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Research article

Uniqueness for a Cauchy problem for the generalized Schrödinger equation

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Abstract: In this work, we consider a Cauchy problem for the generalized Schrödinger equation which has important applications in quantum kinetic theory, water wave problems and ferromagnetism. Due to its multidimensionality, it is important from the point of view of modern physics theories such as quantum field theory and string theory. We prove the uniqueness of the solution of the problem in an unbounded domain by using semigeodesic coordinates. The main tool is a pointwise Carleman estimate. To the authors' best knowledge, this is the first study which deals with the solvability of this problem.

Keywords: uniqueness; Cauchy problem; generalized Schrödinger equation; pointwise Carleman estimate
Mathematics Subject Classification: 35A23, 35A32, 35Q40

1. Introduction

The generalized Schrödinger equation appears in many physical models in quantum kinetic theory [1], in the study of water wave problems and in ferromagnetism. Some of these models are Davey-Stewartson [2], Zakharov-Schulman [3] and Ishimori systems [4]. In these integrable systems, we frequently encounter the nonlinear Schrödinger equation and its generalized form, which provide a description of nonlinear waves such as propagation of a laser beam in a medium whose index of refraction is sensitive to the wave amplitude, water waves at the free surface of an ideal fluid and plasma waves, [5]. A detailed analysis of these types of equations was presented in [5]. Moreover, we refer to [6–10] for the new results related to the existence, uniqueness and asymptotic properties of solutions including radial cases for the nonlinear Schrödinger equation. We also refer to [11–14], where some new integrable systems were introduced based on the non-semisimple Lie algebra.

In this article, we deal with the generalized Schrödinger equation

$$
\begin{aligned}
& i\partial_t u(t,x) + \sum_{s,j=1}^{n} a_{sj}(x)\partial_s\partial_j u(t,x) - \sum_{s,j=n+1}^{n+m} a_{sj}(x)\partial_s\partial_j u(t,x) \\
& + \sum_{j=1}^{n+m} b_j(t,x)\partial_j u(t,x) + b_0(t,x)u(t,x) = f(t,x)
\end{aligned}
\tag{1.1}
$$

in the domain $\Omega = \{(t,x) : t \in \mathbb{R},\ x \in D^{n+m} \subset \mathbb{R}^{n+m}\}$, where the bounded domain D^{n+m} is defined as

$$D^{n+m} = \{x = (x_1, \overline{x}) \in \mathbb{R}^{n+m} : x_1 > 0,\ \overline{x} = (x_2, ..., x_{n+m}) \in \mathbb{R}^{n+m-1}\}$$

and supported by $x_1 = 0$. Throughout the paper, we use the following notations:
$\partial_t u = \frac{\partial u}{\partial t}$, $\partial_1^2 u = \frac{\partial^2 u}{\partial x_1^2}$, $\partial_j u = \frac{\partial u}{\partial x_j}$, $\partial_s\partial_j u = \frac{\partial^2 u}{\partial x_s \partial x_j}$ $(1 \le s,\ j \le n+m)$ and $i = \sqrt{-1}$.

In (1.1), the coefficients a_{sj} $(1 \le s,\ j \le n+m)$ are real-valued, and there exist constants M, M_1 such that $\left\|a_{sj}\right\|_{C^3(\overline{D^{n+m}})} \le M$ and $\left\|fe^{-\kappa t^2}\right\|_{H^2(\mathbb{R})} \le M_1$, where $\kappa > 0$ is a constant. The coefficients $b_j \in L^\infty_{loc}(\Omega)$ $(j = 0, 1, ..., n+m)$ are complex-valued, and there exists a constant M_2 such that $\left|\partial_t^{\beta_0} b_j\right| \le M_2$, where $0 \le \beta_0 \le 2$, $\partial_t^{\beta_0} = \frac{\partial^{\beta_0}}{\partial t^{\beta_0}}$.

We consider the following problem:

Problem 1. Find the function $u(t,x)$ in Ω which satisfies equation (1.1) and the following conditions

$$u(t,0,\overline{x}) = 0,\ \partial_1 u(t,0,\overline{x}) = 0. \tag{1.2}$$

We set

$$\mathcal{U} = \left\{u(t,x) :\ u \in C^3(\Omega),\ \left|\partial_t^{\beta_0}\partial_s^{\beta_1}\partial_j^{\beta_2} u\right| \le C_u \exp(\kappa_u t^2)\right\}, \tag{1.3}$$

where $\beta_0 = 0,1,2,3; \beta_1, \beta_2 = 0,1,2$; $s,\ j = 0,1,2,...,n+m$; and C_u, $\kappa_u > 0$ are constants depending on $u(t,x)$.

In [15,16], the local well-posedness of some initial value problems for the generalized Schrödinger equation was considered in appropriate Sobolev spaces. In [17], a non-standard Cauchy problem for the classical Schrödinger equation was studied, and solvability of the related inverse problem was investigated. A conditional stability estimate for the ultrahyperbolic Schrödinger equation was obtained in [18], where the ultrahyperbolic part of the equation consists of the term $\Delta_y u - \Delta_x u$. In [19], a Hölder stability estimate was established for the inverse problem. Problem 1 is a Cauchy problem for the generalized Schrödinger equation because the data are given on the $x_1 = 0$. We investigate the uniqueness of the solution of the problem, and the main tool is a pointwise Carleman estimate. To the authors' best knowledge, this is the first result devoted to the solvability of Problem 1.

The main result of this paper is given below:

Theorem 1. Assume that there exist two positive constants α_1, α_2 such that

$$\sum_{s,j=2}^{n} \partial_1(a_{sj}(x))\zeta_s\zeta_j \ge \alpha_1|\zeta|^2,\ -\sum_{s,j=n+1}^{n+m} \partial_1(a_{sj}(x))\eta_s\eta_j \ge \alpha_2|\eta|^2 \tag{1.4}$$

for any $\zeta = (\zeta_2, ..., \zeta_n) \in \mathbb{R}^{n-1}$, $\eta = (\eta_{n+1}, ..., \eta_{n+m}) \in \mathbb{R}^m$, $x \in D^{n+m}$. Then, there exists at most one solution $u \in \mathcal{U}$ satisfying (1.1) and (1.2).

Theorem 1 is proved in Section 3 by using a Carleman type inequality which we call "Proposition 1". The proof of Proposition 1 will be given in Section 2.

Here, we suppose that the metric g is given in semigeodesic coordinates, that is, it has the property $g_{11} = 1$, $g_{1j} = 0$, $j = 2, 3, ..., n + m$; so (1.1) takes the form

$$\begin{aligned} Pu \quad \equiv \quad & i\partial_t u + \partial_1^2 u + \sum_{s,j=2}^{n} a_{sj}(x)\partial_s\partial_j u - \sum_{s,j=n+1}^{n+m} a_{sj}(x)\partial_s\partial_j u \\ & + \sum_{j=1}^{n+m} b_j(t,x)\partial_j u(t,x) + b_0(t,x)u = f(t,x). \end{aligned} \tag{1.5}$$

Since we shall prove the uniqueness, we consider the homogeneous version of Problem 1. Now, we define a new function $z = ue^{-\kappa t^2}$, so we can write

$$Pz + 2i\kappa t z = F(t,x), \tag{1.6}$$

$$z(t,0,\overline{x}) = \partial_1 z(t,0,\overline{x}) = 0, \tag{1.7}$$

where $F(t,x) = f(t,x)e^{-\kappa t^2}$.

If we apply the Fourier transform to (1.6) and conditions (1.7) with respect to t, we get

$$-\xi\widehat{z} + \partial_1^2\widehat{z} + \sum_{s,j=2}^{n} a_{sj}\partial_s\partial_j\widehat{z} - \sum_{s,j=n+1}^{n+m} a_{sj}\partial_s\partial_j\widehat{z} + \sum_{j=1}^{n+m}\widehat{b_j\partial_j z} + \widehat{b_0 z} - 2\kappa\partial_\xi\widehat{z} = F_1 + iF_2, \tag{1.8}$$

$$\widehat{z}(\xi,0,\overline{x}) = \partial_1\widehat{z}(\xi,0,\overline{x}) = 0, \tag{1.9}$$

where $\widehat{z}$, $\widehat{F}$, $\widehat{b_j\partial_j z}$, $\widehat{b_0 z}$ are the Fourier transforms with respect to t of the functions z, F, $b_j\partial_j z$, $b_0 z$ ($j = 0, 1, ..., n + m$), respectively. Here, ξ is the parameter of the Fourier transforms and $\partial_\xi z = \frac{\partial z}{\partial \xi}$.

We write $\widehat{z} = w_1(\xi, x) + iw_2(\xi, x)$, $\widehat{b_j\partial_j z} = b_{1j} + ib_{2j}$ and $\widehat{b_0 z} = b_{10} + ib_{20}$ in (1.8), and so we obtain the following system of equations:

$$2\kappa\partial_\xi(w_k) - \partial_1^2(w_k) - \sum_{s,j=2}^{n} a_{sj}\partial_s\partial_j(w_k) + \sum_{s,j=n+1}^{n+m} a_{sj}\partial_s\partial_j(w_k) + \xi w_k = l_k, \tag{1.10}$$

where $k = 1, 2$;

$$l_1 = F_1 - \sum_{j=0}^{n+m} b_{1j}, \; l_2 = F_2 - \sum_{j=0}^{n+m} b_{2j}. \tag{1.11}$$

By (1.9), we have

$$w_1(\xi,0,\overline{x}) = 0, \; \partial_1(w_1)(\xi,0,\overline{x}) = 0, \; w_2(\xi,0,\overline{x}) = 0, \; \partial_1(w_2)(\xi,0,\overline{x}) = 0. \tag{1.12}$$

Thus, we shall show that this homogeneous problem has only zero solution.

2. Pointwise Carleman estimate

T. Carleman [20] established the first Carleman estimate in 1939 for proving the unique continuation for a two-dimensional elliptic equation. A Carleman estimate is an L^2-weighted estimate with large parameters for a solution to a partial differential equation. These parameters play an essential role and are important how to choose a weight function in order to adjust given geometric configurations, [21]. This estimate is used as a powerful tool to establish the uniqueness and stability results for ill-posed Cauchy problems, [22]. In 1954, C. Müller extended Carleman's result to $\mathbb{R}^n$, [23]. After that, A. P. Calderón [24] and L. Hörmander [25] improved these results based on the concept of pseudo-convexity. In [26–28], uniqueness and stability of various problems for the classical Schrödinger equation were studied using Carleman estimates.

The Carleman estimate used in this paper asserts a pointwise inequality, while the conventional Carleman estimates are proved in terms of weighted L^2-integrals of solutions and boundary data. This type of pointwise Carleman estimate also was used in [22, 29, 30] for various equations including hyperbolic, parabolic and ultrahyperbolic equations.

We first reduce (1.10) to a form which is more suitable for a Carleman estimate. For this aim, we define a new variable $\widetilde{x_1} = \sqrt{2x_1} - \mu_0$, $\mu_0 > 0$, and then we define

$$w_k(\xi, x) = w_k(\xi, \frac{1}{2}(\widetilde{x_1} + \mu_0)^2, \overline{x}) \equiv \widetilde{w_k}(\xi, \widetilde{x_1}, \overline{x}).$$

Replacing the notations $\widetilde{w_k}, \widetilde{a}_{sj}, \widetilde{x_1}, \widetilde{b}_r, \widetilde{f}$ with w_k, a_{sj}, x_1, b_r, f for the sake of simplicity, we have

$$P_0 w_k = l'_k,$$

where $l'_k = (x_1 + \mu_0) l_k$, $k = 1, 2$, and

$$\begin{aligned} P_0 w_k \; &= \; (x_1 + \mu_0)^{-1} \partial_1^2 (w_k) + (x_1 + \mu_0)(\sum_{s,j=2}^{n} a_{sj} \partial_s \partial_j (w_k) \\ &\quad - \sum_{s,j=n+1}^{n+m} a_{sj} \partial_s \partial_j (w_k) - 2\kappa \partial_\xi (w_k) - \xi w_k). \end{aligned} \tag{2.1}$$

Next, we define a subdomain of D^{n+m}

$$D_\gamma^{n+m} = \{x : x \in \mathbb{R}^{n+m},\ x_1 > 0,\ \alpha_0 < \psi(x) < \gamma + \alpha_0 < 1\},$$

where $0 < \gamma < 1$, $2\mu_0 \le \min\{\alpha_0, \gamma\}$, and a Carleman weight function

$$\varphi = e^{\lambda \psi^{-\nu}},$$

where $\alpha_0 > 0$, the parameters δ, λ, ν are positive numbers, and

$$\psi(x) = \delta x_1 + \frac{1}{2} \sum_{j=2}^{n+m} (x_j - x_j^0)^2 + \alpha_0. \tag{2.2}$$

Proposition 1. There exist δ_*, $\nu_* = \nu_*(\delta)$ and $\lambda_* = \lambda_*(\nu)$ such that the inequality

$$\begin{aligned}&\psi^{\nu+1}\varphi^2(P_0w)^2 + 2\lambda\nu(x_1+\mu_0)S\varphi^2 w(P_0w)\\ \geq\;& 4\lambda^3\nu^4\psi^{-2\nu-3}\varphi^2w^2 + 2\lambda\nu\varphi^2(\partial_1 w)^2 + \lambda\nu(x_1+\mu_0)\varphi^2\sum_{j=2}^{n+m}(\partial_j w)^2 + \sum_{j=1}^{4} d_j \end{aligned}\tag{2.3}$$

holds for all $\delta > \delta_*$, $\nu > \nu_*$, $\lambda > \lambda_*$ and $w \in C^2(\overline{D_\gamma^{n+m}})$. Here, the constant δ_* depends on $\alpha_1, \alpha_2, M, \psi, \gamma$, and

$$S = \sum_{l,m=2}^{n}\partial_m(a_{lm}\partial_l\psi) - \sum_{l,m=n+1}^{n+m}\partial_m(a_{lm}\partial_l\psi).$$

Lemma 1. Assume that the conditions of Proposition 1 are satisfied. Then, there exist positive constants δ_0, $\nu_0 = \nu_0(\delta)$ and $\lambda_0 = \lambda_0(\nu)$ such that

$$\begin{aligned}\psi^{\nu+1}(P_0w)^2\varphi^2 \;\geq\;& 2\lambda\nu\gamma^{-3}\delta(\partial_1 w)^2\varphi^2 + \lambda^3\nu^4\delta^4\psi^{-2\nu-3}w^2\varphi^2\\ &+\lambda\nu(x_1+\mu_0)\delta(\alpha_1\sum_{j=2}^{n}(\partial_j w)^2 + \alpha_2\sum_{j=n+1}^{n+m}(\partial_j w)^2)\varphi^2\\ &+2\lambda\nu\xi(x_1+\mu_0)^2(\sum_{s,j=2}^{n}\partial_j(a_{sj}\partial_s\psi) - \sum_{s,j=n+1}^{n+m}\partial_j(a_{sj}\partial_s\psi))w^2\varphi^2 + \sum_{j=1}^{3} d_j\end{aligned}\tag{2.4}$$

for all $\delta > \delta_0$, $\nu > \nu_0$, $\lambda > \lambda_0$ and $w \in C^2(\overline{D_\gamma^{n+m}})$. Here, the constant δ_0 depends on $\alpha_1, \alpha_2, M, \psi, \gamma$, and d_j, $1 \leq j \leq 3$, represent some divergence terms which will be given explicitly in the proof below.

Proof. If we define a new function $\vartheta = \varphi w$, then we write

$$\begin{aligned}\psi^{\nu+1}(P_0w)^2\varphi^2 =\;& \varphi^2\psi^{\nu+1}((x_1+\mu_0)^{-1}\varphi^{-1}(\partial_1^2\vartheta + 2\lambda\nu\psi^{-\nu-1}\partial_1\psi\partial_1\vartheta\\ &+(\lambda^2\nu^2\psi^{-2\nu-2}(\partial_1\psi)^2 - \lambda\nu(\nu+1)\psi^{-\nu-2}(\partial_1\psi)^2 + \lambda\nu\psi^{-\nu-1}\partial_1^2\psi)\vartheta)\\ &+(x_1+\mu_0)(\sum_{s,j=2}^{n}a_{sj}\varphi^{-1}(\partial_s\partial_j\vartheta + 2\lambda\nu\psi^{-\nu-1}\partial_s\psi\partial_j\vartheta + (\lambda^2\nu^2\psi^{-2\nu-2}\partial_s\psi\partial_j\psi\\ &-\lambda\nu(\nu+1)\psi^{-\nu-2}\partial_s\psi\partial_j\psi + \lambda\nu\psi^{-\nu-1}\partial_s\partial_j\psi)\vartheta) - \sum_{s,j=n+1}^{n+m}a_{sj}\varphi^{-1}(\partial_s\partial_j\vartheta\\ &+2\lambda\nu\psi^{-\nu-1}\partial_s\psi\partial_j\vartheta + (\lambda^2\nu^2\psi^{-2\nu-2}\partial_s\psi\partial_j\psi - \lambda\nu(\nu+1)\psi^{-\nu-2}\partial_s\psi\partial_j\psi\\ &+\lambda\nu\psi^{-\nu-1}\partial_s\partial_j\psi)\vartheta) - 2\kappa\partial_\xi\vartheta\varphi^{-1} - \xi\vartheta\varphi^{-1}))^2\\ =\;& \psi^{\nu+1}(y_1^2 + (y_2+y_4+y_5)^2 + y_3^2 + 2y_1y_2 + 2y_1y_3\\ &+2y_1(y_4+y_5) + 2y_2y_3 + 2y_3(y_4+y_5)),\end{aligned}\tag{2.5}$$

where

$$\begin{aligned}y_1 &= 2\kappa(x_1+\mu_0)\partial_\xi\vartheta,\\ y_2 &= -((x_1+\mu_0)^{-1}\partial_1^2\vartheta + (x_1+\mu_0)(\sum_{s,j=2}^{n}a_{sj}\partial_s\partial_j\vartheta - \sum_{s,j=n+1}^{n+m}a_{sj}\partial_s\partial_j\vartheta)),\end{aligned}$$

$$
\begin{aligned}
y_3 &= -2\lambda\nu\psi^{-\nu-1}(\partial_1\psi(x_1+\mu_0)^{-1}\partial_1\vartheta+(x_1+\mu_0)(\sum_{s,j=2}^{n}a_{sj}\partial_s\psi\partial_j\vartheta-\sum_{s,j=n+1}^{n+m}a_{sj}\partial_s\psi\partial_j\vartheta)),\\
y_4 &= -K\vartheta,\ y_5=\xi(x_1+\mu_0)\vartheta,
\end{aligned}
$$

and

$$
\begin{aligned}
K &= (x_1+\mu_0)^{-1}(\lambda^2\nu^2\psi^{-2\nu-2}(\partial_1\psi)^2-\lambda\nu(\nu+1)\psi^{-\nu-2}(\partial_1\psi)^2)\\
&\quad+(x_1+\mu_0)\sum_{s,j=2}^{n}a_{sj}(\lambda^2\nu^2\psi^{-2\nu-2}\partial_s\psi\partial_j\psi-\lambda\nu(\nu+1)\psi^{-\nu-2}\partial_s\psi\partial_j\psi+\lambda\nu\psi^{-\nu-1}\delta_{sj})\\
&\quad-(x_1+\mu_0)\sum_{s,j=n+1}^{n+m}a_{sj}(\lambda^2\nu^2\psi^{-2\nu-2}\partial_s\psi\partial_j\psi-\lambda\nu(\nu+1)\psi^{-\nu-2}\partial_s\psi\partial_j\psi+\lambda\nu\psi^{-\nu-1}\delta_{sj}).
\end{aligned}
$$

Next, we calculate the terms in (2.5). Here, we note that $\partial_1\psi=\delta$, $\partial_1^2\psi=0$, $\partial_s\partial_j\psi=\delta_{sj}$, where

$$
\delta_{sj}=\begin{cases}1, & s=j\\ 0, & s\neq j\end{cases},\ 2\le s,\ j\le n+m.
$$

First, we can write

$$
\begin{aligned}
2\psi^{\nu+1}y_1y_2 &= 4\kappa(\nu+1)\psi^{\nu}\partial_1\psi\partial_\xi\vartheta\partial_1\vartheta+4\kappa(\nu+1)(x_1+\mu_0)^2\psi^{\nu}\sum_{s,j=2}^{n}\partial_j\psi a_{sj}\partial_\xi\vartheta\partial_s\vartheta\\
&\quad+4\kappa(x_1+\mu_0)^2\psi^{\nu+1}\sum_{s,j=2}^{n}\partial_j(a_{sj})\partial_\xi\vartheta\partial_s\vartheta-4\kappa(\nu+1)(x_1+\mu_0)^2\psi^{\nu}\sum_{s,j=n+1}^{n+m}\partial_j\psi a_{sj}\partial_\xi\vartheta\partial_s\vartheta\\
&\quad-4\kappa(x_1+\mu_0)^2\psi^{\nu+1}\sum_{s,j=n+1}^{n+m}\partial_j(a_{sj})\partial_\xi\vartheta\partial_s\vartheta+d_1(\vartheta)\\
&= 2\psi^{\nu+1}y_1y_6+d_1(\vartheta),
\end{aligned}
$$

where

$$
\begin{aligned}
y_6 &= (\nu+1)\psi^{-1}((x_1+\mu_0)^{-1}\partial_1\psi\partial_1\vartheta+(x_1+\mu_0)(\sum_{s,j=2}^{n}a_{sj}\partial_s\psi\partial_j\vartheta-\sum_{s,j=n+1}^{n+m}a_{sj}\partial_s\psi\partial_j\vartheta))\\
&\quad+(x_1+\mu_0)(\sum_{s,j=2}^{n}\partial_j(a_{sj})\partial_s\vartheta-\sum_{s,j=n+1}^{n+m}\partial_j(a_{sj})\partial_s\vartheta),
\end{aligned}
$$

$$
\begin{aligned}
d_1 &= d_1(\vartheta)=-4\kappa\partial_1(\psi^{\nu+1}\partial_\xi\vartheta\partial_1\vartheta)+2\kappa\partial_\xi(\psi^{\nu+1}\partial_1^2\vartheta)\\
&\quad-2\kappa(x_1+\mu_0)^2(\sum_{s,j=2}^{n}(2\partial_j(\psi^{\nu+1}a_{sj}\partial_s\vartheta\partial_\xi\vartheta)-\partial_\xi(\psi^{\nu+1}a_{sj}\partial_s\vartheta\partial_j\vartheta))\\
&\quad-\sum_{s,j=n+1}^{n+m}(2(\psi^{\nu+1}a_{sj}\partial_s\vartheta\partial_\xi\vartheta)-\partial_\xi(\psi^{\nu+1}a_{sj}\partial_s\vartheta\partial_j\vartheta))).
\end{aligned}
$$

On the other hand, by the inequality $2pq \geq -p^2 - q^2$, we estimate the first, third, fourth and fifth terms in (2.5), so we have

$$\psi^{\nu+1}(y_1^2 + y_3^2 + 2y_1y_3 + 2y_1y_2) \geq \psi^{\nu+1}(-2y_3y_6 - y_6^2) + d_1(\vartheta).$$

Then, we can write

$$\begin{aligned}
& \psi^{\nu+1}(y_1^2 + y_3^2 + 2y_1y_3 + 2y_1y_2) \\
\geq\ & \psi^{\nu+1}((4\lambda\nu(\nu+1)\psi^{-\nu-2} - (\nu+1)^2\psi^{-2} - 2\lambda\nu\tau_0\psi^{-\nu-1} + (\nu+1)\tau_0\psi^{-1}) \\
& \times((x_1+\mu_0)^{-1}\partial_1\psi\partial_1\vartheta + (x_1+\mu_0)(\sum_{s,j=2}^{n} a_{sj}\partial_s\psi\partial_j\vartheta - \sum_{s,j=n+1}^{n+m} a_{sj}\partial_s\psi\partial_j\vartheta))^2 \\
& -\frac{1}{\tau_0}(2\lambda\nu\psi^{-\nu-1} - (\nu+1)\psi^{-1})(x_1+\mu_0)^2(\sum_{s,j=2}^{n} \partial_j(a_{sj})\partial_s\vartheta - \sum_{s,j=n+1}^{n+m} \partial_j(a_{sj})\partial_s\vartheta)^2 \\
& -(x_1+\mu_0)^2(\sum_{s,j=2}^{n} \partial_j(a_{sj})\partial_s\vartheta - \sum_{s,j=n+1}^{n+m} \partial_j(a_{sj})\partial_s\vartheta)^2) + d_1(\vartheta) \\
\geq\ & -2\lambda\nu\psi(x_1+\mu_0)^2(\sum_{s,j=2}^{n} \partial_j(a_{sj})\partial_s\vartheta - \sum_{s,j=n+1}^{n+m} \partial_j(a_{sj})\partial_s\vartheta)^2 + d_1(\vartheta). \qquad (2.6)
\end{aligned}$$

In (2.6), we used the inequality $2pq \geq -\tau_0 p^2 - \frac{q^2}{\tau_0}$, for $\tau_0 = (\nu+1)\psi^{-1} > 0$. Next, we continue to estimate the other terms in (2.5):

$$\begin{aligned}
2y_1(y_4+y_5)\psi^{\nu+1} &= 2(2\kappa(x_1+\mu_0)\partial_\xi\vartheta)(-K\vartheta + \xi\vartheta(x_1+\mu_0))\psi^{\nu+1} \\
&= \partial_\xi(-2\kappa(x_1+\mu_0)\psi^{\nu+1}K\vartheta^2) + \partial_\xi(2\kappa(x_1+\mu_0)^2\xi\psi^{\nu+1}\vartheta^2) \\
&\quad -2\kappa(x_1+\mu_0)^2\psi^{\nu+1}\vartheta^2. \qquad (2.7)
\end{aligned}$$

Since $\partial_1^2\psi = 0$, we have

$$\begin{aligned}
2y_3y_5\psi^{\nu+1} &= -2\lambda\nu\partial_1(\xi\partial_1\psi\vartheta^2) - 2\lambda\nu(x_1+\mu_0)^2\sum_{s,j=2}^{n}\partial_j(a_{sj}\partial_s\psi\xi\vartheta^2) \\
&\quad +2\lambda\nu(x_1+\mu_0)^2\xi\vartheta^2\sum_{s,j=2}^{n}\partial_j(a_{sj}\partial_s\psi) + 2\lambda\nu(x_1+\mu_0)^2\sum_{s,j=n+1}^{n+m}\partial_j(a_{sj}\partial_s\psi\xi\vartheta^2) \\
&\quad -2\lambda\nu(x_1+\mu_0)^2\xi\vartheta^2\sum_{s,j=n+1}^{n+m}\partial_j(a_{sj}\partial_s\psi), \qquad (2.8)
\end{aligned}$$

and

$$\begin{aligned}
& 2y_2y_3\psi^{\nu+1} \\
=\ & 2\lambda\nu\partial_1\psi\partial_1((x_1+\mu_0)^{-2}(\partial_1\vartheta)^2) + 4\lambda\nu\partial_1\psi(x_1+\mu_0)^{-3}(\partial_1\vartheta)^2 \\
& +2\lambda\nu\partial_1\psi\sum_{s,j=2}^{n}(2\partial_j(a_{sj}\partial_s\vartheta\partial_1\vartheta) - \partial_1(a_{sj}\partial_s\vartheta\partial_j v) - 2\partial_s(a_{sj})\partial_j\vartheta\partial_1\vartheta + \partial_1(a_{sj})\partial_s\vartheta\partial_j\vartheta)
\end{aligned}$$

$$-2\lambda\nu\partial_1\psi\sum_{s,j=n+1}^{n+m}(2\partial_j(a_{sj}\partial_s\vartheta\partial_1\vartheta)-\partial_1(a_{sj}\partial_s\vartheta\partial_j\vartheta)-2\partial_s(a_{sj})\partial_j\vartheta\partial_1\vartheta+\partial_1(a_{sj})\partial_s\vartheta\partial_j\vartheta)$$
$$+2\lambda\nu\sum_{s,j=2}^{n}(2\partial_1(a_{sj}\partial_s\psi\partial_s\vartheta\partial_1\vartheta)-2\partial_1(a_{sj})\partial_s\psi\partial_j\vartheta\partial_1\vartheta-\partial_j(a_{sj}\partial_s\psi(\partial_1\vartheta)^2)+\partial_j(a_{sj})\partial_s\psi(\partial_1\vartheta)^2)$$
$$-2\lambda\nu\sum_{s,j=n+1}^{n+m}(2\partial_1(a_{sj}\partial_s\psi\partial_s\vartheta\partial_1\vartheta)-2\partial_1(a_{sj})\partial_s\psi\partial_j\vartheta\partial_1\vartheta-\partial_j(a_{sj}\partial_s\psi(\partial_1\vartheta)^2)+\partial_j(a_{sj})\partial_s\psi(\partial_1\vartheta)^2)$$
$$+2\lambda\nu(x_1+\mu_0)^2(\sum_{s,j,k,l=2}^{n+m}(2\partial_j(a_{sj}a_{kl}\partial_l\vartheta\partial_s\vartheta\partial_k\psi)-\partial_l(a_{sj}a_{kl}\partial_j\vartheta\partial_s\vartheta\partial_k\psi)$$
$$-2\partial_j(a_{sj}a_{kl}\partial_k\psi)\partial_l\vartheta\partial_s\vartheta+\partial_l(a_{sj}a_{kl}\partial_k\psi)\partial_j\vartheta\partial_s\vartheta))-4\lambda\nu(x_1+\mu_0)^2(\sum_{s,j=2}^{n}\sum_{s,j=n+1}^{n+m}(2\partial_j(a_{sj}a_{kl}\partial_l\vartheta\partial_s\vartheta\partial_k\psi)$$
$$-\partial_l(a_{sj}a_{kl}\partial_j\vartheta\partial_s\vartheta\partial_k\psi)-2\partial_j(a_{sj}a_{kl}\partial_k\psi)\partial_l\vartheta\partial_s\vartheta+\partial_l(a_{sj}a_{kl}\partial_k\psi)\partial_j\vartheta\partial_s\vartheta)).\tag{2.9}$$

The last term has the following form:

$$\begin{aligned}2y_3y_4\psi^{\nu+1} &= 2\lambda\nu\partial_1\psi\partial_1((x_1+\mu_0)^{-1}K\vartheta^2)-2\lambda\nu\partial_1\psi\partial_1((x_1+\mu_0)^{-1}K)\vartheta^2\\ &\quad+2\lambda\nu(x_1+\mu_0)(\sum_{s,j=2}^{n}(\partial_j(a_{sj}\partial_s\psi K\vartheta^2)-\partial_j(a_{sj}\partial_s\psi K)\vartheta^2)\\ &\quad-\sum_{s,j=n+1}^{n+m}(\partial_j(a_{sj}\partial_s\psi K\vartheta^2)-\partial_j(a_{sj}\partial_s\psi K)\vartheta^2)).\end{aligned}\tag{2.10}$$

Then, from (2.6)–(2.10), we can write

$$\begin{aligned}\psi^{\nu+1}(P_0w)^2\varphi^2 \geq\; & -2\lambda\nu\psi(x_1+\mu_0)^2(\sum_{s,j=2}^{n}\partial_j(a_{sj})\partial_s\vartheta-\sum_{s,j=n+1}^{n+m}\partial_j(a_{sj})\partial_s\vartheta)^2+2\lambda\nu(\partial_1\vartheta)^2\sum_{s=2}^{n}a_{ss}\\ & +2\lambda\nu(x_1+\mu_0)^2\xi\vartheta^2(\sum_{s,j=2}^{n}\partial_j(a_{sj}\partial_s\psi)-\sum_{s,j=n+1}^{n+m}\partial_j(a_{sj}\partial_s\psi))-2\lambda\nu(\partial_1\vartheta)^2\sum_{s=n+1}^{n+m}a_{ss}\\ & +4\lambda\nu\partial_1\psi(x_1+\mu_0)^{-3}(\partial_1\vartheta)^2-2\kappa(x_1+\mu_0)^2\psi^{\nu+1}\vartheta^2-2\lambda\nu(\sum_{s,j=2}^{n}(2\partial_s\psi\partial_1(a_{sj})\partial_j\vartheta\partial_1\vartheta\\ & -\partial_s\psi\partial_j(a_{sj})(\partial_1\vartheta)^2+2\partial_1\psi\partial_s(a_{sj})\partial_j\vartheta\partial_1\vartheta-\partial_1\psi\partial_1(a_{sj})\partial_j\vartheta\partial_s\vartheta)\\ & -\sum_{s,j=n+1}^{n+m}(2\partial_s\psi\partial_1(a_{sj})\partial_j\vartheta\partial_1\vartheta-\partial_s\psi\partial_j(a_{sj})(\partial_1\vartheta)^2+2\partial_1\psi\partial_s(a_{sj})\partial_j\vartheta\partial_1\vartheta\\ & -\partial_1\psi\partial_1(a_{sj})\partial_j\vartheta\partial_s\vartheta))+2\lambda\nu(x_1+\mu_0)^2(\sum_{s,j,k,l=2}^{n}(-2\partial_j(a_{sj}a_{kl}\partial_k\psi)\partial_l\vartheta\partial_s\vartheta\\ & +\partial_l(a_{sj}a_{kl}\partial_k\psi)\partial_j\vartheta\partial_s\vartheta)-2\sum_{s,j=2}^{n}\sum_{s,j=n+1}^{n+m}(-2\partial_j(a_{sj}a_{kl}\partial_k\psi)\partial_l\vartheta\partial_j\vartheta\\ & +\partial_l(a_{sj}a_{kl}\partial_k\psi)\partial_j\vartheta\partial_s\vartheta)+\sum_{s,j=n+1}^{n+m}(-2\partial_j(a_{sj}a_{kl}\partial_k\psi)\partial_l\vartheta\partial_j\vartheta+\partial_l(a_{sj}a_{kl}\partial_k\psi)\partial_j\vartheta\partial_s\vartheta))\end{aligned}$$

$$-2\lambda\nu(\partial_1\psi\partial_1((x_1+\mu_0)^{-1}K)+(x_1+\mu_0)\sum_{s,j=2}^{n}(\partial_j(a_{sj}\partial_s\psi K)-\sum_{s,j=n+1}^{n+m}\partial_j(a_{sj}\partial_s\psi\xi))\vartheta^2$$
$$+d_1(\vartheta)+d_2(\vartheta), \tag{2.11}$$

where

$$\begin{aligned} d_2 \;=\; & d_2(\vartheta)=-\partial_\xi(2\kappa(x_1+\mu_0)\psi^{\nu+1}K\vartheta^2)+\partial_\xi(2\kappa(x_1+\mu_0)^2\xi\psi^{\nu+1}\vartheta^2)\\ & -2\lambda\nu\partial_1(\xi\partial_1\psi\vartheta^2)-2\lambda\nu(x_1+\mu_0)^2(\sum_{s,j=2}^{n}\partial_j(a_{sj}\partial_s\psi\xi\vartheta^2)-\sum_{s,j=n+1}^{n+m}\partial_j(a_{sj}\partial_s\psi\xi\vartheta^2))\\ & +2\lambda\nu\partial_1\psi(\partial_1((x_1+\mu_0)^{-2}(\partial_1\vartheta)^2)+\partial_1((x_1+\mu_0)^{-1}K\vartheta^2))\\ & +2\lambda\nu(\sum_{s,j=2}^{n}(2\partial_1(a_{sj}\partial_s\psi\partial_j\vartheta\partial_1\vartheta)-\partial_j(a_{sj}\partial_s\psi(\partial_1\vartheta)^2)\\ & +2\partial_j(\partial_1\psi a_{sj}\partial_1\vartheta\partial_s\vartheta)-\partial_1(\partial_1\psi a_{sj}\partial_j\vartheta\partial_s\vartheta))\\ & -\sum_{s,j=n+1}^{n+m}(2\partial_1(a_{sj}\partial_s\psi\partial_j\vartheta\partial_1\vartheta)-\partial_j(a_{sj}\partial_s\psi(\partial_1\vartheta)^2)\\ & +2\partial_j(\partial_1\psi a_{sj}\partial_1\vartheta\partial_s\vartheta)-\partial_1(\partial_1\psi a_{sj}\partial_j\vartheta\partial_s\vartheta)))\\ & +2\lambda\nu(x_1+\mu_0)^2(\sum_{s,j,k,l=2}^{n+m}(2\partial_j(a_{sj}a_{kl}\partial_l\vartheta\partial_s\vartheta)-\partial_l(a_{sj}a_{kl}\partial_s\psi\partial_j\vartheta\partial_s\vartheta))\\ & -2\sum_{s,j=2}^{n}\sum_{k,l=n+1}^{n+m}(2\partial_j(a_{sj}a_{kl}\partial_l\vartheta\partial_s\vartheta)-\partial_l(a_{sj}a_{kl}\partial_s\psi\partial_j\vartheta\partial_s\vartheta)))\\ & +2\lambda\nu(x_1+\mu_0)(\sum_{s,j=2}^{n}\partial_j(a_{sj}\partial_s\psi K\vartheta^2)-\sum_{s,j=n+1}^{n+m}\partial_j(a_{sj}\partial_s\psi K\vartheta^2)). \end{aligned}$$

Here, we note that $\|a_{sj}\|_{C^1(\overline{D_\gamma^{n+m}})}\le M$, and $\partial_j\psi=(x_j-x_j^0), |\partial_j\psi|=|x_j-x_j^0|\le\sqrt{2\gamma}, (2\le j\le n+m)$ in D_γ^{n+m}. Since $-2pq\ge-\frac{1}{\varepsilon_0(x_1+\mu_0)}p^2-\varepsilon_0(x_1+\mu_0)q^2$, for all $\varepsilon_0>0$ and p,q, we see that

$$4\lambda\nu\delta\sum_{s,j=n+1}^{n+m}\partial_s(a_{sj})\partial_1\vartheta\partial_j\vartheta\ge-2\lambda\nu\delta((x_1+\mu_0)m\varepsilon_0\sum_{j=n+1}^{n+m}(\partial_j\vartheta)^2+m^2\frac{M^2}{\varepsilon_0(x_1+\mu_0)}(\partial_1\vartheta)^2), \tag{2.12}$$

$$4\lambda\nu\sum_{s,j=n+1}^{n+m}\partial_1(a_{sj})\partial_s\psi\partial_1\vartheta\partial_j\vartheta\ge-2\lambda\nu(x_1+\mu_0)\sqrt{2\gamma}m\sum_{j=n+1}^{n+m}(\partial_j\vartheta)^2-2\lambda\nu m^2\frac{M^2\sqrt{2\gamma}}{(x_1+\mu_0)}(\partial_1\vartheta)^2, \tag{2.13}$$

$$-2\lambda\nu\sum_{s,j=n+1}^{n+m}\partial_j(a_{sj})\partial_s\psi(\partial_1\vartheta)^2\ge-2\lambda\nu M\sqrt{2\gamma}m^2(\partial_1\vartheta)^2, \tag{2.14}$$

$$-(\sum_{s,j=2}^{n}\partial_j(a_{sj})\partial_s\vartheta-\sum_{s,j=n+1}^{n+m}\partial_j(a_{sj})\partial_s\vartheta)^2$$

$$\geq \quad -M^2(n-1)((n-1)^2+m)\sum_{j=2}^{n}(\partial_j\vartheta)^2 - M^2m((n-1)+m^2)\sum_{j=n+1}^{n+m}(\partial_j\vartheta)^2, \tag{2.15}$$

$$2\lambda\nu(x_1+\mu_0)^2\sum_{s,j,k,l=2}^{n+m}(\partial_l(a_{sj}a_{kl}\partial_k\psi)\partial_j\vartheta\partial_s\vartheta - 2\partial_j(a_{sj}a_{kl}\partial_k\psi)\partial_l\vartheta\partial_s\vartheta)$$
$$\geq \quad -\lambda\nu(x_1+\mu_0)^2 6M^2(2\sqrt{2\gamma}+1)(n+m-1)^3\sum_{j=2}^{n+m}(\partial_j\vartheta)^2, \tag{2.16}$$

$$2\lambda\nu(x_1+\mu_0)^2\sum_{s,j=2}^{n}\sum_{k,l=n+1}^{n+m}(\partial_l(a_{sj}a_{kl}\partial_k\psi)\partial_j\vartheta\partial_s\vartheta - 2\partial_j(a_{sj}a_{kl}\partial_k\psi)\partial_l\vartheta\partial_s\vartheta)$$
$$\geq \quad -\lambda\nu(x_1+\mu_0)^2 6M^2(2\sqrt{2\gamma}+1)((n-1)\sum_{j=2}^{n}(\partial_j\vartheta)^2 + m\sum_{j=n+1}^{n+m}(\partial_j\vartheta)^2). \tag{2.17}$$

The other terms can be estimated similarly. Hence, we can rewrite (2.11) in the following form:

$$\psi^{\nu+1}(P_0w)^2\varphi^2 \geq E_1(\partial_1\vartheta)^2 + E_2\sum_{j=2}^{n}(\partial_j\vartheta)^2 + E_3\sum_{j=n+1}^{n+m}(\partial_j\vartheta)^2 + E_4\vartheta^2 + d_1 + d_2, \tag{2.18}$$

where

$$\begin{aligned} E_1 \quad = \quad & 4\lambda\nu\delta(x_1+\mu_0)^{-3} - 2\lambda\nu\delta\frac{M^2(n-1)^2}{\varepsilon_0(x_1+\mu_0)} - 2\lambda\nu\frac{M^2(n-1)^2\sqrt{2\gamma}}{(x_1+\mu_0)} - 2\lambda\nu\sqrt{2\gamma}(n-1)^2 \\ & -2\lambda\nu\delta\frac{M^2m^2}{\varepsilon_0(x_1+\mu_0)} - 2\lambda\nu\frac{M^2m^2\sqrt{2\gamma}}{(x_1+\mu_0)} - 2\lambda\nu M\sqrt{2\gamma}m^2 - 2\lambda\nu M(n-1) - 2\lambda\nu Mm, \end{aligned}$$

$$\begin{aligned} E_2 \quad = \quad & 2\lambda\nu(x_1+\mu_0)(\delta\alpha_1 - \delta\varepsilon_0(n-1) - \sqrt{2\gamma}(n-1) - 3M^2(x_1+\mu_0)(2\sqrt{2\gamma}+1)(n-1)^3 \\ & -3M^2(x_1+\mu_0)(2\sqrt{2\gamma}+1)(n-1) + M^2(x_1+\mu_0)(n-1)(m+(n-1)^2)), \end{aligned}$$

$$\begin{aligned} E_3 \quad = \quad & 2\lambda\nu(x_1+\mu_0)(\delta\alpha_2 - \delta\varepsilon_0 m - \sqrt{2\gamma}m - 3M^2(x_1+\mu_0)(2\sqrt{2\gamma}+1)m^3 \\ & -3M^2(x_1+\mu_0)(2\sqrt{2\gamma}+1)m + M^2(x_1+\mu_0)m(m^2+(n-1))), \end{aligned}$$

$$\begin{aligned} E_4 \quad = \quad & -2\lambda\nu(\delta((x_1+\mu_0)^{-1}K)_{x_1} + (x_1+\mu_0)\sum_{s,j=2}^{n}(\partial_j(a_{sj}\partial_s\psi K) - \sum_{s,j=n+1}^{n+m}\partial_j(a_{sj}\partial_s\psi K))) \\ & -2(x_1+\mu_0)^2(k\psi^{\nu+1} - \lambda\nu\xi(\sum_{s,j=2}^{n}\partial_j(a_{sj}\partial_s\psi) - \sum_{s,j=n+1}^{n+m}\partial_j(a_{sj}\partial_s\psi))). \end{aligned}$$

First, we shall estimate the coefficient E_1 :

$$E_1 = 2\lambda\nu(x_1+\mu_0)^{-3}(\delta + E_{11} - E_{12}),$$

where

$$E_{11} = \delta(1 - (x_1 + \mu_0)^2 \frac{M^2}{\varepsilon_0}((n-1)^2 + m^2)),$$

$$E_{12} = (x_1 + \mu_0)^2 \sqrt{2\gamma} M^2((n-1)^2 + m^2) + (x_1 + \mu_0)^3 \sqrt{2\gamma} M((n-1)^2 + m^2) + (x_1 + \mu_0)^3 M(n + m - 1).$$

If we take γ as $0 < \gamma < \frac{2\sqrt{2\varepsilon_0}}{3M\sqrt{(n-1)^2+m^2}}$, then $1 - ((n-1)^2 + m^2)\frac{M^2}{\varepsilon_0}(\frac{3}{4}\gamma)^2 > \frac{1}{2}$, and so we obtain

$$E_{11} > \delta(1 - ((n-1)^2 + m^2)\frac{M^2}{\varepsilon_0}(\frac{3}{4}\gamma)^2) > \frac{1}{2}\delta$$

in D_γ^{n+m}. Setting

$$E_{12}^{'} = \sqrt{2\gamma} M((n-1)^2 + m^2)(M + 1) + M(n + m - 1)$$

for $\delta \geq \delta_1 = 2E_{12}^{'}$, and then $\frac{1}{2}\delta - E_{12}^{'} \geq 0$, which implies that

$$E_1 > 2\lambda\nu(x_1 + \mu_0)^{-3}(\delta + \frac{1}{2}\delta - E_{12}^{'}) > 2\lambda\nu\gamma^{-3}\delta. \tag{2.19}$$

As for the coefficient E_2, if we take ε_0 such that $0 < \varepsilon_0 < \frac{\alpha_1}{4(n-1)}$, then we have

$$\delta(\alpha_1 - \varepsilon_0(n-1)) > \frac{3}{4}\delta\alpha_1. \tag{2.20}$$

Here, we note that

$$\delta_2 = \frac{4}{\alpha_1}(\sqrt{2}(n-1) + 3M^2(2\sqrt{2} + 1)(n-1)^3 + 6M^2(2\sqrt{2} + 1)(n-1) + M^2(n-1)(m + (n-1)^2)).$$

Then, for $\delta \geq \delta_2$, from (2.20) it follows that

$$\begin{aligned} E_2 &= 2\lambda\nu(x_1 + \mu_0)(\delta\alpha_1 - \delta\varepsilon_0(n-1) - \sqrt{2\gamma}(n-1) - 3M^2(x_1 + \mu_0)(2\sqrt{2\gamma} + 1)(n-1)^3 \\ &\quad -3M^2(x_1 + \mu_0)(2\sqrt{2\gamma} + 1)(n-1) - M^2(x_1 + \mu_0)(n-1)(m + (n-1)^2)) \\ &> \lambda\nu\delta\alpha_1(x_1 + \mu_0). \end{aligned} \tag{2.21}$$

Next, we estimate the coefficient E_3. We choose ε_0 such that $0 < \varepsilon_0 < \frac{\alpha_2}{4m}$,

$$\delta(\alpha_2 - \varepsilon_0 m) > \frac{3}{4}\delta\alpha_2. \tag{2.22}$$

Since $\mu_0 < \frac{1}{2}\gamma$, $\delta \geq 4$ and $0 < \delta x_1 < \gamma$ in D_γ^{n+m}, we get $(x_1 + \mu_0) < \frac{3}{4}\gamma$. Here, we set

$$\delta_3 = \frac{4}{\alpha_2}(\sqrt{2}m + 3M^2(2\sqrt{2} + 1)m^3 - 3M^2(2\sqrt{2} + 1)m + M^2 m(m^2 + (n-1))).$$

Then, if we take $\delta \geq \delta_3$, from (2.22) it follows that

$$E_3 > 2\lambda\nu(x_1 + \mu_0)(\frac{3}{4}\delta\alpha_2 - \frac{1}{4}\delta\alpha_2) > \lambda\nu(x_1 + \mu_0)\delta\alpha_2. \tag{2.23}$$

Now, let us estimate the coefficient E_4. By the definition of the function K, we can write E_4 in the following form

$$E_4 = E_{41} + E_{42} + 2(x_1+\mu_0)^2\lambda\nu\xi(\sum_{s,j=2}^{n}\partial_j(a_{sj}\partial_s\psi) - \sum_{s,j=n+1}^{n+m}\partial_j(a_{sj}\partial_s\psi)), \tag{2.24}$$

where

$$\begin{aligned}E_{41} \;=\; & -2\lambda^3\nu^3\delta^3\partial_1((x_1+\mu_0)^{-2}\psi^{-2\nu-2}) - 2\lambda^3\nu^3\delta(\sum_{s,j=2}^{n}\partial_1(a_{sj}\psi^{-2\nu-2})\partial_s\psi\partial_j\psi \\ & -\sum_{s,j=n+1}^{n+m}\partial_1(a_{sj}\psi^{-2\nu-2})\partial_s\psi\partial_j\psi) - 2\lambda^3\nu^3\delta^2(\partial_j(\psi^{-2\nu-2}\sum_{s,j=2}^{n}a_{sj}\partial_s\psi) \\ & -\partial_j(\psi^{-2\nu-2}\sum_{s,j=n+1}^{n+m}a_{sj}\partial_s\psi)) + 4\lambda^3\nu^3(x_1+\mu_0)\sum_{s,j=2}^{n}\sum_{k,l=n+1}^{n+m}\partial_j(a_{sj}a_{kl}\partial_s\psi\psi^{-2\nu-2})\partial_k\psi\partial_l\psi \\ & -2\lambda^3\nu^3(x_1+\mu_0)^2\sum_{s,j,k,l=2}^{n+m}\partial_j(a_{sj}a_{kl}\partial_s\psi\partial_k\psi\partial_l\psi\psi^{-2\nu-2}),\end{aligned}$$

$$E_{42} = 2\lambda^2\nu^2(\delta\partial_1((x_1+\mu_0)^{-1}E_{01}) + (x_1+\mu_0)(\sum_{s,j=2}^{n}\partial_j(a_{sj}\partial_s\psi E_{01}) - \sum_{s,j=n+1}^{n+m}\partial_j(a_{sj}\partial_s\psi E_{01}))) - 2\kappa(x_1+\mu_0)^2\psi^{\nu+1},$$

$$\begin{aligned}E_{01} \;=\; & (x_1+\mu_0)^{-1}(\nu+1)\psi^{-\nu-2}\delta^2 + (x_1+\mu_0)(\sum_{s,j=2}^{n}a_{sj}((\nu+1)\psi^{-\nu-2}\partial_s\psi\partial_j\psi - \psi^{-\nu-1}\delta_{sj}) \\ & -\sum_{s,j=n+1}^{n+m}a_{sj}((\nu+1)\psi^{-\nu-2}\partial_s\psi\partial_j\psi - \psi^{-\nu-1}\delta_{sj})).\end{aligned}$$

We now estimate each of terms in E_{41}, respectively:

$$\begin{aligned}& -2\lambda^3\nu^3\delta^3\partial_1((x_1+\mu_0)^{-2}\psi^{-2\nu-2}) \\ = \;& 4\lambda^3\nu^3\delta^4(\nu+1)(x_1+\mu_0)^{-2}\psi^{-2\nu-3} + 4\lambda^3\nu^3\delta^3(x_1+\mu_0)^{-3}\psi^{-2\nu-2},\end{aligned} \tag{2.25}$$

$$\begin{aligned}& -2\lambda^3\nu^3\delta\sum_{s,j=2}^{n}\partial_1(a_{sj}\psi^{-2\nu-2})\partial_s\psi\partial_j\psi \\ = \;& 4\lambda^3\nu^3\delta^2(\nu+1)\psi^{-2\nu-3}\sum_{s,j=2}^{n}a_{sj}\partial_s\psi\partial_j\psi - 2\lambda^3\nu^3\delta\psi^{-2\nu-2}\sum_{s,j=2}^{n}\partial_1(a_{sj})\partial_s\psi\partial_j\psi,\end{aligned} \tag{2.26}$$

$$2\lambda^3\nu^3\delta^2\partial_j(\psi^{-2\nu-2}\sum_{s,j=n+1}^{n+m}a_{sj}\partial_s\psi)$$

$$= \quad -4\lambda^3\nu^3\delta^2(\nu+1)\psi^{-2\nu-3}\sum_{s,j=n+1}^{n+m} a_{sj}\partial_s\psi + 2\lambda^3\nu^3\delta^2\psi^{-2\nu-2}\sum_{s,j=n+1}^{n+m}(\partial_j(a_{sj})\partial_s\psi + a_{sj}\delta_{sj}), \quad (2.27)$$

$$-2\lambda^3\nu^3(x_1+\mu_0)^2\sum_{s,j,k,l=2}^{n}\partial_j(a_{sj}a_{kl}\partial_s\psi\partial_k\psi\partial_l\psi\psi^{-2\nu-2})$$
$$= \quad 4\lambda^3\nu^3(x_1+\mu_0)^2(\nu+1)\psi^{-2\nu-3}\sum_{s,j,k,l=2}^{n} a_{sj}a_{kl}\partial_s\psi\partial_k\psi\partial_l\psi\partial_j\psi$$
$$-2\lambda^3\nu^3(x_1+\mu_0)^2\psi^{-2\nu-2}\sum_{s,j,k,l=2}^{n}\partial_j(a_{sj}a_{kl}\partial_s\psi\partial_k\psi\partial_l\psi), \quad (2.28)$$

$$4\lambda^3\nu^3(x_1+\mu_0)^2\sum_{s,j=2}^{n}\sum_{k,l=n+1}^{n+m}\partial_j(a_{sj}a_{kl}\partial_s\psi\psi^{-2\nu-2})\partial_k\psi\partial_l\psi$$
$$= \quad -8\lambda^3\nu^3(x_1+\mu_0)^2(\nu+1)\psi^{-2\nu-3}\sum_{s,j=2}^{n}\sum_{k,l=n+1}^{n+m} a_{sj}a_{kl}\partial_s\psi\partial_j\psi\partial_k\psi\partial_l\psi$$
$$+4\lambda^3\nu^3(x_1+\mu_0)^2\psi^{-2\nu-2}\sum_{s,j=2}^{n}\sum_{k,l=n+1}^{n+m}(\partial_j(a_{sj})a_{kl}\partial_s\psi + a_{sj}\partial_j(a_{kl})\partial_s\psi + a_{sj}a_{kl}\delta_{sj})\partial_k\psi\partial_l\psi. \quad (2.29)$$

Then, from these relations we can write

$$E_{41} = 4\lambda^3\nu^3(\nu+1)\psi^{-2\nu-3}(\delta^4(x_1+\mu_0)^{-2} + 2\delta^2\sum_{s,j=2}^{n} a_{sj}\partial_s\psi\partial_j\psi - 2\delta^2\sum_{s,j=n+1}^{n+m} a_{sj}\partial_s\psi\partial_j\psi$$
$$+(x_1+\mu_0)^2\sum_{s,j,k,l=2}^{n+m} a_{sj}a_{kl}\partial_s\psi\partial_k\psi\partial_l\psi\partial_j\psi - 2(x_1+\mu_0)^2\sum_{s,j=2}^{n}\sum_{k,l=n+1}^{n+m} a_{sj}a_{kl}\partial_s\psi\partial_j\psi\partial_k\psi\partial_l\psi)$$
$$+4\lambda^3\nu^3\psi^{-2\nu-2}(\delta^3(x_1+\mu_0)^{-3} - \frac{\delta}{2}\sum_{s,j=2}^{n}\partial_1(a_{sj})\partial_s\psi\partial_j\psi + \frac{\delta}{2}\sum_{s,j=n+1}^{n+m}\partial_1(a_{sj})\partial_s\psi\partial_j\psi$$
$$-\frac{\delta^2}{2}\sum_{s,j=2}^{n}(\partial_j(a_{sj})\partial_s\psi + a_{sj}\delta_{sj}) + \frac{\delta^2}{2}\sum_{s,j=n+1}^{n+m}(\partial_j a_{sj}\partial_s\psi + a_{sj}\delta_{sj})$$
$$-\frac{(x_1+\mu_0)^2}{2}\sum_{s,j,k,l=2}^{n+m}\partial_j(a_{sj}a_{kl}\partial_s\psi\partial_k\psi\partial_l\psi) + (x_1+\mu_0)^2\sum_{s,j=2}^{n}\sum_{k,l=n+1}^{n+m}(\partial_j(a_{sj})a_{kl}\partial_s\psi$$
$$+a_{sj}\partial_j(a_{kl})\partial_s\psi + a_{sj}a_{kl}\delta_{sj})\partial_k\psi\partial_l\psi).$$

Remembering that $(x_1+\mu_0) < \frac{3}{4}\gamma$ in D_γ^{n+m}, we have $\delta^4(x_1+\mu_0)^{-2} > \delta^4$. We set

$$\delta_4 = \frac{1}{2}((4\sqrt{2}+2)M^2(n+m-1)^2 + (2\sqrt{2}+3)M(n+m-1) + 4M^2((n-1)^2+m^2)),$$

then, we see that

$$E_4 > 2\lambda^3\nu^3\delta^4(\nu+1)\psi^{-2\nu-3} + 2\lambda\nu\xi(x_1+\mu_0)^2(\sum_{s,j=2}^{n}\partial_j(a_{sj}\partial_s\psi) - \sum_{s,j=n+1}^{n+m}\partial_j(a_{sj}\partial_s\psi))$$

for $\delta \geq \delta_4$ and $\lambda > \lambda_0$, which yields

$$
\begin{aligned}
\psi^{\nu+1}(P_0 w)^2\varphi^2 \geq\ & 2\lambda\nu\delta\gamma^{-3}(\partial_1\vartheta)^2 + \lambda\nu(x_1+\mu_0)\delta\alpha_1\sum_{j=2}^{n}(\partial_j\vartheta)^2 \\
&+\lambda\nu(x_1+\mu_0)\delta\alpha_2\sum_{j=n+1}^{n+m}(\partial_j\vartheta)^2 + 2\lambda^3\nu^4\delta^4\psi^{-2\nu-3}\vartheta^2 \\
&+2\lambda\nu\xi(x_1+\mu_0)^2\Big(\sum_{s,j=2}^{n}\partial_j(a_{sj}\partial_s\psi) - \sum_{s,j=n+1}^{n+m}\partial_j(a_{sj}\partial_s\psi)\Big)\vartheta^2 + d_1(\vartheta) + d_2(\vartheta). \quad (2.30)
\end{aligned}
$$

Finally, if we write $\vartheta = \varphi w$ in (2.30), then we obtain

$$
\begin{aligned}
& 2\lambda\nu\delta\gamma^{-3}(\partial_1\vartheta)^2 + \lambda\nu\delta\alpha_1(x_1+\mu_0)\sum_{j=2}^{n}(\partial_j\vartheta)^2 + \lambda\nu\delta\alpha_2(x_1+\mu_0)\sum_{j=n+1}^{n+m}(\partial_j\vartheta)^2 + 2\lambda^3\nu^4\delta^4\psi^{-2\nu-3}\vartheta^2 \\
=\ & 2\lambda\nu\gamma^{-3}\delta(\partial_1 w)^2\varphi^2 + \lambda\nu(x_1+\mu_0)\delta\alpha_1\varphi^2\sum_{j=2}^{n}(\partial_j w)^2 + \lambda\nu(x_1+\mu_0)\delta\alpha_2\varphi^2\sum_{j=n+1}^{n+m}(\partial_j w)^2 \\
& +2\lambda^3\nu^4\delta^4\psi^{-2\nu-3}w^2\varphi^2 - 2\lambda^2\nu^2\gamma^{-3}\delta\partial_1(\delta\psi^{-\nu-1}w^2\varphi^2) - \lambda^2\nu^2(x_1+\mu_0)\delta\alpha_1\sum_{j=2}^{n}\partial_j(\psi^{-\nu-1}\partial_j\psi w^2\varphi^2) \\
& -\lambda^2\nu^2(x_1+\mu_0)\delta\alpha_2\sum_{j=n+1}^{n+m}\partial_j(\psi^{-\nu-1}\partial_j\psi w^2\varphi^2) - 2\lambda^3\nu^3\gamma^{-3}\delta^3\psi^{-2\nu-2}w^2\varphi^2 \\
& -\lambda^3\nu^3(x_1+\mu_0)\delta\alpha_1\psi^{-2\nu-2}w^2\varphi^2\sum_{j=2}^{n}(\partial_j w)^2 \\
& -\lambda^3\nu^3(x_1+\mu_0)\delta\alpha_2\psi^{-2\nu-2}w^2\varphi^2\sum_{j=n+1}^{n+m}(\partial_j w)^2 - \lambda^2\nu^2(2\gamma^{-3}\delta^3(\nu+1)\psi^{-\nu-2} \\
& +(x_1+\mu_0)\delta\alpha_1(\nu+1)\psi^{-\nu-2}\sum_{j=2}^{n}(\partial_j w)^2 + (x_1+\mu_0)\delta\alpha_2(\nu+1)\psi^{-\nu-2}\sum_{j=n+1}^{n+m}(\partial_j w)^2 \\
& -(x_1+\mu_0)\delta\alpha_1\psi^{-\nu-1}(n-1) - (x_1+\mu_0)\delta\alpha_2\psi^{-\nu-1}m)w^2\varphi^2. \quad (2.31)
\end{aligned}
$$

Choosing $\nu \geq \nu_0 = 2\gamma^{-3}\delta^{-1} + \frac{3}{2}\gamma^2\delta^{-3}(\alpha_1+\alpha_2) + 1$, we have

$$
\begin{aligned}
& w^2\varphi^2\psi^{-2\nu-2}(\lambda^3\nu^4\delta^4\psi^{-1}\varphi^2 - 2\lambda^3\nu^3\gamma^{-3}\delta^3 - \lambda^3\nu^3(x_1+\mu_0)\delta(\alpha_1\sum_{j=2}^{n}(\partial_j w)^2 + \alpha_2\sum_{j=n+1}^{n+m}(\partial_j w)^2)) \\
\geq\ & \lambda^3\nu^3\delta^4 w^2\varphi^2\psi^{-2\nu-2}(\nu - 2\gamma^{-3}\delta^{-1} - \frac{3}{2}\gamma^2\delta^{-3}(\alpha_1+\alpha_2)) \\
\geq\ & \lambda^3\nu^3\delta^4 w^2\varphi^2\psi^{-2\nu-2}. \quad (2.32)
\end{aligned}
$$

On the other hand, if $\lambda \geq \lambda_0 = 2\gamma^{-3}(\nu+1) + \frac{3}{2}\gamma^2(\nu+1)(\alpha_1+\alpha_2)$, then we get

$$
\lambda^3\nu^3\delta^4\psi^{-2\nu-2}w^2\varphi^2 - 2\lambda^2\nu^2\gamma^{-3}\delta^3(\nu+1)\psi^{-\nu-2}w^2\varphi^2
$$

$$-\lambda^2\nu^2(x_1+\mu_0)\delta(\alpha_1+\alpha_2)(\nu+1)\psi^{-\nu-2}w^2\varphi^2\sum_{j=2}^{n+m}(\partial_j w)^2$$

$$\geq \lambda^2\nu^3\delta^4\psi^{-2\nu-2}w^2\varphi^2(\lambda-2\gamma^{-3}(\nu+1)-\frac{3}{2}\gamma^2(\nu+1)(\alpha_1+\alpha_2))\geq 0. \tag{2.33}$$

Therefore, for $\delta>\delta_0$, $\nu\geq\nu_0$ and $\lambda\geq\lambda_0$, from (2.30)-(2.33) we have

$$\begin{aligned}
&2\lambda\nu\gamma^{-3}\delta(\partial_1\vartheta)^2+\lambda\nu(x_1+\mu_0)\delta\alpha_1\sum_{j=2}^{n}(\partial_j\vartheta)^2+\lambda\nu(x_1+\mu_0)\delta\alpha_2\sum_{j=n+1}^{n+m}(\partial_j\vartheta)^2+2\lambda^3\nu^4\delta^4\psi^{-2\nu-3}\vartheta^2\\
\geq\ &2\lambda\nu\gamma^{-3}\delta(\partial_1 w)^2\varphi^2+\lambda\nu(x_1+\mu_0)\delta\alpha_1\varphi^2\sum_{j=2}^{n}(\partial_j w)^2+\lambda\nu(x_1+\mu_0)\delta\alpha_2\varphi^2\sum_{j=n+1}^{n+m}(\partial_j w)^2\\
&+\lambda^3\nu^4\delta^4\psi^{-2\nu-3}w^2\varphi^2+d_3(w),
\end{aligned}\tag{2.34}$$

where $\delta_0=\max\{4,\delta_1,\delta_2,\delta_3,\delta_4\}$, and

$$\begin{aligned}
d_3 &= d_3(w)=-2\lambda^2\nu^2\gamma^{-3}\delta\partial_1(\psi^{-\nu-1}\delta w^2\varphi^2)-\lambda^2\nu^2(x_1+\mu_0)\delta\alpha_1\sum_{j=2}^{n}\partial_j(\psi^{-\nu-1}\partial_j\psi w^2\varphi^2)\\
&\quad-\lambda^2\nu^2(x_1+\mu_0)\delta\alpha_2\sum_{j=n+1}^{n+m}\partial_j(\psi^{-\nu-1}\partial_j\psi w^2\varphi^2).
\end{aligned}$$

From relations (2.30) and (2.34), the proof of Lemma 1 is completed. □

Lemma 2. If λ exceeds some constant, then for all $w\in C^2(D_\gamma^{n+m})$, we have

$$\begin{aligned}
&2\lambda\nu(x_1+\mu_0)\varphi^2 S w(P_0 w)\\
\geq\ &-4S^0\lambda^3\nu^3\psi^{-2\nu-2}(\delta^2+2\gamma M(n-1)^2+m^2)w^2\varphi^2\\
&-2\lambda\nu S^0\varphi^2((\partial_1 w)^2+(x_1+\mu_0)^2M((n-1)\sum_{j=2}^{n}(\partial_j w)^2+m\sum_{j=n+1}^{n+m}(\partial_j w)^2))\\
&-C\lambda^2\nu^2\psi^{-\nu-1}\delta^2\varphi^2w^2-2\lambda\nu S\xi(x_1+\mu_0)^2\varphi^2w^2+d_4(w),
\end{aligned}\tag{2.35}$$

where $C>0$ is a constant depending on a_{sj} and dimension $n+m$,

$$\begin{aligned}
d_4 &= d_4(w)=\partial_\xi(-2\lambda\nu\kappa S\varphi^2(x_1+\mu_0)^2)+\lambda\nu\partial_1(2\varphi^2 S w\partial_1 w-\partial_1(\varphi^2 S)w^2)\\
&\quad-\lambda\nu(x_1+\mu_0)^2\sum_{s,j=2}^{n}(\partial_j(\partial_s(\varphi^2 S a_{sj})w^2)-2\partial_s(\varphi^2 S a_{sj}w\partial_j w))\\
&\quad+\lambda\nu(x_1+\mu_0)^2\sum_{s,j=n+1}^{n+m}(\partial_j(\partial_s(\varphi^2 S a_{sj})w^2)-2\partial_s(\varphi^2 S a_{sj}w\partial_j w)),\\
S &= \sum_{l,m=2}^{n}\partial_m(a_{lm}\partial_l\psi)-\sum_{l,m=n+1}^{n+m}\partial_m(a_{lm}\partial_l\psi),\\
S^0 &= (n-1)M((n-1)\sqrt{2\gamma}+2)+mM(m\sqrt{2\gamma}+2).
\end{aligned}$$

Proof. By using the equality

$$\begin{aligned}
&2\lambda\nu(x_1+\delta_0)\varphi^2 S w(P_0 w)\\
=\;& 2\lambda\nu\varphi^2 S w\partial_1^2 w + 2\lambda\nu(x_1+\delta_0)^2\varphi^2 S w(\sum_{s,j=2}^{n} a_{sj}\partial_s\partial_j w - \sum_{s,j=n+1}^{n+m} a_{sj}\partial_s\partial_j w)\\
&-4\lambda\nu(x_1+\delta_0)^2\varphi^2 S w\kappa\partial_\xi w - 2\lambda\nu(x_1+\delta_0)^2\varphi^2 S w^2\xi
\end{aligned}$$

and taking into account $\partial_s p\partial_j q = \partial_j((\partial_s p)q) - \partial_j(\partial_s p)q$, we have

$$\begin{aligned}
&2\lambda\nu(x_1+\mu_0)\varphi^2 S w(P_0 w)\\
=\;& -2\lambda\nu\varphi^2 S((\partial_1 w)^2 + (x_1+\mu_0)^2(\sum_{s,j=2}^{n} a_{sj}\partial_s w\partial_j w - \sum_{s,j=n+1}^{n+m} a_{sj}\partial_s w\partial_j w))\\
&+S_1 w^2 - 2\lambda\nu(x_1+\mu_0)^2\varphi^2 S\xi w^2 + d_4(w),
\end{aligned}\tag{2.36}$$

where

$$S_1 = \lambda\nu(\partial_1^2(\varphi^2 S) + (x_1+\mu_0)^2(\partial_s\partial_j(\varphi^2 S\sum_{s,j=2}^{n} a_{sj}) - \partial_s\partial_j(\varphi^2 S\sum_{s,j=n+1}^{n+m} a_{sj}))).$$

Remembering that $|S| \le S^0$ and $\|a_{sj}\|_{C^1(D_\gamma^{n+m})} \le M$ $(2 \le s, j \le n+m)$, we see that the first term in (2.36) can be written as

$$\begin{aligned}
&-2\lambda\nu\varphi^2 S((\partial_1 w)^2 + (x_1+\mu_0)^2(\sum_{s,j=2}^{n} a_{sj}\partial_s w\partial_j w - \sum_{s,j=n+1}^{n+m} a_{sj}\partial_s w\partial_j w))\\
\ge\;& -2\lambda\nu S^0\varphi^2((\partial_1 w)^2 + (x_1+\mu_0)^2 M((n-1)\sum_{j=2}^{n}(\partial_j w)^2 - m\sum_{j=n+1}^{n+m}(\partial_j w)^2)).
\end{aligned}\tag{2.37}$$

Next, as for the coefficient in the second term, we write

$$\begin{aligned}
S_1 =\;& \lambda\nu(\partial_1^2(\varphi^2)S + 2\partial_1(\varphi^2)\partial_1 S + \varphi^2\partial_1^2 S\\
&+(x_1+\mu_0)^2(\sum_{s,j=2}^{n}(S\partial_s(\partial_j(\varphi^2)a_{sj}) + \partial_s(\varphi^2)(2\partial_j S a_{sj} + S\partial\partial_j(a_{sj}))\\
&+\varphi^2(\partial_s\partial_j S a_{sj} + 2\partial_j S\partial_s(a_{sj}) + S\partial_s\partial_j(a_{sj})))\\
&-\sum_{s,j=n+1}^{n+m}(S\partial_s(\partial_j(\varphi^2)a_{sj}) + \partial_s(\varphi^2)(2\partial_j S a_{sj} + S\partial\partial_j(a_{sj}))\\
&+\varphi^2(\partial_s\partial_j S a_{sj} + 2\partial_j S\partial_s(a_{sj}) + S\partial_s\partial_j(a_{sj}))))).
\end{aligned}\tag{2.38}$$

Since

$$\begin{aligned}
\partial_1^2(\varphi^2) =\;& 2\lambda\nu\varphi^2(2\lambda\nu\delta^2\psi^{-2\nu-2} + (\nu+1)\delta^2\psi^{-\nu-2}),\\
\partial_j(a_{sj}\partial_s(\varphi^2)) =\;& 2\lambda\nu\varphi^2(a_{sj}(2\lambda\nu\psi^{-2\nu-2}\partial_s\psi\partial_j\psi + (\nu+1)\psi^{-\nu-2}\partial_s\psi\partial_j\psi\\
&-\psi^{-\nu-1}\partial_s\partial_j\psi) - \partial_s(a_{sj})\psi^{-\nu-1}\partial_j\psi),
\end{aligned}\tag{2.39}$$

$$|S| \leq S^0,\ |\partial_j S| \leq S^0, |\partial_s\partial_j S| \leq (n-1)M((n-1)\sqrt{2\gamma}+3) + mM(m\sqrt{2\gamma}+3), \tag{2.40}$$

we see that

$$\begin{aligned}|S_1| \ \leq\ & 4\lambda^3\nu^3\varphi^2\psi^{-2\nu-2}S^0(\delta^2+(x_1+\delta_0)^2(\sum_{s,j=2}^{n+m}\left|a_{sj}\partial_s\psi\partial_j\psi\right|) + \lambda^2\nu^2\psi^{-\nu-1}\delta^2\varphi^2 S^0(2(\nu+1)\psi^{-1}+\frac{4}{\delta}\\ &+\frac{(x_1+\delta_0)^2}{\delta^2}(\sum_{s,j=2}^{n+m}\left|2(\nu+1)\varphi^2 a_{sj}\psi^{-1}\partial_s\psi\partial_j\psi\right| + \left|2a_{sj}\partial_s\partial_j\psi\right|\\ &+\left|2\partial_s(a_{sj})\partial_j\psi\right| + \left|4\partial_s\psi a_{sj}\right| + \left|2\partial_s\psi\partial_s(a_{sj})\right|)))\\ &+\lambda\nu\varphi^2(\sum_{s,j=2}^{n+m}\left|2\partial_s(a_{sj})\right|S^0 + \left|\partial_s\partial_j(a_{sj})\right|S^0) + \left|\partial_1^2 S\right| + (x_1+\delta_0)^2(\sum_{s,j=2}^{n+m}\left|a_{sj}\right|\left|\partial_s\partial_j S\right|)).\end{aligned}$$

If we choose that

$$\lambda \geq \lambda_1 = \sum_{s,j=2}^{n+m}(\left|2\partial_s(a_{sj})\right| + \left|\partial_s\partial_j(a_{sj})\right|)S^0 + \left|\partial_1^2 S\right| + (x_1+\delta_0)^2(\sum_{s,j=2}^{n+m}\left|a_{sj}\right|\left|\partial_s\partial_j S\right|),$$

and $\nu \geq 1$, then we have

$$|S_1| \leq 4\lambda^3\nu^3\varphi^2\psi^{-2\nu-2}S^0(\delta^2+(x_1+\delta_0)^2(\sum_{s,j=2}^{n+m}\left|a_{sj}\partial_s\psi\partial_j\psi\right|)) + C\lambda^2\nu^2\psi^{-\nu-1}\delta^2\varphi^2,$$

where

$$\begin{aligned}C \ =\ & S^0(2(\nu+1)\psi^{-1}+\frac{4}{\delta}+\frac{(x_1+\delta_0)^2}{\delta^2}(\sum_{s,j=2}^{n+m}(\left|2(\nu+1)a_{sj}\psi^{-1}\partial_s\psi\partial_j\psi\right|\\ &+\left|2a_{sj}\partial_s\partial_j\psi\right| + \left|2\partial_s(a_{sj})\partial_j\psi\right| + \left|4\partial_s\psi a_{sj}\right| + \left|2\partial_s\psi\partial_s(a_{sj})\right|))) + \frac{\psi^{\nu+1}}{\delta^2}.\end{aligned}$$

Here, we note that $\left|a_{sj}\right| \leq M$ and $\left|\partial_j\psi\right| = \sqrt{2\gamma}, (2 \leq s, j \leq n+m)$. We see that

$$S_1 \geq -4\lambda^3\nu^3\varphi^2\psi^{-2\nu-2}S^0(\delta^2+2\gamma M(n-1)^2+2\gamma Mm^2) - C\lambda^2\nu^2\delta^2\psi^{-\nu-1}\varphi^2. \tag{2.41}$$

Then, by relations (2.36) and (2.41), we write

$$\begin{aligned}2\lambda\nu(x_1+\mu_0)\varphi^2 S w(P_0 w) \ \geq\ & -4S^0\lambda^3\nu^3\psi^{-2\nu-2}(\delta^2+2\gamma M(n-1)^2+m^2)w^2\varphi^2 - 2\lambda\nu S^0\varphi^2((\partial_1 w)^2\\ &+(x_1+\mu_0)^2 M((n-1)\sum_{j=2}^{n}(\partial_j w)^2 + m\sum_{j=n+1}^{n+m}(\partial_j w)^2))\\ &-C\lambda^2\nu^2\psi^{-\nu-1}\delta^2\varphi^2 w^2 - 2\lambda\nu S\xi(x_1+\mu_0)^2\varphi^2 w^2 + d_4(w).\end{aligned}$$

□

Proof of Proposition 1.

We sum inequalities (2.4) and (2.35) and then we obtain

$$\begin{aligned}
&\psi^{\nu+1}\varphi^2(P_0w)^2 + 2\lambda\nu(x_1+\mu_0)\varphi^2 Sw(P_0w)\\
\geq\ & \sum_{j=1}^{4} d_j + 2\lambda\nu(\partial_1 w)^2\varphi^2(\delta\gamma^{-3} - S^0) + \lambda\nu(x_1+\mu_0)(\delta\alpha_1 - 2S^0M(n-1)(x_1+\mu_0))\varphi^2\sum_{s,j=2}^{n}(\partial_j w)^2\\
&+\lambda\nu(x_1+\mu_0)(\delta\alpha_2 - 2S^0Mm(x_1+\mu_0))\varphi^2\sum_{s,j=n+1}^{n+m}(\partial_j w)^2 + 4\lambda^3\nu^3\psi^{-2\nu-3}w^2\varphi^2(\frac{1}{4}\nu\delta^4\\
&-\psi S^0(\delta^2 + 2\gamma M((n-1)^2+m^2)) - \frac{C\delta^2\psi^{\nu+2}}{\lambda\nu}).
\end{aligned}$$

By choosing $\delta_* = \max\{\delta_0, \delta_5\}$, $\delta_5 = \max\{(1+S^0)\gamma^3, (1+2S^0\gamma M(n-1))/\alpha_1, (1+2S^0\gamma Mm)/\alpha_2\}$, $\lambda_* \geq \max\{\lambda_0, \lambda_1, C\}$, $\nu_* \geq \max\{\nu_0, \frac{4}{\delta^4}(1+\psi S^0(\delta^2 + 2M\gamma((n-1)^2+m^2)+\delta^2)\}$, we have (2.3) for $\delta > \delta_*$, $\nu > \nu_*$, $\lambda > \lambda_*$. This completes the proof.

3. Proof of Theorem 1

In order to prove Theorem 1, by (1.11) and the equality $l'_k = (x_1+\mu_0)l_k$, we write

$$(l'_1)^2 + (l'_2)^2 \leq 4(F_1^2 + F_2^2)(x_1+\mu_0)^2 + 4(n+m+1)\sum_{j=0}^{n+m}(b_{1j}^2 + b_{2j}^2). \tag{3.1}$$

Using (3.1), we have

$$\begin{aligned}
&\sum_{k=1}^{2}(\psi^{\nu+1}(P_0w_k)^2\varphi^2 + 2\lambda\nu(x_1+\mu_0)Sw_k(P_0w_k)\varphi^2)\\
\leq\ & \sum_{k=1}^{2}(\psi^{\nu+1}(P_0w_k)^2\varphi^2 + (P_0w_k)^2\varphi^2 + \lambda^2\nu^2(x_1+\mu_0)^2S^2w_k^2\varphi^2)\\
=\ & \sum_{k=1}^{2}((\psi^{\nu+1}+1)(P_0w_k)^2\varphi^2 + \lambda^2\nu^2(x_1+\mu_0)^2S^2w_k^2\varphi^2)\\
=\ & (\psi^{\nu+1}+1)((l'_1)^2 + (l'_2)^2)\varphi^2 + \lambda^2\nu^2(x_1+\mu_0)^2S^2(w_1^2+w_2^2)\varphi^2\\
\leq\ & (\psi^{\nu+1}+1)(x_1+\mu_0)^2(4(F_1^2+F_2^2) + 4(n+m+1)\sum_{j=0}^{n+m}(b_{1j}^2+b_{2j}^2))\varphi^2\\
&+\lambda^2\nu^2(x_1+\mu_0)^2S^2(w_1^2+w_2^2)\varphi^2.
\end{aligned}\tag{3.2}$$

Next, we apply the Carleman estimate in (3.2):

$$\begin{aligned}
&\sum_{k=1}^{2}(\psi^{\nu+1}(P_0w_k)^2\varphi^2 + 2\lambda\nu(x_1+\mu_0)Sw_k(P_0w_k))\varphi^2\\
\geq\ & 2\lambda\nu\varphi^2\sum_{k=1}^{2}(\partial_1(w_k))^2 + \lambda\nu(x_1+\mu_0)\varphi^2\sum_{k=1}^{2}\sum_{j=2}^{n+m}(\partial_j(w_k))^2
\end{aligned}$$

$$+4\lambda^3\nu^3\psi^{-2\nu-3}\varphi^2\sum_{k=1}^{2}(w_k)^2+\sum_{k=1}^{2}\sum_{j=1}^{4}d_j(w_k). \tag{3.3}$$

By (3.2) and (3.3), we write

$$\begin{aligned}
&\sum_{k=1}^{2}(2\lambda\nu\varphi^2(\partial_1(w_k))^2+\lambda(x_1+\mu_0)\varphi^2\sum_{j=2}^{n+m}(\partial_j(w_k))^2+4\lambda^3\nu^3\psi^{-2\nu-3}\varphi^2(w_k)^2)+\sum_{k=1}^{2}\sum_{j=1}^{4}d_j(w_k)\\
\leq\ &(\psi^{\nu+1}+1)(x_1+\mu_0)^2(4(F_1^2+F_2^2)+4(n+m+1)\sum_{j=0}^{n+m}(b_{1j}^2+b_{2j}^2))\varphi^2\\
&+\lambda^2\nu^2(x_1+\mu_0)^2S^2(w_1^2+w_2^2)\varphi^2. \qquad (3.4)
\end{aligned}$$

If we multiply (3.4) by $(1+\xi^2)^2$ and integrate with respect to ξ over the interval $(-\infty,\infty)$, we have

$$\begin{aligned}
&\sum_{k=1}^{2}(2\lambda\nu(\partial_1(w_k))^2+\lambda(x_1+\mu_0)\sum_{j=2}^{n+m}(\partial_j(w_k))^2+4\lambda^3\nu^3\psi^{-2\nu-3}(w_k)^2)\varphi^2(1+\xi^2)^2d\xi\\
&+\sum_{k=1}^{2}\sum_{j=1}^{4}\int_{-\infty}^{\infty}d_j(w_k)(1+\xi^2)^2d\xi\\
\leq\ &(\psi^{\nu+1}+1)(x_1+\mu_0)^2(4M_1+8(n+m+1)M_3\sum_{k=1}^{2}\sum_{j=0}^{n+m}\int_{-\infty}^{\infty}(\partial_j(w_k))^2(1+\xi^2)^2d\xi)\varphi^2\\
&+\lambda^2\nu^2\varphi^2(x_1+\mu_0)^2S^2\sum_{k=1}^{2}\int_{-\infty}^{\infty}w_k^2d\xi. \qquad (3.5)
\end{aligned}$$

In (3.5), we used the following relation by the Plancherel's theorem:

$$\begin{aligned}
\sum_{j=0}^{n+m}\int_{-\infty}^{\infty}(1+\xi^2)^2\left|\widehat{b_j\partial_j z}\right|^2d\xi\ &\leq\ 2\sum_{j=0}^{n+m}\int_{-\infty}^{\infty}\sum_{0\leq\beta_0\leq 2}|\partial_t^{\beta_0}(b_j\partial_j z)|^2dt\\
&\leq\ 2M_3\sum_{j=0}^{n+m}\int_{-\infty}^{\infty}(1+\xi^2)^2|\widehat{\partial_j z}|^2d\xi\\
&=\ 2M_3\sum_{k=1}^{2}\sum_{j=0}^{n+m}\int_{-\infty}^{\infty}(1+\xi^2)^2(\partial_j(w_k))^2d\xi,
\end{aligned}$$

where M_3 is a constant depending on M_2. Then, the inequality (3.5) takes the form

$$\begin{aligned}
&E_5\sum_{k=1}^{2}\int_{-\infty}^{\infty}(\partial_1(w_k))^2(1+\xi^2)^2\varphi^2d\xi+E_6\sum_{k=1}^{2}\sum_{j=2}^{n+m}\int_{-\infty}^{\infty}(\partial_j(w_k))^2(1+\xi^2)^2\varphi^2d\xi\\
&+E_7\sum_{k=1}^{2}\int_{-\infty}^{\infty}(w_k)^2(1+\xi^2)^2\varphi^2d\xi\leq-\sum_{k=1}^{2}\sum_{j=1}^{4}\int_{-\infty}^{\infty}d_j(w_k)(1+\xi^2)^2d\xi, \qquad (3.6)
\end{aligned}$$

where

$$\begin{aligned}E_5 &= 2\lambda\nu - 8(n+m+1)(\psi^{\nu+1}+1)(x_1+\mu_0)^2 M_3,\\ E_6 &= \lambda\nu(x_1+\mu_0) - 8(n+m+1)(\psi^{\nu+1}+1)(x_1+\mu_0)^2 M_3,\\ E_7 &= 4\lambda^3\nu^3\psi^{-2\nu-3} - (\lambda^2\nu^2 S^2 - 8(n+m+1)(\psi^{\nu+1}+1)M_3 + 4(\psi^{\nu+1}+1)M_1)(x_1+\mu_0)^2.\end{aligned}$$

We estimate the coefficients E_5, E_6, E_7 :

For $0 < \psi < \eta < 1$, and $\lambda > 1$, we have

$$E_5 = 2\lambda\nu - 16(n+m+1)M_3 > \lambda \tag{3.7}$$

for $\nu \geq \nu_1 = \frac{1}{2} + 8(n+m+1)M_3$, and

$$E_6 = (x_1+\mu_0)^2(\lambda\nu - 16(n+m+1))M_3 > (x_1+\mu_0)^2\lambda \tag{3.8}$$

for $\nu \geq \nu_2 = 1 + 16(n+m+1)M_3 + C$.

If $\lambda > \max\{\lambda_2, \lambda_3\}$, $\lambda_2 = (S^0)^2$, $\lambda_3 = 16(n+m+1)M_3 + 8M_1$, and then we have

$$E_7 \geq 4\lambda^3\nu^3\psi^{-2\nu-3} - \lambda^2\nu^2(S^0)^2 - 16(n+m+1)M_3 - 8M_1 \geq 2\lambda^3\nu^3\psi^{-2\nu-3}. \tag{3.9}$$

Hence, for $\lambda > 1$ and $\nu > \max\{\nu_1, \nu_2\}$, (3.6)–(3.9) yield

$$\begin{aligned}&\sum_{k=1}^{2}\int_{-\infty}^{\infty}(\lambda(\partial_1(w_k))^2 + \lambda(x_1+\mu_0)^2\sum_{j=2}^{n+m}(\partial_j(w_k))^2 + 2\lambda^3\nu^3\psi^{-2\nu-3}(w_k)^2)(1+\xi^2)^2\varphi^2 d\xi\\ \leq\; & -\sum_{k=1}^{2}\sum_{j=1}^{4}\int_{-\infty}^{\infty} d_j(w_k)(1+\xi^2)^2 d\xi.\end{aligned} \tag{3.10}$$

Since $2(1+\xi^2)^2 > 1$ and $\varphi^2 > 1$ on D_γ^{n+m}, we write

$$\sum_{k=1}^{2}\int_{-\infty}^{\infty}(w_k)^2 d\xi \leq -\frac{1}{\lambda^3\nu^3}\sum_{k=1}^{2}\sum_{j=1}^{4}\int_{-\infty}^{\infty} d_j(w_k)(1+\xi^2)^2 d\xi. \tag{3.11}$$

Integrating inequality (3.11) over the domain D_γ^{n+m} and passing to the limit as $\lambda \to \infty$, we get $w_1 = w_2 = 0$,, that is, $\widehat{z} = 0$ which implies $w(x) = 0$ in D_γ^{n+m}. Thus, Theorem 1 is proved.

4. Conclusions

In this study, we consider a Cauchy problem for the generalized Schrödinger equation. We prove the uniqueness of the solution of the problem by using the Carleman estimate. By similar arguments, a stability estimate can be obtained. Due to its multidimensional structure, the equation has an important place in some modern physics theories such as M-theory and twistor theory. In the future, we are planning to investigate the existence of the solution of the problem.

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Conflict of interest

The authors declare no conflict of interest.

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