal{N} \in \mathbf{S}^n$ denotes the sets of symmetric $n \times n$ matrix. $\mathcal{P} > 0$ means that the matrix $\mathcal{P}$ is a real symmetric positive definite matrix. I and I_n denote the identity matrix with appropriate dimension and $n \times n$ identity matrix, respectively. 0_m and $0_{m\times k}$ denote the $m \times m$ and $m \times k$ sizes zero matrices, respectively. $\|\cdot\|$ refers to the Euclidean vector norm and the induced matrix norm. $diag\{\cdots\}$ denotes the block diagonal matrix. The symmetric elements will be denoted by $*$. For a given matrix $X \in \mathbf{R}^{m\times n}$, such that $rank(X) = r$, we define $X^{\perp} \in \mathbf{R}^{n\times(n-r)}$ as the right orthogonal complement of X; i.e., $XX^{\perp} = 0$. $X_{[\alpha,\ \beta(t)]}$ represents the value of function X is dependent on the scalar α and scalar function $\beta(t)$. $Sym\{X\}$ denotes $X + X^T$. $col\{\cdots\}$ is the column matrices.

2. Problem formation

Consider the following bounded peaked input linear systems with time-varying delay [7, 42, 43]

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A(t))\, x(t) + (D + \Delta D(t))\, x(t - \eta(t)) + (B + \Delta B(t))\, w(t), \quad \forall\ t > 0, \\ x(s) &\equiv 0, \quad \forall s \in [-\eta,\ 0], \quad \eta > 0, \end{aligned} \tag{2.1}$$

where $x(t) \in \mathbf{R}^n$ is the state vector. $A,\ \Delta A(t) \in \mathbf{R}^{n\times n}$, $D,\ \Delta D(t) \in \mathbf{R}^{n\times n}$ and $B,\ \Delta B(t) \in \mathbf{R}^{n\times m}$. $A,\ D,\ B$ are known constant matrices. $\Delta A(t),\ \Delta D(t),\ \Delta B(t)$ are uncertain matrices which belong to a given polytope by a linear convex-hull of matrices $A_{u,k},\ D_{u,k}$ and $B_{u,k}$

$$\Lambda(t) = \sum_{k=1}^{N} \lambda_k(t)\Lambda_k \tag{2.2}$$

where $\lambda_k(t) \in [0, 1]$, $\sum_{k=1}^{N} \lambda_k(t) = 1$, $\Lambda(t) = [\Delta A(t),\ \Delta D(t),\ \Delta B(t)]$. And $\Lambda_k = [\mathcal{A}_{u,k},\ \mathcal{D}_{u,k},\ \mathcal{B}_{u,k}]$ $(k = 1,\ \ldots,\ N)$ are the vertices. $\mathcal{A}_{u,k},\ \mathcal{D}_{u,k},\ \mathcal{B}_{u,k}$ are known constant matrices with appropriate dimensions. $w(t) \in \mathbf{R}^m$ is the input in the form

$$w^T(t)w(t) \le w_m^2. \tag{2.3}$$

The time-varying delay $\eta(t)$ is considered in this paper. It is assumed that the time-varying delays have the following conditions as:

Case 1.

$$0 \le \eta(t) \le \eta, \quad -\infty \le \dot{\eta}(t) \le \mu, \quad \forall t > 0, \tag{2.4}$$

Case 2.

$$0 \le \eta_m \le \eta(t) \le \eta, \quad -\mu \le \dot{\eta}(t) \le \mu, \quad \forall t > 0. \tag{2.5}$$

A reachable set that bounds the state of systems (2.1) is defined by $\mathcal{R}_x \triangleq \{x(t) \in \mathbf{R}^n : x(t)$ and $w(t)$ satisfy Eqs (2.1) and (2.3)$\}$. As Boyd and Wang pointed out [11, 12], this RSE problem has the same meaning as the problem of finding an ellipsoid bound to the $\mathcal{R}_x$. The ellipsoid can be defined by

$$\mathcal{R}_\epsilon \triangleq \{x \in \mathbf{R}^n,\ x^T\mathcal{P}x \le 1\}, \tag{2.6}$$

where matrix $\mathcal{P}$ has constant value and positive definite. Thus, instead of solving the optimized RSE problem, our work can focus on finding a solution for the smallest ellipsoid bound problem.

The main goal of this paper is to calculate the optimized RS bounds of the system (2.1) with $w(t)$. With the following adopted lemmas, the next section derives the construction of suitable LKFs for system (2.1) and sufficient conditions.

Lemma 1. *[11] Let $V(x(t))$ be a LKF for system (2.1) with $V(x(0)) = 0$ and $w^T(t)w(t) \leq w_m^2$. If*

$$\dot{V}(x(t)) + \alpha V(x(t)) - \frac{\alpha}{w_m^2} w^T(t)w(t) \leq 0, \ \alpha > 0, \tag{2.7}$$

then, $V(x(t)) \leq 1, \ \forall \ t \geq 0$.

Lemma 2. *[36]* $((l = 0, \ 2), \ k = 0)$ *For a matrix $\mathcal{M} > 0$, the following inequality is satisfied:*

$$\Phi_0(\alpha) \ \ \geq \ \ \sum_{i=0}^{l} (2i+1)^T \ \Upsilon_i^T(\alpha)\mathcal{M}\Upsilon_i(\alpha), \tag{2.8}$$

where $\Phi_0(\alpha) = (b-a)\int_a^b \alpha^T(s)\mathcal{M}\alpha(s)ds$, and the $\Upsilon_0(\alpha) = \int_a^b \alpha(s)ds$, $\Upsilon_1(\alpha) = \frac{2}{b-a}\int_a^b\int_s^b \alpha(v)dvds - \int_a^b \alpha(s)ds$ and $\Upsilon_2(\alpha) = \frac{12}{(b-a)^2}\int_a^b\int_s^b\int_v^b \alpha(z)dzdvds - \frac{6}{b-a}\int_a^b\int_s^b \alpha(v)dvds + \int_a^b \alpha(s)ds$.

Lemma 3. *[31] Let $\mathcal{N}_1 \in \mathbf{R}^{n\times n}$, $\mathcal{N}_2 \in \mathbf{R}^{m\times m}$ be positive definite matrices, if there exists any matrix $\mathcal{X} \in \mathbf{R}^{n\times m}$ such that $\begin{bmatrix} \mathcal{N}_1 & \mathcal{X} \\ \mathcal{X}^T & \mathcal{N}_2 \end{bmatrix} \geq 0$, then the inequality*

$$\begin{bmatrix} \frac{1}{\alpha}\mathcal{N}_1 & 0 \\ 0 & \frac{1}{1-\alpha}\mathcal{N}_2 \end{bmatrix} \geq \begin{bmatrix} \mathcal{N}_1 & \mathcal{X} \\ \mathcal{X}^T & \mathcal{N}_2 \end{bmatrix} \tag{2.9}$$

holds for all $\alpha \in (0, \ 1)$.

Lemma 4. *[38] Let $S \subseteq \mathbf{R}^d$, $\zeta \in \mathbf{R}^n$, $\Phi = \Phi^T \in \mathbf{S}^n$, and $B \in \mathbf{R}^{m\times n}$. For each $s \in S$, the following statements are equivalent:*
(i) $\zeta^T\Phi(s)\zeta < 0, \ \forall B(s)\zeta = 0, \ \zeta \neq 0$,
(ii) $\exists\Psi(s) \in \mathbf{R}^{n\times m} : \Phi(s) + \Psi(s)B(s) + B^T(s)\Psi^T(s) < 0$,
(iii) $(B^{\perp}(s))^T\Phi(s)B^{\perp}(s) < 0$.

3. Main results

The RSE of the system (2.1) is proposed in this section. To avoid complicated expression, some notations of matrices are defined as

$$\xi(t) \ = \ \begin{bmatrix} \frac{1}{\eta^2(t)}\int_{t-\eta(t)}^{t}\int_s^t\int_u^t x(v)dvduds \\ \hline \frac{1}{(\eta-\eta(t))^2}\int_{t-\eta}^{t-\eta(t)}\int_s^{t-\eta(t)}\int_u^{t-\eta(t)} x(v)dvduds \\ \hline \int_{t-\eta(t)}^{t} x(s)ds \\ \hline \int_{t-\eta}^{t-\eta(t)} x(s)ds \end{bmatrix},$$

$$\zeta(t) \ = \ col\left\{\begin{bmatrix} x(t) \\ \hline x(t-\eta(t)) \\ \hline x(t-\eta) \\ \hline \dot{x}(t) \\ \hline \dot{x}(t-\eta) \end{bmatrix}, \begin{bmatrix} \frac{1}{\eta(t)}\int_{t-\eta(t)}^{t} x(s)ds \\ \hline \frac{1}{\eta-\eta(t)}\int_{t-\eta}^{t-\eta(t)} x(s)ds \\ \hline \frac{1}{\eta^2(t)}\int_{t-\eta(t)}^{t}\int_s^t x(u)duds \\ \hline \frac{1}{(\eta-\eta(t))^2}\int_{t-\eta}^{t-\eta(t)}\int_s^{t-\eta(t)} x(u)duds \\ \hline \frac{1}{\eta(t)}\int_{t-\eta(t)}^{t}\int_s^t x(u)duds \\ \hline \frac{1}{\eta-\eta(t)}\int_{t-\eta}^{t-\eta(t)}\int_s^{t-\eta(t)} x(u)duds \end{bmatrix}, \xi(t), \ w(t)\right\}, \ \mathfrak{Q} = \begin{bmatrix} \mathfrak{Q}_1 & 0_n \\ 0_n & \mathfrak{Q}_2 \end{bmatrix},$$

$$\begin{aligned}
\mathcal{Q}_k &= \mathfrak{Q} + \frac{1}{e^{-\alpha\eta}}\begin{bmatrix} -\alpha\mathcal{N} & \mathcal{P}_k \\ * & -\frac{\alpha}{\eta}(2-k)\mathcal{G} \end{bmatrix}(k=1,\ 2),\ \mathcal{Q}_{aug,k} = \begin{bmatrix} \mathcal{Q}_k & 0_{2n} & 0_{2n} \\ * & 3\mathcal{Q}_k & 0_{2n} \\ * & * & 5\mathcal{Q}_k \end{bmatrix}(k=1,\ 2),\\
\Omega &= \begin{bmatrix} \mathcal{Q}_{aug,1} & \mathcal{S} \\ * & \mathcal{Q}_{aug,2} \end{bmatrix},\ \mathcal{P}_{aug} = \begin{bmatrix} \mathcal{P}_1 & 0_n & 0_n \\ * & \mathcal{P}_2 - \mathcal{P}_1 & 0_n \\ * & * & -\mathcal{P}_2 \end{bmatrix},\ \tilde{\mathcal{R}}_1 = diag\{\mathcal{R}_1,\ 0_n,\ 0_n,\ 0_n\},\\
A_j &= A + \mathcal{A}_{u,j},\ D_j = D + \mathcal{D}_{u,j},\ B_j = B + \mathcal{B}_{u,j}\ (j = 1,\ \ldots,\ N),\\
e_k &= [0_{n\times(k-1)n},\ I_n,\ 0_{n\times((15-k)n+m)}]^T\ (k = 1,\ 2,\ \ldots,\ 15),\ e_{16} = [0_{m\times 15n},\ I_m]^T,\\
\Upsilon &= [e_1,\ e_2,\ \ldots,\ e_{16}],\\
\varepsilon_1 &= e_{14} + e_{15},\ \varepsilon_2 = \eta e_1 - e_{14} - e_{15},\\
\Gamma_{j,[\eta(t)]} &= col\{-e_4^T + A_j e_1^T + D_j e_2^T + B_j e_{16}^T,\ \eta(t)e_8^T - e_{10}^T,\ (\eta-\eta(t))e_9^T - e_{11}^T,\ \eta(t)e_6^T - e_{14}^T,\\
&\quad (\eta - \eta(t))e_7^T - e_{15}^T\}\ (j = 1,\ \ldots,\ N),\\
\Lambda_1 &= [e_1 - e_2,\ e_{14},\ e_1 + e_2 - 2e_6,\ e_{14} - 2e_{10},\ e_1 - e_2 + 6e_6 - 12e_8,\ e_{14} - 6e_{10} + 12e_{12}],\\
\Lambda_2 &= [e_2 - e_3,\ e_{15},\ e_2 + e_3 - 2e_7,\ e_{15} - 2e_{11},\ e_2 - e_3 + 6e_7 - 12e_9,\ e_{15} - 6e_{11} + 12e_{13}],\\
\tilde{\Xi}_1 &= Sym\left\{[e_1,\ e_3,\ \varepsilon_1,\ \varepsilon_2]\mathcal{R}[e_4,\ e_5,\ e_1 - e_3,\ \eta e_4 - (e_1 - e_3)]^T\right\},\\
\Xi_1 &= \tilde{\Xi}_1 + \alpha[e_1,\ e_3,\ \varepsilon_1,\ \varepsilon_2]\mathcal{R}[e_1,\ e_3,\ \varepsilon_1,\ \varepsilon_2]^T,\\
\Xi_2 &= e_1\mathcal{G}e_1^T - (1-\mu)e_2\mathcal{G}e_2^T + \eta e_4\mathcal{N}e_4^T - \eta e_5\mathcal{N}e_5^T,\ \tilde{\Xi}_3 = \eta^2[e_4,\ e_1]\mathfrak{Q}[e_4,\ e_1]^T,\\
\Xi_3 &= \tilde{\Xi}_3 - e^{-\alpha\eta}[\Lambda_1,\ \Lambda_2]\Omega[\Lambda_1,\ \Lambda_2]^T + \eta[e_1,\ e_2,\ e_3]\mathcal{P}_{aug}[e_1,\ e_2,\ e_3]^T,\\
\Xi &= \Xi_1 + \Xi_2 + \Xi_3 - \frac{\alpha}{w_m^2}e_{16}e_{16}^T.
\end{aligned} \tag{3.1}$$

Now, we have the following theorem.

Theorem 1. *For given scalars $\eta > 0$, $w_m^2 > 0$ and $\mu > 0$, the RSs of the system (2.1) with delay conditions of (2.4) are bounded by an ellipsoid $\mathcal{R}_\epsilon$ (2.6), if there exist matrices $\mathcal{R} \in \mathbf{S}_+^{4n}$, $\mathcal{R}_1$, $\mathcal{N}$, $\mathcal{G}$, $\mathfrak{Q}_k \in \mathbf{S}_+^n$ $(k = 1,\ 2)$, $\mathcal{P}_i$ $(i = 1,\ 2)$ $\in \mathbf{S}^n$, any matrix $\mathcal{S} \in \mathbf{R}^{6n\times 6n}$, and a scalar $\alpha > 0$, such that the following linear matrix inequalities(LMIs) hold:*

$$\begin{gathered}
\left(\Gamma_{j,[0]}^{\perp}\right)^T \Xi \left(\Gamma_{j,[0]}^{\perp}\right) \leq 0,\\
\left(\Gamma_{j,[\eta]}^{\perp}\right)^T \Xi \left(\Gamma_{j,[\eta]}^{\perp}\right) \leq 0\ (j = 1,\ \ldots,\ N),\\
\mathcal{Q}_1 \geq 0,\ \mathcal{Q}_2 \geq 0,\ \Omega \geq 0,\ \mathcal{R} \geq \tilde{\mathcal{R}}_1,
\end{gathered} \tag{3.2}$$

where Ξ and $\Gamma_{j,[\eta(t)]}$ are defined in (3.1), respectively.

Proof. For positive matrices $\mathcal{R}$, $\mathcal{R}_1$, $\mathcal{N}$, $\mathcal{G}$, $\mathfrak{Q}_k$ $(k = 1,\ 2)$, let us choose the following LKFs candidate as

$$V_T(t) = \sum_{k=1}^{3} V_{Tk}(t), \tag{3.3}$$

where

$$V_{T1}(t) = \mathfrak{B}_1^T(t)\mathcal{R}\mathfrak{B}_1(t),$$

$$
\begin{aligned}
V_{T2}(t) &= \eta\int_{t-\eta}^{t}\dot{x}^T(s)\mathcal{N}\dot{x}(s)ds+\int_{t-\eta(t)}^{t}x^T(s)\mathcal{G}x(s)ds,\\
V_{T3}(t) &= \eta\int_{t-\eta}^{t}\int_{s}^{t}e^{\alpha(u-t)}\dot{x}^T(u)\mathfrak{Q}_1\dot{x}(u)duds\\
&\quad+\eta\int_{t-\eta}^{t}\int_{s}^{t}e^{\alpha(u-t)}x^T(u)\mathfrak{Q}_2x(u)duds,
\end{aligned}
$$

where $\mathfrak{B}_1(t) = col\{x(t),\ x(t-\eta),\ \int_{t-\eta}^{t}x(s)ds,\ \int_{t-\eta}^{t}\int_{s}^{t}\dot{x}(u)duds\}$. The $\dot{V}_{T1}(t)$ and $\dot{V}_{T2}(t)$ can be calculated as

$$
\dot{V}_{T1}(t) = 2\mathfrak{B}_1^T(t)\mathcal{R}\dot{\mathfrak{B}}_1(t) = \zeta^T(t)\tilde{\Xi}_1\zeta(t)\underbrace{-\alpha V_{T1}(t)+\alpha V_{T1}(t)}_{=0} = \zeta^T(t)\Xi_1\zeta(t)-\alpha V_{T1}(t), \tag{3.4}
$$

$$
\begin{aligned}
\dot{V}_{T2}(t) &= \eta\dot{\mathfrak{B}}_2^T(t,\ t-\eta)\begin{bmatrix}\mathcal{N} & 0_n\\ * & -\mathcal{N}\end{bmatrix}\dot{\mathfrak{B}}_2(t,\ t-\eta)+\mathfrak{B}_2^T(t,\ t-\eta(t))\begin{bmatrix}\mathcal{G} & 0_n\\ * & -(1-\dot{\eta}(t))\mathcal{G}\end{bmatrix}\\
&\quad\times\mathfrak{B}_2(t,\ t-\eta(t)) \le \zeta^T(t)\Xi_2\zeta(t),
\end{aligned} \tag{3.5}
$$

where $\mathfrak{B}_2(t,\ s) = col\{x(t),\ x(s)\}$.

Then, the $\dot{V}_{T3}(t)$ can be expressed as

$$
\begin{aligned}
\dot{V}_{T3}(t) &= \eta\int_{t-\eta}^{t}e^{\alpha(t-t)}\mathfrak{B}_3^T(t)\mathfrak{Q}\mathfrak{B}_3(t)ds-\eta\int_{t-\eta}^{t}e^{\alpha(s-t)}\mathfrak{B}_3^T(s)\mathfrak{Q}\mathfrak{B}_3(s)ds\\
&\quad+\eta\int_{t-\eta}^{t}\int_{s}^{t}e^{\alpha(s-t)}\frac{d}{dt}\left(\mathfrak{B}_3^T(u)\mathfrak{Q}\mathfrak{B}_3(u)\right)duds-\alpha V_{T3}(t)\\
&= \underbrace{\eta^2\mathfrak{B}_3^T(t)\mathfrak{Q}\mathfrak{B}_3(t)}_{\zeta^T(t)\tilde{\Xi}_3\zeta(t)}-\eta\int_{t-\eta}^{t}e^{\alpha(s-t)}\mathfrak{B}_3^T(s)\mathfrak{Q}\mathfrak{B}_3(s)ds-\alpha V_{T3}(t)\underbrace{-\alpha V_{T2}(t)+\alpha V_{T2}(t)}_{=0}\\
&= \zeta^T(t)\tilde{\Xi}_3\zeta(t)-\eta\int_{t-\eta(t)}^{t}e^{\alpha(s-t)}\mathfrak{B}_3^T(s)\mathfrak{Q}\mathfrak{B}_3(s)ds+\alpha V_{T21}(t)+\alpha V_{T23}(t)\\
&\quad-\eta\int_{t-\eta}^{t-\eta(t)}e^{\alpha(s-t)}\mathfrak{B}_3^T(s)\mathfrak{Q}\mathfrak{B}_3^T(s)ds+\alpha V_{T22}(t)-\alpha V_{T2}(t)-\alpha V_{T3}(t),
\end{aligned} \tag{3.6}
$$

where $V_{T2}(t) = \underbrace{\eta\int_{t-\eta(t)}^{t}\dot{x}^T(s)\mathcal{N}\dot{x}(s)ds}_{V_{T21}(t)}+\underbrace{\eta\int_{t-\eta}^{t-\eta(t)}\dot{x}^T(s)\mathcal{N}\dot{x}(s)ds}_{V_{T22}(t)}+\underbrace{\int_{t-\eta(t)}^{t}x^T(s)\mathcal{G}x(s)ds}_{V_{T23}(t)}$, and $\mathfrak{B}_3(u) = col\{\dot{x}(u),\ x(u)\}$.

Inspired by [44], the zero equations with matrices $\mathcal{P}_i = \mathcal{P}_i^T$ $(i = 1,\ 2)$ are considered as

$$
0 = \eta\mathfrak{B}_2^T(t,t-\eta(t))\begin{bmatrix}\mathcal{P}_1 & 0_n\\ 0_n & -\mathcal{P}_1\end{bmatrix}\mathfrak{B}_2(t,\ t-\eta(t))-\eta\int_{t-\eta(t)}^{t}\mathfrak{B}_3^T(s)\begin{bmatrix}0_n & \mathcal{P}_1\\ \mathcal{P}_1^T & 0_n\end{bmatrix}\mathfrak{B}_3(s)ds, \tag{3.7}
$$

$$
0 = \eta\mathfrak{B}_2^T(t-\eta(t),t-\eta)\begin{bmatrix}\mathcal{P}_2 & 0_n\\ 0_n & -\mathcal{P}_2\end{bmatrix}\mathfrak{B}_2(t-\eta(t),\ t-\eta)-\eta\int_{t-\eta}^{t-\eta(t)}\mathfrak{B}_3^T(s)\begin{bmatrix}0_n & \mathcal{P}_2\\ \mathcal{P}_2^T & 0_n\end{bmatrix}\mathfrak{B}_3(s)ds. \tag{3.8}
$$

An upper bound of the first integral term of (3.6) with $\alpha V_{T21}(t)$, $\alpha V_{T23}(t)$ and Eq (3.7) can be obtained as

$$\begin{aligned}
&-\eta\int_{t-\eta(t)}^{t} e^{\alpha(s-t)}\mathfrak{B}_3^T(s)\mathfrak{Q}\mathfrak{B}_3(s)ds+\alpha V_{T21}(t)+\alpha V_{T23}(t)\\
=\;&-\eta\int_{t-\eta(t)}^{t} e^{\alpha(s-t)}\mathfrak{B}_3^T(s)\mathfrak{Q}\mathfrak{B}_3(s)ds-\eta\int_{t-\eta(t)}^{t}\mathfrak{B}_3^T(s)\begin{bmatrix} 0_n & \mathcal{P}_1\\ \mathcal{P}_1^T & 0_n\end{bmatrix}\mathfrak{B}_3(s)ds\\
&+\eta\mathfrak{B}_2^T(t,\ t-\eta(t))\begin{bmatrix} \mathcal{P}_1 & 0_n\\ 0_n & -\mathcal{P}_1\end{bmatrix}\mathfrak{B}_2(t,\ t-\eta(t))+\alpha V_{T21}(t)+\alpha V_{T23}(t)\\
\leq\;&-\eta e^{-\alpha\eta}\int_{t-\eta(t)}^{t}\mathfrak{B}_3^T(s)\mathfrak{Q}\mathfrak{B}_3(s)ds-\eta e^{-\alpha\eta}\int_{t-\eta(t)}^{t}\mathfrak{B}_3^T(s)\left(\frac{1}{e^{-\alpha\eta}}\begin{bmatrix} -\alpha\mathcal{N} & \mathcal{P}_1\\ \mathcal{P}_1^T & -\frac{\alpha}{\eta}\mathcal{G}\end{bmatrix}\right)\mathfrak{B}_3(s)ds\\
&+\eta\mathfrak{B}_2^T(t,\ t-\eta(t))\begin{bmatrix} \mathcal{P}_1 & 0_n\\ 0_n & -\mathcal{P}_1\end{bmatrix}\mathfrak{B}_2(t,\ t-\eta(t)).
\end{aligned}\tag{3.9}$$

With the Lemma 2 ($l=2$) and an LMI condition, integral terms of (3.9) can be bounded as

$$\begin{aligned}
&-\eta e^{-\alpha\eta}\int_{t-\eta(t)}^{t}\mathfrak{B}_3^T(s)\mathfrak{Q}\mathfrak{B}_3(s)ds-\eta e^{-\alpha\eta}\int_{t-\eta(t)}^{t}\mathfrak{B}_3^T(s)\left(\frac{1}{e^{-\alpha\eta}}\begin{bmatrix} -\alpha\mathcal{N} & \mathcal{P}_1\\ \mathcal{P}_1^T & -\frac{\alpha}{\eta}\mathcal{G}\end{bmatrix}\right)\mathfrak{B}_3(s)ds\\
&+\eta\mathfrak{B}_2^T(t,\ t-\eta(t))\begin{bmatrix} \mathcal{P}_1 & 0_n\\ 0_n & -\mathcal{P}_1\end{bmatrix}\mathfrak{B}_2(t,\ t-\eta(t))\\
\leq\;&-\frac{\eta e^{-\alpha\eta}}{\eta(t)}\left(\Lambda_{1,1}^T(t)\left(\underbrace{\mathfrak{Q}+\frac{1}{e^{-\alpha\eta}}\begin{bmatrix} -\alpha\mathcal{N} & \mathcal{P}_1\\ \mathcal{P}_1^T & -\frac{\alpha}{\eta}\mathcal{G}\end{bmatrix}}_{Q_1}\right)\Lambda_{1,1}(t)+3\Lambda_{1,2}^T(t)Q_1\Lambda_{1,2}(t)+5\Lambda_{1,3}^T(t)Q_1\Lambda_{1,3}(t)\right)\\
&+\eta\mathfrak{B}_2^T(t,\ t-\eta(t))\begin{bmatrix} \mathcal{P}_1 & 0_n\\ 0_n & -\mathcal{P}_1\end{bmatrix}\mathfrak{B}_2(t,\ t-\eta(t)),
\end{aligned}\tag{3.10}$$

where $\Lambda_{1,1}(t)=\int_{t-\eta(t)}^{t}\mathfrak{B}_3(s)ds$, $\Lambda_{1,2}(t)=\int_{t-\eta(t)}^{t}\mathfrak{B}_3(s)ds-\frac{2}{\eta(t)}\int_{t-\eta(t)}^{t}\int_{s}^{t}\mathfrak{B}_3(u)duds$ and $\Lambda_{1,3}(t)=\int_{t-\eta(t)}^{t}\mathfrak{B}_3(s)ds-\frac{6}{\eta(t)}\int_{t-\eta(t)}^{t}\int_{s}^{t}\mathfrak{B}_3(u)duds+\frac{12}{\eta^2(t)}\int_{t-\eta(t)}^{t}\int_{s}^{t}\int_{u}^{t}\mathfrak{B}_3(v)dvduds$.

Likewise, the another integral term of (3.6) with $\alpha V_{T22}(t)$ and Eq (3.8) are bounded as

$$\begin{aligned}
&-\eta\int_{t-\eta}^{t-\eta(t)} e^{\alpha(s-t)}\mathfrak{B}_3^T(s)\mathfrak{Q}\mathfrak{B}_3(s)ds+\alpha V_{T22}(t)\\
\leq\;&-\eta e^{-\alpha\eta}\int_{t-\eta}^{t-\eta(t)}\mathfrak{B}_3^T(s)\mathfrak{Q}\mathfrak{B}_3(s)ds-\eta e^{-\alpha\eta}\int_{t-\eta}^{t-\eta(t)}\mathfrak{B}_3^T(s)\left(\frac{1}{e^{-\alpha\eta}}\begin{bmatrix} -\alpha\mathcal{N} & \mathcal{P}_2\\ \mathcal{P}_2^T & 0_n\end{bmatrix}\right)\mathfrak{B}_3(s)ds\\
&+\eta\mathfrak{B}_2^T(t-\eta(t),\ t-\eta)\begin{bmatrix} \mathcal{P}_2 & 0_n\\ 0_n & -\mathcal{P}_2\end{bmatrix}\mathfrak{B}_2(t-\eta(t),\ t-\eta).
\end{aligned}\tag{3.11}$$

With the Lemma 2 ($l=2$) and an LMI condition, integral terms of (3.11) can be bounded as

$$-\eta e^{-\alpha\eta}\int_{t-\eta}^{t-\eta(t)}\mathfrak{B}_3^T(s)\mathfrak{Q}\mathfrak{B}_3(s)ds-\eta e^{-\alpha\eta}\int_{t-\eta}^{t-\eta(t)}\mathfrak{B}_3^T(s)\left(\frac{1}{e^{-\alpha\eta}}\begin{bmatrix} -\alpha\mathcal{N} & \mathcal{P}_2\\ \mathcal{P}_2^T & 0_n\end{bmatrix}\right)\mathfrak{B}_3(s)ds$$

$$+\eta\mathfrak{B}_2^T(t-\eta(t),\ t-\eta)\begin{bmatrix} \mathcal{P}_2 & 0_n \\ 0_n & -\mathcal{P}_2 \end{bmatrix}\mathfrak{B}_2(t-\eta(t),\ t-\eta)$$

$$\leq -\frac{\eta e^{-\alpha\eta}}{\eta-\eta(t)}\left(\Lambda_{2,1}^T(t)\left(\underbrace{\mathfrak{Q}+\frac{1}{e^{-\alpha\eta}}\begin{bmatrix} -\alpha\mathcal{N} & \mathcal{P}_2 \\ \mathcal{P}_2^T & 0_n \end{bmatrix}}_{Q_2}\right)\Lambda_{2,1}(t)+3\Lambda_{2,2}^T(t)Q_2\Lambda_{2,2}(t)+5\Lambda_{2,3}^T(t)Q_2\Lambda_{2,3}(t)\right)$$

$$+\eta\mathfrak{B}_2^T(t-\eta(t),\ t-\eta)\begin{bmatrix} \mathcal{P}_2 & 0_n \\ 0_n & -\mathcal{P}_2 \end{bmatrix}\mathfrak{B}_2(t-\eta(t),\ t-\eta), \tag{3.12}$$

where $\Lambda_{2,1}(t) = \int_{t-\eta}^{t-\eta(t)}\mathfrak{B}_3(s)ds$, $\Lambda_{2,2}(t) = \int_{t-\eta}^{t-\eta(t)}\mathfrak{B}_3(s)ds - \frac{2}{\eta-\eta(t)}\int_{t-\eta}^{t-\eta(t)}\int_{s}^{t-\eta(t)}\mathfrak{B}_3(u)duds$ and $\Lambda_{2,3}(t) = \int_{t-\eta}^{t-\eta(t)}\mathfrak{B}_3(s)ds - \frac{6}{\eta-\eta(t)}\int_{t-\eta}^{t-\eta(t)}\int_{s}^{t-\eta(t)}\mathfrak{B}_3(u)duds + \frac{12}{(\eta-\eta(t))^2}\int_{t-\eta}^{t-\eta(t)}\int_{s}^{t-\eta(t)}\int_{u}^{t-\eta(t)}\mathfrak{B}_3(v)dvduds$.

And then, by utilizing Lemma 3, the sum of (3.10) and (3.12) can be written and bounded as

$$-\frac{\eta e^{-\alpha\eta}}{\eta(t)}\underbrace{\begin{bmatrix} \Lambda_{1,1}(t) \\ \Lambda_{1,2}(t) \\ \Lambda_{1,3}(t) \end{bmatrix}^T}_{\Lambda_1^T(t)}\underbrace{\begin{bmatrix} Q_1 & 0_{2n} & 0_{2n} \\ * & 3Q_1 & 0_{2n} \\ * & * & 5Q_1 \end{bmatrix}}_{Q_{aug,1}}\Lambda_1(t) - \frac{\eta e^{-\alpha\eta}}{\eta-\eta(t)}\underbrace{\begin{bmatrix} \Lambda_{2,1}(t) \\ \Lambda_{2,2}(t) \\ \Lambda_{2,3}(t) \end{bmatrix}^T}_{\Lambda_2^T(t)}\underbrace{\begin{bmatrix} Q_2 & 0_{2n} & 0_{2n} \\ * & 3Q_2 & 0_{2n} \\ * & * & 5Q_2 \end{bmatrix}}_{Q_{aug,2}}\Lambda_2(t)$$

$$+\eta\mathfrak{B}_4^T(t)\underbrace{\begin{bmatrix} \mathcal{P}_1 & 0_n & 0_n \\ * & \mathcal{P}_2-\mathcal{P}_1 & 0_n \\ * & * & -\mathcal{P}_2 \end{bmatrix}}_{\mathcal{P}_{aug}}\mathfrak{B}_4(t)$$

$$\leq -e^{-\alpha\eta}\begin{bmatrix} \Lambda_1(t) \\ \Lambda_2(t) \end{bmatrix}^T\underbrace{\begin{bmatrix} Q_{aug,1} & \mathcal{S} \\ \mathcal{S}^T & Q_{aug,2} \end{bmatrix}}_{\Omega}\begin{bmatrix} \Lambda_1(t) \\ \Lambda_2(t) \end{bmatrix} + \eta\mathfrak{B}_4^T(t)\mathcal{P}_{aug}\mathfrak{B}_4(t). \tag{3.13}$$

From (3.6) to (3.13), an upper bound of $\dot{V}_{T3}(t)$ can be obtained as

$$\begin{aligned}\dot{V}_{T3}(t) &\leq \zeta^T(t)\tilde{\Xi}_3\zeta(t) - e^{-\alpha\eta}\begin{bmatrix} \Lambda_1(t) \\ \Lambda_2(t) \end{bmatrix}^T\Omega\begin{bmatrix} \Lambda_1(t) \\ \Lambda_2(t) \end{bmatrix} + \eta\mathfrak{B}_4^T(t)\mathcal{P}_{aug}\mathfrak{B}_4(t) \\ &\quad -\alpha V_{T2}(t) - \alpha V_{T3}(t) \\ &= \zeta^T(t)\Xi_3\zeta(t) - \alpha V_{T2}(t) - \alpha V_{T3}(t), \end{aligned}\tag{3.14}$$

where $\mathfrak{B}_4(t) = col\{x(t),\ x(t-\eta(t)),\ x(t-\eta)\}$ and other notations are defined in (3.1).

The system (2.1) with polytope expression (2.2) and some equations with vector $\zeta(t)$ can be considered as

$$\underbrace{\begin{bmatrix} -e_4^T + A_je_1^T + D_je_2^T + B_je_{16}^T \end{bmatrix}}_{\Gamma_{1,j}}\zeta(t) = 0\ (j=1,\ \ldots,N), \tag{3.15}$$

$$\underbrace{\begin{bmatrix} \eta(t)e_8^T - e_{10}^T \end{bmatrix}}_{\Gamma_{2,[\eta(t)]}}\zeta(t) = 0, \tag{3.16}$$

$$\underbrace{\left[\ (\eta-\eta(t))\,e_9^T - e_{11}^T\ \right]}_{\Gamma_{3,[\eta(t)]}}\zeta(t) = 0, \tag{3.17}$$

$$\underbrace{\left[\ \eta(t)e_6^T - e_{14}^T\ \right]}_{\Gamma_{4,[\eta(t)]}}\zeta(t) = 0, \tag{3.18}$$

$$\underbrace{\left[\ (\eta-\eta(t))\,e_7^T - e_{15}^T\ \right]}_{\Gamma_{5,[\eta(t)]}}\zeta(t) = 0, \tag{3.19}$$

where $\Gamma_{1,j}\zeta(t)$ $(j = 1,\ \dots,\ N)$, $\Gamma_{i,[\eta(t)]}\zeta(t)$ $(i = 2,\ \dots,\ 5)$ represent the zero equations.
By combining the (3.15)–(3.19) and the free-weighting matrices $\phi_{1,j}$ (j=1, ..., N), $\phi_{i,k}$ (i=2 ..., 5, k=1, 2) $\in R^{(15n+m)\times n}$, the zero equations can be obtained as

$$\zeta^T(t)\left(\Phi_{j,k}\Gamma_{j,[\eta(t)]} + \Gamma^T_{j,[\eta(t)]}\Phi^T_{j,k}\right)\zeta(t) = 0\ (j = 1,\ \dots,\ N,\ k = 1,\ 2), \tag{3.20}$$

with the relation $\Phi_{j,k} = \Upsilon[\phi_{1,j},\ \phi_{2,k},\ \dots,\ \phi_{5,k}]$, $\Gamma_{j,[\eta(t)]} = col\left\{\Gamma_{1,j},\ \Gamma_{2[\eta(t)]},\ \dots,\ \Gamma_{5[\eta(t)]}\right\}$.

Therefore, from (3.4)–(3.14) and (3.20), the $\dot{V}_T(t) + \alpha V_T(t) - \frac{\alpha}{w_m^2}w^T(t)w(t)$ can be bounded as

$$\begin{aligned} &\dot{V}_T(t) + \alpha V_T(t) - \frac{\alpha}{w_m^2}w^T(t)w(t) \\ \leq\ & \zeta^T(t)(\Xi_1 + \Xi_2 + \Xi_3 - \frac{\alpha}{w_m^2}e_{16}e_{16}^T + Sym\{\Phi_{j,k}\Gamma_{j,[\eta(t)]}\})\zeta(t). \end{aligned} \tag{3.21}$$

If the following inequality is negative definite, the chosen LKFs (3.3) can satisfy the condition $V_T(t) \leq\ 1$ by Lemma 1.

$$\Xi_1 + \Xi_2 + \Xi_3 - \frac{\alpha}{w_m^2}e_{16}e_{16}^T + Sym\{\Phi_{j,k}\Gamma_{j,[\eta(t)]}\} \leq 0\ \text{ subject to } \Gamma_{j,[\eta(t)]}\zeta(t) = 0, \tag{3.22}$$

where the $\Phi_{k,1}$ with $\eta(t) = 0$ and the $\Phi_{k,2}$ with $\eta(t) = \eta$ are defined, respectively.

Since the $\Gamma_{j,[\eta(t)]}$ is dependent on system polytope and time-delay $\eta(t) \in [0, \eta]$, if the following inequalities:

$$\Xi_1 + \Xi_2 + \Xi_3 - \tfrac{\alpha}{w_m^2}e_{16}e_{16}^T + Sym\{\Phi_{j,1}\Gamma_{j,[0]}\} \leq 0, \tag{3.23}$$

$$\Xi_1 + \Xi_2 + \Xi_3 - \tfrac{\alpha}{w_m^2}e_{16}e_{16}^T + Sym\{\Phi_{j,2}\Gamma_{j,[\eta]}\} \leq 0\ (j = 1,\ \dots,\ N) \tag{3.24}$$

hold, then inequality (3.22) is satisfied.

Finally, by utilizing of AZEA according to Lemma 4, (3.23) and (3.24) can be described as

$$\left(\Gamma^{\perp}_{j,[0]}\right)^T\left(\underbrace{\Xi_1 + \Xi_2 + \Xi_3 - \frac{\alpha}{w_m^2}e_{16}e_{16}^T}_{\Xi}\right)\left(\Gamma^{\perp}_{j,[0]}\right) \leq 0, \tag{3.25}$$

$$\left(\Gamma^{\perp}_{j,[\eta]}\right)^T\left(\underbrace{\Xi_1 + \Xi_2 + \Xi_3 - \frac{\alpha}{w_m^2}e_{16}e_{16}^T}_{\Xi}\right)\left(\Gamma^{\perp}_{j,[\eta]}\right) \leq 0\ (j = 1,\ \dots,\ N). \tag{3.26}$$

For satisfying definition (2.6), a matrix $\tilde{\mathcal{R}}_1 \in \mathbf{R}^{4n\times 4n}$, which was inspired by Wang et al. [12], is defined as

$$\mathcal{R} \geq \begin{bmatrix} \mathcal{R}_1 & 0_{n\times 3n} \\ * & 0_{3n\times 3n} \end{bmatrix} = \tilde{\mathcal{R}}_1. \tag{3.27}$$

From (3.3)–(3.27) and Lemma 1, it is easy to guarantee that $x^T(t)\mathcal{R}_1 x(t) \leq V_{T1}(t) \leq V_T(t) \leq 1$. Thus, the following condition

$$\begin{bmatrix} \bar{\delta} I_n & I_n \\ * & \mathcal{R}_1 \end{bmatrix} \geq 0 \tag{3.28}$$

can be derived from $x^T(t)\mathcal{R}_1 x(t) \leq 1$. Then, if conditions (3.2) and (3.28) satisfy, the RS of the system (2.1) is contained in the ellipsoid (2.6). By the Schur complement, the condition (3.28) represents $\mathcal{R}_1 \geq 1/\bar{\delta} I_n = \delta I_n$ where δ is an ellipsoidal parameter. This completes our proof. □

Remark 1. In (3.21), adding zero equalities with free-weighting matrices can derive improved estimation results. However, the free-weighting matrices $\Phi_{j,k}$ have an enormous amount of computational burden. This fatal disadvantage makes the researcher calculate the trade-off. As past work [41] showed, AZEA can eliminate the concerns of the computational burden by applying finsler's lemma. So, a more optimized reachable bounding set is proposed with AZEA.

In the following corollary, the time-delay condition when an upper bound of $\dot{\eta}(t)$ is *unknown* can be considered. To prove them with simplicity, the following LKFs and notations are rewritten.

$$\begin{aligned}
Q_{C,k} &= Q + \frac{1}{e^{-\alpha\eta}}\left(-\alpha\mathcal{N}_C + \begin{bmatrix} 0_n & \mathcal{P}_i \\ * & 0_n \end{bmatrix}\right)(k = 1,\ 2),\\
Q_{C\ aug,k} &= \begin{bmatrix} Q_{C,k} & 0_{2n} & 0_{2n} \\ * & 3Q_{C,k} & 0_{2n} \\ * & * & 5Q_{C,k} \end{bmatrix}(k = 1,\ 2),\\
\Omega_C &= \begin{bmatrix} Q_{C\ aug,1} & \mathcal{S} \\ * & Q_{C\ aug,2} \end{bmatrix},\\
\Xi_{C2} &= \eta[e_4,\ e_1]\mathcal{N}_C[e_4,\ e_1]^T - \eta[e_5,\ e_3]\mathcal{N}_C[e_5,\ e_3]^T,\\
\Xi_{C3} &= \tilde{\Xi}_3 - e^{-\alpha\eta}[\Lambda_1,\ \Lambda_2]\Omega_C[\Lambda_1,\ \Lambda_2]^T + \eta[e_1,\ e_2,\ e_3]\mathcal{P}_{aug}[e_1,\ e_2,\ e_3]^T,\\
\Xi_C &= \Xi_1 + \Xi_{C2} + \Xi_{C3} - \frac{\alpha}{w_m^2}e_{16}e_{16}^T.
\end{aligned} \tag{3.29}$$

Corollary 1. *For a given scalar* $\eta > 0$*, the RSs of the system (2.1) with time-delay condition* $0 \leq \eta(t) \leq \eta$ *is bounded by an ellipsoid* $\mathcal{R}_\epsilon$ *(2.6), if there exist matrices* $\mathcal{R} \in \mathbf{S}_+^{4n}$, $\mathcal{R}_1 \in \mathbf{S}_+^n$, $\mathcal{N}_C$, $Q \in \mathbf{S}_+^{2n}$, $\mathcal{P}_i$ $(i = 1,\ 2) \in \mathbf{S}^n$*, any matrix* $\mathcal{S} \in \mathbf{R}^{6n\times 6n}$*, and a scalar* $\alpha > 0$*, such that the following LMIs hold:*

$$\begin{aligned}
\left(\Gamma_{j,[0]}^{\perp}\right)^T \Xi_C \left(\Gamma_{j,[0]}^{\perp}\right) &\leq 0,\\
\left(\Gamma_{j,[\eta]}^{\perp}\right)^T \Xi_C \left(\Gamma_{j,[\eta]}^{\perp}\right) &\leq 0\ (j = 1,\ \ldots,\ N),\\
Q_{C,1} \geq 0,\ Q_{C,2} \geq 0,\ \Omega_C &\geq 0,\ \mathcal{R} \geq \tilde{\mathcal{R}},
\end{aligned} \tag{3.30}$$

where Ξ_C, $Q_{C,k}$ *and* Ω_C *are defined in (3.29), respectively. Other notations are defined in (3.1).*

Proof. In order to discuss the delay conditions $0 \le \eta(t) \le \eta$ and *unknown* $\dot{\eta}(t)$, let us choose the LKFs candidate as

$$V_C(t) = \sum_{k=1}^{3} V_{Ck}(t), \tag{3.31}$$

where

$$\begin{aligned} V_{C1}(t) &= V_{T1}(t), \\ V_{C2}(t) &= \eta \int_{t-\eta}^{t} \mathfrak{B}_3^T(s) \mathcal{N}_C \mathfrak{B}_3(s) ds, \\ V_{C3}(t) &= \eta \int_{t-\eta}^{t} \int_{s}^{t} e^{\alpha(u-t)} \mathfrak{B}_3^T(u) \mathcal{Q} \mathfrak{B}_3(u) du ds. \end{aligned}$$

The $\dot{V}_{C2}(t)$ can be calculated as

$$\begin{aligned} \dot{V}_{C2}(t) &= \eta \mathfrak{B}_3^T(t) \mathcal{N}_C \mathfrak{B}_3(t) - \eta \mathfrak{B}_3^T(t-\eta) \mathcal{N}_C \mathfrak{B}_3(t-\eta) \\ &= \zeta^T(t) \Xi_{C2} \zeta(t). \end{aligned} \tag{3.32}$$

And the expressions of $\dot{V}_{C1}(t)$ and $\dot{V}_{C3}(t)$ from (3.31) are similar to (3.4) and (3.6), respectively.

The $V_{C2}(t)$ derives more simple results than $V_{T2}(t)$. In (3.10), the absence of intergral term, which has interval from $t - \eta(t)$ to t makes difference. So, integral term of $V_{C3}(t)$ can be written as

$$\begin{aligned} & -\eta e^{-\alpha\eta} \int_{t-\eta(t)}^{t} \mathfrak{B}_2^T(s) \mathcal{Q} \mathfrak{B}_2(s) ds - \eta e^{-\alpha\eta} \int_{t-\eta(t)}^{t} \mathfrak{B}_2^T(s) \frac{1}{e^{-\alpha\eta}} \left(-\alpha \mathcal{N}_C + \begin{bmatrix} 0_n & \mathcal{P}_1 \\ \mathcal{P}_1^T & 0_n \end{bmatrix} \right) \mathfrak{B}_2(s) ds \\ \le & -\frac{\eta e^{-\alpha\eta}}{\eta(t)} \left(\Lambda_{1,1}^T(t) \mathcal{Q}_{C,1} \Lambda_{1,1}(t) + 3\Lambda_{1,2}^T(t) \mathcal{Q}_{C,1} \Lambda_{1,2}(t) + 5\Lambda_{1,3}^T(t) \mathcal{Q}_{C,1} \Lambda_{1,3}(t) \right) \\ & + \eta \mathfrak{B}_2^T(t,\ t-\eta(t)) \begin{bmatrix} \mathcal{P}_1 & 0_n \\ 0_n & -\mathcal{P}_1 \end{bmatrix} \mathfrak{B}_2^T(t,\ t-\eta(t)) \end{aligned} \tag{3.33}$$

where $\mathcal{Q}_{C,1} = \mathcal{Q} + \frac{1}{e^{-\alpha\eta}} \left(-\alpha \mathcal{N}_C + \begin{bmatrix} 0_n & \mathcal{P}_1 \\ \mathcal{P}_1^T & 0_n \end{bmatrix} \right)$.

And other integral term has same process. Thus, the upper bound of $\dot{V}_{C3}(t)$ can be obtained that

$$\begin{aligned} \dot{V}_{C3}(t) \le\ & \zeta^T(t) \tilde{\Xi}_3 \zeta(t) - e^{-\alpha\eta} \begin{bmatrix} \Lambda_1(t) \\ \Lambda_2(t) \end{bmatrix}^T \underbrace{\begin{bmatrix} \mathcal{Q}_{C\ aug,1} & \mathcal{S} \\ \mathcal{S}^T & \mathcal{Q}_{C\ aug,2} \end{bmatrix}}_{\Omega_C} \begin{bmatrix} \Lambda_1(t) \\ \Lambda_2(t) \end{bmatrix} \\ & + \mathfrak{B}_4^T(t) \mathcal{P}_{aug} \mathfrak{B}_4(t) - \alpha V_{C2}(t) - \alpha V_{C3}(t) \\ =\ & \zeta^T(t) \Xi_{C3} \zeta(t) - \alpha V_{C2}(t) - \alpha V_{C3}(t). \end{aligned} \tag{3.34}$$

Likewise, the (3.15)–(3.19) can be combined, and the $\dot{V}_C(t) + \alpha V_C(t) - \frac{\alpha}{w_m^2} w^T(t) w(t)$ can be bounded as

$$\dot{V}_C(t) + \alpha V_C(t) - \frac{\alpha}{w_m^2} w^T(t) w(t) \le \zeta^T(t) \left(\Xi_C + Sym\{\Phi_{j,k} \Gamma_{j,[\eta(t)]}\} \right) \zeta(t). \tag{3.35}$$

Lastly, similar to Theorem 1, LMIs (3.30) can be obtained. So rest proof is omitted. □

Remark 2. In past literature, constructing integral LKFs for RSE was limited by the presence of e^{s-t}. However, inspired by Kwon et al. [35], the authors tried to choose the integral Lyapunov function $V_{C2}(t)$ and $V_{C3}(t)$ with augmented vectors $\begin{bmatrix} \dot{x}^T(s) & x^T(s) \end{bmatrix}^T$. Although some definite conditions about $Q_{C,k}$ and Ω_C are needed to combine inequality lemmas, the result can lead to get a tighter upper bound of RSE condition. To the author's knowledge, this trial is the first time in RSE.

In the following theorem, an interval time-delay condition and a lower bound condition about $\dot{\eta}(t)$ are investigated, the following new LKFs and notations are introduced to prove them.

$$
\begin{aligned}
\hat{\xi}(t) &= \begin{bmatrix} \frac{1}{\eta^2(t)}\int_{t-\eta(t)}^{t}\int_{s}^{t}\int_{u}^{t} x(v)dvduds \\ \frac{1}{(\eta-\eta(t))^2}\int_{t-\eta}^{t-\eta(t)}\int_{s}^{t-\eta(t)}\int_{u}^{t-\eta(t)} x(v)dvduds \\ x(t-\eta_m) \\ \dot{x}(t-\eta_m) \\ \int_{t-\eta_m}^{t} x(s)ds \\ \int_{t-\eta(t)}^{t-\eta_m} x(s)ds \\ \int_{t-\eta}^{t-\eta(t)} x(s)ds \end{bmatrix}, \\
\hat{\zeta}(t) &= col\left\{ \begin{bmatrix} x(t) \\ x(t-\eta(t)) \\ x(t-\eta) \\ \dot{x}(t) \\ \dot{x}(t-\eta) \end{bmatrix}, \begin{bmatrix} \frac{1}{\eta(t)}\int_{t-\eta(t)}^{t} x(s)ds \\ \frac{1}{\eta-\eta(t)}\int_{t-\eta}^{t-\eta(t)} x(s)ds \\ \frac{1}{\eta^2(t)}\int_{t-\eta(t)}^{t}\int_{s}^{t} x(u)duds \\ \frac{1}{(\eta-\eta(t))^2}\int_{t-\eta}^{t-\eta(t)}\int_{s}^{t-\eta(t)} x(u)duds \\ \frac{1}{\eta(t)}\int_{t-\eta(t)}^{t}\int_{s}^{t} x(u)duds \\ \frac{1}{\eta-\eta(t)}\int_{t-\eta}^{t-\eta(t)}\int_{s}^{t-\eta(t)} x(u)duds \end{bmatrix}, \hat{\xi}(t),\ w(t) \right\}, \\
\hat{Q}_k &= Q + \frac{1}{e^{-\alpha\eta}}\begin{bmatrix} -\alpha\mathcal{N}_2 & \mathcal{P}_k \\ * & -\frac{\alpha}{\eta}\mathcal{G}_k \end{bmatrix}(k=1,\ 2),\ \hat{Q}_{aug,k} = \begin{bmatrix} \hat{Q}_k & 0_{2n} & 0_{2n} \\ * & 3\hat{Q}_k & 0_{2n} \\ * & * & 5\hat{Q}_k \end{bmatrix}(k=1,\ 2), \\
Q_3 &= \mathcal{U} - \frac{\alpha}{e^{-\alpha\eta_m}}\mathcal{N}_1,\ \hat{\Omega} = \begin{bmatrix} \hat{Q}_{aug,1} & \mathcal{S} \\ * & \hat{Q}_{aug,2} \end{bmatrix},\ \tilde{\mathcal{R}}_2 = diag\{\mathcal{R}_2,\ 0_n,\ 0_n,\ 0_n,\ 0_n,\ 0_n,\ 0_n\}, \\
\hat{e}_i &= [0_{n\times(k-1)n},\ I_n,\ 0_{n\times((18-k)n+m)}]^T\ (k=1,\ 2,\ \ldots,\ 18),\ \hat{e}_{19} = [0_{m\times 18n},\ I_m]^T, \\
\hat{\Upsilon} &= [\hat{e}_1,\ \hat{e}_2,\ \ldots,\ \hat{e}_{19}], \\
\hat{\Gamma}_{j,[\eta(t)]} &= col\{-\hat{e}_4^T + A_j\hat{e}_1^T + D_j\hat{e}_2^T + B_j\hat{e}_{19}^T,\ \eta(t)\hat{e}_8^T - \hat{e}_{10}^T,\ (\eta-\eta(t))\hat{e}_9^T - \hat{e}_{11}^T,\ \eta(t)\hat{e}_6^T - \hat{e}_{16}^T - \hat{e}_{17}^T, \\
&\quad (\eta-\eta(t))\hat{e}_7 - \hat{e}_{18}\}\ (j=1,\ \ldots,\ N), \\
\hat{\varepsilon}_1 &= [\hat{e}_{16},\ \hat{e}_{17},\ \hat{e}_{18},\ \eta\hat{e}_1 - (\hat{e}_{16}+\hat{e}_{17}+\hat{e}_{18})], \\
\hat{\varepsilon}_{2[\dot{\eta}(t)]} &= [\hat{e}_1 - \hat{e}_{14},\ \hat{e}_{14} - (1-\dot{\eta}(t))\,\hat{e}_2,\ (1-\dot{\eta}(t))\,\hat{e}_2 - \hat{e}_3,\ \eta\hat{e}_4 - (\hat{e}_1 - \hat{e}_3)], \\
\hat{\Lambda}_1 &= [\hat{e}_1 - \hat{e}_2,\ \hat{e}_{16} + \hat{e}_{17},\ \hat{e}_1 + \hat{e}_2 - 2\hat{e}_6,\ \hat{e}_{16} + \hat{e}_{17} - 2\hat{e}_{10},\ \hat{e}_1 - \hat{e}_2 + 6\hat{e}_6 - 12\hat{e}_8, \\
&\quad \hat{e}_{16} + \hat{e}_{17} - 6\hat{e}_{10} + 12\hat{e}_{12}], \\
\hat{\Lambda}_2 &= [\hat{e}_2 - \hat{e}_3,\ \hat{e}_{18},\ \hat{e}_2 + \hat{e}_3 - 2\hat{e}_7,\ \hat{e}_{18} - 2\hat{e}_{11},\ \hat{e}_2 - \hat{e}_3 + 6\hat{e}_7 - 12\hat{e}_9,\ \hat{e}_{18} - 6\hat{e}_{11} + 12\hat{e}_{13}], \\
\bar{\Xi}_{1[\dot{\eta}(t)]} &= Sym\left\{[\hat{e}_1,\ \hat{e}_3,\ \hat{e}_{14},\ \hat{\varepsilon}_1]\,\hat{\mathcal{R}}\left[\hat{e}_4,\ \hat{e}_5,\ \hat{e}_{15},\ \hat{\varepsilon}_{2[\dot{\eta}(t)]}\right]^T\right\}, \\
\hat{\bar{\Xi}}_{1[\dot{\eta}(t)]} &= \bar{\Xi}_{1[\dot{\eta}(t)]} + \alpha\,[\hat{e}_1,\ \hat{e}_3,\ \hat{e}_{14},\ \hat{\varepsilon}_1]\,\hat{\mathcal{R}}\,[\hat{e}_1,\ \hat{e}_3,\ \hat{e}_{14},\ \hat{\varepsilon}_1]^T,
\end{aligned}
$$

$$\hat{\Xi}_{2[\dot{\eta}(t)]} = \eta_m \hat{e}_4 \mathcal{N}_1 \hat{e}_4^T - \eta_m \hat{e}_{15} \mathcal{N}_1 \hat{e}_{15}^T + \eta \hat{e}_4 \mathcal{N}_2 \hat{e}_4^T - \eta \hat{e}_5 \mathcal{N}_2 \hat{e}_5^T + \hat{e}_1 \mathcal{G}_1 \hat{e}_1^T - (1-\dot{\eta}(t)) \hat{e}_2 \mathcal{G}_1 \hat{e}_2^T + (1-\dot{\eta}(t)) \hat{e}_2 \mathcal{G}_2 \hat{e}_2^T - \hat{e}_3 \mathcal{G}_2 \hat{e}_3^T,$$

$$\bar{\Xi}_3 = \eta^2 [\hat{e}_4,\ \hat{e}_1] \mathcal{Q} [\hat{e}_4,\ \hat{e}_1]^T + \eta_m^2 \hat{e}_4 \mathcal{U} \hat{e}_4^T,$$

$$\hat{\Xi}_3 = \bar{\Xi}_3 - e^{-\alpha\eta} [\hat{\Lambda}_1,\ \hat{\Lambda}_2] \hat{\Omega} [\hat{\Lambda}_1,\ \hat{\Lambda}_2]^T + \eta [\hat{e}_1,\ \hat{e}_2,\ \hat{e}_3] \mathcal{P}_{aug} [\hat{e}_1,\ \hat{e}_2,\ \hat{e}_3]^T - [\hat{e}_1 - \hat{e}_{14}] \mathcal{Q}_3 [\hat{e}_1 - \hat{e}_{14}]^T,$$

$$\hat{\Xi}_{[\dot{\eta}(t)]} = \hat{\Xi}_{1[\dot{\eta}(t)]} + \hat{\Xi}_{2[\dot{\eta}(t)]} + \hat{\Xi}_3 - \frac{\alpha}{w_m^2} \hat{e}_{19} \hat{e}_{19}^T. \quad (3.36)$$

Theorem 2. *For given scalars $\eta_m > 0$, $\eta > 0$ and $\mu > 0$, the system (2.1), the RSs of the system (2.1) with time-delay condition (2.5) are bounded by an ellipsoid $\mathcal{R}_\epsilon$ (2.6), if there exist matrices $\hat{\mathcal{R}} \in \mathbf{S}_+^{7n}$, $\mathcal{R}_2$, $\mathcal{N}_i$, $\mathcal{G}_i$ $(i = 1,\ 2)$, $\mathcal{U} \in \mathbf{S}_+^n$, $\mathcal{Q} \in \mathbf{S}_+^{2n}$, $\mathcal{P}_i$ $(i = 1,\ 2)$ $\in \mathbf{S}^{n\times n}$, any matrix $\mathcal{S} \in \mathbf{R}^{6n\times 6n}$, and a scalar $\alpha > 0$, such that the following LMIs hold:*

$$\left(\hat{\Gamma}_{j,[\eta_m]}^{\perp}\right)^T \hat{\Xi}_{[-\mu]} \left(\hat{\Gamma}_{j,[\eta_m]}^{\perp}\right) \le 0,$$
$$\left(\hat{\Gamma}_{j,[\eta_m]}^{\perp}\right)^T \hat{\Xi}_{[\mu]} \left(\hat{\Gamma}_{j,[\eta_m]}^{\perp}\right) \le 0,$$
$$\left(\hat{\Gamma}_{j,[\eta]}^{\perp}\right)^T \hat{\Xi}_{[-\mu]} \left(\hat{\Gamma}_{j,[\eta]}^{\perp}\right) \le 0,$$
$$\left(\hat{\Gamma}_{j,[\eta]}^{\perp}\right)^T \hat{\Xi}_{[\mu]} \left(\hat{\Gamma}_{j,[\eta]}^{\perp}\right) \le 0 \ (j = 1,\ \ldots,\ N),$$
$$\hat{\mathcal{Q}}_1 \ge 0,\ \hat{\mathcal{Q}}_2 \ge 0,\ \mathcal{Q}_3 \ge 0,\ \hat{\Omega} \ge 0,\ \hat{\mathcal{R}} \ge \tilde{\mathcal{R}}_2, \quad (3.37)$$

where $\hat{\Xi}_{[\dot{\eta}(t)]}$ and $\hat{\Gamma}_{j,[\eta(t)]}$ are defined in (3.36), respectively.

Proof. For the condition (2.5), positive matrices $\hat{\mathcal{R}}$, $\mathcal{R}_2$, $\mathcal{N}_i$, $\mathcal{G}_i$ $(i = 1,\ 2)$, $\mathcal{U}$, $\mathcal{Q}$, and the following LKFs candidate are chosen as

$$\hat{V}_T(t) = \sum_{k=1}^{3} \hat{V}_{Tk}(t), \quad (3.38)$$

where

$$\hat{V}_{T1}(t) = \hat{\mathfrak{B}}_1^T(t) \hat{\mathcal{R}} \hat{\mathfrak{B}}_1(t),$$
$$\hat{V}_{T2}(t) = \eta_m \int_{t-\eta_m}^{t} \dot{x}^T(s) \mathcal{N}_1 \dot{x}(s) ds + \eta \int_{t-\eta}^{t} \dot{x}^T(s) \mathcal{N}_2 \dot{x}(s) ds + \int_{t-\eta(t)}^{t} x^T(s) \mathcal{G}_1 x(s) ds + \int_{t-\eta}^{t-\eta(t)} x^T(s) \mathcal{G}_2 x(s) ds,$$
$$\hat{V}_{T3}(t) = \eta \int_{t-\eta}^{t} \int_{s}^{t} e^{\alpha(u-t)} \mathfrak{B}_3^T(u) \mathcal{Q} \mathfrak{B}_3(u) du ds + \eta_m \int_{t-\eta_m}^{t} \int_{s}^{t} e^{\alpha(u-t)} \dot{x}^T(u) \mathcal{U} \dot{x}(u) du ds,$$

where $\hat{\mathfrak{B}}_1(t) = col\{x(t),\ x(t-\eta_m),\ x(t-\eta),\ \int_{t-\eta_m}^{t} x(s)ds,\ \int_{t-\eta(t)}^{t-\eta_m} x(s)ds,\ \int_{t-\eta}^{t-\eta(t)} x(s)ds,\ \int_{t-\eta}^{t}\int_{s}^{t} \dot{x}(u)duds\}$.

By constructing the LKFs as (3.38), the RSE about conditions (2.5) can be proved. And the $\mathfrak{B}_2(t)$, $\mathfrak{B}_3(t)$ are noticed in Theorem 1.

The $\dot{\hat{V}}_{T1}$ and $\dot{\hat{V}}_{T2}$ can be written as

$$\dot{\hat{V}}_{T1}(t) = \hat{\zeta}^T(t) \hat{\Xi}_{1[\dot{\eta}(t)]} \hat{\zeta}(t) - \alpha \hat{V}_{T1}(t), \quad (3.39)$$

$$\begin{aligned}\dot{\hat{V}}_{T2}(t) &= \dot{\hat{\mathfrak{B}}}_2^T(t,\ t-\eta_m,\ t-\eta)\begin{bmatrix} \eta_m\mathcal{N}_1+\eta\mathcal{N}_2 & 0_n & 0_n \\ * & -\eta_m\mathcal{N}_1 & 0_n \\ * & * & -\eta\mathcal{N}_2 \end{bmatrix}\dot{\hat{\mathfrak{B}}}_2(t,\ t-\eta_m,\ t-\eta) \\ &\quad +\hat{\mathfrak{B}}_2^T(t,\ t-\eta(t),\ t-\eta)\begin{bmatrix} \mathcal{G}_1 & 0_n & 0_n \\ * & (1-\dot{\eta}(t))(\mathcal{G}_2-\mathcal{G}_1) & 0_n \\ * & * & -\mathcal{G}_2 \end{bmatrix}\hat{\mathfrak{B}}_2(t,\ t-\eta(t),\ t-\eta) \\ &= \hat{\zeta}^T(t)\hat{\Xi}_{2[\dot{\eta}(t)]}\hat{\zeta}(t), \end{aligned} \tag{3.40}$$

where $\hat{\mathfrak{B}}_2(t,\ u,\ s) = col\{x(t),\ x(u),\ x(s)\}$.
Calculating the time-derivative of $\hat{V}_{T3}$ leads to

$$\begin{aligned}\dot{\hat{V}}_{T3}(t) &= \hat{\zeta}^T(t)\bar{\Xi}_3\hat{\zeta}(t) - \eta\int_{t-\eta(t)}^{t} e^{\alpha(s-t)}\mathfrak{B}_2^T(s)\mathcal{Q}\mathfrak{B}_2(s)ds + \alpha\left(\hat{V}_{T21}(t)+\hat{V}_{T23}(t)\right) \\ &\quad -\eta\int_{t-\eta}^{t-\eta(t)} e^{\alpha(s-t)}\mathfrak{B}_3^T(s)\mathcal{Q}\mathfrak{B}_3(s)ds + \alpha\left(\hat{V}_{T22}(t)+\hat{V}_{T24}(t)\right) - \int_{t-\eta_m}^{t}\dot{x}^T(s)\mathcal{U}\dot{x}(s)ds \\ &\quad +\alpha\hat{V}_{T20}(t) - \alpha\hat{V}_{T2}(t) - \alpha\hat{V}_{T3}(t), \end{aligned} \tag{3.41}$$

where

$$\begin{aligned}\hat{V}_{T2}(t) &= \underbrace{\eta_m\int_{t-\eta_m}^{t}\dot{x}^T(s)\mathcal{N}_1\dot{x}(s)ds}_{\hat{V}_{T20}(t)} + \underbrace{\eta\int_{t-\eta(t)}^{t}\dot{x}^T(s)\mathcal{N}_2\dot{x}(s)ds}_{\hat{V}_{T21}(t)} + \underbrace{\eta\int_{t-\eta}^{t-\eta(t)}\dot{x}^T(s)\mathcal{N}_2\dot{x}(s)ds}_{\hat{V}_{T22}(t)} \\ &\quad + \underbrace{\int_{t-\eta(t)}^{t} x^T(s)\mathcal{G}_1x(s)ds}_{\hat{V}_{T23}(t)} + \underbrace{\int_{t-\eta}^{t-\eta(t)} x^T(s)\mathcal{G}_2x(s)ds}_{\hat{V}_{T24}(t)}. \end{aligned}$$

Same as (3.7) and (3.8), the zero equations with P_i $(i = 1,\ 2)$ are considered. An upper bound of the first integral term of (3.41) with $\alpha\hat{V}_{T21}(t),\ \alpha\hat{V}_{T23}(t)$ and zero Eq (3.7) can be obtained as

$$\begin{aligned} & -\eta\int_{t-\eta(t)}^{t} e^{\alpha(s-t)}\mathfrak{B}_3^T(s)\mathcal{Q}\mathfrak{B}_3^T(s)ds + \alpha\left(\hat{V}_{T21}(t)+\hat{V}_{T23}(t)\right) \\ \le\ & -\eta e^{-\alpha\eta}\int_{t-\eta(t)}^{t}\mathfrak{B}_3^T(s)\mathcal{Q}\mathfrak{B}_3(s)ds - \eta e^{-\alpha\eta}\int_{t-\eta(t)}^{t}\mathfrak{B}_3^T(s)\left(\frac{1}{e^{-\alpha\eta}}\begin{bmatrix} -\alpha\mathcal{N}_2 & \mathcal{P}_1 \\ \mathcal{P}_1^T & -\frac{\alpha}{\eta}\mathcal{G}_1 \end{bmatrix}\right)\mathfrak{B}_3(s)ds \\ & +\eta\mathfrak{B}_2^T(t,\ t-\eta(t))\begin{bmatrix} \mathcal{P}_1 & 0_n \\ 0_n & -\mathcal{P}_1 \end{bmatrix}\mathfrak{B}_2(t,\ t-\eta(t)). \end{aligned} \tag{3.42}$$

With the Lemma 2 $(l = 2)$ and an LMI condition, the integral terms of (3.42), which have interval from $t-\eta(t)$ to t, can be bounded as

$$\begin{aligned} & -\eta e^{-\alpha\eta}\int_{t-\eta(t)}^{t}\mathfrak{B}_3^T(s)\mathcal{Q}\mathfrak{B}_3(s)ds - \eta e^{-\alpha\eta}\int_{t-\eta(t)}^{t}\mathfrak{B}_3^T(s)\left(\frac{1}{e^{-\alpha\eta}}\begin{bmatrix} -\alpha\mathcal{N}_2 & \mathcal{P}_1 \\ \mathcal{P}_1^T & -\frac{\alpha}{\eta}\mathcal{G}_1 \end{bmatrix}\right)\mathfrak{B}_3^T(s)ds \\ \le\ & -\frac{\eta e^{-\alpha\eta}}{\eta(t)}\left(\Lambda_{1,1}^T(t)\hat{\mathcal{Q}}_1\Lambda_{1,1}(t) + 3\Lambda_{1,2}^T(t)\hat{\mathcal{Q}}_1\Lambda_{1,2}(t) + 5\Lambda_{1,3}^T(t)\hat{\mathcal{Q}}_1\Lambda_{1,3}(t)\right), \end{aligned} \tag{3.43}$$

where $\hat{Q}_1 = Q + \frac{1}{e^{-\alpha\eta}} \begin{bmatrix} -\alpha\mathcal{N}_2 & \mathcal{P}_1 \\ \mathcal{P}_1^T & -\frac{\alpha}{\eta}\mathcal{G}_1 \end{bmatrix}$.

Likewise, the another integral term of (3.42) with $\alpha\hat{V}_{T22}(t)$, $\alpha\hat{V}_{T24}(t)$ and Eq (3.8) are bounded as

$$\begin{aligned} &-\eta\int_{t-\eta}^{t-\eta(t)} e^{\alpha(s-t)}\mathfrak{B}_3^T(s)Q\mathfrak{B}_3(s)ds + \alpha\left(\hat{V}_{T22}(t) + \hat{V}_{T24}(t)\right) \\ \leq\ & -\eta e^{-\alpha\eta}\int_{t-\eta}^{t-\eta(t)} \mathfrak{B}_3^T(s)Q\mathfrak{B}_3(s)ds - \eta e^{-\alpha\eta}\int_{t-\eta}^{t-\eta(t)} \mathfrak{B}_3^T(s)\left(\frac{1}{e^{-\alpha\eta}}\begin{bmatrix} -\alpha\mathcal{N}_2 & \mathcal{P}_2 \\ \mathcal{P}_2^T & -\frac{\alpha}{\eta}\mathcal{G}_2 \end{bmatrix}\right)\mathfrak{B}_3(s)ds \\ &+\eta\mathfrak{B}_2^T(t-\eta(t),\ t-\eta)\begin{bmatrix} \mathcal{P}_2 & 0_n \\ 0_n & -\mathcal{P}_2 \end{bmatrix}\mathfrak{B}_2(t-\eta(t),\ t-\eta). \end{aligned} \tag{3.44}$$

With the Lemma 2 ($l = 2$) and an LMI condition, integral terms of (3.44), which have interval from $t-\eta$ to $t-\eta(t)$, can be bounded as

$$\begin{aligned} &-\eta e^{-\alpha\eta}\int_{t-\eta}^{t-\eta(t)} \mathfrak{B}_3^T(s)Q\mathfrak{B}_3(s)ds - \eta e^{-\alpha\eta}\int_{t-\eta}^{t-\eta(t)} \mathfrak{B}_3^T(s)\left(\frac{1}{e^{-\alpha\eta}}\begin{bmatrix} -\alpha\mathcal{N}_2 & \mathcal{P}_2 \\ \mathcal{P}_2^T & -\frac{\alpha}{\eta}\mathcal{G}_2 \end{bmatrix}\right)\mathfrak{B}_3(s)ds \\ \leq\ & -\frac{\eta e^{-\alpha\eta}}{\eta-\eta(t)}\left(\Lambda_{2,1}^T(t)\hat{Q}_2\Lambda_{2,1}(t) + 3\Lambda_{2,2}^T(t)\hat{Q}_2\Lambda_{2,2}(t) + 5\Lambda_{2,3}^T(t)\hat{Q}_2\Lambda_{2,3}(t)\right), \end{aligned} \tag{3.45}$$

where $\hat{Q}_2 = Q + \frac{1}{e^{-\alpha\eta}} \begin{bmatrix} -\alpha\mathcal{N} & \mathcal{P}_2 \\ \mathcal{P}_2^T & -\frac{\alpha}{\eta}\mathcal{G}_2 \end{bmatrix}$.

And then, by utilizing Lemma 3, sum of (3.43) and (3.45) can be rewritten and bounded as

$$\begin{aligned} &-\frac{\eta e^{-\alpha\eta}}{\eta(t)}\Lambda_1^T(t)\underbrace{\begin{bmatrix} \hat{Q}_1 & 0_{2n} & 0_{2n} \\ * & 3\hat{Q}_1 & 0_{2n} \\ * & * & 5\hat{Q}_1 \end{bmatrix}}_{\hat{Q}_{aug,1}}\Lambda_1(t) - \frac{\eta e^{-\alpha\eta}}{\eta-\eta(t)}\Lambda_2^T(t)\underbrace{\begin{bmatrix} \hat{Q}_2 & 0_{2n} & 0_{2n} \\ * & 3\hat{Q}_2 & 0_{2n} \\ * & * & 5\hat{Q}_2 \end{bmatrix}}_{\hat{Q}_{aug,2}}\Lambda_2(t) \\ \leq\ & -e^{-\alpha\eta}\begin{bmatrix} \Lambda_1(t) \\ \Lambda_2(t) \end{bmatrix}^T\underbrace{\begin{bmatrix} \hat{Q}_{aug,1} & \mathcal{S} \\ \mathcal{S}^T & \hat{Q}_{aug,2} \end{bmatrix}}_{\hat{\Omega}}\begin{bmatrix} \Lambda_1(t) \\ \Lambda_2(t) \end{bmatrix}. \end{aligned} \tag{3.46}$$

Finally, after Lemma 2 ($l = 0$) about $-\eta_m\int_{t-\eta_m}^{t} \dot{x}^T(s)\mathcal{U}\dot{x}(s)ds$ in (3.41) with $\alpha\hat{V}_{T20}(t)$, the $\dot{\hat{V}}_{T3}(t)$ has an upper bound as

$$\dot{\hat{V}}_{T3}(t) \ \leq\ \hat{\zeta}^T(t)\hat{\Xi}_3\hat{\zeta}(t) - \alpha\hat{V}_{T2}(t) - \alpha\hat{V}_{T3}(t), \tag{3.47}$$

where $\hat{\Xi}_{1[\dot{\eta}(t)]}$, $\hat{\Xi}_{2[\dot{\eta}(t)]}$, $\hat{\Xi}_3$ and $\hat{\zeta}(t)$ are defined in (3.36).

Similarly with Theorem 1, the zero equations can be obtained as

$$\hat{\zeta}^T(t)\left(\hat{\Phi}_{j,k}\hat{\Gamma}_{j,[\eta(t)]} + \hat{\Gamma}_{j,[\eta(t)]}^T\hat{\Phi}_{j,k}^T\right)\hat{\zeta}(t) = 0 \quad (j = 1,\ \ldots,\ N,\ k = 1,\ 2), \tag{3.48}$$

where $\hat{\Phi}_{j,k} = \hat{\Upsilon}[\hat{\phi}_{1,j},\ \hat{\phi}_{2,k},\ \ldots,\ \hat{\phi}_{5,k}]$, $\hat{\Gamma}_{j,[\eta(t)]} = col\left\{\hat{\Gamma}_{1,j},\ \hat{\Gamma}_{2[\eta(t)]},\ \ldots,\ \hat{\Gamma}_{5[\eta(t)]}\right\}$.

Therefore, from (3.39)–(3.47) with adding the (3.48), the following RSE condition can be written as

$$\dot{\hat{V}}_T(t) + \alpha\hat{V}_T(t) - \frac{\alpha}{w_m^2}w^T(t)w(t) \ \leq\ \hat{\zeta}^T(t)\left(\hat{\Xi}_{[\dot{\eta}(t)]} + Sym\{\hat{\Phi}_{j,k}\hat{\Gamma}_{j,[\eta(t)]}\}\right)\hat{\zeta}(t) \leq 0. \tag{3.49}$$

The inequality (3.49) can be satisfied with the following equality condition holds:

$$\hat{\Xi}_{1[\dot{\eta}(t)]} + \hat{\Xi}_{2[\dot{\eta}(t)]} + \hat{\Xi}_3 - \frac{\alpha}{w_m^2}\hat{e}_{19}\hat{e}_{19}^T + Sym\{\hat{\Phi}_{j,k}\hat{\Gamma}_{j,[\eta(t)]}\} \leq 0. \tag{3.50}$$

And then, the LMIs conditions (3.37) are obtained by the same processes as Theorem 1. This completes our proof. □

Remark 3. In RSE studies, past literature constructed the non-integral Lyapunov functional candidate $V_1(t)$ by choosing only $x(t)$. And recently, Wang et al. [12] introduced an augmented method of non-integral Lyapunov functional candidate for RSE. With this expanded Lyapunov functional, a less conservative result can be obtained. However, it still has limitations for deriving various time-delay conditions without time-varying delay information in $V_1(t)$. In this paper, by choosing the appropriate state vectors in the proposed $\hat{V}_{T1}(t)$, the time-varying delay condition (2.5) can be considered effectively.

Remark 4. For investigating the practical conditions, various time-varying delay conditions should be considered. Case 1(2.4) is the time condition for Theorem 1 where $0 \leq \eta(t) \leq \eta$, $-\infty \leq \dot{\eta}(t) \leq \mu$. Case 2(2.5) is the time condition for Theorem 2 where $0 \leq \eta_m \leq \eta(t) \leq \eta$, $-\mu \leq \dot{\eta}(t) \leq \mu$. The time condition, when an upper bound of $\dot{\eta}(t)$ is *unknown* is investigated in Corollary 1. Moreover, by constructing the $\hat{V}_{T1}(t)$ with augmented vectors about $t-\eta_m$, and the $\hat{V}_{T2}(t)$ with integral intervals from $t-\eta_m$ to t and from $t-\eta$ to $t-\eta(t)$, the delay condition Case 2(2.5) can be investigated in Theorem 2.

4. Numerical examples

In this section, we provide two examples to show the improved RSE with optimized ellipsoidal bound parameter $\bar{\delta} = \delta^{-1}$.

Example 1. Consider the system (2.1) with the following parameters which have been studied for polytopic uncertainty:

$$\begin{aligned}
&A + \mathcal{A}_{u,1} = \begin{bmatrix} -2 & 0 \\ 0 & -0.7 \end{bmatrix},\ D + \mathcal{D}_{u,1} = \begin{bmatrix} -1 & 0 \\ -1 & -0.9 \end{bmatrix},\ B + \mathcal{B}_{u,1} = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix},\\
&A + \mathcal{A}_{u,2} = \begin{bmatrix} -2 & 0 \\ 0 & -1.1 \end{bmatrix},\ D + \mathcal{D}_{u,2} = \begin{bmatrix} -1 & 0 \\ -1 & -1.1 \end{bmatrix},\ \mathcal{B}_{u,2} = \mathcal{B}_{u,1},\ w^T(t)w(t) \leq w_m^2 = 1,\\
&A_j = A + \mathcal{A}_{u,j},\ D_j = D + \mathcal{D}_{u,j},\ B_j = B + \mathcal{B}_{u,j}\ (j = 1,\ 2),\ \eta_m = 0,\ \eta = 0.7,\ \eta_m \leq \eta(t) \leq \eta.
\end{aligned} \tag{4.1}$$

By definition (2.2), system uncertainties are expressed in polytope expression. So polytopic uncertainties about the system are introduced with A_j, D_j, B_j ($j = 1, 2$). For comparison with past literature [7, 35, 42, 45, 46], the delay conditions in (4.1) are utilized. In Table 1, our computed results are listed with various conditions of time-varying delay. It should be noted that our result with unknown $\dot{\eta}(t)$ represents a smaller $\bar{\delta}$ than the result of the delay condition (2.4). This difference is derived from the augmented $\begin{bmatrix} \dot{x}^T(s) & x^T(s) \end{bmatrix}^T$ in $V_{C2}(t)$, $V_{C3}(t)$. It is well-known that for getting less conservative results, the LKFs should be constructed with bounded conditions about $\eta(t)$ and $\dot{\eta}(t)$. Thus, a smaller ellipsoidal bound can be obtained with a simple augmented approach in Corollary 1.

Moreover, the number of decision variables for Table 1(Example 1) is listed in Table 2. And Figure 1 shows that comparison of results when delay conditions (2.5) where $\eta_m = 0$, $\eta = 0.7$ are selected.

Table 1. The sizes if $\bar{\delta}$ bound with $\eta = 0.7$ and different μ (Example 1).

μ	0.1	0.3	0.6	0.9	*unknown*	$\dot{\eta}(t)$
Kwon [35](Thm1)	-	-	-	-	2.1139	$-\infty \le \dot{\eta}(t) \le \infty$
Sheng [45](Thm1)	-	-	-	-	1.61	$-\infty \le \dot{\eta}(t) \le \infty$
Our result(Cor1)	-	-	-	-	1.2277	$-\infty \le \dot{\eta}(t) \le \infty$
Kwon [35](Thm2)	1.9475	2.0182	2.1123	2.1139	-	$-\infty \le \dot{\eta}(t) \le \mu$
Chen [42](Thm10)	1.84	1.95	2.06	3.09	-	$-\infty \le \dot{\eta}(t) \le \mu$
Our result(Thm1)	1.4683	1.4844	1.4844	1.4844	-	$-\infty \le \dot{\eta}(t) \le \mu$
Zuo [7](Thm7)	1.94	2.08	2.60	3.51	-	$-\mu \le \dot{\eta}(t) \le \mu$
Kwon [35](Thm4)	1.7728	1.8199	1.8674	1.9433	-	$-\mu \le \dot{\eta}(t) \le \mu$
Chen [46](Thm2)	1.51	1.63	1.94	2.05	-	$-\mu \le \dot{\eta}(t) \le \mu$
Our result(Thm2, $\eta_m = 0$)	1.1766	1.2091	1.2462	1.2607	-	$-\mu \le \dot{\eta}(t) \le \mu$

Table 2. The number of decision variables for Example 1.

Methods	**NoDVs**
Zuo [7](Thm7)	$2n^2 + 2n + 5$
Kwon [35](Thm1)	$6n^2 + 2n$
Kwon [35](Thm2)	$6.5n^2 + 2.5n$
Kwon [35](Thm4)	$11.5n^2 + 4.5n$
Sheng [45](Thm1)	$12.5n^2 + (3.5 + m)n + 1$
Chen [42](Thm10)	$7n^2 + 5n + 1$
Chen [46](Thm2)	$6n^2 + 4n + 1$
Our result(Thm1)	$47.5n^2 + 5.5n + 1$
Our result(Cor1)	$49.5n^2 + 5.5n + 1$
Our result(Thm2)	$66.5n^2 + 8.5n + 1$
*m: dimension of disturbance	

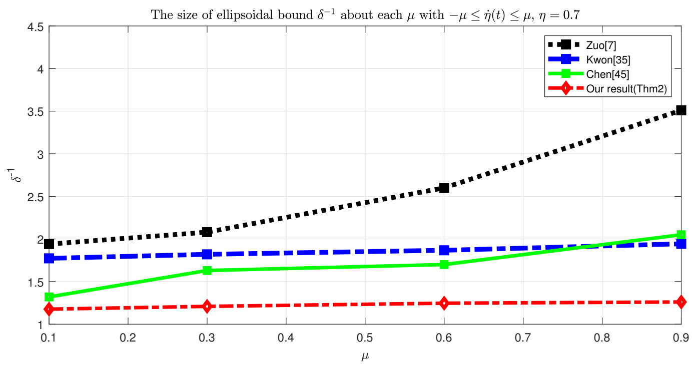

Figure 1. The graph undervalues of Table 1 ($-\mu \le \dot{\eta}(t) \le \mu$) (Example 1, Theorem 2).

Example 2. Consider the system (2.1) with the following parameters:

$$A_j = \begin{bmatrix} -1 & 0.5 \\ 1 & -4 \end{bmatrix},\ D_j = \begin{bmatrix} 0.1 & 0.2 \\ -0.4 & 2 \end{bmatrix},\ B_j = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}\ (j = 1),$$
$$\eta_m = 0.1,\ \eta = 0.5,\ \mu = 0.2,\ w^T(t)w(t) \le w_m^2 = 1. \tag{4.2}$$

Theorem 2 focused on investigating the interval time delay $\eta(t)$ and lower bound of $\dot{\eta}(t)$ conditions. The obtained ellipsoid matrices about not only *Optimized* but also constant parameters $\alpha = 0.3,\ 0.6,\ 0.9$, which mean matrix $\mathcal{P}$ in (2.6), are listed in Table 3. Here, the *fminsearch.m*, which *Kim* in [6] introduced, can be used for getting a local optimum α. The system state trajectory and guaranteed RS bound in [25] are compared with our obtained results in Figure 2. Finally, the number of decision variables for Table 3 (Example 2) is listed in Table 4.

Table 3. Ellipse parameters and the comparisons of ellipse size for different α (Example 2).

α	0.3	0.6	0.9	Optimized
Zhang [43](Thm1)	$\begin{bmatrix} 3.739 & -0.307 \\ -0.307 & 3.053 \end{bmatrix}$	$\begin{bmatrix} 3.984 & -0.038 \\ -0.038 & 3.9511 \end{bmatrix}$	$\begin{bmatrix} 1.013 & -0.008 \\ -0.008 & 1.024 \end{bmatrix}$	-
Ding [25](Cor2)	$\begin{bmatrix} 3.948 & -0.386 \\ -0.386 & 3.083 \end{bmatrix}$	$\begin{bmatrix} 4.333 & -0.266 \\ -0.266 & 4.101 \end{bmatrix}$	$\begin{bmatrix} 1.048 & -0.056 \\ -0.056 & 1.123 \end{bmatrix}$	$\begin{bmatrix} 4.932 & -0.572 \\ -0.572 & 4.293 \end{bmatrix}, \bar{\delta} = 0.253$
Our result(Thm2)	$\begin{bmatrix} 3.493 & -0.016 \\ -0.016 & 3.463 \end{bmatrix}$	$\begin{bmatrix} 4.952 & -0.001 \\ -0.001 & 4.951 \end{bmatrix}$	$\begin{bmatrix} 1.722 & -0.202 \\ -0.202 & 1.944 \end{bmatrix}$	$\begin{bmatrix} 5.184 & -0.345 \\ -0.345 & 4.804 \end{bmatrix}, \bar{\delta} = 0.217$

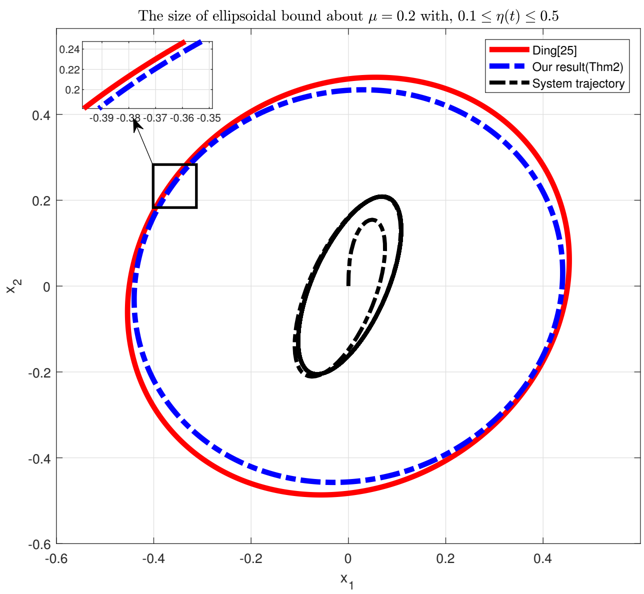

Figure 2. The system trajectories under values $\eta_m = 0.1,\ \eta = 0.5,\ \mu = 0.2,\ \eta(t) = \mu sin(t) + \eta - \mu$ and $w^T(t) = [cos(t),\ sin(t)]$ (Example 2, Theorem 2).

Table 4. The number of decision variables for Example 2.

Methods	NoDVs
Zhang [43](Thm1)	$4.5n^2 + 3.5n + 2$
Ding [25](Cor1)	$7.5n^2 + 3.5n + 1$
Our result(Thm2)	$66.5n^2 + 8.5n + 1$

5. Conclusions

In this paper, the augmented integral LKFs methods for RSE problems about time-delay linear systems with uncertainty and peak input value were proposed. For the various delay conditions, Theorem 1 and Theorem 2 constructed appropriate LKFs, and they utilized the WBII and RCA methods with an augmented zero equality approach. And Corollary 1 introduced the augmented LKFs method in integral terms with considering $e^{(s-t)}$. Finally, the superiorities of our methods are represented in tables and figures. Based on the proposed idea, the authors will try to investigate the various systems of reachable set estimations, such as neural networks, discrete-time models, sampled-data control systems, and so on.

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Conflict of interest

The authors declare no conflict of interest.

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