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Research article

Penalty approach for KT-pseudoinvex multidimensional variational control problems

Preeti[1], **Poonam Agarwal**[2], **Savin Treanţă**[3,4,5] **and Kamsing Nonlaopon**[6,*]

[1] Department of Mathematics-SAS, VIT Vellore, Vellore, Tamil Nadu, India
[2] Department of Applied Science and Humanities, Inderprastha Engineering College, Ghaziabad, U.P, India
[3] Department of Applied Mathematics, University Politehnica of Bucharest, 060042 Bucharest, Romania
[4] Academy of Romanian Scientists, 54 Splaiul Independentei, 050094 Bucharest, Romania
[5] "Fundamental Sciences Applied in Engineering" Research Center (SFAI), University Politehnica of Bucharest, 060042 Bucharest, Romania
[6] Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand

* **Correspondence:** Email: nkamsi@kku.ac.th; Tel: +66866421582.

Abstract: The present paper is the result of a contemplative study of a multi-time control problem (MCP) by considering its associated equivalent auxiliary control problem $(\text{MCP})_\varsigma$ via the exact l_1 penalty method. Further study reveals that the solution set of the considered problem and the auxiliary problem exhibits an equivalence under the KT-pseudoinvexity hypothesis. Moreover, the study is extended towards the saddle point defined for (MCP) to establish the relationship between the solution set of multi-time control problem (MCP) and its associated equivalent auxiliary control problem $(\text{MCP})_\varsigma$. Finally, we present an illustrative application to authenticate the results presented in this paper.

Keywords: exact l_1 penalty method; KT-pseudoinvexity; control problem
Mathematics Subject Classification: 26A51, 49J20, 90C30, 90C46

1. Introduction

The formulation of several methods for solving nonlinear programming problems with the help of transformation into equivalent auxiliary problem techniques has always grabbed the attention of various mathematicians. The class of exact penalty methods has proved to be of great interest as it enables the transformation of a constrained problem into a single unconstrained problem via the penalty parameter that penalizes any violation of the constraints. Zangwill [21], and Erimen [5] laid the foundation for finding solutions for nonlinear constrained problems via the penalty method. Antczak extensively worked on the exact penalty method for different types of mathematical programming problems under various suitable assumptions [2–4]. Further, the penalty approach was applied to multi-dimensional control problems in the presence of convexity by Jayswal and Preeti [7, 8].

On the other hand, the notion of convexity paved the way for establishing a number of concepts to facilitate real applications with the help of optimization problems. The concept of invexity is one such concept that proved to be a trailblazer since the moment it was introduced by Hanson [6]. Invexity theory has come a long way since its inception for scalar-constrained optimization problems. Due to its varied range of applications in optimization theory, a plethora of mathematicians have worked extensively in this direction, and they have come up with a variety of generalizations of this concept. To name, the concept of pre-invexity was introduced by Weir, and Mond [20]. Strong and weak convexity was introduced by Jeyakumar [9]. Nahak and Nanda [14] extended the duality results of variational problems to pseudo-invex functions. Noor and Noor [15] introduced strongly α-pre-invex functions. Also, they laid the foundation of the relationship among strongly α-preinvex, strongly α-invex and $\alpha\eta$-monotonicities under appropriate conditions.

Further, the concept of KT-invexity emerged from that of invexity when Martin maintained the sufficiency of Kuhn-Tucker (KT) conditions. This also led to the result that KT-invexity is a necessary and sufficient condition for a KT-point to be a global minimizer. Then, the notion of KT-pseudoinvexity was established by Treanţă and Arana-Jiménez [17] and it was shown that KT-pseudoinvex multi-dimensional control problem is formulated in such a way that KT-point serves as an optimal solution. After that, Treanţă efficiently generalized KT-pseudoinvexity and derived interesting results under various hypotheses for control problem [18, 19]. For other but connected points of view, the reader can consult Jiang et al. [10], Lin [11] and Lin et al. [12].

The present work is organized as follows: Section 2 includes some definitions, preliminaries, and notations, which will help to understand the developed results in this paper. Section 3 addresses the unconstrained problem corresponding to (MCP) and demonstrates that the solution set of considered constrained problem and its corresponding unconstrained problem coincide under KT-pseudoinvexity. Further, we furnish the developed results via a non-trivial example in Section 4. Finally, the conclusion of the paper is summarized in Section 5.

2. Fundamental concepts

Let $\mathbb{R}^l, \mathbb{R}^u$ and $\mathbb{R}^v$ be three Euclidean spaces of dimensions l, u and v, respectively. Let $\varphi_{s_0,s_1} \subset \mathbb{R}^l$ be a hyper-parallelepiped joint by $s_0 = \left(s_0^\varrho\right)$ and $s_1 = \left(s_1^\varrho\right), \varrho = \overline{1,l}$, where $s_0 = \left(s_0^\varrho\right)$ and $s_1 = \left(s_1^\varrho\right)$ are points situated diagonally opposite to each other. The point $s = \left((s^\varrho), \varrho = \overline{1,l}\right) \in \varphi_{s_0,s_1} \subset \mathbb{R}^l$ is known as *multi-time*. Let $\mathcal{A}$ be the space of state functions (piecewise smooth) $a : \varphi_{s_0,s_1} \subset \mathbb{R}^l \to \mathbb{R}^u$, $a = (a^\kappa) \in \mathbb{R}^u$. Let $\mathcal{B}$ be the space of control functions (piecewise continuous) $b : \varphi_{s_0,s_1} \subset \mathbb{R}^l \to \mathbb{R}^v$, $b = (b^j) \in \mathbb{R}^v$. $ds = ds^1 \wedge \cdots \wedge ds^l$ is the volume element on $\mathbb{R}^l \supset \varphi_{s_0,s_1}$.

Mathematically, the multi-time control problem is modulated as follows:

$$\text{(MCP)} \quad \min_{(a(\cdot),b(\cdot))} \int_{\varphi_{s_0,s_1}} \psi(s, \mathrm{a}(s), \mathrm{b}(s))\, ds$$
$$\text{subject to} \quad \zeta_m(s, a(s), b(s)) \leq 0, \quad m = \overline{1, r},$$
$$\frac{\partial a^\kappa}{\partial s^\varrho} = \eta_\varrho^\kappa(s, a(s), b(s)), \quad \kappa \in \overline{1, u}, \varrho = \overline{1, l},$$
$$a(s_0) = a_0, \quad a(s_1) = a_1,$$

where $s \in \varphi_{s_0,s_1}, \psi : \varphi_{s_0,s_1} \times \mathcal{A} \times \mathcal{B} \to \mathbb{R}$, $\zeta_m : \varphi_{s_0,s_1} \times \mathcal{A} \times \mathcal{B} \to \mathbb{R}$, $m = \overline{1, r}$, $\eta_\varrho = \left(\eta_\varrho^\kappa\right) : \varphi_{s_0,s_1} \times \mathcal{A} \times \mathcal{B} \to \mathbb{R}^u$ and $\varrho = \overline{1, l}$ represent C^∞-class functionals. The function η_ϱ fulfils complete integrability conditions (closeness conditions)

$$D_\beta \eta_\varrho = D_\varrho \eta_\beta, \ \varrho, \ \beta = \overline{1, l}, \ \varrho \neq \beta,$$

where D_β is the total derivative.

Let

$$\vartheta = \Big\{(a, b) \in \mathcal{A} \times \mathcal{B} : \ \zeta_m(s, a(s), b(s)) \leq 0, \frac{\partial a^\kappa}{\partial s^\varrho} = \eta_\varrho^\kappa(s, a(s), b(s)), a(s_0) = a_0,$$
$$a(s_1) = a_1, s \in \varphi_{s_0,s_1}, m = \overline{1, r}, \kappa \in \overline{1, u}, \varrho = \overline{1, l}\Big\}$$

be the feasible set for (MCP).

To simplify the representation of the paper, we signify some notations as:

$a = a(s), \bar{a} = \bar{a}(s), \hat{a} = \hat{a}(s), b = b(s), \bar{b} = \bar{b}(s), \hat{b} = \hat{b}(s), \pi = (s, a(s), b(s)), \bar{\pi} = (s, \bar{a}(s), \bar{b}(s)), \hat{\pi} = (s, \hat{a}(s), \hat{b}(s)), \varsigma = \varsigma(s), \bar{\varsigma} = \bar{\varsigma}(s)$ and $\varphi = \varphi_{s_0,s_1}$.

Definition 2.1. A solution $(\bar{a}, \bar{b}) \in \vartheta$ is said to be an optimal solution to (MCP), if

$$\int_\varphi \psi(\bar{\pi}) ds \leq \int_\varphi \psi(\pi) ds, \quad \forall (a, b) \in \vartheta.$$

Definition 2.2. [17] The multi-time control problem (MCP) is said to be KT-pseudoinvex at $(\bar{a}, \bar{b}) \in \vartheta$ if for all Lagrange multipliers (piecewise smooth) $\bar{v}_m = (\bar{v}_m(s)) \in \mathbb{R}_+, m = \overline{1, r}$, $\bar{\gamma}_\varrho^\kappa = (\bar{\gamma}_\varrho^\kappa(s)) \in \mathbb{R}, \kappa = \overline{1, u}, \varrho = \overline{1, l}$, there exist $\theta : \varphi \times \mathcal{A} \times \mathcal{B} \times \mathbb{R}^r \times \mathbb{R}^{ul} \to \mathbb{R}^u$ of C^1-class with $\theta|_{\partial\phi}$ and $\xi : \varphi \times \mathcal{A} \times \mathcal{B} \times \mathbb{R}^r \times \mathbb{R}^{ul} \to \mathbb{R}^v$ of C^0-class with $\xi|_{\partial\phi}$ such that for all $(a, b) \in \mathcal{A} \times \mathcal{B}$

$$\int_\varphi \psi(\pi) ds - \int_\varphi \psi(\bar{\pi}) ds < 0.$$

Then, we obtain

$$\int_\varphi \Big[\psi_a(\bar{\pi}) + \bar{v}_m (\zeta_m)_a(\bar{\pi}) + \bar{\gamma}_\varrho^\kappa (\eta_\varrho^\kappa)_a(\bar{\pi})\Big] \theta ds - \int_\varphi \bar{\gamma}_\varrho^\kappa D_\varrho \theta ds$$
$$+ \int_\varphi \Big[\psi_b(\bar{\pi}) + \bar{v}_m (\zeta_m)_b(\bar{\pi}) + \bar{\gamma}_\varrho^\kappa (\eta_\varrho^\kappa)_b(\bar{\pi})\Big] \xi ds < 0,$$

or equivalently

$$\int_{\varphi}\left[\psi_a(\bar{\pi}) + \bar{\nu}_m(\zeta_m)_a(\bar{\pi}) + \bar{\gamma}^{\kappa}_{\varrho}(\eta^{\kappa}_{\varrho})_a(\bar{\pi})\right]\theta ds - \int_{\varphi}\bar{\gamma}^{\kappa}_{\varrho}D_{\varrho}\theta ds$$
$$+ \int_{\varphi}\left[\psi_b(\bar{\pi}) + \bar{\nu}_m(\zeta_m)_b(\bar{\pi}) + \bar{\gamma}^{\kappa}_{\varrho}(\eta^{\kappa}_{\varrho})_b(\bar{\pi})\right]\xi ds \geq 0.$$

Therefore, we have

$$\int_{\varphi}\psi(\pi)ds - \int_{\varphi}\psi(\bar{\pi})ds \geq 0.$$

Definition 2.3. [17] [Necessary Optimality Conditions] A solution $(\bar{a}, \bar{b}) \in \vartheta$ is said to be a KT-point to (MCP) if there exist Lagrange multipliers (piecewise smooth) $\bar{\nu}_m = (\bar{\nu}_m(s)) \in \mathbb{R}_+, m = \overline{1, r}$, $\bar{\gamma}^{\kappa}_{\varrho} = (\bar{\gamma}^{\kappa}_{\varrho}(s)) \in \mathbb{R}, \kappa = \overline{1, u}$ and $\varrho = \overline{1, l}$ such that

$$\frac{\partial\psi}{\partial a^{\kappa}}(\bar{\pi}) + \bar{\nu}_m\frac{\partial\zeta_m}{\partial a^{\kappa}}(\bar{\pi}) + \bar{\gamma}^{\kappa}_{\varrho}\frac{\partial\eta^{\kappa}_{\varrho}}{\partial a^{\kappa}}(\bar{\pi}) + \frac{\partial\bar{\gamma}^{\kappa}_{\varrho}}{\partial s^{\varrho}} = 0, \quad \kappa = \overline{1, u}, \tag{2.1}$$

$$\frac{\partial\psi}{\partial b^{j}}(\bar{\pi}) + \bar{\nu}_m\frac{\partial\zeta_m}{\partial b^{j}}(\bar{\pi}) + \bar{\gamma}^{\kappa}_{\varrho}\frac{\partial\eta^{\kappa}_{\varrho}}{\partial b^{j}}(\bar{\pi}) = 0, \quad j = \overline{1, v}, \tag{2.2}$$

$$\bar{\nu}_m\zeta_m(\bar{\pi}) = 0, \quad \bar{\nu}_m \geq 0 \tag{2.3}$$

for all $s \in \varphi$, except at discontinuities.

The theorem mentioned below states that the KT point indeed serves as a necessary optimality condition for being an optimal solution for (MCP).

Theorem 2.1. *If* $(\bar{a}, \bar{b})$ *is a normal optimal solution for* (MCP)*, then* $(\bar{a}, \bar{b})$ *is a KT-point.*

Proof. The proof of this theorem follows in the same manner as in Theorem 1 of [17]. Hence, it is omitted. □

3. The exact l_1 penalty method

Now, we modulate an equivalent unconstrained multi-time control problem associated with (MCP) via an exact l_1 penalty method as:

$$(\text{MCP})_{\varsigma} \ \min_{(a(\cdot),b(\cdot))}\int_{\varphi}\chi(s, a(s), b(s), \varsigma(s))ds = \int_{\varphi}\Bigg\{\psi(s, a(s), b(s))$$
$$+ \varsigma(s)\left[\sum_{m=1}^{r}\zeta^{+}_{m}(s, a(s), b(s)) + \sum_{\kappa=1}^{u}\sum_{\varrho=1}^{l}\left|\eta^{\kappa}_{\varrho}(s, a(s), b(s)) - \frac{\partial a^{\kappa}}{\partial s^{\varrho}}\right|\right]\Bigg\}ds,$$

where the penalty parameter $\varsigma(s) > 0$ and the penalty function $\zeta^{+}_{m}(s, a(s), b(s))$ is defined as:

$$\zeta^{+}_{m}(s, a(s), b(s)) = \begin{cases} 0, & \text{if } \zeta_m(s, a(s), b(s)) \leq 0; \\ \zeta_m(s, a(s), b(s)), & \text{if } \zeta_m(s, a(s), b(s)) > 0. \end{cases} \tag{3.1}$$

Definition 3.1. A solution $(\bar{a}, \bar{b}) \in \mathcal{A} \times \mathcal{B}$ is called a minimizer to $(\text{MCP})_\varsigma$, if

$$\int_\varphi \chi(\bar{\pi}, \varsigma) ds \leq \int_\varphi \chi(\pi, \varsigma) ds, \quad \forall (a, b) \in \mathcal{A} \times \mathcal{B}.$$

Now, we shall prove the relationship between the solution set of constrained problems and its associated unconstrained problem under KT-pseudoinvexity.

Theorem 3.1. *Let* $\left(\bar{a}, \bar{b}\right)$ *be an optimal solution to* (MCP) *and assume that the considered problem* (MCP) *is KT-pseudoinvex at* $\left(\bar{a}, \bar{b}\right)$ *on* $\mathcal{A} \times \mathcal{B}$. *If* $\varsigma \geq \max\left\{\bar{v}_m, m = \overline{1, r}, \ |\bar{\gamma}_\varrho^\kappa|, \kappa = \overline{1, u}, \varrho = \overline{1, l}\right\}$, *then* $\left(\bar{a}, \bar{b}\right)$ *is also a minimizer of* $(\text{MCP})\varsigma$.

Proof. Let $\left(\bar{a}, \bar{b}\right)$ be not a minimizer of $(\text{MCP})_\varsigma$. Then, there exists a solution $\left(\hat{a}, \hat{b}\right) \in \mathcal{A} \times \mathcal{B}$ such that

$$\int_\varphi \chi\left(\hat{\pi}, \varsigma\right) ds < \int_\varphi \chi\left(\bar{\pi}, \varsigma\right) ds,$$

or

$$\int_\varphi \left\{\psi(\hat{\pi}) + \varsigma \left[\sum_{m=1}^{r} \zeta_m^+(\hat{\pi}) + \sum_{\kappa=1}^{u} \sum_{\varrho=1}^{l} \left|\eta_\varrho^\kappa(\hat{\pi}) - \frac{\partial a^\kappa}{\partial s^\varrho}\right|\right]\right\} ds$$
$$< \int_\varphi \left\{\psi(\bar{\pi}) + \varsigma \left[\sum_{m=1}^{r} \zeta_m^+(\bar{\pi}) + \sum_{\kappa=1}^{u} \sum_{\varrho=1}^{l} \left|\eta_\varrho^\kappa(\bar{\pi}) - \frac{\partial a^\kappa}{\partial s^\varrho}\right|\right]\right\} ds.$$

The feasibility of $\left(\bar{a}, \bar{b}\right)$ along with the above inequality and relation (3.1) produces

$$\int_\varphi \left\{\psi(\hat{\pi}) + \varsigma \left[\sum_{m=1}^{r} \zeta_m^+(\hat{\pi}) + \sum_{\kappa=1}^{u} \sum_{\varrho=1}^{l} \left|\eta_\varrho^\kappa(\hat{\pi}) - \frac{\partial a^\kappa}{\partial s^\varrho}\right|\right]\right\} ds < \int_\varphi \psi(\bar{\pi}) ds.$$

Since the penalty parameter $\varsigma \geq \max\left\{\bar{v}_m, m = \overline{1, r}, \ |\bar{\gamma}_\varrho^\kappa|, \kappa = \overline{1, u}, \varrho = \overline{1, l}\right\} > 0$ and the penalty function is also positive, therefore, it follows that

$$\int_\varphi \psi(\hat{\pi}) ds < \int_\varphi \psi(\bar{\pi}) ds. \tag{3.2}$$

On the other hand, by hypothesis, (MCP) is KT-pseudoinvex at $\left(\bar{a}, \bar{b}\right)$ on $\mathcal{A} \times \mathcal{B}$. Therefore, there exist $\theta : \varphi \times \mathcal{A} \times \mathcal{B} \times \mathbb{R}^r \times \mathbb{R}^{ul} \to \mathbb{R}^u$ of C^1-class with $\theta|_{\partial\varphi} = 0$ and $\xi : \varphi \times \mathcal{A} \times \mathcal{B} \times \mathbb{R}^r \times \mathbb{R}^{ul} \to \mathbb{R}^v$ of C^0-class with $\xi|_{\partial\varphi} = 0$ such that

$$\int_\varphi \left[\psi_a(\bar{\pi}) + \bar{v}_m (\zeta_m)_a(\bar{\pi}) + \bar{\gamma}_\varrho^\kappa (\eta_\varrho^\kappa)_a(\bar{\pi})\right] \theta ds - \int_\varphi \bar{\gamma}_\varrho^\kappa D_\varrho \theta ds$$
$$+ \int_\varphi \left[\psi_b(\bar{\pi}) + \bar{v}_m (\zeta_m)_b(\bar{\pi}) + \bar{\gamma}_\varrho^\kappa (\eta_\varrho^\kappa)_b(\bar{\pi})\right] \xi ds < 0, \quad \forall (a, b) \in \mathcal{A} \times \mathcal{B}. \tag{3.3}$$

By well-established results, we get

$$D_\varrho \left[\theta \gamma_\varrho\right] = \gamma_\varrho D_\varrho \theta + \theta D_\varrho \gamma_\varrho,$$

$$\int_{\varphi} \theta D_{\varrho}\gamma_{\varrho} ds = \int_{\varphi} D_{\varrho}\left[\theta\gamma_{\varrho}\right] ds - \int_{\varphi} \gamma_{\varrho} D_{\varrho}\theta ds.$$

Using $\theta|_{\partial}\varphi = 0$ along with flow-divergence formula, we obtain

$$\int_{\varphi} D_{\varrho}\left[\theta\gamma_{\varrho}\right] ds = \int_{\partial\varphi} \left[\theta\gamma_{\varrho}\right] \overrightarrow{n} ds = 0,$$

where $\overrightarrow{n} = (\overrightarrow{n}_{\varrho}), \varrho = \overline{1,u}$, is the normal unit vector to the hypersurface $\partial\varphi$, hence

$$\int_{\varphi} \theta D_{\varrho}\gamma_{\varrho} ds = -\int_{\varphi} \gamma_{\varrho} D_{\varrho}\theta ds. \tag{3.4}$$

By using the condition (3.4), (3.3) is reduced as

$$\int_{\varphi} \left[\psi_a(\bar{\pi}) + \bar{\nu}_m(\zeta_m)_a(\bar{\pi}) + \bar{\gamma}^{\kappa}_{\varrho}(\eta^{\kappa}_{\varrho})_a(\bar{\pi})\right] \theta ds + \int_{\varphi} \left[D_{\varrho}\bar{\gamma}^{\kappa}_{\varrho}\right] \theta ds$$
$$+ \int_{\varphi} \left[\psi_b(\bar{\pi}) + \bar{\nu}_m(\zeta_m)_b(\bar{\pi}) + \bar{\gamma}^{\kappa}_{\varrho}(\eta^{\kappa}_{\varrho})_b(\bar{\pi})\right] \xi ds < 0.$$

Since, the conditions (2.1)–(2.3) are fulfilled at $\left(\bar{a}, \bar{b}\right)$, therefore, the above inequality can be written as

$$\int_{\varphi} \left[\psi_a(\bar{\pi}) + \bar{\nu}_m(\zeta_m)_a(\bar{\pi}) + \bar{\gamma}^{\kappa}_{\varrho}(\eta^{\kappa}_{\varrho})_a(\bar{\pi})\right] \theta ds + \int_{\varphi} \left[D_{\varrho}\bar{\gamma}^{\kappa}_{\varrho}\right] \theta ds$$
$$+ \int_{\varphi} \left[\psi_b(\bar{\pi}) + \bar{\nu}_m(\zeta_m)_b(\bar{\pi}) + \bar{\gamma}^{\kappa}_{\varrho}(\eta^{\kappa}_{\varrho})_b(\bar{\pi})\right] \xi ds = 0 < 0,$$

which is a contradiction and the proof is completed. □

Now, we authenticate Theorem 3.1 with the help of following example.

Example 3.1. Let $l = 2, u = 1$ and $v = 1$. φ_{s_0,s_1} is a rectangle joint by the diagonally opposite points $s_0 = \left(s^1_0, s^2_0\right)$ and $s_1 = \left(s^1_1, s^2_1\right)$ in $\mathbb{R}^2$ (in particular cases, φ_{s_0,s_1} is a square).

Let us formulate a multi-time control problem (MCP) as below:

$$\text{(MCP1)} \quad \min_{(a(\cdot),b(\cdot))} \int_{\varphi_{0,4}} (b^2 - 5b + 16)\, ds^1 ds^2$$
$$\text{subject to} \quad 25 - a^2 \leq 0,$$
$$\frac{\partial a}{\partial s^1} = 2 - b,$$
$$\frac{\partial a}{\partial s^2} = 2 - b,$$
$$a(0,0) = 0,\ a(4,4) = 10,$$

where $s = \left(s^1, s^2\right) \in \varphi_{0,4}$ and the multi-time objective functional represents the mass of $\varphi_{0,4}$ with the density $\left(b^2 - 5b + 16\right)$ that depends on the current point, and the controlled dynamical system

$\frac{\partial a}{\partial s^1} = \frac{\partial a}{\partial s^2} = 2 - b$ is a neuron activation system which describes the controlled behavior of an artificial neural system from the initial point $a(0,0) = 0$ to the endpoint $a(4,4) = 10$.

By computation, we find $\left(\bar{a}, \bar{b}\right) = \left(\frac{5}{4}\left(s^1 + s^2\right), \frac{3}{4}\right)$ is an optimal solution to (MCP1) at which (2.1)–(2.3) are also fulfilled with Lagrange multiplier $\bar{\nu} = 0, \bar{\gamma}_1^1 = \frac{-35}{2}$ and $\bar{\gamma}_2^1 = \frac{-35}{2}$.

Now, we formulate $(\text{MCP1})_\varsigma$ associated to (MCP1) via an exact l_1 penalty method as follows:

$$(\text{MCP1})_\varsigma \min_{(a(\cdot),b(\cdot))} \int_{\varphi_{0,1}} \left\{\left(b^2 - 5b + 16\right) + \varsigma\left[\max\left\{0, 25 - a^2\right\} + \left|\frac{\partial a}{\partial s^1} - 2 + b\right| + \left|\frac{\partial a}{\partial s^2} - 2 + b\right|\right]\right\} ds^1 ds^2.$$

The following inequality

$$\int_{\varphi_{0,1}} \left\{\left(b^2 - 5b + 16\right) + \varsigma\left[\max\left\{0, 25 - a^2\right\} + \left|\frac{\partial a}{\partial s^1} - 2 + b\right| + \left|\frac{\partial a}{\partial s^2} - 2 + b\right|\right]\right\} ds^1 ds^2$$
$$- \int_{\varphi_{0,1}} \left\{\left(\bar{b}^2 - 5\bar{b} + 16\right) + \varsigma\left[\max\left\{0, 25 - \bar{a}^2\right\} + \left|\frac{\partial a}{\partial s^1} - 2 + \bar{b}\right| + \left|\frac{\partial a}{\partial s^2} - 2 + \bar{b}\right|\right]\right\} ds^1 ds^2 \geq 0$$

holds at $\left(\bar{a}, \bar{b}\right) = \left(\frac{5}{4}\left(s^1 + s^2\right), \frac{3}{4}\right)$ with $\varsigma \geq \frac{35}{2}$ and for all $(a, b) \in \mathcal{A} \times \mathcal{B}$. Further, the fact that the problem (MCP1) is a KT- pseudoinvex problem is quite evident at $(\bar{a}, \bar{b})$ on $\mathcal{A} \times \mathcal{B}$. Thus, all the assumptions of Theorem 3.1 are fulfilled. Hence, we conclude that $(\bar{a}, \bar{b}) = \left(\frac{5}{4}\left(s^1 + s^2\right), \frac{3}{4}\right)$ is also a minimizer of $(\text{MCP1})_\varsigma$.

Proposition 3.1. *Let* $\left(\bar{a}, \bar{b}\right)$ *be a minimizer to* $(\text{MCP})_{\bar{\varsigma}}$. *Then*

$$\int_\varphi \psi(\bar{\pi}) ds \leq \int_\varphi \psi(\pi) ds, \quad \forall (a, b) \in \vartheta.$$

Proof. Let $\left(\bar{a}, \bar{b}\right)$ be a minimizer to $(\text{MCP})_{\bar{\varsigma}}$. Therefore, by Definition 3.1, for all $(a, b) \in \mathcal{A} \times \mathcal{B}$, we have

$$\int_\varphi \chi(\bar{\pi}, \bar{\varsigma}) ds \leq \int_\varphi \chi(\pi, \bar{\varsigma}) ds,$$

or

$$\int_\varphi \left\{\psi(\bar{\pi}) + \bar{\varsigma}\left[\sum_{m=1}^r \zeta_m^+(\bar{\pi}) + \sum_{\kappa=1}^u \sum_{\varrho=1}^l \left|\eta_\varrho^\kappa(\bar{\pi}) - \frac{\partial a^\kappa}{\partial s^\varrho}\right|\right]\right\} ds$$
$$\leq \int_\varphi \left\{\psi(\pi) + \bar{\varsigma}\left[\sum_{m=1}^r \zeta_m^+(\pi) + \sum_{\kappa=1}^u \sum_{\varrho=1}^l \left|\eta_\varrho^\kappa(\pi) - \frac{\partial a^\kappa}{\partial s^\varrho}\right|\right]\right\} ds.$$

Since $\vartheta \subseteq \mathcal{A} \times \mathcal{B}$, then, by the relation (3.1), we get the following inequality for all $(a, b) \in \vartheta$

$$\int_\varphi \left\{\psi(\bar{\pi}) + \bar{\varsigma}\left[\sum_{m=1}^r \zeta_m^+(\bar{\pi}) + \sum_{\kappa=1}^u \sum_{\varrho=1}^l \left|\eta_\varrho^\kappa(\bar{\pi}) - \frac{\partial a^\kappa}{\partial s^\varrho}\right|\right]\right\} ds \leq \int_\varphi \psi(\pi) ds.$$

Again, using the relation (3.1), we have

$$\int_\varphi \psi(\bar{\pi}) ds \leq \int_\varphi \psi(\pi) ds, \quad \forall (a, b) \in \vartheta.$$

□

Theorem 3.2. *Let* $\left(\bar{a}, \bar{b}\right)$ *be a minimizer to* (MCP)$_{\bar{\varsigma}}$. *Considering the fact that the inequality*

$$\int_{\varphi} \chi(\pi, \varsigma) ds \geq \int_{\varphi} \chi(\bar{\pi}, \varsigma) ds$$

holds for any $\varsigma \geq \bar{\varsigma}$ *and for all* $(a, b) \in \vartheta$. *Further, assume that the considered problem* (MCP) *is KT-pseudoinvex on* $\mathcal{A} \times \mathcal{B}$. *If* $\varsigma \geq \max\left\{\bar{v}_m, m = \overline{1, r},\ |\bar{\gamma}^{\kappa}_{\varrho}|, \kappa = \overline{1, u}, \varrho = \overline{1, l}\right\}$, *then* $(\bar{a}, \bar{b})$ *is also an optimal solution to* (MCP).

Proof. Since $\left(\bar{a}, \bar{b}\right)$ is a minimizer to (MCP)$_{\bar{\varsigma}}$, therefore, by Proposition 3.1, we get

$$\int_{\varphi} \psi(\bar{\pi}) ds \leq \int_{\varphi} \psi(\pi) ds, \quad \forall (a, b) \in \vartheta. \tag{3.5}$$

To prove that $\left(\bar{a}, \bar{b}\right)$ is an optimal solution to (MCP), first, we prove that $\left(\bar{a}, \bar{b}\right) \in \vartheta$. By contradiction, we assume that $\left(\bar{a}, \bar{b}\right) \notin \vartheta$. Therefore, by (3.1), we have

$$\int_{\varphi} \left\{ \sum_{m=1}^{r} \zeta_m^{+}(\bar{\pi}) + \sum_{\kappa=1}^{u} \sum_{\varrho=1}^{l} \left| \eta^{\kappa}_{\varrho}(\bar{\pi}) - \frac{\partial a^{\kappa}}{\partial s^{\varrho}} \right| \right\} ds > 0. \tag{3.6}$$

Let (a, b) be any feasible solution to (MCP). By assumption, the following inequality

$$\int_{\varphi} \chi(\pi, \varsigma) ds \geq \int_{\varphi} \chi(\bar{\pi}, \varsigma) ds$$

holds for any $\varsigma \geq \bar{\varsigma}$ and for all $(a, b) \in \vartheta$, or

$$\begin{aligned} &\int_{\varphi} \left\{ \psi(\pi) + \varsigma \left[\sum_{m=1}^{r} \zeta_m^{+}(\pi) + \sum_{\kappa=1}^{u} \sum_{\varrho=1}^{l} \left| \eta^{\kappa}_{\varrho}(\pi) - \frac{\partial a^{\kappa}}{\partial s^{\varrho}} \right| \right] \right\} ds \\ &\geq \int_{\varphi} \left\{ \psi(\bar{\pi}) + \varsigma \left[\sum_{m=1}^{r} \zeta_m^{+}(\bar{\pi}) + \sum_{\kappa=1}^{u} \sum_{\varrho=1}^{l} \left| \eta^{\kappa}_{\varrho}(\bar{\pi}) - \frac{\partial a^{\kappa}}{\partial s^{\varrho}} \right| \right] \right\} ds. \end{aligned} \tag{3.7}$$

However, if we set

$$\varsigma > \max \left\{ \frac{\int_{\varphi} \{\psi(\pi) - \psi(\bar{\pi})\} ds}{\int_{\varphi} \left\{ \sum_{m=1}^{r} \zeta_m^{+}(\bar{\pi}) + \sum_{\kappa=1}^{u} \sum_{\varrho=1}^{l} \left| \eta^{\kappa}_{\varrho}(\bar{\pi}) - \frac{\partial a^{\kappa}}{\partial s^{\varrho}} \right| \right\} ds}, \bar{\varsigma}; (a, b) \in \vartheta \right\},$$

the inequalities (3.5) and (3.6) conclude that ς is a positive real number. Therefore, we can write the above inequality as:

$$\int_{\varphi} \left\{ \psi(\bar{\pi}) + \varsigma \left[\sum_{m=1}^{r} \zeta_m^{+}(\bar{\pi}) + \sum_{\kappa=1}^{u} \sum_{\varrho=1}^{l} \left| \eta^{\kappa}_{\varrho}(\bar{\pi}) - \frac{\partial a^{\kappa}}{\partial s^{\varrho}} \right| \right] \right\} ds > \int_{\varphi} \psi(\pi) ds.$$

The above inequality together with the feasibility of (a, b) in (MCP) and the relation (3.1) yields

$$\int_{\varphi} \left\{ \psi(\bar{\pi}) + \varsigma \left[\sum_{m=1}^{r} \zeta_m^{+}(\bar{\pi}) + \sum_{\kappa=1}^{u} \sum_{\varrho=1}^{l} \left| \eta^{\kappa}_{\varrho}(\bar{\pi}) - \frac{\partial a^{\kappa}}{\partial s^{\varrho}} \right| \right] \right\} ds$$

$$> \int_{\varphi}\left\{\psi(\pi)+\varsigma\left[\sum_{m=1}^{r}\zeta_m^{+}(\pi)+\sum_{\kappa=1}^{u}\sum_{\varrho=1}^{l}\left|\eta_{\varrho}^{\kappa}(\pi)-\frac{\partial a^{\kappa}}{\partial s^{\varrho}}\right|\right]\right\}ds,$$

which contradicts the inequality (3.7). Therefore, $\left(\bar{a},\bar{b}\right)\in\vartheta$ and its optimality in (MCP) follows directly from (3.5). □

4. Saddle point optimality criterion

In this section, the concept of the saddle point criterion is elaborately discussed with the motive of establishing the relationship between the solution set of (MCP) along with the $(\mathrm{MCP})_{\varsigma}$.

To begin with, we mention here the definitions of Lagrange functional along with saddle point for an (MCP).

Definition 4.1. The Lagrange functional $\mathcal{L}(a,b,\nu,\gamma)$ defined for (MCP) is given as:

$$\mathcal{L}(a,b,\nu,\gamma)=\int_{\varphi}\left\{\psi(\pi)+\sum_{m=1}^{r}\nu_m\zeta_m(\pi)+\sum_{\kappa=1}^{u}\sum_{\varrho=1}^{l}\gamma_{\varrho}^{\kappa}\left(\frac{\partial a^{\kappa}}{\partial s^{\varrho}}-\eta_{\varrho}^{\kappa}(\pi)\right)\right\}ds,$$

where $\nu=(\nu_m)\in\mathbb{R}_+^r$ and $\gamma=(\gamma_{\varrho}^{\kappa})\in\mathbb{R}_+^{ul}$.

Definition 4.2. A point $\left(\bar{a},\bar{b},\bar{\nu},\bar{\gamma}\right)\in\vartheta\times\mathbb{R}^r\times\mathbb{R}^{ul}$ is called a saddle point of the Lagrange functional defined for (MCP), if there exist Lagrange multipliers (piecewise smooth functions) $\bar{\nu}=(\bar{\nu}_m)\in\mathbb{R}_+^r$ and $\bar{\gamma}=(\bar{\gamma}_{\varrho}^{\kappa})\in\mathbb{R}_+^{ul}$ such that the following inequalities hold:

(i) $\mathcal{L}\left(\bar{a},\bar{b},\nu,\gamma\right)\le\mathcal{L}\left(\bar{a},\bar{b},\bar{\nu},\bar{\gamma}\right),\quad\forall\nu\in\mathbb{R}_+^r,\forall\gamma\in\mathbb{R}_+^{ul};$

(ii) $\mathcal{L}\left(a,b,\bar{\nu},\bar{\gamma}\right)\ge\mathcal{L}\left(\bar{a},\bar{b},\bar{\nu},\bar{\gamma}\right),\quad\forall(a,b)\in\vartheta.$

Next, with the help of the notion of the saddle point, we establish the fact that an equivalence lies in the problems (MCP) and $(\mathrm{MCP})_{\varsigma}$ under KT-psedoinvexity.

Theorem 4.1. *Let* $\left(\bar{a},\bar{b},\bar{\nu},\bar{\gamma}\right)$ *be a saddle point of the Lagrange functional defined for* (MCP). *If we consider the penalty parameter* ς *to be sufficiently large in the sense that*

$$\varsigma\ge\max\left\{\bar{\nu}_m, m=\overline{1,r},|\bar{\gamma}_{\varrho}^{\kappa}|\;\middle|\;\varrho=\overline{1,l},\kappa=\overline{1,u}\right\},$$

then $\left(\bar{a},\bar{b}\right)$ *is a minimizer of* $(\mathrm{MCP})_{\varsigma}$.

Proof. Let us prove the result with the help of contradiction and assume that $\left(\bar{a},\bar{b}\right)$ is not a minimizer of $(\mathrm{MCP})_{\varsigma}$. Then, there exists $\left(\hat{a},\hat{b}\right)\in\mathcal{A}\times\mathcal{B}$ such that

$$\int_{\varphi}\chi\left(\hat{a},\hat{b},\varsigma\right)ds<\int_{\varphi}\chi\left(\bar{a},\bar{b},\varsigma\right)ds.$$

From the definition of the penalized problem $(\mathrm{MCP})_{\varsigma}$, we get

$$\int_{\varphi}\left\{\psi(\hat{\pi})+\varsigma\left[\sum_{m=1}^{r}\zeta_m^{+}(\hat{\pi})+\sum_{\kappa=1}^{u}\sum_{\varrho=1}^{l}\left|\frac{\partial a^{\kappa}}{\partial s^{\varrho}}-\eta_{\varrho}^{\kappa}(\hat{\pi})\right|\right]\right\}ds$$

$$< \int_{\varphi} \left\{ \psi(\bar{\pi}) + \varsigma \left[\sum_{m=1}^{r} \zeta_m^+(\bar{\pi}) + \sum_{\kappa=1}^{u} \sum_{\varrho=1}^{l} \left| \frac{\partial a^\kappa}{\partial s^\varrho} - \eta_\varrho^\kappa(\bar{\pi}) \right| \right] \right\} ds.$$

The feasibility of $\left(\bar{a}, \bar{b}\right)$ in (MCP) and (3.1) imply that

$$\int_{\varphi} \left\{ \psi(\hat{\pi}) + \varsigma \left[\sum_{m=1}^{r} \zeta_m^+(\hat{\pi}) + \sum_{\kappa=1}^{u} \sum_{\varrho=1}^{l} \left| \frac{\partial a^\kappa}{\partial s^\varrho} - \eta_\varrho^\kappa(\hat{\pi}) \right| \right] \right\} ds < \int_{\varphi} \psi(\bar{\pi}) ds.$$

As, the penalty parameter

$$\varsigma \geq \max \left\{ \bar{\nu}_m, |\bar{\gamma}_\varrho^\kappa| \,\middle|\, m = \overline{1, r}, \varrho = \overline{1, l}, \kappa = \overline{1, u} \right\}.$$

From the above inequality, we have

$$\int_{\varphi} \left\{ \psi(\hat{\pi}) + \sum_{m=1}^{r} \bar{\nu}_m \zeta_m^+(\hat{\pi}) + \sum_{\kappa=1}^{u} \sum_{\varrho=1}^{l} \left| \bar{\gamma}_\varrho^\kappa \left(\frac{\partial a^\kappa}{\partial s^\varrho} - \eta_\varrho^\kappa(\hat{\pi}) \right) \right| \right\} ds < \int_{\varphi} \psi(\bar{\pi}) ds.$$

Since $\left(\hat{a}, \hat{b}\right)$ is not a feasible solution in (MCP), by using (3.1), we get

$$\int_{\varphi} \left\{ \psi(\hat{\pi}) + \sum_{m=1}^{r} \bar{\nu}_m \zeta_m(\hat{\pi}) + \sum_{\kappa=1}^{u} \sum_{\varrho=1}^{l} \bar{\gamma}_\varrho^\kappa \left(\frac{\partial a^\kappa}{\partial s^\varrho} - \eta_\varrho^\kappa(\hat{\pi}) \right) \right\} ds < \int_{\varphi} \psi(\bar{\pi}) ds. \tag{4.1}$$

On the other hand, by assumption, $\left(\bar{a}, \bar{b}, \bar{\nu}, \bar{\gamma}\right)$ is a saddle point of the Lagrange functional defined for (MCP). Then, by Definition 4.2(i), we have

$$\mathcal{L}\left(\bar{a}, \bar{b}, \nu, \gamma\right) \leq \mathcal{L}\left(\bar{a}, \bar{b}, \bar{\nu}, \bar{\gamma}\right), \quad \forall \nu = (\nu_m) \in \mathbb{R}_+^r, \gamma = (\gamma_\varrho^\kappa) \in \mathbb{R}_+^{ul}.$$

By Definition 4.1, we have

$$\int_{\varphi} \left\{ \psi(\bar{\pi}) + \sum_{m=1}^{r} \nu_m \zeta_m(\bar{\pi}) + \sum_{\varrho=1}^{m} \sum_{\varrho=1}^{n} \gamma_\varrho^\kappa \left(\frac{\partial a^\kappa}{\partial s^\varrho} - \eta_\varrho^\kappa(\bar{\pi}) \right) \right\} ds$$
$$\leq \int_{\varphi} \left\{ \psi(\bar{\pi}) + \sum_{m=1}^{r} \bar{\nu}_m \zeta_m(\bar{\pi}) + \sum_{\varrho=1}^{m} \sum_{\varrho=1}^{n} \bar{\gamma}_\varrho^\kappa \left(\frac{\partial a^\kappa}{\partial s^\varrho} - \eta_\varrho^\kappa(\bar{\pi}) \right) \right\} ds.$$

Taking $\nu_m = 0, m = \overline{1, r}$ for the preceding inequality, we get

$$\int_{\varphi} \sum_{m=1}^{r} \bar{\nu}_m \zeta_m(\bar{\pi}) ds \geq 0. \tag{4.2}$$

By the feasibility of $(\bar{a}, \bar{b})$ in (MCP), we get

$$\int_{\varphi} \sum_{m=1}^{r} \bar{\nu}_m \zeta_m(\bar{\pi}) ds \leq 0. \tag{4.3}$$

Thus, the inequalities (4.2) and (4.3) together imply that

$$\int_{\varphi}\sum_{m=1}^{r}\bar{\nu}_m\zeta_m(\bar{\pi})ds = 0.$$

Further, from the feasibility of $\left(\bar{a},\bar{b}\right)$ in (MCP), we also have

$$\int_{\varphi}\left(\eta^{\kappa}_{\varrho}(\bar{\pi}) - \frac{\partial a^{\kappa}}{\partial s^{\varrho}}\right)ds = 0.$$

Therefore, the inequality (4.1) can be rewritten as

$$\int_{\varphi}\left\{\psi(\hat{\pi}) + \sum_{m=1}^{r}\bar{\nu}_m\zeta_m(\hat{\pi}) + \sum_{\kappa=1}^{u}\sum_{\varrho=1}^{l}\bar{\gamma}^{\kappa}_{\varrho}\left(\frac{\partial a^{\kappa}}{\partial s^{\varrho}} - \eta^{\kappa}_{\varrho}(\hat{\pi})\right)\right\}ds$$
$$< \int_{\varphi}\left\{\psi(\bar{\pi}) + \sum_{m=1}^{r}\bar{\nu}_m\zeta_m(\bar{\pi}) + \sum_{\kappa=1}^{u}\sum_{\varrho=1}^{l}\bar{\gamma}^{\kappa}_{\varrho}\left(\frac{\partial a^{\kappa}}{\partial s^{\varrho}} - \eta^{\kappa}_{\varrho}(\bar{\pi})\right)\right\}ds.$$

By Definition 4.1, we get

$$\mathcal{L}\left(\hat{a},\hat{b},\bar{\nu},\bar{\gamma}\right) < \mathcal{L}\left(\bar{a},\bar{b},\bar{\nu},\bar{\gamma}\right), \quad \forall(\hat{a},\hat{b}) \in \vartheta.$$

But, it is contrary to Definition 4.2(*ii*). This completes the proof. □

Next, we construct a non-convex optimization control problem to authenticate the Theorem 4.1 under KT-pseudoinvexity assumptions.

Example 4.1. Let $l = 2$ $\mathcal{A} = [0,2]$ and $\mathcal{B} = [0,2]$. φ_{s_0,s_1} is a rectangle joint by the diagonally opposite points $s_0 = \left(s^1_0, s^2_0\right)$ and $s_1 = \left(s^1_1, s^2_1\right)$ in $\mathbb{R}^2$ (in particular cases, φ_{s_0,s_1} is a square). Let us formulate a multi-time control problem (MCP) as below:

$$\text{(MCP2)} \quad \min_{(a(\cdot),b(\cdot))}\int_{\varphi_{0,2}}\left(-2b - b^3\right)ds^1ds^2$$
$$\text{subject to} \quad 16 - a^2 \leq 0,$$
$$\frac{\partial a}{\partial s^1} = 2b,$$
$$\frac{\partial a}{\partial s^2} = 2b,$$
$$a(0,0) = 0,\ a(2,2) = 16,$$

where $s = \left(s^1, s^2\right) \in \varphi_{0,2}$ and the multi-time objective functional represents the mass of $\varphi_{0,2}$ with the density $\left(-2b - b^3\right)$ that depends on the current point, and the controlled dynamical system $\frac{\partial a}{\partial s^1} = \frac{\partial a}{\partial s^2} = 2b$ is a neuron activation system which describes the controlled behavior of an artificial neural system from the initial point $a(0,0) = 0$ to the end point $a(2,2) = 16$. Note that

$$\vartheta = \left\{(a,b) \in \mathcal{A}\times\mathcal{B} : -4 \leq a \leq 4, \frac{\partial a}{\partial s^1} - 2b, \frac{\partial a}{\partial s^2} - 2b, a(0,0) = 0, a(2,2) = 16\right\}$$

is the set of the all feasible solution to (MCP2). By computation, we find $\left(\bar{a}, \bar{b}\right) = \left(4\left(s^1 + s^2\right), 2\right)$ is an optimal solution to (MCP2) at which (2.1)–(2.3) are also fulfilled with Lagrange multiplier $\bar{\nu} = 0, \bar{\gamma}_1^1 + \bar{\gamma}_2^1 = 7$.

Now, we frame the Lagrange functional for (MCP2) as follows:

$$\mathcal{L}(a, b, \mu, \gamma) = \int_{\varphi_{0,2}} \left\{\left(-2b - b^3\right) + \varsigma\left(16 - a^2\right) + \gamma_1\left[\frac{\partial a}{\partial s^1} - 2b\right] + \gamma_2\left[\frac{\partial a}{\partial s^2} - 2b\right]\right\} ds^1 ds^2$$

and it can be observed that the following inequalities:

(i) $\mathcal{L}\left(\bar{a}, \bar{b}, \nu, \gamma\right) \leq \mathcal{L}\left(\bar{a}, \bar{b}, \bar{\nu}, \bar{\gamma}\right), \quad \forall \nu \in \mathbb{R}_+, \forall \gamma \in \mathbb{R}_+^2;$

(ii) $\mathcal{L}\left(a, b, \bar{\nu}, \bar{\gamma}\right) \geq \mathcal{L}\left(\bar{a}, \bar{b}, \bar{\nu}, \bar{\gamma}\right), \quad \forall (a, b) \in \vartheta$

are satisfied at $\left(\bar{a}, \bar{b}\right) = \left(4\left(s^1 + s^2\right), 2\right)$ with Lagrange multiplier $\bar{\nu} = 0, \bar{\gamma}_1^1 + \bar{\gamma}_2^1 = 7$.

Next, we formulate $(\text{MCP2})_\varsigma$ associated to (MCP2) via an exact l_1 penalty method as follows:

$$(\text{MCP2})_\varsigma \min_{(a(\cdot), b(\cdot))} \int_{\varphi_{0,1}} \left\{\left(-2b - b^3\right) + \varsigma\left[\max\left\{0, 16 - a^2\right\} + \left|\frac{\partial a}{\partial s^1} - 2b\right| + \left|\frac{\partial a}{\partial s^2} - 2b\right|\right]\right\} ds^1 ds^2.$$

Further, the fact that the problem (MCP2) is a KT- pseudoinvex problem is quite evident at $\left(\bar{a}, \bar{b}\right)$ on $\mathcal{A} \times \mathcal{B}$. Thus, all the assumptions of Theorem 4.1 are fulfilled. Hence, we conclude that $\left(\bar{a}, \bar{b}\right) = \left(4\left(s^1 + s^2\right), 2\right)$ is also a minimizer of $(\text{MCP2})_\varsigma$.

Theorem 4.2. *Let $\left(\bar{a}, \bar{b}\right)$ be a minimizer of the penalized problem* $(\text{MCP})_\varsigma$. *Assume that ϑ is a compact subset of $\mathbb{R}^u \times \mathbb{R}^v$ and for any $\varsigma > \bar{\varsigma}$, the inequality*

$$\int_\varphi \chi(a, b, \varsigma) ds \nless \int_\varphi \chi(\bar{a}, \bar{b}, \varsigma) ds$$

is satisfied for all $(a, b) \in \vartheta$. Further, if the considered problem (MCP) *is KT-pseudoinvex at $\left(\bar{a}, \bar{b}\right)$ on $\mathcal{A} \times \mathcal{B}$, then $\left(\bar{a}, \bar{b}, \bar{\nu}, \bar{\gamma}\right)$ is a saddle point of the Lagrange functional defined for* (MCP).

Proof. To show that $\left(\bar{a}, \bar{b}, \bar{\nu}, \bar{\gamma}\right)$ is a saddle point of the Lagrange functional defined for (MCP), first, we prove that $\left(\bar{a}, \bar{b}\right)$ is an optimal solution of (MCP). By hypothesis, $\left(\bar{a}, \bar{b}\right)$ is a minimizer to $(\text{MCP})_\varsigma$, then by Proposition 3.1, we have

$$\int_\varphi \psi(\bar{\pi}) ds \leq \int_\varphi \psi(\pi) ds, \quad \forall (a, b) \in \vartheta, \tag{4.4}$$

which indicates that, on the set of all feasible solutions, the value of $\int_\varphi \psi(\pi) ds$ is bounded below in a compact set ϑ.

Now, we shall prove that $\left(\bar{a}, \bar{b}\right)$ is an optimal solution to (MCP). Let us proceed by contradiction and assume that $\left(\bar{a}, \bar{b}\right)$ is not a feasible solution to (MCP). Then, by (3.1), it follows that

$$\int_\varphi \left\{\sum_{m=1}^{r} \zeta_m^+(\bar{\pi}) + \sum_{\kappa=1}^{u} \sum_{\varrho=1}^{l} \left|\frac{\partial a^\kappa}{\partial s^\varrho} - \eta_\varrho^\kappa(\bar{\pi})\right|\right\} ds > 0. \tag{4.5}$$

Further, if we consider a point $\left(\hat{a}, \hat{b}\right)$ as a feasible solution of (MCP) then, for any $\varsigma > \bar{\varsigma}$, we get

$$\int_{\varphi} \chi\left(\hat{a}, \hat{b}, \varsigma\right) ds \nless \int_{\varphi} \chi\left(\bar{a}, \bar{b}, \varsigma\right) ds.$$

According to the definition of the penalized problem $(\text{MCP})_{\varsigma}$, the above inequality implies that

$$\begin{aligned} &\int_{\varphi} \left\{ \psi(\hat{\pi}) + \varsigma \left[\sum_{m=1}^{r} \zeta_m^{+}(\hat{\pi}) + \sum_{\kappa=1}^{u} \sum_{\varrho=1}^{l} \left| \frac{\partial a^{\kappa}}{\partial s^{\varrho}} - \eta_{\varrho}^{\kappa}(\hat{\pi}) \right| \right] \right\} ds \\ &\quad \nless \int_{\varphi} \left\{ \psi(\bar{\pi}) + \varsigma \left[\sum_{m=1}^{r} \zeta_m^{+}(\bar{\pi}) + \sum_{\kappa=1}^{u} \sum_{\varrho=1}^{l} \left| \frac{\partial a^{\kappa}}{\partial s^{\varrho}} - \eta_{\varrho}^{\kappa}(\bar{\pi}) \right| \right] \right\} ds. \end{aligned} \tag{4.6}$$

Also, on considering

$$\varsigma > \max \left\{ \frac{\int_{\varphi} \psi(\hat{\pi}) ds - \int_{\varphi} \psi(\bar{\pi}) ds}{\int_{\varphi} \left\{ \sum_{m=1}^{r} \zeta_m^{+}(\bar{\pi}) + \sum_{\kappa=1}^{u} \sum_{\varrho=1}^{l} \left| \frac{\partial a^{\kappa}}{\partial s^{\varrho}} - \eta_{\varrho}^{\kappa}(\bar{\pi}) \right| \right\} ds}, \bar{\varsigma} \right\}, \tag{4.7}$$

the inequalities (4.4), (4.5) and (4.7) imply that ς is a finite non-negative real number and we obtain the following inequality

$$\int_{\varphi} \left\{ \psi(\bar{\pi}) + \varsigma \left[\sum_{m=1}^{r} \zeta_m^{+}(\bar{\pi}) + \sum_{\kappa=1}^{u} \sum_{\varrho=1}^{l} \left| \frac{\partial a^{\kappa}}{\partial s^{\varrho}} - \eta_{\varrho}^{\kappa}(\bar{\pi}) \right| \right] \right\} ds > \int_{\varphi} \psi(\hat{\pi}) ds. \tag{4.8}$$

From the feasibility of $\left(\hat{a}, \hat{b}\right)$ in (MCP) and (3.1), we have

$$\varsigma \left[\sum_{m=1}^{r} \zeta_m^{+}(\hat{\pi}) + \sum_{\kappa=1}^{u} \sum_{\varrho=1}^{l} \left| \frac{\partial a^{\kappa}}{\partial s^{\varrho}} - \eta_{\varrho}^{\kappa}(\hat{\pi}) \right| \right] = 0.$$

Thus, the inequality (4.8) can be rewritten as

$$\begin{aligned} &\int_{\varphi} \left\{ \psi(\bar{\pi}) + \varsigma \left[\sum_{m=1}^{r} \zeta_m^{+}(\bar{\pi}) + \sum_{\kappa=1}^{u} \sum_{\varrho=1}^{l} \left| \frac{\partial a^{\kappa}}{\partial s^{\varrho}} - \eta_{\varrho}^{\kappa}(\bar{\pi}) \right| \right] \right\} ds \\ &\quad > \int_{\varphi} \left\{ \psi(\hat{\pi}) + \varsigma \left[\sum_{m=1}^{r} \zeta_m^{+}(\hat{\pi}) + \sum_{\kappa=1}^{u} \sum_{\varrho=1}^{l} \left| \frac{\partial a^{\kappa}}{\partial s^{\varrho}} - \eta_{\varrho}^{\kappa}(\hat{\pi}) \right| \right] \right\} ds. \end{aligned}$$

The above expression is in contradiction to inequality (4.6). It leads to the fact that $\left(\bar{a}, \bar{b}\right)$ is a feasible solution of (MCP). This result in association with (4.4) leads to the conclusion that (MCP) attains its optimal solution at $\left(\bar{a}, \bar{b}\right)$.

Now, we show that $\left(\bar{a}, \bar{b}, \bar{\nu}, \bar{\gamma}\right)$ is a saddle point of the Lagrange functional defined for (MCP). Since $\left(\bar{a}, \bar{b}\right)$ is an optimal solution to (MCP), there exist Lagrange multipliers $\bar{\nu} \in \mathbb{R}^{r}$ and $\bar{\gamma} \in \mathbb{R}^{ul}$ such that the necessary optimality conditions (2.1)–(2.3) are satisfied at $\left(\bar{a}, \bar{b}\right)$ on $\mathcal{A} \times \mathcal{B}$. From the necessary optimality condition (2.3) and the feasibility of $\left(\bar{a}, \bar{b}\right)$ in (MCP), we have

$$\int_{\varphi} \nu_m \zeta_m(\bar{\pi}) ds \leq \int_{\varphi} \bar{\nu}_m \zeta_m(\bar{\pi}) ds, \quad \forall \nu_m \in \mathbb{R}^r_+.$$

Equivalently,

$$\int_{\varphi} \left\{ \psi(\bar{\pi}) + \sum_{m=1}^{r} \nu_m \zeta_m(\bar{\pi}) + \sum_{\kappa=1}^{u} \sum_{\varrho=1}^{l} \gamma_{\varrho}^{\kappa} \left(\frac{\partial a^{\kappa}}{\partial s^{\varrho}} - \eta_{\varrho}^{\kappa}(\bar{\pi}) \right) \right\} ds$$
$$\leq \int_{\varphi} \left\{ \psi(\bar{\pi}) + \sum_{m=1}^{r} \bar{\nu}_m \zeta_m(\bar{\pi}) + \sum_{\kappa=1}^{u} \sum_{\varrho=1}^{l} \bar{\gamma}_{\varrho}^{\kappa} \left(\frac{\partial a^{\kappa}}{\partial s^{\varrho}} - \eta_{\varrho}^{\kappa}(\bar{\pi}) \right) \right\} ds.$$

By Definition 4.1, we get

$$\mathcal{L}\left(\bar{a}, \bar{b}, \nu, \gamma\right) \leq \mathcal{L}\left(\bar{a}, \bar{b}, \bar{\nu}, \bar{\gamma}\right), \quad \forall \nu \in \mathbb{R}^r_+, \gamma \in \mathbb{R}^{ul}_+.$$

Next, we prove the second condition of the saddle point of the Lagrange functional defined for (MCP). On the other hand, by hypothesis, (MCP) is KT-pseudoinvex at $\left(\bar{a}, \bar{b}\right)$ on $\mathcal{A} \times \mathcal{B}$. Therefore, we get the following inequality

$$\int_{\varphi} \left[\psi_a(\bar{\pi}) + \bar{\nu}_m (\zeta_m)_a(\bar{\pi}) + \bar{\gamma}_{\varrho}^{\kappa} (\eta_{\varrho}^{\kappa})_a(\bar{\pi}) \right] \theta ds - \int_{\varphi} \bar{\gamma}_{\varrho}^{\kappa} D_{\varrho} \theta ds$$
$$+ \int_{\varphi} \left[\psi_b(\bar{\pi}) + \bar{\nu}_m (\zeta_m)_b(\bar{\pi}) + \bar{\gamma}_{\varrho}^{\kappa} (\eta_{\varrho}^{\kappa})_b(\bar{\pi}) \right] \xi ds \geq 0,$$

implies

$$\int_{\varphi} \psi(\pi) ds - \int_{\varphi} \psi(\bar{\pi}) ds \geq 0, \quad \forall (a, b) \in \mathcal{A} \times \mathcal{B}.$$

Since $\vartheta \subset \mathcal{A} \times \mathcal{B}$, therefore, the following inequality is also satisfied for all $(a, b) \in \vartheta$,

$$\int_{\varphi} \psi(\pi) ds - \int_{\varphi} \psi(\bar{\pi}) ds \geq 0,$$

or

$$\int_{\varphi} \psi(\pi) ds \geq \int_{\varphi} \psi(\bar{\pi}) ds.$$

If we set $\bar{\nu}_m = 0, \forall m = \overline{1, r}$ then the above inequality can be written as

$$\int_{\varphi} \left\{ \psi(\pi) + \sum_{m=1}^{r} \bar{\nu}_m \zeta_m(\pi) + \sum_{\kappa=1}^{u} \sum_{\varrho=1}^{l} \bar{\gamma}_{\varrho}^{\kappa} \left(\frac{\partial a^{\kappa}}{\partial s^{\varrho}} - \eta_{\varrho}^{\kappa}(\pi) \right) \right\} ds$$
$$\geq \int_{\varphi} \left\{ \psi(\bar{\pi}) + \sum_{m=1}^{r} \bar{\nu}_m \zeta_m(\bar{\pi}) + \sum_{\kappa=1}^{u} \sum_{\varrho=1}^{l} \bar{\gamma}_{\varrho}^{\kappa} \left(\frac{\partial a^{\kappa}}{\partial s^{\varrho}} - \eta_{\varrho}^{\kappa}(\bar{\pi}) \right) \right\} ds.$$

By Definition 4.1, we get

$$\mathcal{L}\left(a, b, \bar{\nu}, \bar{\gamma}\right) \geq \mathcal{L}\left(\bar{a}, \bar{b}, \bar{\nu}, \bar{\gamma}\right), \quad \forall (a, b) \in \vartheta.$$

Therefore, this completes the proof. □

5. Conclusions

Through the study of (MCP), we led to the conclusion that the exact l_1 penalty method serves as the best tool for the construction of an equivalent auxiliary control problem $(\text{MCP})_\varsigma$ associated with (MCP). Further, a relationship between an optimal solution of (MCP) and a minimizer of $(\text{MCP})\varsigma$ was established under KT-pseudoinvexity assumptions. Furthermore, we discussed the optimality of the considered problem (MCP) with the help of saddle point criteria. Also, we provided a non-trivial example to strengthen the results demonstrated in this paper.

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Conflict of interest

The authors declare that they have no competing interests.

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