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Finite groups for which all proper subgroups have consecutive character degrees

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Abstract: Huppert and Qian et al. classified finite groups for which all irreducible character degrees are consecutive. The aim of this paper is to determine the structure of finite groups whose irreducible character degrees of their proper subgroups are all consecutive.

Keywords: simple group; proper subgroup; character degree; solvable group
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1. Introduction

Throughout this paper, all considered groups are finite. Denote the set of all complex irreducible characters of a group G by $\mathrm{Irr}(G)$, denote the linear characters of G by $\mathrm{Lin}(G)$, and write $\mathrm{cd}(G)$ for the set of all complex irreducible character degrees of G, i.e.,

$$\mathrm{cd}(G) = \{\chi(1) : \chi \in \mathrm{Irr}(G)\}.$$

We use usually the *degrees* of G instead of the more precise *irreducible character degrees*.

Huppert determined the structure of finite groups with consecutive character degrees; see Section 32 of [14]. In order to argue in short, we give a conception.

Definition 1.1. *Let G be a finite group. If the degrees from* $\mathrm{cd}(G)-\{1\}$ *are consecutive, then we call G a* CCD-*group (or a non-*CCD-*group otherwise). We also assume that an abelian group is a* CCD-*group.*

Qian in [26] improved the result of Huppert [14] and considered CCD-groups.

Let $d_i(G)$ or d_i without confusion be the smallest i-th degree in $\mathrm{cd}(G) - \{1\}$. Tiep and Zalesskii in [32] determined the first three smallest character degrees of classical groups. Recently, Liu, Lei and Li in [21] considered non-solvable CCD-groups by using the result of [32].

On the other hand, some authors considered the influence of subgroups with certain properties on the structure of finite groups. In particular, the structures of finite groups are determined if the proper subgroups are all abelian [1] or solvable [27, 31] or supersolvable [6] or nilpotent [18] or minimal non-nilpotent [2].

Recently some scholars combined the representation theory of finite groups with subgroup properties. In particular, the structures of finite groups are determined if all their subgroups have prime power degrees or degrees divisible by two primes; see [20, 22].

In this paper, we consider the influence of proper (maximal) subgroups whose degrees are consecutive on the structure of finite groups. Let $A * B$ be the central product of two groups A and B. To argue in short, we begin with the following definition.

Definition 1.2. *Let $\sum(G)$ be the set of all proper subgroups of a group G. If for each $M \in \sum(G)$, M is a* CCD*-group, then we call G an* SCCD*-group (or a non-*SCCD*-group otherwise). We always assume that an abelian group is an* SCCD*-group.*

Theorem 1.3. *Let G be a non-solvable* SCCD*-group. Then, G is isomorphic to one of the following groups:*

(1) $\mathrm{PSL}_2(q)$ *where $q \geq 4$ is even or $q \geq 5$ is an odd prime or $q \in \{3^p, 5^p\}$ with p a prime.*

(2) $S \times H$ where $S \in \{A_5, S_5\}$ and H is abelian.

(3) $\mathrm{SL}_2(q)$ *where $q \geq 7$ is an odd prime or $q \in \{3^p, 5^p\}$ with p a prime.*

(4) $\mathrm{SL}_2(5) * H$ *with H/C_2 abelian.*

We call a group G a CC-group if $\mathrm{cd}(G) = \{1, 2, 3, \cdots, k\}$. Huppert investigated the structure of finite CC-groups G; see Sec. 32 of [14]. Let m be a positive integer. Then, what can we see if for each $H \in \sum G$, $\mathrm{cd}(H) = \{1, 2, 3, \cdots, m\}$? We also call a group G an SCC-group if for each $H \in \sum G$, H is a CC-group. As an application of Theorem 1.3, we have the following result.

Theorem 1.4. *Let G be an* SCC*-group. Then, G is solvable. In particular, one of the following statements holds:*

(i) $\mathrm{cd}(G) \subseteq \{1, 2\}$.

(ii) G is nilpotent, and G is isomorphic to a metabelian p-group or a direct product of a 2-group P and $2'$-group H where $\mathrm{cd}(P) = \{1, 2\}$ *and* $\mathrm{cd}(H) = \{1\}$.

(iii) G is a Frobenius group with kernel F and complement H, respectively, such that

(iii.a) $\mathrm{cd}(F) = \{1, 2\}$ *and H is of a prime order such that for each non-identity $W \in \sum G$,* $|W| \not\equiv 1 \pmod{|H|}$.

(iii.b) H is of prime order or 4; F is abelian. Furthermore, if H is of order 4, then F is isomorphic to $C_p \times C_q$ with $q \equiv p \equiv -1 \pmod 4$ *or C_p with* $p \equiv 1 \pmod 4$*; if H is of prime order > 2, then* $|H| \,\|\, (|F| - 1)$.

(iv) G is a direct product of two groups H_1 and H_2 where H_1 and H_2 are metabelian with $\mathrm{cd}(H_i) = \{1, 2\}$, $i = 1, 2$*; or only one of H_1 and H_2 is metabelian, say, H with* $\mathrm{cd}(H) = \{1, 2\}$.

The rest of the paper is organized as follows. In Section 2, we assemble some basic results which will be used throughout this paper. In Section 3, we determine the structure of the simple CCD-groups. In Section 4, we classify the non-solvable SCCD-groups. In Section 5, we give an application of Theorem 1.3 and also determine the structure of an SCC-group.

Let $A : B$ or $A.B$ be an extension of a group A by a group B. Let $\mathrm{GL}_n(q)$ be the general linear group of dimension n over a finite field of order q. Let m, n be positive integers, denote by $m \| (n-1)$ if $m|(n-1)$ but $m \nmid (\frac{n}{p} - 1)$ for all nontrivial divisors p of n. The other notation is standard; please see [7, 15], for instance.

2. Basic results

In this section, some needed results are shown. First, some information about Frobenius groups is given.

Lemma 2.1 (Theorems 13.3 and 13.8 of [9])**.** *Let $G = K : H$ be a Frobenius group with kernel K and complement H. Then,*

(i) $|H| \big| (|K| - 1)$.

(ii) Any subgroup of H of order p^2 or pq is cyclic where p and q are primes.

(iii) If $|H|$ is even, K is abelian.

(iv) In any case, K is nilpotent.

(v) Assume that K has $h(K)$ conjugacy classes, H has $h(H)$ conjugacy classes, and G has characters with the following forms:

(a) $h(H)$ irreducible characters $\chi_1, \cdots, \chi_{h(H)}$ with K in their kernel. If $\mu_1, \cdots, \mu_{h(H)}$ are the irreducible characters of H, then these satisfy

$$\chi_i(hk) = \mu_i(h) \text{ for all } h \in H,\ k \in K.$$

(b) Whenever $\tau \neq 1_K$ is an irreducible character of K, then τ^G is an irreducible character of G. This gives $[h(K) - 1]/|H|$ irreducible characters of G with K not in their kernel. Such τ^G satisfies

$$\tau^G|_H = \tau(1)\rho_H$$

where ρ_H is the regular character of H.

Related to Frobenius groups, we need some character degrees for special cases to determine the group structure. From [19, Table 1] and [24, Theorem] and using the notation of [3], we get that ${}^2B_2(q)$ with $q = 2^{2m+1}$ and ${}^2G_2(q)$ with $q = 3^{2m+1}$ have maximal subgroups of the form $K : C_{q-1}$, where K is a Sylow 2 or 3-subgroup of ${}^2B_2(q)$ or ${}^2G_2(q)$, respectively. Now, we have the following result by using the above Lemma.

Lemma 2.2. *Let $G \in \{{}^2B_2(q)$ with $q = 2^{2m+1} \geq 2^3$, ${}^2G_2(q)$ with $q = 3^{2m+1} \geq 3^3\}$ and let K be a Sylow $\pi(q)$-subgroup of G. Then, G has a maximal subgroup of the form $K : C_{q-1}$ that is a non-*CCD*-group.*

Proof. The first claim follows from above arguments.

Now, we prove that $K : C_{q-1}$ is a non-CCD-group. Let $H := C_{q-1}$. Now, $K : H$ is a Frobenius group. So, we need to prove that $K : H$ is not a CCD-group. Assume the contrary, and then $K : H$ is a CCD-group.

To get the desired result, we consider two cases in what follows.

Case 1: $G ={}^2 B_2(q)$.

Let $\tau \in \mathrm{Irr}(K) - \mathrm{Lin}(K)$. Note that K is a Sylow 2-subgroup of ${}^2B_2(q)$ since $|K| = |{}^2B_2(q)|_2 = q^2$. We see that $Z(K)$ is of order q, so $K/Z(K)$ of order q is abelian.

For any $1 \neq d \in \mathrm{cd}(K)$, we have that $d \mid 2^m$. In fact, $|K/Z(K)| = q$, so for each $d \in \mathrm{cd}(K)$, one has $d \Big| \sqrt{q} = 2^{\frac{2m+1}{2}}$, i.e., $d \Big| 2^m$.

On the other hand, $G/K \cong H$ is cyclic, so $(q-1) \in \mathrm{cd}(G)$, as the cyclic group H acts faithfully on the nilpotent group K.

Lemma 2.1(v)(b) implies that $\mathrm{cd}(K:H) = \{1, d, q-1, d(q-1)\}$ with $d \neq 1$. Hypothesis and $d \Big| 2^m$ show that $d + 2 = d(q-1)$, i.e., $d = \frac{2}{q-2}$, so $1 \leq q-2 \leq 2$ implies $q = 4 \ngeq 8$, a contradiction to the hypothesis.

From now on, we have proved that $K : H$ is a non-CCD-group when $G =^2 B_2(q)$.

Case 2: $G =^2 G_2(q)$.

Similarly, we can get that K is a Sylow 3-subgroup of ${}^2G_2(q))$ since $|K| = |{}^2G_2(q)|_3 = q^3$. By a Theorem of [34], one has that the center $Z(K)$ of K is of order q.

Let $d \in \mathrm{cd}(K) - \mathrm{Lin}(K)$. Then, $d \Big| q$, and in particular, $d \leq q$. Note that $(K:H)/K \cong H$ is a cyclic group, so $(q-1) \in \mathrm{cd}(G)$, as the cyclic group H acts faithfully on the nilpotent group K. Now, Lemma 2.1(v)(b) implies that $\mathrm{cd}(K:H) = \{1, q-1, d, d(q-1)\}$. Hypothesis forces that $q(q-1)-1 = q$ for $d = q$ or $d + 2 = d(q-1)$ for $d \Big| q$, so $q = 3 \ngeq 3^3$, a contradiction. Therefore, $K : H$ is a non-CCD-group when $G =^2 G_2(q)$.

Now, the lemma is proved. □

Let $\max G$ or $\max(G)$ be the set of all proper maximal subgroups with respect to their subgroup-order divisibility. Then, $\max G \subseteq \sum G$.

Lemma 2.3. *(1) G is an* SCCD*-group if and only if every proper subgroup H of G is a* CCD*-group.*

(2) Let N be a non-trivial normal subgroup of an SCCD*-group G. Then, G/N is a* CCD*-group.*

Let $\Phi(G)$ be the Frattini subgroup of a group G.

Proof. (1) By Definition of an SCCD-group, we can get this.

(2) Since N is non-trivial and normal in G, we assume that N is minimal. Assume that G/N is a non-CCD-group. If $N \nleq \Phi(G)$, then choose a maximal subgroup $M \in \max G$ with $N \nleq M$, so $G = MN$. Note that $M \cap N = N$ or $M \cap N = 1$. If the former, then $N \leq M$, a contradiction to the choice of M. Thus, G/N is isomorphic to M, which is a CCD-group, a contradiction also. If $N \leq \Phi(G)$, then for $M \in \max G$, $N \leq M$. Since G/N is a non-CCD-group, we have that G/N is a non-SCCD-group, and so in G/N there is a non-CCD-subgroup, say, M/N. It follows that M is a non-CCD-group, a contradiction to $M \in \max G$ (note that for each $M \in \max G$, M is a CCD-group). □

In general, the product of two CCD-groups is not a an CCD-group.

Example 2.4. *Let A_1 be any maximal proper subgroup of a non-abelian p-group A of order p^4 with $\mathrm{cd}(A_1) = \{1, p\}$, and let B be a q-group with $\mathrm{cd}(B) = \{1, q\}$, where q, p are different primes. Let $G = A \times B$. Then, G is not an* SCCD*-group. We will show this by contradiction. Assume that G is an* SCCD*-group. Then, $A_1 \times B$ is a maximal subgroup of G, and $\mathrm{cd}(A_1 \times B) = \{1, p, q, pq\}$ by Theorem 4.21 of [15]. Say $p < q$. Then, $p + 1 = q$ by assumption, so $p = 2$, and $q = 3$. Thus, $\mathrm{cd}(A_1 \times B) = \{1, 2, 3, 6\}$, and $A_1 \times B$ is not a* CCD*-group, a contradiction.*

There is a CCD-group that is not an SCCD-group.

Example 2.5. *Let $G = \mathrm{PGL}_2(q)$ with $q > 5$ odd. Then, $\mathrm{cd}(G) = \{1, q-1, q, q+1\}$ by [30], so G is a* CCD*-group. Obviously,* $\mathrm{PSL}_2(q)$ *is a normal subgroup of* $\mathrm{PGL}_2(q)$ *with index* $\gcd(2, q-1)$*, and*

$$\mathrm{cd}(\mathrm{PSL}_2(q)) = \{1, \tfrac{q+(-1)^{(q-1)/2}}{2}, q-1, q, q+1\}.$$

If G is an SCCD*-group, then* $\mathrm{PSL}_2(q)$ *is a* CCD*-group. On the other hand, hypothesis forces* $\frac{q+(-1)^{(q-1)/2}}{2} + 1 = q - 1$ *or* $1 + 1 = \frac{q+(-1)^{(q-1)/2}}{2}$*, so the two equations have solutions* $q = 3, 5$*, a contradiction to the hypothesis. It follows that* $\mathrm{PSL}_2(q)$ *is not a* CCD*-group, so G is not an* SCCD*-group.*

An extension of an SCCD-group by an SCCD-group is not an SCCD-group usually.

Example 2.6. *Let S_5 be the symmetric group of degree 5. Then, S_5 is an* SCCD*-group (we can check it by [7]). Also, 2^4, an elementary group of order 2^4, is an* SCCD*-group. Now, let $G = 2^4 : S_5$. It is easy to see that $2^4 : A_5$, say, H, is a proper subgroup of G. If G is an* SCCD*-group, then H is a* CCD*-group by Lemma 2.3. On the other hand, by [4]* $\mathrm{cd}(H) = \{1, 3, 4, 5, 15\}$*, which means that H is not a* CCD*-group. It follows that G is not an* SCCD*-group.*

Let $\mathrm{PSL}^+_n(q) = \mathrm{PSL}_n(q)$ and $\mathrm{PSL}^-_n(q) = \mathrm{PSU}_n(q)$. Let $\mathrm{SL}^+_n(q) = \mathrm{SL}_n(q)$ and $\mathrm{SL}^-_n(q) = \mathrm{SU}_n(q)$. In order to prove our main result, we need some information about the structure of some special subgroup of a non-abelian simple group due to Liu [20].

Lemma 2.7 (Lemma 2 of [20])**.** *Let q be a prime power, and let $\epsilon = \pm$.*

(1) Let $n \geq 8$. Then, A_n has a subgroup A_{n-1}.

(2) Let $n \geq 4$. Then, $\mathrm{PSL}^\epsilon_n(q)$ *has a subgroup isomorphic to* $\mathrm{SL}^\pm_{n-1}(q)$ *or* $\mathrm{PSL}^\pm_{n-1}(q)$*, and* $\mathrm{SL}^\epsilon_n(q)$ *has a subgroup of the form* $\mathrm{SL}^\epsilon_{n-1}(q)$*.*

(3) Let $n \geq 2$. Then, $\mathrm{PSp}_{2n}(q)$ *has a subgroup* $\mathrm{PSp}_{2(n-1)}(q)$*.*

(4) Let $n \geq 3$, with q odd. Then, $\Omega_{2n+1}(q)$ contains a subgroup $\Omega_{2n-1}(q)$.

(5) Let $n \geq 4$. Then, $\mathrm{P}\Omega^\epsilon_{2n}(q)$ *has a subgroup* $\mathrm{P}\Omega^\epsilon_{2n-2}(q)$ *with q odd or* $\mathrm{PSp}_{2n-2}(q)$ *with q even.*

3. Simple SCCD-groups

By the classification theorem of finite simple groups, a non-abelian simple group is isomorphic to an alternating group A_n, a Sporadic simple group or Tits group, an exceptional group of Lie type or a classical group of Lie type. Now, we will determine the structure of the simple SCCD-groups by the following Lemmas.

Lemma 3.1. *Let G be an alternating A_n of degree $n \geq 5$. Assume that G is an* SCCD*-group, and then G is isomorphic to A_5 or A_6.*

Proof. We assume that G is an SCCD-group, so we need to check whether its subgroups are CCD-groups or not.

Let $n = 5$. Then, by [7, pp. 2], $\max A_5 = \{A_4, D_{10}, S_3\}$. Note that $\mathrm{cd}(A_4) = \{1, 3\}$, $\mathrm{cd}(D_{10}) = \mathrm{cd}(S_3) = \{1, 2\}$. It follows that all the subgroups of A_5 are CCD-groups and A_5 is an SCCD-group, so $G \cong A_5$.

Let $n = 6$, and then by [7, pp. 4], $\max A_6 = \{A_5, 3^2 : 4, S_4\}$. As $\mathrm{cd}(A_5) = \{1, 3, 4, 5\}$, $\mathrm{cd}(3^2 : 4) = \{1, 4\}$ and $\mathrm{cd}(S_4) = \{1, 2, 3\}$, one has that the subgroups $A_5, 3^2 : 4, S_4$ of A_6 are CCD-groups, and so A_6 is an SCCD-group. Thus, $G \cong A_6$.

Let $n = 7$, and then by [7, pp. 10], A_7 contains a subgroup A_6, but by [7, pp. 4] $\mathrm{cd}(A_6) = \{1, 5, 8, 9, 10\}$. Now, A_6 is not a CCD-group. It follows that A_7 is a non-SCCD-group.

If $n \geq 8$, then we get that A_n has a subgroup A_{n-1} by Lemma 2.7, so one has the subgroup series

$$A_7 < A_8 < \cdots A_{n-1} < A_n.$$

It follows that A_n with $n \geq 8$ is not an SCCD-group since A_7 is a non-CCD-group. □

Lemma 3.2. *There does not exist an* SCCD-*group for a Sporadic simple group.*

Proof. By [7], there is a subgroup $H \in \max G$ which is a non-CCD-group. □

Lemma 3.3. *There does not exist a simple exceptional* SCCD-*group of Lie type.*

Proof. It is well-known in this case that, for a simple group G of exceptional Lie type, G is isomorphic to ${}^2B_2(q)$ with $q = 2^{2m+1} \geq 8$, ${}^2G_2(q)$ with $q = 3^{2m+1}$, $m \geq 1$, $G_2(q)$, ${}^3D_4(q)$, $F_4(q)$, ${}^2F_4(q^2)$, $E_6^{\epsilon}(q)$, $E_7(q)$, or $E_8(q)$. We deal with these case by case.

The following two subgroup series are from Table 1:

$$G_2(q) > \mathrm{SU}_3(q^2).2 > \mathrm{SU}_3(q^2)$$

and

$${}^2F_4(q^2) > \mathrm{SU}_3(q^2).2 > \mathrm{SU}_3(q^2).$$

However, by [23], one has

$$\begin{cases} d_2(\mathrm{SU}_3(q^2)) = q^2 - q + 1, d_3(\mathrm{SU}_3(q^2)) = (q-1)(q^2 - q + 1), & \text{for } q \equiv 0, 1 \pmod 3 \\ d_2(\mathrm{SU}_3(q^2)) = q^2 - q + 1, d_3(\mathrm{SU}_3(q^3)) = \frac{1}{3}(q-1)(q^2 - q + 1), & \text{for } q \equiv 2 \pmod 3, \end{cases}$$

and hypothesis $d_2 + 1 = d_3$ implies $q = 2$ or 4, a contradiction. It follows that $G_2(q)$ and ${}^2F_2(q^2)$ are non-SCCD-groups. Note that ${}^2F_4(2)'$ is a non-SCCD-group by [7]. Also, we can show from Table 1 that ${}^3D_4(q)$, $F_4(q)$, and $E_6^{\epsilon}(q)$ are non-SCCD-groups. Now, if $\mathrm{P}\Omega_{12}^{+}(q)$ is a non-CCD-group, then $E_7(q)$ and $E_8(q)$ are non-SCCD-groups. In fact,

$$\mathrm{P}\Omega_{12}^{+}(q) > \cdots > \mathrm{P}\Omega_{4}^{+}(q) \cong \mathrm{PSL}_2(q) \times \mathrm{PSL}_2(q).$$

Now, by Theorem 4.21 of [15], $\mathrm{PSL}_2(q)\times\mathrm{PSL}_2(q)$ is not a CCD-group, so $\mathrm{P}\Omega_{12}^{+}(q)$ is a non-CCD-group. Thus, $E_7(q)$ and $E_8(q)$ are non-SCCD-groups.

Table 1. Exceptional groups of Lie type.

G	$H \in \max G$	Reference
$G_2(q)$	$\mathrm{SL}_3(q).2$, $\mathrm{SU}_3(q^2).2$	[19]
${}^3D_4(q)$	$G_2(q)$	[19]
$F_4(q)$	${}^3D_4(q^3)$	[19]
$E_6^{\epsilon}(q)$	$F_4(q)$	[19]
$E_7(q)$	$\mathrm{P}\Omega_{12}^{+}(q)$	[19]
$E_8(q)$	$E_7(q))$	[19]
${}^2F_4(q^2)$	$\mathrm{SU}_3(q^2).2$	[24]

Now, by Lemma 2.2, ${}^2B_2(q)$ with $q = 2^{2m+1} \geq 8$ and ${}^2G_2(q)$ with $q = 3^{2m+1}$, $m \geq 1$, are non-SCCD-groups. □

Proposition 3.4. *Let G be a simple classical* SCCD-*group of Lie type. Then, G is isomorphic to* $\mathrm{PSL}_2(q)$ *where q is even, or q is an odd prime, or* $q \in \{3^p, 5^p\}$ *with p a prime.*

Proof. The Classification Theorem of Finite Simple Groups shows that G may be isomorphic to one of the following groups: $\mathrm{PSL}_{l+1}(q)$, $\mathrm{PSU}_{l+1}(q)$, $\Omega_{2l+1}(q)$, $\mathrm{PSp}_{2l}(q)$, $\mathrm{P}\Omega^{+}_{2l}(q)$ or $\mathrm{P}\Omega^{-}_{2l}(q)$, where q is a power of a prime. Now, we need to show whether these groups are SCCD-groups or not.

Case 1: $\mathrm{PSL}_{l+1}(q)$ **with** $l \geq 1$**.**

Subcase 1: $l = 1$.

Note from Table 2 that $\mathrm{cd}(E_q : C_{(q-1)/k}) = \{1, \frac{q-1}{k}\}$, $\mathrm{cd}(D_{2(q\pm1)/k}) = \{1, 2\}$, $\mathrm{cd}(S_4) = \{1, 2, 3\}$, $\mathrm{cd}(A_4) = \{1, 3\}$, and $\mathrm{cd}(A_5) = \{1, 3, 4, 5\}$. Note that $E_q : C_{(q-1)/k}$, $D_{2(q\pm1)/k}$, S_4, A_4 and A_5 are CCD-groups, so in the following, we only need to consider maximal subgroup $H := \mathrm{PSL}_2(q_0).\gcd(\gcd(2, q-1), b)$ with $q = q_0^b$, $q_0 \neq 2$ and b a prime.

Table 2. $\mathrm{PSL}_2(q)$, $q \geq 5$ (Chap II Theo. 8.27 [12]).

max $\mathrm{PSL}_2(q)$	Condition
$E_q : C_{(q-1)/k}$	$k = \gcd(q-1, 2)$
$D_{2(q-1)/k}$	$q \notin \{5, 7, 9, 11\}$
$D_{2(q+1)/k}$	$q \notin \{7, 9\}$
$\mathrm{PSL}_2(q_0).(k, b)$	$q = q_0^b$, b a prime, $q_0 \neq 2$
S_4	$q = p \equiv \pm1 \pmod 8$
A_4	$q = p \equiv 3, 5, 13, 27, 37 \pmod{40}$
A_5	$q \equiv \pm1 \pmod{10}$, $F_q = F_p[\sqrt{5}]$

If $q = 2^n$, then $\gcd(\gcd(2, q-1), p) = 1$, H is isomorphic $\mathrm{PSL}_2(q_0)$, and $\mathrm{PSL}_2(q_0)$ is a CCD-group because $\mathrm{cd}(\mathrm{PSL}_2(q_0)) = \{1, q_0 - 1, q_0, q_0 + 1\}$. Thus, $\mathrm{PSL}_2(q)$ is an SCCD-group for q even.

Let $q = r^n$ be odd, where r is a prime. If $n = 1$, then $G \cong \mathrm{PSL}_2(q)$ with q a prime, as in this case, and $\mathrm{PSL}_2(q)$ possibly contains $E_q : C_{(q-1)/k}$, $D_{2(q-1)/k}$, $D_{2(q+1)/k}$, S_4, A_4, A_5 as its maximal subgroups. Note that all these possible maximal subgroups are CCD-groups. If $n > 1$, then $\gcd(2, q-1) = 2$, so G has a maximal subgroup $\mathrm{PSL}_2(q_0).(2, b)$ with $b \big| n$.

- If $b = 2$, then by Theorem A of [35], we can assume that $\mathrm{PGL}_2(q_0) \cong \mathrm{PSL}_2(q_0).2$ or $\mathrm{PGL}_2(q_0) \ncong \mathrm{PSL}_2(q_0).2$. Observe that $\mathrm{PSL}_2(q_0)$ is a normal subgroup of $\mathrm{PGL}_2(q_0)$ or $\mathrm{PSL}_2(q_0).2$. Then, we have by [8] that $\mathrm{cd}(\mathrm{PSL}_2(q_0)) = \{1, \frac{q_0+\varepsilon}{2}, q_0 - 1, q_0, q_0 + 1\}$ with $\varepsilon = (-1)^{(q-1)/2}$. It follows from hypothesis that $1 = \frac{q_0+\varepsilon}{2}$, or $\frac{q_0+\varepsilon}{2} + 1 = q_0 - 1$, and so $q_0 = 3$ or 5.
- If $b \geq 3$ is a prime, then $\gcd(2, b) = 1$, and $\mathrm{PSL}_2(q)$ contains a maximal subgroup $\mathrm{PSL}_2(q_0)$ by Table 2. As $\mathrm{cd}(\mathrm{PSL}_2(q_0)) = \{1, \frac{q_0+\varepsilon}{2}, q_0 - 1, q_0, q_0 + 1\}$ with $q = q_0^b$ and $\varepsilon = (-1)^{(q-1)/2}$, one gets from hypothesis that $q_0 = 3$ or 5, since the possible maximal subgroups $E_q : Z_{(q-1)/k}$, $D_{(q-1)/k}$, $D_{2(q+1)/k}$, S_4, A_4, A_5 for $\mathrm{PSL}_2(q)$ are CCD-groups.

It follows that $\mathrm{PSL}_2(r^b)$ is an SCCD-group where $r \in \{3, 5\}$ and $b \geq 2$ is a prime.

Subcase 2: $l = 2$.

If $q = q_0^b$ with b a prime, then by Table 3, G contains a maximal subgroup $\mathrm{PSL}_3(q_0)$. Note from [29] that $\mathrm{cd}(\mathrm{PSL}_3(q_0)) = \{1, q_0(q_0+1), q_0^2+q_0+1, (q_0+1)(q_0-1)^2, q_0^3-1, q_0^3, q_0(q_0^2+q_0+1), (q_0+1)(q_0^2+q_0+1)\}$. It is easy to see that the degrees of $\mathrm{PSL}_3(q_0)$ are not consecutive. Thus, q is a prime.

Table 3. $\mathrm{PSL}_3(q)$ and $\mathrm{PSU}_3(q)$ ([17]).

	max $\mathrm{PSL}_3(q)$	Condition	max $\mathrm{PSU}_3(q)$	Condition
$\mathcal{C}_1$	$E_{q^2} : \frac{1}{k}\mathrm{GL}_2(q)$		$E_q^{1+2} : Z_{\frac{q^2-1}{l}}$	
			$\frac{1}{l}\mathrm{GU}_2(q)$	
$\mathcal{C}_2$	$Z_{\frac{(q-1)^2}{k}}.S_3$	$q \geq 5$	$Z_{\frac{(q+1)^2}{l}}.S_3$	$q \neq 5$
$\mathcal{C}_3$	$Z_{\frac{q^2+q+1}{k}}.3$	$q \neq 4$	$Z_{(q^2-q+1)/l}.3$	$q \notin \{3,5\}$
$\mathcal{C}_5$	$\mathrm{PSL}_3(q_0).(k,b)$	$q = q_0^b$, b prime	$\mathrm{PSU}_3(q_0).(b,l)$	$q = q_0^b, b \geq 3$ prime
			$\mathrm{SO}_3(q)$	$q \geq 7$ odd
$\mathcal{C}_6$	$3^2.\mathrm{SL}_2(3)$	$q = p \equiv 1(9)$	$3^2.\mathrm{SL}_2(3)$	$q = p \equiv 8(9)$
	$3^2.Q_8$	$q = p \equiv 4,7(9)$	$3^2.Q_8$	$11 \leq q = p \equiv 2,5(9)$
$\mathcal{C}_8$	$\mathrm{SO}_3(q)$	q odd		
	$\mathrm{PSU}_3(q_0)$	$q = q_0^2$		
$\mathcal{S}$	A_6	$p \equiv 1,2,4,7,8,13(15)$	A_6	$p = q \equiv 11,14(15)$
		$F_q = F_p[\sqrt{5}, \sqrt{-3}]$		
	$\mathrm{PSL}_2(7)$	$2 < q = p \equiv 1,2,4(7)$	M_{10}	$q = 5$
			$\mathrm{PSL}_2(7)$	$5 \neq q = p \equiv 3,5,6(7)$
			A_7	$q = 5$

$k = \gcd(3, q-1)$, and $l = \gcd(3, q+1)$. Here, $m \equiv r(n)$ denotes $m \equiv r \pmod{n}$.

Since q is an odd prime, we can get by Table 3 that G contains a maximal subgroup $\mathrm{SO}_3(q) \cong \mathrm{SL}_2(q)$. We know from [16] that $\mathrm{cd}(\mathrm{SL}_2(q)) = \{1, \frac{q-1}{2}, \frac{q+1}{2}, q-1, q, q+1\}$, so by hypothesis, $q = 5$. By [7, pp. 38], $\mathrm{PSL}_3(5)$ contains a maximal subgroup $4^2 : S_3$, and by [4], $\mathrm{cd}(4^2 : S_3) = \{1, 2, 3, 6\}$, a contradiction to hypothesis. It follows that $q = 2$, and so $G \cong \mathrm{PSL}_3(2) \cong \mathrm{PSL}_2(7)$ is considered above.

Subcase 3: $l = 3$.

By Lemma 2.7, $\mathrm{PSL}_4(q)$ contains a subgroup isomorphic to either $\mathrm{PSL}_3(q)$ or $\mathrm{SL}_3(q)$. By [29], we obtain that either

$$d_2(\mathrm{PSL}_3(q)) = q^2 + q + 1 \text{ and } d_3(\mathrm{PSL}_3(q)) = (q-1)^2(q+1)$$

or

$$d_2(\mathrm{SL}_3(q)) = q^2 + q + 1 \text{ and } d_3(\mathrm{SL}_3(q)) \in \{(q-1)^2(q+1), (q-1)^2(q+1)/3\},$$

so hypothesis forces $d_2 + 1 = d_3$, but the equation has no solution in $\mathbb{N}$, the nonnegative integer set. Now, $\mathrm{PSL}_3(q)$ is not a CCD-group, and so $\mathrm{PSL}_4(q)$ is not an SCCD-group.

Subcase 4: $l \geq 4$.

Applying Lemma 2.7 repeatedly, we have that

$$\text{either } \mathrm{PSL}_3(q) < \mathrm{PSL}_{l+1}(q) \text{ or } \mathrm{SL}_3(q) < \mathrm{PSL}_{l+1}(q).$$

Note that $\mathrm{PSL}_3(q)$ and $\mathrm{SL}_3(q)$ are not CCD-groups by Subcase 3 of Case 1. Thus, $\mathrm{PSL}_{l+1}(q)$ is not an SCCD-group.

Case 2: $\mathrm{PSU}_{l+1}(q)$ with $l \geq 2$.

Note that $\mathrm{PSU}_2(q) \cong \mathrm{PSL}_2(q)$. We deal with this by two cases: $l = 2$ and $l \geq 3$.

• Let $l = 2$.

We know that $\mathrm{PSU}_3(2) \cong 3^2.Q_8$ is solvable, so $q \geq 3$. $\mathrm{PSU}_3(q)$ has a subgroup of the form: $E_q^{1+2} : C_{q^2-1}$. By the Main theorem of [28], $\mathrm{cd}(E_q^{1+2}) = \{1, q\}$. By Lemma 2.1, we get that either

$$\mathrm{cd}(E_q^{1+2} : C_{q^2-1}) = \{1, q, q^2 - 1, q(q^2 - 1)\} \text{ for } (3, q + 1) = 1$$

or

$$\mathrm{cd}(E_q^{1+2} : C_{(q^2-1)/3}) = \{1, q, (q^2 - 1)/3, q(q^2 - 1)/3\} \text{ for } (3, q + 1) = 3,$$

so hypothesis shows that $q + 1 = q^2 - 1$ or $q + 1 = \frac{q^2-1}{3}$. We have $q = 2 \not> 3$ or $q = 4$. Thus, by ATLAS [7], $\mathrm{PSU}_3(4)$ contains a maximal subgroup $5^2 : S_3$, and by [4], $\mathrm{cd}(5^2 : S_3) = \{1, 2, 3, 6\}$, a contradiction to hypothesis.

Now, we have that, for $q \geq 3$, $\mathrm{PSU}_3(q)$ is not an SCCD-group; in particular, $\mathrm{PSU}_3(q)$ is not a CCD-group.

• Let $l \geq 3$.

Applying Lemma 2.7 repeatedly, we get that

$$\text{either } \mathrm{PSU}_3(q) < \mathrm{PSU}_{l+1}(q) \text{ or } \mathrm{SU}_3(q) < \mathrm{PSU}_{l+1}(q),$$

and so $\mathrm{PSU}_{l+1}(q)$ is not an SCCD-group.

Case 3: $\mathrm{PSp}_{2l}(q)$ with $l \geq 2$.

Let $l = 2$. Since $\mathrm{PSp}_4(2) \cong S_6$, where S_n is a symmetric group of degree n, is not simple, we assume that $q \geq 3$. By [17, pp. 209], $\mathrm{PSp}_4(q)$ has a maximal subgroup $(\mathrm{PSp}_2(q) \times \mathrm{PSp}_2(q)).2 > \mathrm{PSp}_2(q) \times \mathrm{PSp}_2(q)$. Note that $\mathrm{PSp}_2(q) \cong \mathrm{PSL}_2(q)$ and that S_6 is a non-CCD-group.

If q is odd or $q \geq 4$ is even, then $q - 1, q, q + 1 \in \mathrm{cd}(\mathrm{PSL}_2(q))$, so by Theorem 4.21 of [15], one has that $(q + 1)^2, q(q + 1) \in \mathrm{cd}(\mathrm{SL}_2(q) \times \mathrm{SL}_2(q))$. By hypothesis, $(q + 1)q + 1 = (q + 1)^2$, and so $q = 2$, a contradiction.

Now, we get that $\mathrm{PSp}_4(q)$ is not an SCCD-group.

Let $l \geq 3$. Then, by Lemma 2.7, we get a subgroup series

$$\mathrm{PSp}_4(q) < \mathrm{PSp}_6(q) < \cdots < \mathrm{PSp}_{2(l-1)}(q) < \mathrm{PSp}_{2l}(q),$$

and $\mathrm{PSp}_{2l}(q)$ is not an SCCD-group.

Case 4: $\Omega_{2l+1}(q)$ with $l \geq 2$.

If $l = 2$, then $\Omega_5(q) \cong \mathrm{PSp}_4(q)$ is a non-SCCD-group by Case 3.

If $l = 3$, then $\Omega_5(q) < \Omega_7(q)$ by Lemma 2.7. We know from Case 3 that $\Omega_5(q)$ is a non-SCCD-group, and so is $\Omega_7(q)$.

Let $l \geq 4$. Then, Lemma 2.7 shows that $\Omega_{2l+1}(q)$ has a subgroup series

$$\Omega_5(q) < \Omega_7(q) < \cdots < \Omega_{2l-1}(q) < \Omega_{2l+1}(q),$$

so $\Omega_{2l+1}(q)$ is a non-SCCD-group.

Case 5: $\mathrm{P}\Omega_{2l}^{\varepsilon}(q)$ with $l \geq 4$, $\varepsilon = \pm$.

We first consider $\mathrm{P}\Omega_{2l}^{+}(q)$. If $l = 4$, then by [17, pp. 214], $\mathrm{P}\Omega_8^{+}(q)$ contains a subgroup $\mathrm{P}\Omega_7(q)$. It follows from Case 4 that $\mathrm{P}\Omega_8^{+}(q)$ is a non-SCCD-group.

If $l \geq 5$, then using Lemma 2.7, $\mathrm{P}\Omega_{2l}^{+}(q)$ contains a subgroup isomorphic to

$$\mathrm{P}\Omega_{2(l-1)}(q) \text{ for } q \text{ odd or } \mathrm{PSp}_{2(l-1)}(q) \text{ for } q \text{ even},$$

so Cases 3 and 4 imply that $\mathrm{P}\Omega_{2l}^{+}(q)$ is a non-SCCD-group.

Similarly we can see that $\mathrm{P}\Omega_{2l}^{-}(q)$ with $l \geq 4$ is a non-SCCD-group. □

Now, we can prove the following result.

Theorem 3.5. *Let G be a non-abelian simple* SCCD*-group. Then, G is isomorphic to* $\mathrm{PSL}_2(q)$ *where $q \geq 4$ is even or $q \geq 5$ is an odd prime or $q \in \{3^p, 5^p\}$ with p a prime.*

Note $A_5 \cong \mathrm{PSL}_2(5) \cong \mathrm{PSL}_2(4)$, and $A_6 \cong \mathrm{PSL}_2(3^2)$.

Proof. Lemmas 3.1– 3.3 and Proposition 3.4 imply the result. □

4. Non-solvable SCCD-groups

In this section, we first show the structure of almost simple SCCD-groups and then give that of non-solvable SCCD-groups.

Note that $\mathrm{PGL}_2(5) \cong S_5$, and let $\mathrm{Out}(A)$ be the outer-automorphism group of a group A.

Theorem 4.1. *Let G be an* SCCD*-group, and let S be a non-abelian simple group such that $S \leq G \leq$* $\mathrm{Aut}(S)$*. Then, G is isomorphic to one of the following groups:*

(1) $\mathrm{PSL}_2(q)$ *where $q \geq 4$ is even or $q \geq 5$ is an odd prime or $q \in \{3^p, 5^p\}$ with p a prime,*

(2) S_5.

Proof. Let G be an almost simple SCCD-group with socle S. Thus, S is also a simple SCCD-group, and by Theorem 3.5, $S \cong \mathrm{PSL}_2(q)$ where $q \geq 4$ is even or q is an odd prime or $q \in \{3^p, 5^p\}$ with p a prime.

If $q = 2^f$ for some $f \geq 2$, then G is isomorphic to $\mathrm{PSL}_2(2^f).d$ with $d \big| f$. If G is a simple group, then G is isomorphic to $\mathrm{PSL}_2(q)$ as $\mathrm{PSL}_2(q) \cong \mathrm{PGL}_2(q) \cong \mathrm{SL}_2(q)$. If $f = 2$, then $G \cong S_5$ is an SCCD-group. If $f \geq 3$, then $E_{2^f} : C_{2^f-1} : C_f$ is a subgroup of $\mathrm{PSL}_2(2^f).f$ (see [11]). By Lemma 2.1, one has $\mathrm{cd}(E_{2^f} : C_{2^f-1} : f) = \{1, f, 2^f - 1\}$. By hypothesis, we get $f + 1 = 2^f - 1$, so $f = 2 \ngeq 3$, a contradiction.

Now, q is odd, and by Theorem 3.5, these cases will be considered: $q \geq 5$ is an odd prime, or $q \in \{3^p, 5^p\}$ with p a prime.

Case 1: $q \geq 5$ is an odd prime.

Note from [7, pp. xvi] that $|\mathrm{Out}(\mathrm{PSL}_2(q))| = 2$ for q an odd prime. It follows that G is possibly isomorphic to $\mathrm{PGL}_2(q)$. Note that $\mathrm{PSL}_2(q) \in \max \mathrm{PGL}_2(q)$, and

$$\mathrm{cd}(\mathrm{PSL}_2(q)) = \{1, \tfrac{q+(-1)^{\frac{q-1}{2}}}{2}, q-1, q, q+1\}.$$

Lemma 2.3 gives that $\frac{q+(-1)^{\frac{q-1}{2}}}{2} + 1 = q - 2$, so $q = 5$. As $\mathrm{PSL}_2(5) \cong \mathrm{PSL}_2(4) \cong A_5$, $\mathrm{PGL}_2(5) \cong S_5$ is considered above.

Case 2: $q = 3^p$ or 5^p with p a prime.

Let $\mathfrak{S} = \{\mathrm{PGL}_2(q), \mathrm{PSL}_2(q).2, \mathrm{PSL}_2(q).p, \mathrm{PGL}_2(q).p, \mathrm{PSL}_2(q).(2p)\}$.

In this case, we obtain that $|\mathrm{Out}(\mathrm{PSL}_2(q))| = 2 \cdot p$, so G is possibly isomorphic to $S \in \mathfrak{S}$ by Corollary 6.2 of [35]. Observe that for each $S \in \mathfrak{S}$, S has a subgroup isomorphic to $\mathrm{PSL}_2(q)$. Note that

$$\mathrm{cd}(\mathrm{PSL}_2(q)) = \{1, \tfrac{q+\varepsilon}{2}, q-1, q, q+1\} \text{ with } \varepsilon = (-1)^{\frac{q-1}{2}},$$

so $\mathrm{PSL}_2(q)$ is not a CCD-group, as $q > 5$ is a power of an odd prime. It follows that for every $S \in \mathfrak{S}$, S is a non-SCCD-group. □

Now, we will give the proof of Theorem 1.3.

Proof of Theorem 1.3.

Proof. As G is non-solvable, we have a normal subgroup series $1 \leq H \leq K \leq G$ such that K/H is isomorphic to a direct product of isomorphic non-abelian simple groups and that $|G/K|$ divides the order of the outer-automorphism group $\mathrm{Out}(K/H)$; (see [36]).

Assume that

$$K/H \cong \underbrace{S \times S \times S \cdots \times S}_{m \text{ times}}$$

where S is isomorphic to one of the groups as listed in Theorem 3.5. If $m \geq 2$, then by Theorem 4.21 of [15], $L \times S \times S \cdots \times S$ is a non-CCD-group where L is a non-abelian subgroup of S. Now, $m = 1$, and so K/H is a simple group isomorphic to $S \in \mathfrak{S}$, where $\mathfrak{S}$ is a set which consists of the groups $\mathrm{PSL}_2(q)$ where $q \geq 4$ is even or $q \geq 5$ is an odd prime or $q \in \{3^p, 5^p\}$ with p a prime.

Case 1: K/H is isomorphic to $\mathrm{PSL}_2(q)$ with $q = 2^f$, $f \geq 2$.

We know that when q is even, $\mathrm{PGL}_2(q) \cong \mathrm{PSL}_2(q) \cong \mathrm{SL}_2(q)$. Thus, by Theorem 4.1, G/H is possibly isomorphic to $\mathrm{PSL}_2(q).f$. However, $E_q : C_{q-1} : C_f$ is a subgroup of $\mathrm{PSL}_2(q).f$, so $q = 4$, and G/H is isomorphic to A_5 or S_5, as the proof of Theorem 4.1. We see from Chap 2, Theorem 6.10 of [12] that $G'/H \cong A_5$. We know that the order of the Schur multiplier of A_5 is 2, and $|\mathrm{Out}(A_5)| = 2$.

If $[G', H] = 1$, then $G \cong H \times A_5$ or $G \cong H \times S_5$. If H is non-abelian, then we can assume that $\mathrm{cd}(H) = \{1, m\}$. Now, by [7], $H \times A_4$ and $H \times S_4$ are subgroups of $H \times A_5$ and $H \times S_5$ respectively, so $\mathrm{cd}(H \times A_4) = \{1, m, 3, 3m\}$ and $\mathrm{cd}(H \times S_4) = \{1, 2, 3, m, 2m, 3m\}$. We see that $H \times A_4$ and $S_4 \times H$ are CCD-subgroups, so $3 + 1 = 3m$ or $m + 1 = 3m$, but the two equations have no solutions in N. So, H is abelian.

If $|[G', H]| \neq 1$, then $G \cong \mathrm{SL}_2(5) * H$, and $H \geq C_2$. We similarly get that H/C_2 is abelian as the case $[G', H] = 1$.

Case 2: K/H is isomorphic to $\mathrm{PSL}_2(q)$ where $q \geq 5$ is an odd prime or $q \in \{3^p, 5^p\}$ with p a prime.

Where N1 maximal under $< \delta >$ with $|\delta| = (q-1, 2)$; N2 maximal under subgroups not contained in $< \varphi >$ with $|\varphi| = e$, $q = p^e$, p a prime.

From Case 1, we see that $q \geq 7$. By Theorem 4.1, G/H is isomorphic to $\mathrm{PSL}_2(q)$ where $q \geq 7$ is an odd prime or $q \in \{3^p, 5^p\}$ with p a prime. Now, by [12, Chap 2, Theorem 6.10], $G'/H \cong \mathrm{PSL}_2(q)$.

If $[G', H] = 1$, then $G \cong H \times \mathrm{PSL}_2(q)$. We can show similarly that H is abelian in this case. If $H \neq 1$, then $\mathrm{PSL}_2(q)$ is a proper subgroup of G, and so $\mathrm{PSL}_2(q)$ is a CCD-group. We see that $\mathrm{cd}(\mathrm{PSL}_2(q)) = \{1, \frac{q+(-1)^{(q-1)/2}}{2}, q-1, q, q+1\}$, so $q - 1 = \frac{q+(-1)^{(q-1)/2}}{2} + 1$. Now, $q = 5$, a contradiction. So, $G \cong \mathrm{PSL}_2(q)$, as desired.

If $[G', H] \neq 1$, then $H \geq C_2(= Z(\mathrm{SL}_2(q)))$, and G is isomorphic to $\mathrm{SL}_2(q) * H$. If $H > C_2$, then $\mathrm{SL}_2(q)$ is a proper subgroup of G, so it is a CCD-subgroup. We see that

$$\mathrm{cd}(\mathrm{SL}_2(q)) = \{1, \tfrac{q-1}{2}, \tfrac{q+1}{2}, q-1, q, q+1\},$$

so $q = 5$, a contradiction. Thus, $H = C_2$, and $\mathrm{SL}_2(q)C_2 \cong \mathrm{SL}_2(q)$. Furthermore, if q is prime, then by Table 4, $\mathrm{SL}_2(q)$ possibly has subgroups $E_q : C_{q-1}$, $D_{2(q\pm1)}$, $2.S_4$, $2.A_4$ and $2.A_5$, which are CCD-groups. If $q \in \{3^p, 5^p\}$ with p a prime, then Table 4 shows that $\mathrm{SL}_2(q)$ possibly subgroups $E_q : C_{q-1}$, $D_{2(q\pm1)}$, $\mathrm{SL}_2(q_0)$, $\mathrm{PSL}_2(q_0)$ and $2.A_5$, which are CCD-groups, too. Thus, G is isomorphic to $\mathrm{SL}_2(q)$, as wanted. □

Table 4. $\mathrm{SL}_2(q)$, $q \geq 4$([3, pp. 377]).

$\max \mathrm{SL}_2(q)$	Condition
$E_q : C_{q-1}$	
$Q_{2(q-1)}$	$q \neq 5, 7, 9, 11$; q odd
	N1 if $q = 7, 11$; N2 if $q = 9$
$D_{2(q-1)}$	q even
$Q_{2(q+1)}$	$q \neq 7, 9$; q odd
	N1 if $q = 7$; N2 if $q = 9$
$D_{2(q+1)}$	q even
$\mathrm{SL}_2(q_0).2$	$q = q_0^2$, q odd
$\mathrm{SL}_2(q_0)$	$q = q_0^r$, q odd, r odd prime
$\mathrm{PSL}_2(q_0)$	$q = q_0^r$, q even, $q_0 \neq 2$, r prime
$2.S_4$	$q = p \equiv \pm 1 \pmod 8$
$2.A_4$	$q = p \equiv \pm 3, 5, \pm 13 \pmod{40}$
	N1 if $q = p \equiv \pm 11, \pm 19 \pmod{40}$
$2.A_5$	$q = p \equiv \pm 1 \pmod{10}$
	$q = p^2$, $p \equiv \pm 3 \pmod{10}$

5. Applications

In this section, we will prove that an SCC-group is solvable (see Theorem 5.1) and then determine the structure of an SCC-group.

Theorem 5.1. *Let G be a finite* SCC*-group. Then, G is solvable.*

Proof. Assume that the result is not true, and then G is a non-solvable group. It is easy to get that a CC-group must be a CCD-group, so by hypothesis, for each $H \in \sum G$, H is a CCD-group. Thus, we have that G is an SCCD-group. Now, Theorem 1.3 shows that G has one of the following structure:

(1) $\mathrm{PSL}_2(q)$ where $q \geq 4$ is even or $q \geq 5$ is an odd prime or $q \in \{3^p, 5^p\}$ with p a prime.
(2) $S \times H$ where $S \in \{A_5, S_5\}$ and H is abelian.
(3) $\mathrm{SL}_2(q)$ where $q \geq 7$ is an odd prime or $q \in \{3^p, 5^p\}$ with p a prime.
(4) $\mathrm{SL}_2(5) * H$ with H/C_2 abelian.

So, four cases are dealt with in what follows.

Case 1: $\mathrm{PSL}_2(q)$ where $q \geq 4$ is even or $q \geq 5$ is an odd prime or $q \in \{3^p, 5^p\}$ with p a prime.

By Table 2, $E_q : C_{(q-1)/k} \in \max \mathrm{PSL}_2(q)$, so $\mathrm{cd}(E_q : C_{(q-1)/k}) = \{1, (q-1)/k\} = \{1, 2\}$. It follows that $q = 5$. If $q = 5$, then by [7, pp. 2], $A_4 \in \max A_5$. We know that $\mathrm{cd}(A_4) = \{1, 3\}$, a contradiction.

Case 2: $S \times H$ where $S \in \{A_5, S_5\}$ and H is abelian.

Obviously, A_4 is a subgroup of S. The transitivity of subgroups shows that A_4 is a subgroup of G, so $\mathrm{cd}(A_4) = \{1, 3\}$ gives a contradiction.

Case 3: $\mathrm{SL}_2(q)$ where $q \geq 7$ is an odd prime or $q \in \{3^p, 5^p\}$ with p a prime.

Now, Table 4 means that $E_q : C_{q-1} \in \max \mathrm{SL}_2(q)$, and hypothesis shows that $\mathrm{cd}(E_q : C_{q-1}) = \{1, q-1\} = \{1, 2\}$, so $q = 3 \ngeq 7$, a contradiction.

Case 4: $\mathrm{SL}_2(5) * H$ with H/C_2 abelian.

We see that $\mathrm{SL}_2(5)$ has a subgroup of the form $E_5 : C_4$, and $\mathrm{cd}(E_5 : C_4) = \{1, 4\}$, a contradiction. It follows from the four cases that G is solvable, as desired. □

Let $\rho(G)$ be the set of prime divisors of degrees of the irreducible characters of a group G, that is,

$$\rho(G) = \{p : p\big|d, d \in \mathrm{cd}(G)\}.$$

Lemma 5.2. *Let G be a non-abelian* SCC*-group. Then, $|\rho(G)| \leq 2$.*

Proof. We know that for each non-abelian $A \in \sum G$, A is a CC-group, so $2 \in \mathrm{cd}(A)$. It follows that $2 \in \rho(G)$. We know from Theorem 5.1 that G is solvable, so the character degree graph $\mathrm{T}(G)$ of G has at most two connected components; see [25]. Let $\rho(G) = \{2, p_1, p_2, \cdots, p_s\}$ with $p_1 < p_2 < \cdots < p_s$. Let $s \geq 2$, and let H be a Hall $\{p_1, p_2, \cdots, p_s\}$-subgroup of G. Then H is a CC-group, so $\mathrm{cd}(H) \subseteq \{1, p_1, p_2, p_1p_2, \cdots\}$. We get a contradiction as all p_is are ≥ 3. So, H is abelian and non-normal in G by Theorem 7.1 of [10]. As $s \geq 2$, Hall $\{2, p_2\}$-subgroup H_{2,p_2} of G is a CC-group, so $\mathrm{cd}(H_{2,p_2}) = \{1, 2, p_2, \cdots\}$. It follows that $p_2 = 3$, a contradiction, as $p_2 > p_1 \geq 3$. It follows that the Sylow p_2-subgroup H_{p_2} of H_{2,p_2} is normal and abelian in H_{2,p_2}. We also can get that the Sylow p_i-subgroup H_{p_i} of the Hall $\{2, p_i\}$-subgroup H_{2,p_i} of G is abelian and normal in H_{2,p_i}. It follows that the Hall $\{p_2, \cdots, p_s\}$-subgroups $H_{p_2,p_3,\cdots,p_s}$ of $H_{2,p_2,\cdots,p_s}$ are abelian and normal in $H_{2,p_2,\cdots,p_s}$. We see that $G = H_{p_1,p_2,\cdots,p_s}H_{2,p_2,\cdots,p_s}$ and $H_{p_1,p_2,\cdots,p_s} \cap H_{2,p_2,\cdots,p_s} = H_{p_2,\cdots,p_s}$, so $H_{p_2,\cdots,p_s}$ is abelian and normal in G, a contradiction to Theorem 7.1 of [10]. It follows that $s = 1$, the desired result. □

Lemma 5.3. *Let G be a nilpotent non-abelian* SCC*-group. Then, G is isomorphic to a metabelian p-group or a direct product of a 2-group P and $2'$-group H where* $\mathrm{cd}(P) = \{1, 2\}$ *and* $\mathrm{cd}(H) = \{1\}$.

Proof. Let P_i be Sylow p_i-subgroups of G with primes p_i, and $p_1 < p_2 < \cdots < p_s$. As G is nilpotent, write $G = P_1 \times P_2 \times \cdots \times P_s$. By Lemma 5.2, $\rho(G) = \{p\}$, or $\rho(G) = \{p, q\}$ with different primes p, q.

• $\rho(G) = \{p\}$.

If $s = 1$ and $p = 2$, then G is a 2-group. We know that for each $H \in \sum(G)$, H is a CC-group, so $\mathrm{cd}(H) = \{1\}$ or $\mathrm{cd}(H) = \{1, 2\}$. Thus, G is a 2-group whose proper subgroups all have degrees 1 or 2. If $s \geq 2$ and $p = 2$, then G is a direct product of a non-abelian 2-subgroup P and a Hall $2'$-subgroup H. As $H \neq 1$, we have $\mathrm{cd}(P) = \{1, 2\}$. If H is non-abelian, then $Z(P) \times H$ is a non-abelian CC-group, so $\mathrm{cd}(Z(P) \times H) = \{1, p', \cdots\}$. Hypothesis shows that $p' = 2$, a contradiction. Hence, H is abelian. We now draw a conclusion that $G = P \times H$ with $\mathrm{cd}(P) = \{1, 2\}$, and H abelian.

Now, we consider $p > 2$. If $s = 1$, then assumption gives that G is a metabelian p-group. If $s \geq 2$, then $G = P \times H$ where P is a p-subgroup and H is a p'-subgroup. If P is non-abelian, then P is a CC-subgroup of G, and so $\mathrm{cd}(P) = \{1, p, \cdots\}$. Now, hypothesis gives that $p = 2 \ngeq 3$, a contradiction. Hence, P is abelian, and so is G. Otherwise, there is a subgroup which is not a CC-subgroup.

• $\rho(G) = \{p, q\}$.

In this case, $G = P \times H$ with $\rho(P) = \{p\}$ and $\rho(H) = \{q\}$. By the above arguments, we can assume that $p = 2$ and $\mathrm{cd}(P) = \{1, 2\}$. Thus, $C_2 \times H$ is a CC-subgroup of G, and $\mathrm{cd}(C_2 \times H) = \{1, q, \cdots\}$. Hence, $q = 2$, too, a contradiction. So, this case can not occur. □

Lemma 5.4. *Let $G = F : H$ be a Frobenius group with kernel F and complement H, respectively. If G is an* SCC*-group, then one of the following holds:*

(i) $\mathrm{cd}(F) = \{1, 2\}$*, and H is a prime order such that for each non-identity $W \in \sum F$, $|W| \not\equiv 1$* (mod $|H|$).

(ii) F is abelian, and H is of order a prime or 4. Furthermore, if H is of order 4, then F is isomorphic to $C_p \times C_q$ with $q \equiv p \equiv -1 \pmod 4$ or C_p with $p \equiv 1 \pmod 4$. If H is of prime order > 2, then $|H|||(|F| - 1)$.

Proof. We divide the proofs into two cases: F is non-abelian, and F is abelian.

Case 1: F is non-abelian

Then, F is a CC-subgroup of G, and so $2 \in \mathrm{cd}(F)$. We know that F is nilpotent, so by Lemma 5.3, $\mathrm{cd}(F) = \{1, 2\}$. Thus, F is nilpotent with $\mathrm{cd}(F) = \{1, 2\}$.

If H is non-abelian, then Lemma 2.1 gives that H is of odd order, and so by Lemma 5.3 H is metabelian. Let H_p with $|H_p| = p$ be a proper subgroup of H with certain prime p. Then, $F : H_p$ is also a Frobenius subgroup of G. By assumption, $F : H_p$ is a CC-group, and so by Lemma 2.1, $\mathrm{cd}(F : H_p) = \{1, 2, p, 2p\}$, a contradiction. Thus, H is abelian. We claim that H is of prime order. In fact, if $|H| = pq$ with primes p, q, then $F : H_q$ and $F : H_p$ are Frobenius subgroups of G, so they are CC-groups. By Lemma 2.1, we have that $\mathrm{cd}(F : H_p) = \{1, 2, p, 2p\}$, and $\mathrm{cd}(F : H_q) = \{1, 2, q, 2q\}$, a contradiction. This means that H is of prime order, say, q.

By Lemma 2.1, $|F| \equiv 1 \pmod q$. Let $F_{p_i}, i = 1, 2, \cdots, n$, be the Sylow p_i-subgroup of F. If $|F_1F_2 \cdots F_s| \equiv 1 \pmod q$, then $|F_{s+1} \cdots F_n| \equiv 1 \pmod q$, so $W := F_{s+1} \cdots F_n : H$ is a Frobenius group. Now, $\mathrm{cd}(W) = \{1, q\}$, a contradiction, as W is a CC-subgroup of G. Thus, $|K| \not\equiv 1 \pmod q$ for all $K \in \sum F$, but $|F| \equiv 1 \pmod q$.

Case 2: F is abelian.

We claim that

$$H \text{ is abelian and of order prime } p \text{ or order 2 or 4.}$$

In fact, if H is non-abelian, and H has two subgroups C_p and C_q with primes p, q, then $\mathrm{cd}(F : C_p) = \{1, p\}$, and $\mathrm{cd}(F : C_q) = \{1, q\}$. By hypothesis, $F : C_p$ and $F : C_q$ are CC-groups, so we have that $p = q = 2$ is possible. It follows that either $p = 2$ and H is a 2-group of order ≤ 4 or H is of odd prime order.

If H is of order 2, then F is of odd order, and for all $W \in \sum G$, W is a CC-group, since $\mathrm{cd}(W) = \{1\}$ or $\{1, 2\}$.

If H is of order 4, then F is of odd order, too.

- If F is a p-group, and then say $|F| = p^s$. Let $s \geq 3$, then G has a subgroup $C_{p^2} : C_4$ which is a CC-group. We know that $\mathrm{cd}(C_{p^2} : C_4) = \{1, 4\}$, a contradiction. Thus, $s \leq 2$: namely, $|F| = p$ or p^2. If $|F| = p$, then $p \equiv 1 \pmod 4$; if $|F| = p^2$, then $p^2 \equiv 1 \pmod 4$. If $|F| = p^2$ and $p \equiv 1 \pmod 4$, then G has a Frobenius subgroup $C_p : C_4$ which is not a CC-group. Thus, $|F| = p^2$ with $p \equiv -1 \pmod 4$, or $|F| = p$ with $p \equiv 1 \pmod 4$.
- If F is a non-p-group, then F has two subgroups C_p and C_q. Note that p, q are odd, so $p \equiv \pm 1 \pmod 4$ and $q \equiv \pm 1 \pmod 4$. If $p \equiv q \equiv 1 \pmod 4$, then G has a subgroup of the form $C_p : C_4$ which is not a CC-group, a contradiction. If $q \equiv -1 \pmod 4$ and $p \equiv 1 \pmod 4$, or $q \equiv 1 \pmod 4$ and $p \equiv -1 \pmod 4$, then $q^2 p \equiv 1 \pmod 4$. However, $C_p : C_4$ is a Frobenius group, so it is a CC-group, a contradiction (note in this case that $pq \equiv 1 \pmod 2$, but $pq \equiv -1 \pmod 4$). Now, we draw a conclusion that $p \equiv q \equiv -1 \pmod 4$. If $|F| = (pq)^{2s} q^t$ or $(pq)^{2s} p^t$ with $s \geq 1, t \geq 0$, then a CC-subgroup $C_{pq} : C_4$ is a Frobenius group, a contradiction. Thus, $|F| = pq$, and F is isomorphic to $C_p \times C_q$, as F is nilpotent.

Let H be of prime order $r > 2$. Note that the nilpotence of F gives that

$$F = F_{p_1} \times F_{p_2} \times \cdots \times F_{p_s}.$$

As with the above arguments, two cases are considered: $s = 1$ and $s \geq 2$.

- $s = 1$. Then, F is a p-group for certain prime p, so $r \equiv 1 \pmod{|F|}$. If F has a subgroup K such that $K : H$ is a Frobenius group, then $K : H$ is not a CC-group. Thus, $r||(|F| - 1)$.
- $s \geq 2$. Then, similarly, we can show that $r||(|F| - 1)$.

Now, the lemma is complete. □

Lemma 5.5. *Let G be a non-abelian* SCC*-group. Assume that G is a direct product of at least two* CC*-groups. Then, G is isomorphic to $H_1 \times H_2$ where at least one of H_1 and H_2 is metabelian with* $\mathrm{cd}(H_i) \subseteq \{1, 2\}$.

Proof. Let $G = H_1 \times H_2 \times \cdots H_s$ where all H_i are CC-groups. If H_i are all abelian, then G is abelian, so we assume that for some i, H_i is non-abelian, say, H_1. If the other H_i for $i = 2, \cdots s$ are abelian, then G is an SCC-group when $\mathrm{cd}(H_1) = \{1, 2\}$. If $\mathrm{cd}(H_1) = \{1, p\}$ for some prime $p > 2$, then G is not an SCC-group when $H_i \neq 1$, $i = 2, 3, \cdots, s$. It follows that $G = H_1$, a contradiction. Thus, we assume that $G = H_1 \times H_2 \times \cdots \times H_s$ with $\mathrm{cd}(H_1) = \mathrm{cd}(H_2) = \{1, 2\}$. If $s \geq 3$, then $H_1 \times H_2$ is a CC-subgroup of G, but by Theorem 4.21 of [15], $\mathrm{cd}(H_1 \times H_2) = \{1, 2, 4\}$, a contradiction. Thus, $s = 2$, and $G = H_1 \times H_2$, where $\mathrm{cd}(H_1) = \mathrm{cd}(H_2) = \{1, 2\}$, and H_1 and H_2 are metabelian. We similarly can get that $G = H_1 \times H_2 \cdots H_k \times H_{k+1} \times \cdots H_s$ is not an SCC-group if $\mathrm{cd}(H_i) = \{1, 2\}$, $i = 1, 2, \cdots, k$, and $k \geq 3$. □

Proof of Theorem 1.4.

Proof. If G is an SCC-group, then G is solvable by Theorem 5.1. If G is a nilpotent group or a Frobenius group or a direct product of two CC-groups, then we can get (ii), (iii) and (iv) by Lemmas 5.3, 5.4 and 5.5. Thus, in what follows, we assume that G is neither a nilpotent group, nor a Frobenius group nor a direct product of two CC-groups. If G is abelian or $\mathrm{cd}(G) = \{1, 2\}$, then for all $H \in \sum G$, H is always a CC-group, so we get (i). Thus, we assume that $\mathrm{cd}(G) \neq \{1\}$ and $\mathrm{cd}(G) \neq \{1, 2\}$. Since G is a solvable SCC-group, we get that G has a factor; say that N/M is such a factor. Note that $N \in \sum G$, so N is a CC-group. Now,

$$\mathrm{cd}(N) = \{1\}, \{1, 2\}, \{1, 2, 3, \} \text{ or } \{1, 2, 3, 4\};$$

see [13] or [14, Sec. 32]. By pp. 168 of [13],

$$N/M \text{ is isomorphic to } S_3 \text{ or } A_4 \text{ or } 3^2 : 2^2.$$

It is easy to see that M is abelian. (In fact, let $\theta \in \mathrm{Irr}(M) \setminus \mathrm{Lin}(M)$. Then, $I := \mathrm{I}_N(\theta)$, the inertia subgroup of θ in N, is equal to N or KM where $K \in \sum N/M$. If $I = KM$, then $\theta^N(1) = |N : I|\theta(1)$, so $\theta(1)$, $\theta^N(1) \in \mathrm{cd}(N)$. Note that N is a CC-group, so $\mathrm{cd}(N) = \{1, \theta(1), 2, \theta^N(1)\}$ or $\mathrm{cd}(N) = \{1, \theta(1), 3, \theta^N(1)\}$ or $\mathrm{cd}(N) = \{1, \theta(1), 4, \theta^N(1)\}$, contradicting Theorem of [13].)

Let $\overline{G} = G/M$ and let $\overline{N} = N/M$. Then,

$$\overline{N} \cong \overline{N}C_{\overline{G}}(\overline{N})/C_{\overline{G}}(\overline{N}) \leq \overline{G}/C_{\overline{G}}(\overline{N}) = N_{\overline{G}}(\overline{N})/C_{\overline{G}}(\overline{N}) \leq \mathrm{Aut}(\overline{N}).$$

Set $L = \{xM : xM \in C_{\overline{G}}(\overline{N}\}$, and then $G/L \cong \overline{G}/C_{\overline{G}}(\overline{N})$. Notice that L is also abelian as the proof as M is abelian. So, we have that $\mathrm{Aut}(\overline{N}) = S_3$, S_4 and $3^2 : D_8$ when $\overline{N}$ is isomorphic to S_3, A_4 and $3^2 : 2^2$, respectively; see [33]. Observe that the Schur multipliers of $\overline{N}$ are trivial, so G is possibly isomorphic to $S_3 \times L$, $A_4 \times L$, $S_4 \times L$, $(3^2 : C_4) \times L$ or $(3^2 : D_8) \times L$. If G is isomorphic to $S_3 \times L$, then it is contained in (i); if G is isomorphic to $S_4 \times L$ or $3^2 : D_8$, then $A_4 \in \sum S_4$ and $3^3 : 2^2 \in \sum(3^2 : D_8) \times L$ are CC-groups, a contradiction. Thus, A_4 and $3^2 : 2^2$ are not proper subgroups of G, and so $L = 1$. Now, G is isomorphic to A_4 or $3^2 : C_4$, as desired. □

6. Conclusions

From Theorem 1.3, we know that a group whose non-linear character degrees are consecutive is possibly non-solvable, but a group whose irreducible character degrees is consecutive are solvable. Chen in [5] gave the information of groups whose irreducible character degrees are arithmetic numbers. In comparison with Theorem 5.1, is the group solvable if proper subgroups all have this property?

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Conflict of interest

The authors declare that he has no conflict of interest.

References

1. H. R. Brahana, On the metabelian groups which contain a given group H as a maximal invariant abelian subgroup, *Amer. J. Math.*, **56** (1934), 490–510. https://doi.org/10.2307/2370950
2. R. Brandl, Groups with few non-nilpotent subgroups, *J. Algebra Appl.*, **16** (2017), 1750188. https://doi.org/10.1142/S0219498817501882
3. J. N. Bray, D. F. Holt, C. M. Roney-Dougal, The maximal subgroups of the low-dimensional finite classical groups, In: *London mathematical society lecture note series*, Cambridge: Cambridge University Press, 2013. https://doi.org/10.1017/CBO9781139192576
4. T. Breuer, The GAP character table library, 2012. http://www.math.rwth-aachen.de/T̃homas.Breuer/ctbllib .
5. S. A. Chen, Finite solvable groups with irreducible character degrees of arithmetic number, *Acta Math. Sinica Chin. Ser.*, **56** (2013), 31–40.
6. Z. M. Chen, Inner- and outer-supersolvable groups, and sufficient conditions for a group to be supersolvable, *Acta Math. Sinica Chin. Ser.*, **27** (1984), 694–703.

7. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, *Atlas of finite groups*, Eynsham: Oxford University Press, 1985.
8. L. E. Dickson, *Linear groups: with an exposition of the Galois field theory*, New York: Dover Publications, 1958.
9. L. Dornhoff, *Group representation theory. Part A: Ordinary representation theory*, New York: Marcel Dekker, 1971.
10. M. Geck, G. Malle, The character theory of finite groups of Lie type, In: *Cambridge studies in advanced mathematics*, Cambridge: Cambridge University Press, 2020.
11. M. Giudici, Maximal subgroups of almost simple groups with socle $PSL(2,q)$, *arXiv: Group Theory*, 2007, 0703685. https://doi.org/10.48550/arXiv.math/0703685
12. B. Huppert, Die grundlehren der mathematischen wissenschaften, band 134, In: *Endliche gruppen. I*, Berlin: Springer, 1967.
13. B. Huppert, A characterization of GL$(2,3)$ and SL$(2,5)$ by the degrees of their representations, *Forum Math.*, **1** (1989), 167–183. https://doi.org/10.1515/form.1989.1.167
14. B. Huppert, Character theory of finite groups, In: *De gruyter expositions in mathematics*, Berlin: Walter de Gruyter & Co., 1998. https://doi.org/10.1515/9783110809237
15. I. M. Isaacs, *Character theory of finite groups*, New York: Dover Publications, 1994.
16. H. E. Jordan, Group-characters of various types of linear groups, *American J. Math.*, **29** (1907), 387–405. https://doi.org/10.2307/2370015
17. P. B. Kleidman, *The subgroup structure of some finite simple groups*, Ph.D Thesis, University of Cambridge, 1987.
18. Q. Li, X. Guo, On generalization of minimal non-nilpotent groups, *Lobachevskii J. Math.*, **31** (2010), 239–243. https://doi.org/10.1134/S1995080210030078
19. M. W. Liebeck, J. Saxl, On the orders of maximal subgroups of the finite exceptional groups of Lie type, *Proc. London Math. Soc.*, **55** (1987), 299–330.
20. S. Liu, On groups whose irreducible character degrees of all proper subgroups are all prime powers, *J. Math.*, **2021** (2021), 6345386. https://doi.org/10.1155/2021/6345386
21. S. Liu, D. Lei, X. Li, On groups with consecutive three smallest character degrees, *ScienceAsia*, **45** (2019), 474–481.
22. S. Liu, X. Tang, Nonsolvable groups whose degrees of all proper subgroups are the direct products of at most two prime numbers, *J. Math.*, **2022** (2022), 1455299. https://doi.org/10.1155/2022/1455299
23. Rwth-Aachen University, Character degrees and their multiplicities for some groups of lie type of rank $<$ 9, 2007. Available from: http://www.math.rwth-aachen.de/ Frank.Luebeck/chev/DegMult/index.html?LANG=en .
24. G. Malle, The maximal subgroups of ${}^2F_4(q^2)$, *J. Algebra*, **139** (1991), 52–69. https://doi.org/10.1016/0021-8693(91)90283-E
25. O. Manz, W. Willems, T. R. Wolf, The diameter of the character degree graph, *J. Reine Angew. Math.*, **402** (1989), 181–198.

26. G. Qian, Finite groups with consecutive nonlinear character degrees, *J. Algebra*, **285** (2005), 372–382. https://doi.org/10.1016/j.jalgebra.2004.11.021

27. J. W. Randolph, Finite groups with solvable maximal subgroups, *Proc. Amer. Math. Soc.*, **23** (1969), 490–492. https://doi.org/10.2307/2036570

28. J. Sangroniz, Character degrees of the Sylow p-subgroups of classical groups, Cambridge: Cambridge University Press, 2003. https://doi.org/10.1017/CBO9780511542787.016

29. W. A. Simpson, J. S. Frame, The character tables for SL(3, q), SU(3, q^2), PSL(3, q), PSU(3, q^2), *Canad. J. Math.*, **25** (1973), 486–494. https://doi.org/10.4153/CJM-1973-049-7

30. R. Steinberg, The representations of GL(3, q), GL(4, q), PGL(3, q), and PGL(4, q), *Canad. J. Math.*, **3** (1951), 225–235. https://doi.org/10.4153/cjm-1951-027-x

31. J. G. Thompson, Nonsolvable finite groups all of whose local subgroups are solvable, *Bull. Amer. Math. Soc.*, **74** (1968), 383–437. https://doi.org/10.1090/S0002-9904-1968-11953-6

32. P. H. Tiep, A. E. Zalesskii, Minimal characters of the finite classical groups, *Comm. Algebra*, **24** (1996), 2093–2167. https://doi.org/10.1080/00927879608825690

33. L. Wang, On the automorphism groups of Frobenius groups, *Comm. Algebra*, **48** (2020), 5330–5342. https://doi.org/10.1080/00927872.2020.1788045

34. H. N. Ward, On Ree's series of simple groups, *Trans. Amer. Math. Soc.*, **121** (1966), 62–89. https://doi.org/10.2307/1994333

35. D. L. White, Character degrees of extensions of $\mathrm{PSL}_2(q)$ and $\mathrm{SL}_2(q)$, *J. Group Theory*, **16** (2013), 1–33. https://doi.org/10.1515/jgt-2012-0026

36. H. Xu, G. Chen, Y. Yan, A new characterization of simple K_3-groups by their orders and large degrees of their irreducible characters, *Comm. Algebra*, **42** (2014), 5374–5380. https://doi.org/10.1080/00927872.2013.842242

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