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Research article

On fixed point results for some generalized nonexpansive mappings

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Abstract: We investigate an Ishikawa iteration process in the set up of generalized α- nonexpansive mappings. Approximation of these two mappings to a common fixed point by $\Delta-$convergence and strong convergence of the scheme in hyperbolic space are also illustrated. The presented results amplify and polish many recent ideas put forward in uniformly convex Banach spaces, including CAT(0) spaces.

Keywords: common fixed point; generalized α- nonexpansive mappings; hyperbolic space
Mathematics Subject Classification: 47H10, 54H25

1. Introduction

Let $\mathcal{K}$ be a subset of a metric space (X, d). Design the set of common fixed points of P and S by $Fix(P) \cap Fix(S)$, where a common fixed point (in short, comm.f.p.) of a pair of self mappings P and S given on $\mathcal{K}$ [4] is a point $\zeta \in \mathcal{K}$ for which $P(\zeta) = \zeta = S(\zeta)$. Recall that a mapping $P : \mathcal{K} \to \mathcal{K}$ is termed:
(1) Nonexpansive given that $d(P(u), P(v)) \leq d(u, v)$, for all $u, v \in \mathcal{K}$.
(2) Quasi-nonexpansive given that $Fix(P) \neq \phi$ and for all $u \in \mathcal{K}$ and $v \in Fix(P)$, the following assertion holds: $d(P(u), v) \leq d(u, v)$.
It is obvious that each nonexpansive mappings with just a fixed point (in short, f.p.) is a quasi-nonexpansive mapping.

In 2011, the notion of $\alpha-$ nonexpansive mapping was put forward in Banach space by Aoyama and Kohsaka [1]. In 2017, this notion was partially extended to the notion of generalized (in short, grz.) $\alpha-$nonexpansive mapping in Banach space by Pant and Shukla [22] as: consider a Banach space X with its nonempty subset $\mathcal{K}$, the mapping $P : \mathcal{K} \longrightarrow \mathcal{K}$ is considered a grz. $\alpha-$nonexpansive provided there exists $\alpha \in [0, 1)$ such that for all $u, v \in \mathcal{K}$, $\frac{1}{2}\|u - P(u)\| \leq \|u - v\|$ implies $\|P(u) - P(v)\| \leq \alpha \|P(u) - v\| + \alpha \|P(v) - u\| + (1 - 2\alpha) \|u - v\|$.

The fixed point theory has become one of the fields that have gained wide and rapid development in recent years, due to the progress and diversity of the various iterative processes used to approximate fixed points of nonexpansive mappings and the wider classes of nonexpansive mappings, (see [7, 10–12, 21, 23, 28–30]). Very famous Mann iteration process and Ishikawa iteration process among others. In 1953, Mann [19] brought up the fundamental way for approximating f.p. of continuous transformation in Banach space as:

$$\begin{cases} u_1 \in \mathcal{K}, \\ u_{n+1} = t_n P(u_n) + (1 - t_n)u_n, & n \in \mathbb{N}, \end{cases}$$

where $\{t_n\}$ is a sequence belonging to $[0, 1]$.

Moreover, in 1974, Ishikawa [14] refined Mann's iterative process from one to two-step iterations; he also put up an iterative process to approximate f.p. of psedu-contractive compact mapping in Hilbert space given below:

$$\begin{cases} u_1 \in \mathcal{K}, \\ v_n = (1 - s_n)u_n + s_n P(u_n), \\ u_{n+1} = (1 - t_n)u_n + t_n P(v_n), & n \in \mathbb{N}, \end{cases}$$

with $\{t_n\}$ and $\{s_n\}$ denoting sequences lying in $[0, 1]$ and obeying some criteria. It can be seen that Mann iteration is a special case of Ishikawa iteration when $s_n = 0, \forall n \in \mathbb{N}$.

There are three basic rules on which the fixed point theory is based. One of them is the type of mappings for which we want to find the fixed points, the other is the iteration process used to approximate the fixed point, and the last is the space on which the mapping is given and studied. Banach spaces have been extensively studied by researchers as is evident in the literature, due to the fact that Banach spaces always have convex structures. While metric spaces do not naturally have this structure. That is why the need arose to launch convex structure to it. The notion of convex metric spaces was first launched by Takahashi [27], he scrutinized the fixed point theory for nonexpansive mappings in the convex metric spaces. Then, there were respective ventures to present a convex structure on metric space. Hyperbolic space is an example of metric spaces with a convex structure, respective views have emerged, (see [8, 18, 24]). The class of hyperbolic spaces given by Kohlenbach in [18] is more general than the class of hyperbolic space set up by Reich and Shafrir in [24], however it is more restrictive than the class of hyperbolic space launched by Goebel and Kirk in [8]. This class of hyperbolic space includes Banach spaces, Cartesian products of Hilbert balls, Hadamard manifords, CAT(0) spaces, $\mathbb{R}$-trees and Hilbert ball with the hyperbolic metric. For additional examples and information of hyperbolic spaces, (see [8, 9, 18, 24]).

In 2014, Khan [16] investigated $\Delta-$convergence as well as strong convergence of a three-step iterative process, which was mentioned in his paper for nonexpansive mappings on a nonlinear domain of hyperbolic spaces. Fukhar-ud-din and Khamsi [5] established strong convergence and $\Delta-$convergence results to a comm.f.p. in hyperbolic space, using an Ishikawa iteration scheme associated to a pair of nonexpansive mappings, which was given by Das and Debata [3] as:

$$\begin{cases} v_n = s_n P(u_n) + (1 - s_n)u_n \\ u_{n+1} = t_n S(v_n) + (1 - t_n)u_n, & n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\{t_n\}$ and $\{s_n\}$ are sequences in $[0, 1]$ enjoying certain criteria.

In 2018, Mebawondo and Izuchukwu [20] launched the class of grz. $\alpha-$ nonexpansive mapping in hyperbolic space as: in a metric space X, a mapping $P : \mathcal{K} \subset X \to \mathcal{K}$ is named grz. $\alpha-$nonexpansive given that for all $u, v \in \mathcal{K}$ there exists $\alpha \in [0, 1)$ such that
$\frac{1}{2}d(u, P(u)) \leq d(u, v)$ implies
$d(P(u), P(v)) \leq \alpha d(P(u), v) + \alpha d(P(v), u) + (1 - 2\alpha)d(u, v)$.
Further, they scrutinized some fixed points properties and demiclosedness principle for grz. $\alpha-$nonexpansive mappings in the uniformly convex hyperbolic spaces and they set up strong and $\Delta-$convergence results by using iteration process, known as Picard Normal S-iteration process, which is mentioned in their paper.

Then, In 2019, Fukhar-ud-din scrutinized $\Delta-$convergence and strong convergence of the Ishikawa iteration (1.1) to the comm.f.p. of an $\alpha-$ nonexpansive mapping and a mapping enjoying criterion (C) in the framework of a convex metric space. Suanoom et al. [26] established $\Delta-$convergence results and strong convergence results for a grz. $\alpha-$nonexpansive mappings in a hyperbolic space by using the iteration process which is mentioned in their paper.

In 2020, Bantaojai et al. [2] launched the class of a square $\alpha-$ nonexpansive mapping (or $\alpha-$nonexpansive mapping) in hyperbolic space as: in a metric space X, a mapping $P : \mathcal{K} \subset X \to \mathcal{K}$ is named square $\alpha-$nonexpansive mapping given that $\alpha < 1$ such that

$$d^2(P(u), P(v)) \leq \alpha d^2(P(u), v) + \alpha d^2(u, P(v)) + (1 - 2\alpha)d^2(u, v),$$

for all $u, v \in \mathcal{K}$.

The aim of this paper is to bring up a conceptual theoretical bodywork based on studying the strong and $\Delta-$convergence results of two grz. $\alpha-$ nonexpansive mappings P and S to a comm.f.p. in uniformly convex hyperbolic space. The results put up in this paper are new and extend the corresponding results for uniformly convex Banach spaces as well as CAT(0) spaces.

2. Preliminaries

During this paper, suppose X is a hyperbolic space which was introduced by Kohlenbach [18], as:

Definition 2.1. *A hyperbolic space (in short, hbc.s.) is a metric space (X, d) with a mapping $W : X^2 \times [0, 1] \to X$ enjoying the following criteria.*
$(H1)$ $d(p, W(u, v, \alpha)) \leq (1 - \alpha)d(p, u) + \alpha d(p, v)$;
$(H2)$ $d(W(u, v, \alpha), W(u, v, \beta)) = |\alpha - \beta|\, d(u, v)$;
$(H3)$ $W(u, v, \alpha) = W(v, u, 1 - \alpha)$;
$(H4)$ $d(W(u, z, \alpha), W(v, w, \alpha)) \leq (1 - \alpha)d(u, v) + \alpha d(z, w)$,
for all $u, v, z, w \in X$ *and* $\alpha, \beta \in [0, 1]$.

Now, we recall some definitions on hbc.s..

Definition 2.2. *([25]) Suppose that X is a hbc.s. with a mapping $W : X^2 \times [0, 1] \to X$.*
(1) *A nonempty subset $\mathcal{K}$ of X is named convex if $W(u, v, \lambda) \in \mathcal{K}$,*
for all $u, v \in \mathcal{K}$ and $\lambda \in [0, 1]$.
(2) *X is named uniformly convex (in short, u.c.) if for any $r > 0$ and $\varepsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that for all $u, v, z \in X$*

$$d(W(u, v, \frac{1}{2})) \leq (1 - \delta)r,$$

provided $d(u,z) \leq r$, $d(v,z) \leq r$ *and* $d(u,v) \geq \varepsilon r$.
(3) *A map* $\eta : (0,\infty) \times (0,2] \to (0,1]$ *which provides such a* $\delta = \eta(r,\varepsilon)$ *for a given* $r > 0$ *and* $\varepsilon \in (0,2]$ *is known as a modulus of uniform convexity of X. The mapping* η *is said to be monotone, if it decreases with r (for a fixed* ε*).*

Definition 2.3. *([25]) Let* $\{u_n\}$ *be a bounded sequence in a hbc.s. X. For* $u \in X$*, we define a continuous functional* $r(\cdot, \{u_n\}) : X \to [0,\infty)$ *by*

$$r(u, \{u_n\}) = \limsup_{n\to\infty} d(u, u_n).$$

The asymptotic radius $r(\{u_n\})$ *of* $\{u_n\}$ *is given by*

$$r(\{u_n\}) = inf \{r(u, u_n) : u \in X\}.$$

A point $u \in \mathcal{K}$ *is named an asymptotic center of the sequence* $\{u_n\}$ *with respect to* $\mathcal{K} \subseteq X$ *if*

$$r(u, \{u_n\}) = inf \{r(v, \{u_n\}) : v \in \mathcal{K}\}.$$

The set of all asymptotic centers of $\{u_n\}$ *with respect to* $\mathcal{K}$ *is denoted by* $A_{\mathcal{K}}(\{u_n\})$*. Simply, denoted by* $r(\{u_n\})$ *and* $A(\{u_n\})$ *to the asymptotic radius and the asymptotic center which are taken with respect to X, respectively.*

In u.c. Banach space and CAT(0) space, we have the following property: the bounded sequences have unique asymptotic centers with respect to closed convex subsets. In the case of hbc.s., we have the following result.

Lemma 2.1. *([25]) Let X be a complete u.c.hbc.s. with monotone modulus of uniform convexity* η*. Then every bounded sequence* $\{u_n\}$ *in X has a unique asymptotic center with respect to any nonempty closed convex subset* $\mathcal{K}$ *of X.*

Definition 2.4. *([17]) A sequence* $\{u_n\}$ *in X is said to* Δ*-converge to* $u \in X$*, if u is the unique asymptotic center of* $\{u_{n_k}\}$ *for every subsequence* $\{u_{n_k}\}$ *of* $\{u_n\}$*. In this case, we write* $\Delta - \lim_{n\to\infty} u_n = u$*.*

Lemma 2.2. *([25]) Let X be a u.c.hbc.s. with monotone modulus of uniform convexity* η*. Let* $u \in X$ *and* $\{\alpha_n\}$ *be a sequence in* $[a,b]$ *for some* $a, b \in (0,1)$*. If* $\{u_n\}$ *and* $\{v_n\}$ *are sequences in X such that* $\limsup_{n\to\infty} d(u_n, u) \leq c$*,* $\limsup_{n\to\infty} d(v_n, u) \leq c$ *and* $\limsup_{n\to\infty} d(W(u_n, v_n, \alpha_n), u) = c$ *for some* $c \geq 0$*. Then* $\lim_{n\to\infty} d(u_n, v_n) = 0$*.*

Definition 2.5. *([13]) Let* $\mathcal{K}$ *be a nonempty subset of a hbc.s. X and* $\{u_n\}$ *be a sequence in X. Then* $\{u_n\}$ *is termed a Fejèr monotone sequence with respect to* $\mathcal{K}$ *if for all* $u \in \mathcal{K}$ *and* $n \geq 1$*,*

$$d(u_{n+1}, u) \leq d(u_n, u).$$

In our main results, in the case, when we examine the strong convergence result, an additional criterion on the mappings P and S will be added. This criterion is termed condition (AV) and is given as:

Definition 2.6. *([15]) Two self-mappings P and S on* $\mathcal{K}$ *with a nonempty subset* $Fix(P) \cap Fix(S)$ *of* $\mathcal{K}$ *are said to obey criterion (AV) if one can locate a nondecreasing function g on* $[0,\infty)$ *with* $g(0) = 0$ *and* $g(t) > 0$ *for all* $t \in (0,\infty)$ *such that*

$$\frac{1}{2}\left[d(u, P(u)) + d(u, S(u))\right] \geq g(d(u, Fix(P) \cap Fix(S))) \quad \text{for all} \quad u \in \mathcal{K}.$$

3. Main results

In this section, we discuss the convergence behavior of two grz. $\alpha-$ nonexpansive mappings to a comm.f.p. by using Ishikawa iteration. We start by proving the following lemmas that are useful in our results.

Lemma 3.1. *Suppose that P, S are two self grz. $\alpha-$ nonexpansive mappings given on a nonempty subset $\mathcal{K}$ of a metric space X with a comm.f.p. of P and S, say ζ. Then P and S are quasi-nonexpansive mappings and the comm.f.p. set is a closed subset of X.*

Proof. Let $u \in \mathcal{K}$ and $\zeta \in Fix(P) \cap Fix(S)$. From the definition of grz. $\alpha-$ nonexpansive mapping, we have

$$\begin{aligned} d(P(u), \zeta) &= d(P(u), P(\zeta)) \\ &\leq \alpha d(P(u), \zeta) + \alpha d(u, P(\zeta)) + (1 - 2\alpha) d(u, \zeta) \\ &= \alpha d(P(u), \zeta) + \alpha d(u, \zeta) + (1 - 2\alpha) d(u, \zeta) \\ &= \alpha d(P(u), \zeta) + (1 - \alpha) d(u, \zeta). \end{aligned}$$

Now, we get

$$(1 - \alpha) d(P(u), \zeta) \leq (1 - \alpha) d(u, \zeta),$$

since $\alpha \in [0, 1)$, then we round off that

$$d(P(u), \zeta) \leq d(u, \zeta). \tag{3.1}$$

Whence, P is quasi-nonexpansive mapping.

In the same way, we can deduce that S is quasi-nonexpansive mapping, i.e.,

$$(S(u), \zeta) \leq d(u, \zeta). \tag{3.2}$$

Next, we prove that the comm.f.p. set $Fix(P) \cap Fix(S)$ is a closed set. Suppose $\{\zeta_n\}$ is a sequence in $Fix(P) \cap Fix(S)$ such that $\zeta_n \longrightarrow \zeta$ as $n \longrightarrow \infty$. And we want to show that $\zeta \in Fix(P) \cap Fix(S)$.

From (3.1), we get $d(P(\zeta), \zeta_n) \leq d(\zeta, \zeta_n) \longrightarrow 0$ as $n \longrightarrow \infty$, and by the uniqueness of the limit point, we have $P(\zeta) = \zeta$. And from (3.2), we obtain $d(S(\zeta), \zeta_n) \leq d(\zeta, \zeta_n) \longrightarrow 0$ as $n \longrightarrow \infty$, Whence, by the uniqueness of the limit point, we have $S(\zeta) = \zeta$. Whence, $\zeta \in Fix(P) \cap Fix(S)$. This means that $Fix(P) \cap Fix(S)$ is a closed set. □

Lemma 3.2. *Suppose that P, S are two self grz. $\alpha-$ nonexpansive mappings given on a nonempty closed convex subset $\mathcal{K}$ of a hbc.s. X. Then for the sequence $\{u_n\}$, given by Ishikawa iteration* (1.1)*, the following statements are obeyed:*
1) *$\{u_n\}$ is a Fejèr monotone sequence with respect to $Fix(P) \cap Fix(S)$.*
2) *$\lim_{n\to\infty} d(u_n, \zeta)$ exists for each $\zeta \in Fix(P) \cap Fix(S)$.*
3) *$\lim_{n\to\infty} d(u_n, Fix(P) \cap Fix(S))$ exists.*

Proof. 1) By using Lemma 3.1 for a comm.f.p. ζ of P and S, we consider

$$d(u_{n+1}, \zeta) = d(W(S(v_n), u_n, t_n), \zeta)$$

$$
\begin{aligned}
&\leq t_n d(S(v_n),\zeta) + (1-t_n)d(u_n,\zeta)\\
&\leq t_n d(v_n,\zeta) + (1-t_n)d(u_n,\zeta)\\
&= t_n d(W(P(u_n),u_n,s_n),\zeta) + (1-t_n)d(u_n,\zeta)\\
&\leq t_n s_n d(P(u_n),\zeta) + t_n(1-s_n)d(u_n,\zeta) + (1-t_n)d(u_n,\zeta)\\
&\leq t_n s_n d(u_n,\zeta) + t_n d(u_n,\zeta) - t_n s_n d(u_n,\zeta) + (1-t_n)d(u_n,\zeta)\\
&= d(u_n,\zeta).
\end{aligned}
$$

Then, we round off that

$$d(u_{n+1},\zeta) \leq d(u_n,\zeta).$$

Whence $\{u_n\}$ is a Fejèr monotone sequence with respect to $Fix(P) \cap Fix(S)$.
2) From (1), we have $\{d(u_n,\zeta)\}$ is non-increasing sequence and bounded. Whence, $\lim_{n\to\infty} d(u_n,\zeta)$ exists for each $\zeta \in Fix(P) \cap Fix(S)$.
3) It is considered a direct conclusion from (1) and (2). □

Lemma 3.3. *Suppose that P, S are two self grz. $\alpha-$ nonexpansive mappings given on a nonempty convex closed subset $\mathcal{K}$ of a complete u.c.hbc.s. with a monotone modulus of uniform convexity η. Suppose that $Fix(P) \cap Fix(S) \neq \phi$. Then for the sequence $\{u_n\}$, which given by Ishikawa iteration* (1.1)*, we have*

$$\lim_{n\to\infty} d(u_n, S(u_n)) = 0 = \lim_{n\to\infty} d(u_n, P(u_n)).$$

Proof. Let $\zeta \in Fix(P) \cap Fix(S)$, from Lemma 3.2, we round off $\lim_{n\to\infty} d(u_n,\zeta)$ exists for each $\zeta \in Fix(P) \cap Fix(S)$.
Suppose

$$\lim_{n\to\infty} d(u_n,\zeta) = c, \tag{3.3}$$

where $c \geq 0$ is a real number. From Lemma 3.1, since $d(P(u_n),\zeta) \leq d(u_n,\zeta)$, we have

$$\limsup_{n\to\infty} d(P(u_n),\zeta) \leq c. \tag{3.4}$$

Moreover,

$$
\begin{aligned}
d(v_n,\zeta) &= d(W(P(u_n),u_n,s_n),\zeta)\\
&\leq s_n d(P(u_n),\zeta) + (1-s_n)d(u_n,\zeta)\\
&\leq s_n d(u_n,\zeta) + (1-s_n)d(u_n,\zeta)\\
&= d(u_n,\zeta).
\end{aligned}
$$

Therefor, we have

$$\limsup_{n\to\infty} d(v_n,\zeta) \leq c. \tag{3.5}$$

From Lemma 3.1, since $d(S(v_n),\zeta) \leq d(v_n,\zeta)$, we have

$$\limsup_{n\to\infty} d(S(v_n),\zeta) \leq c. \tag{3.6}$$

From (3.3), we get that

$$\lim_{n\to\infty} d(W(S(v_n), u_n, t_n), \zeta) = c.$$

Now, from Lemma 2.2, we round off

$$\lim_{n\to\infty} d(S(v_n), u_n) = 0. \tag{3.7}$$

By Lemma 3.1, we have

$$\begin{aligned} d(u_n, \zeta) &\leq d(u_n, S(v_n)) + d(S(v_n), \zeta) \\ &\leq d(u_n, S(v_n)) + d(v_n, \zeta). \end{aligned}$$

Taking the limit infimum in the above inequality, we obtain

$$c \leq \liminf_{n\to\infty} d(v_n, \zeta). \tag{3.8}$$

From (3.5) and (3.8), we deduce

$$\lim_{n\to\infty} d(v_n, \zeta) = c.$$

From Ishikawa iteration (1.1), we have

$$\lim_{n\to\infty} d(W(P(u_n), u_n, s_n), \zeta) = c.$$

Then, by Lemma 2.2, we round off

$$\lim_{n\to\infty} d(P(u_n), u_n) = 0. \tag{3.9}$$

From Ishikawa iteration (1.1), we have

$$\begin{aligned} d(P(u_n), v_n) &= d(P(u_n), W(P(u_n), u_n, s_n)) \\ &\leq s_n d(P(u_n), P(u_n)) + (1 - s_n) d(P(u_n), u_n) \\ &= (1 - s_n) d(P(u_n), u_n). \end{aligned}$$

Then, we obtain

$$\lim_{n\to\infty} d(P(u_n), v_n) = 0. \tag{3.10}$$

Now, from definition of grz. $\alpha-$ nonexpansive mapping, we have

$$\begin{aligned} d(S(u_n), u_n) &\leq d(S(u_n), S(v_n)) + d(S(v_n), u_n) \\ &\leq \alpha d(S(u_n), v_n) + \alpha d(u_n, S(v_n)) + (1 - 2\alpha) d(u_n, v_n) + d(S(v_n), u_n) \\ &= \alpha d(S(u_n), v_n) + (1 + \alpha) d(u_n, S(v_n)) + (1 - 2\alpha) d(u_n, v_n) \\ &\leq \alpha d(S(u_n), u_n) + \alpha d(u_n, v_n) + (1 + \alpha) d(u_n, S(v_n)) + (1 - 2\alpha) d(u_n, v_n). \end{aligned}$$

Then, we obtain $(1-\alpha)d(S(u_n), u_n) \leq (1-\alpha)d(u_n, v_n) + (1+\alpha)d(u_n, S(v_n))$, since $\alpha \in [0, 1)$, we deduce

$$d(S(u_n), u_n) \leq d(u_n, v_n) + \frac{(1+\alpha)}{(1-\alpha)} d(u_n, S(v_n))$$

$$\leq d(u_n, P(u_n)) + d(P(u_n), v_n) + \frac{(1+\alpha)}{(1-\alpha)} d(u_n, v_n).$$

From (3.7), (3.9) and (3.10), we round off

$$\lim_{n\to\infty} d(S(u_n), u_n) = 0.$$

□

The following result corresponds to the result known as demiclosedness principle, which due to Gohde in u.c. Banach space.

Lemma 3.4. *Suppose that P is a self grz. $\alpha-$ nonexpansive mapping given on a nonempty convex closed subset $\mathcal{K}$ of a complete u.c.hbc.s. X. Let $\{u_n\} \in \mathcal{K}$ be an approximate f.p. sequence of P, i.e., $\lim_{n\to\infty} d(u_n, P(u_n)) = 0$. If $x \in \mathcal{K}$ is the asymptotic center of $\{u_n\}$ with respect to $\mathcal{K}$, then u is a f.p. of P. In particular, if $\{u_n\} \in \mathcal{K}$ is an approximate f.p. sequence of P, such that $\Delta - \lim_{n\to\infty} u_n = u$, then $u \in Fix(P)$.*

Proof. Assume that $u \in \mathcal{K}$ is the unique asymptotic center of $\{u_n\}$ with respect to $\mathcal{K}$. And assume that $\{u_n\}$ is an approximate f.p. sequence of P. Since,

$$\begin{aligned} d(P(u), u_n) &\leq d(P(u), P(u_n)) + d(P(u_n), u_n) \\ &\leq \alpha d(P(u), u_n) + \alpha d(u, P(u_n)) + (1-2\alpha) d(u, u_n) + d(P(u_n), u_n). \end{aligned}$$

Then, we have

$$\begin{aligned} (1-\alpha) d(P(u), u_n) &\leq \alpha d(u, u_n) + \alpha d(u_n, P(u_n)) + (1-2\alpha) d(u, u_n) + d(P(u_n), u_n) \\ &= (1-\alpha) d(u, u_n) + (1+\alpha) d(u_n, P(u_n)). \end{aligned}$$

Then,

$$d(P(u), u_n) \leq d(u, u_n) + \frac{(1+\alpha)}{(1-\alpha)} d(u_n, P(u_n)).$$

Whence, we obtain

$$\begin{aligned} r(P(u), \{u_n\}) &= \limsup_{n\to\infty} d(P(u), u_n) \\ &\leq \limsup_{n\to\infty} d(u, u_n) + \limsup_{n\to\infty} \frac{(1+\alpha)}{(1-\alpha)} d(u_n, P(u_n)) \\ &= \limsup_{n\to\infty} d(u, u_n) = r(u, \{u_n\}). \end{aligned}$$

By Lemma 2.1, we deduce

$$P(u) = u.$$

Whence, u is a f.p. of P. □

Theorem 3.1. *(**$\Delta-$ convergent result**) Suppose that P, S are two self grz. $\alpha-$ nonexpansive mappings given on a nonempty closed convex subset $\mathcal{K}$ of a complete u.c.hbc.s. X with monotone modulus of uniform convexity η, such that $Fix(P) \cap Fix(S)$ is nonempty. Then the sequence $\{u_n\}$, which given by Ishikawa iteration* (1.1) *$\Delta-$ converges to a comm.f.p. of P and S.*

Proof. Step 1: by Lemma 3.2, we have $\lim_{n\to\infty} d(u_n, \zeta)$ exists, for each $\zeta \in Fix(P) \cap Fix(S)$.
Step 2: by Lemma 3.3, we attain

$$\lim_{n\to\infty} d(u_n, S(u_n)) = 0 = \lim_{n\to\infty} d(u_n, P(u_n)).$$

Step 3: suppose $\psi_\Delta(u_n)$ is the set of all $\Delta-$ limits of the sequence $\{u_n\}$. Assume $\psi_\Delta(u_n)$, where the union is taken over all subsequences $\{u_{n_k}\}$ of $\{u_n\}$.

Now, we prove that $\psi_\Delta(u_n) \subset Fix(P) \cap Fix(S)$.

Let ζ is $\Delta-$limit of the sequence $\{u_n\}$. Since by Lemma 3.2 $\{u_n\}$ is an approximate f.p. sequence of P and S, then by Lemma 3.4 we have $\zeta \in Fix(P) \cap Fix(S)$. Whence, $\psi_\Delta(u_n) \subset Fix(P) \cap Fix(S)$.

Then, we prove that $\psi_\Delta(u_n)$ is a singleton set. Let l and $\mathcal{K}$ are $\Delta-$limits of the subsequences $\{u_{n_l}\}$ and $\{u_{n_k}\}$ of $\{u_n\}$, respectively.

By Lemma 2.1, $A_k(\{u_{n_k}\}) = \{k\}$ and $A_l(\{u_{n_l}\}) = \{l\}$.

By Lemma 3.3, we have $\lim_{k\to\infty} d(u_{n_k}, S(u_{n_k})) = 0 = \lim_{k\to\infty} d(u_{n_k}, P(u_{n_k}))$.

And $\lim_{l\to\infty} d(u_{n_l}, S(u_{n_l})) = 0 = \lim_{l\to\infty} d(u_{n_l}, P(u_{n_l}))$.

By Lemma 3.4, we round off that l and k are comm.f.p.(s) of P and S.

Now, we prove that $k = l$. Suppose $k \neq l$, then by the uniqueness of asymptotic center, we obtain

$$\begin{aligned}
\limsup_{n\to\infty} d(u_n, k) &= \limsup_{k\to\infty} d(u_{n_k}, k) \\
&< \limsup_{k\to\infty} d(u_{n_k}, l) \\
&= \limsup_{n\to\infty} d(u_n, l) \\
&= \limsup_{l\to\infty} d(u_{n_l}, l) \\
&< \limsup_{l\to\infty} d(u_{n_l}, k) \\
&= \limsup_{n\to\infty} d(u_n, k),
\end{aligned}$$

arrives at a contradiction. Whence $k = l$.

Thus, the sequence $\{u_n\}$ $\Delta-$ converges to a comm.f.p. of P and S. □

Using the criterion (AV) on the mappings P and S, we set up the following result.

Theorem 3.2. ***(Strong convergent result)*** *Suppose that P, S are two self grz. $\alpha-$ nonexpansive mappings given on a nonempty closed convex subset $\mathcal{K}$ of a complete u.c.hbc.s. X with a monotone modulus of uniform convexity η. If $Fix(P) \cap Fix(S)$ is nonempty and P and S obey criterion (AV), then the sequence $\{u_n\}$, which is given by Ishikawa iteration* (1.1) *converges strongly to a comm.f.p. of P and S.*

Proof. By Lemma 3.1, $Fix(P) \cap Fix(S)$ is a closed subset of X. By using criterion (AV) and Lemma 3.3, we obtain $\lim_{n\to\infty} d(u_n, Fix(P) \cap Fix(S)) = 0$.

By Lemma 3.2, we have $u_n \to \zeta$ as $n \to \infty$, for some $\zeta \in Fix(P) \cap Fix(S)$. □

Theorem 3.3. ***(Strong convergent result)*** *Suppose that P, S are two self grz. $\alpha-$ nonexpansive mappings given on a nonempty closed convex subset $\mathcal{K}$ of a complete u.c.hbc.s. X with a monotone* *modulus of uniform convexity η, let $\{u_n\}$ be a sequence given by Ishikawa iteration* (1.1). *If $Fix(P) \cap Fix(S)$ is nonempty, then the sequence $\{u_n\}$ converges strongly to a comm.f.p. of P and S iff $\liminf_{n\to\infty} d(u_n, Fix(P) \cap Fix(S)) = 0$, where $d(u, Fix(P) \cap Fix(S))$ is the distance from u to the comm.f.p. set $Fix(P) \cap Fix(S)$.*

Proof. Assume that $\{u_n\}$ converges strongly to $\zeta \in Fix(P) \cap Fix(S)$. Whence, it is clear that $\liminf_{n\to\infty} d(u_n, Fix(P) \cap Fix(S)) = 0$. Conversely, suppose that $\liminf_{n\to\infty} d(u_n, Fix(P) \cap Fix(S)) = 0$. As by Lemma 3.2, $\lim_{n\to\infty} d(u_n, Fix(P) \cap Fix(S))$ exists, then $\lim_{n\to\infty} d(u_n, Fix(P) \cap Fix(S)) = 0$.

Now, without loss of generality, assume $\{u_{n_k}\}$ to be a subsequence of $\{u_n\}$, such that $d(u_{n_k}, \zeta_k) < \frac{1}{2^k}$, for all $k \geq 1$, where $\{\zeta_k\}$ is a sequence in $Fix(P) \cap Fix(S)$. By Lemma 3.2, we obtain

$$d(u_{n_{k+1}}, \zeta_k) \leq d(u_{n_k}, \zeta_k) \leq \frac{1}{2^k} \tag{3.11}$$

Next, we prove that $\{\zeta_k\}$ is a Cauchy sequence in $Fix(P) \cap Fix(S)$. From (3.11), we deduce

$$\begin{aligned} d(\zeta_{k+1}, \zeta_k) &\leq d(\zeta_{k+1}, u_{n_{k+1}}) + d(u_{n_{k+1}}, \zeta_k) \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}. \end{aligned}$$

This shows that $\{\zeta_k\}$ is a Cauchy sequence in $Fix(P) \cap Fix(S)$. As by Lemma 3.1, $Fix(P) \cap Fix(S)$ is a closed subset of X, $\{\zeta_k\}$ converges to a comm.f.p. ζ of P and S. Since $d(u_{n_k}, \zeta) \leq d(u_{n_k}, \zeta_k) + d(\zeta_k, \zeta) \to 0$ as $k \to \infty$,
then

$$\lim_{k\to\infty} d(u_{n_k}, \zeta) = 0.$$

By Lemma 3.2, $\lim_{n\to\infty} d(u_n, \zeta)$ exists, whence, the sequence $\{u_n\}$ is convergent to ζ. □

In 2017, Pant and Shukla shown an interesting example of the grz. $\alpha-$ nonexpansive mapping in the linear case of hyperbolic metric space and presented certain comparative convergence behaviors with regards to some powerful iteration procedures including the famous Mann and Ishikawa iterations among others. The next example is given to support our results. We use MATLAB to check the convergence of the mappings P and S to a common fixed point. Thereafter we expose the results in the following Table 1 and Figure 1.

Table 1. Influence of corfficients and initial guesses.

Influence of initial gusses with $t_n = \frac{1}{n+5}$ and $s_n = \frac{n}{n+5}$ coefficients	
Initial gusses	Convergence in number of iterations
$u_1 = \frac{1}{4}$	101
$u_1 = \frac{1}{8}$	101
$u_1 = \frac{1}{10}$	101
Influence of initial gusse $u_1 = \frac{1}{4}$	
Coefficients	Convergence in number of iterations
$t_n = \frac{1}{(3n+7)^{\frac{3}{2}}}$, $s_n = \frac{2n}{7n+10}$	29
$t_n = \frac{2n}{(5n+2)^3}$, $s_n = \frac{5n}{(7n+1)^{\frac{5}{4}}}$	2
$t_n = \frac{\sqrt{n}}{(n+5)^{\frac{5}{2}}}$, $s_n = \frac{1}{\sqrt{n+5}}$	288

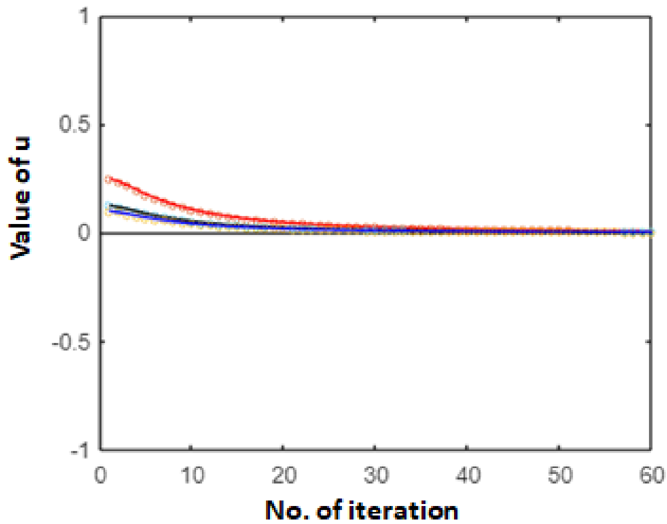

Figure 1. Convergence behavior of two grz. α- nonexpansive mappings.

Example 3.1. *Consider the real line* $\mathbb{R}$ *as a hyperbolic metric space and* $\mathcal{K}$ *be the subset of* $\mathbb{R}$, $\mathcal{K} = [-1, 1]$ *with the usual norm* $|.|$ *and let* $P : \mathcal{K} \to \mathcal{K}$ *be defined as:*

$$P(u) = \begin{cases} \frac{u}{2}, & if \;\; u \in [-1, 0) \\ -u, & if \;\; u \in [0, 1] \setminus \left\{\frac{1}{2}\right\} \\ 0, & if \;\; u = \frac{1}{2}. \end{cases} \quad and \quad S(u) = \begin{cases} u, & if \;\; u \in [-1, 1) \\ \frac{1}{2}, & if \;\; u = 1. \end{cases}$$

Then P and S are grz. $\frac{1}{2}-$ *nonexpansive mapping. Moreover, Fix(P)=*$\{0\}$ *and Fix(S)=*$[-1, 1)$*, then we deduce Fix(P)* $\cap$ *Fix (S)=* $\{0\}$*, i.e., 0 is a comm.f.p. of P and S.*

Proof. *First, we show that P is a grz.* $\frac{1}{2}$*-nonexpansive mapping, by following the same steps in [22]. With* $\alpha = \frac{1}{2}$*, we have a different cases:*
Case 1: when $u, v \in [-1, 0)$*, we get*

$$\begin{aligned}\alpha d(P(u), v) + \alpha d(P(v), u) + (1-2\alpha)d(u, v) &= \frac{1}{2}\left|\frac{u}{2} - v\right| + \frac{1}{2}\left|\frac{v}{2} - u\right| \\ &= \frac{1}{2}\left|\frac{u}{2} - v\right| + \frac{1}{2}\left|u - \frac{v}{2}\right| \\ &\geq \frac{1}{2}\left|\frac{u}{2} - v + u - \frac{v}{2}\right| \\ &= \frac{3}{4}|u - v| \\ &\geq \frac{1}{2}|u - v| = d(P(u) - P(v)).\end{aligned}$$

Case 2: when $u \in [-1, 0)$*,* $v \in [0, 1] \setminus \left\{\frac{1}{2}\right\}$*, we get*
since $u < 0$ *and* $v \geq 0$*,*

$$d(P(u), P(v)) = \left|\frac{u}{2} - v\right| = \begin{cases} \frac{u}{2} + v, & \text{if } \frac{|u|}{2} < v, \\ -\frac{u}{2} - v, & \text{if } \frac{|u|}{2} \geq v. \end{cases}$$

In the first case, we conclude

$$\begin{aligned}\alpha d(P(u), v) + \alpha d(P(v), u) + (1-2\alpha)d(u, v) &= \frac{1}{2}\left|\frac{u}{2} - v\right| + \frac{1}{2}|-v - u| \\ &\geq \frac{1}{2}\left(v - \frac{u}{2}\right) + \frac{1}{2}(v + u) \\ &= \frac{u}{4} + v \\ &\geq \frac{u}{2} + v = d(P(u), P(v)).\end{aligned}$$

In the second case, we deduce

$$\begin{aligned}\alpha d(P(u), v) + \alpha d(P(v), u) + (1-2\alpha)d(u, v) &= \frac{1}{2}\left|\frac{u}{2} - v\right| + \frac{1}{2}|-v - u| \\ &\geq \frac{1}{2}\left(v - \frac{u}{2}\right) + \frac{1}{2}(-v - u) \\ &= -\frac{3}{4}u \\ &\geq -\frac{u}{2} - v = d(P(u), P(v)).\end{aligned}$$

Case 3: when $u \in [-1, 0)$ *and* $v = \frac{1}{2}$*, then*

$$\alpha d(P(u), v) + \alpha d(P(v), u) + (1-2\alpha)d(u, v) = \frac{1}{2}\left|\frac{u}{2} - \frac{1}{2}\right| + \frac{1}{2}|0 - u|$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\frac{1}{2} - \frac{u}{2}\right) - \frac{1}{2}u \\
&= \frac{3}{4}u + \frac{1}{4} \\
&\geq -\frac{u}{2} = d(P(u), P(v)).
\end{aligned}
$$

Case 4: when $u, v \in [0,1] \setminus \left\{\frac{1}{2}\right\}$ and $u \leq v$, we have

$$
\begin{aligned}
\alpha d(P(u), v) + \alpha d(P(v), u) + (1-2\alpha) d(u, v) &= \frac{1}{2}\left|-u - v\right| + \frac{1}{2}\left|-v - u\right| \\
&= \frac{1}{2}(u + v) + \frac{1}{2}(v + u) \\
&= (v + u) \geq v - u = d(P(u), P(v)).
\end{aligned}
$$

Case 5: when $u \in [0,1] \setminus \left\{\frac{1}{2}\right\}$, $v = \frac{1}{2}$, we have

$$
\begin{aligned}
\alpha d(P(u), v) + \alpha d(P(v), u) + (1-2\alpha) d(u, v) &= \frac{1}{2}\left|-u - \frac{1}{2}\right| + \frac{1}{2}\left|0 - u\right| \\
&= \frac{1}{2}\left(u + \frac{1}{2}\right) + \frac{1}{2}u \\
&= u + \frac{1}{4} \geq u = d(P(u), P(v)).
\end{aligned}
$$

Second, we show that S is a grz. $\frac{1}{2}$-nonexpansive mapping. With $\alpha = \frac{1}{2}$, we have the following cases:
Case 1: when $u, v \in [-1, 1)$ then

$$
\begin{aligned}
d(S(u), S(v)) &= |u - v|, \\
d(S(u), v) &= |u - v|, \\
d(S(v), u) &= |v - u|, \\
d(u, v) &= |u - v|.
\end{aligned}
$$

Thus,

$$
d(S(u), S(v)) = |u - v| \leq \frac{1}{2}|u - v| + \frac{1}{2}|u - v| = |u - v|.
$$

Case 2: when $u \in [-1, 1)$, $v = 1$, then

$$
\begin{aligned}
d(S(u), S(v)) &= \left|u - \frac{1}{2}\right| \leq |u| + \frac{1}{2}, \\
d(S(u), v) &= |u - 1| \leq |u| + 1, \\
d(S(v), u) &= \left|\frac{1}{2} - u\right| \leq \frac{1}{2} + |u|, \\
d(u, v) &= |u - 1| \leq |u| + 1.
\end{aligned}
$$

Thus,

$$\begin{aligned} d(S(u), S(v)) &\leq \frac{1}{2} + \frac{|u|}{2} + \frac{1}{4} + \frac{|u|}{2} \\ &= \frac{3}{4} + |u|. \end{aligned}$$

Case 3: when $u = 1, v = 1$*, then*

$$\begin{aligned} d(S(u), S(v)) &= \left|\frac{1}{2} - \frac{1}{2}\right| = 0, \\ d(S(u), v) &= \left|\frac{1}{2} - 1\right| = \frac{1}{2}, \\ d(S(v), u) &= \left|\frac{1}{2} - 1\right| = \frac{1}{2}, \\ d(u, v) &= |1 - 1| = 0. \end{aligned}$$

Thus,

$$\begin{aligned} d(S(u), S(v)) &\leq \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right) \\ &= \frac{1}{4} + \frac{1}{4} \\ &= \frac{1}{2}. \end{aligned}$$

□

4. Conclusions

This paper focused on investigating the strong and $\Delta-$ convergence of the Ishikawa iteration process which was proposed in [3] to a common fixed point of two generalized $\alpha-$ nonexpansive mappings. A $\Delta-$ convergence result in Theorem 3.1 generalized Theorem 3.8 in [2] as well as two strong convergence results in Theorem 3.2 and Theorem 3.3 are presented by appending additional condition on the mappings. These results correspond to the idea of Theorem 2.7 in [6] and Theorem 3.11 in [26], in that order. Furthermore, we illustrate our results by exhibiting a numerical example. A graphical representation showed the convergence behavior by using Ishikawa iteration of two generalized $\alpha-$ nonexpansive mappings. We noted that our results achieved herein can be refined to larger spaces.

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Conflict of interest

The authors declare that they have no competing interests.

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