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Research article

Numerical analysis of fractional-order nonlinear Gardner and Cahn-Hilliard equations

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Abstract: In this study, the suggested residual power series transform method is used to compute the numerical solution of the fractional-order nonlinear Gardner and Cahn-Hilliard equations and the result is discovered in a fast convergent series. The leverage and efficacy of the suggested technique are demonstrated by the test examples provided. The achieved results are proved graphically. The current method handles the series solution in a sizable admissible domain in a powerful way. It provides a simple means of modifying the solution's convergence zone. Results with graphs expressly demonstrate the effectiveness and abilities of the suggested method.

Keywords: residual power series method; Laplace transform; Gardner and Cahn-Hilliard equations; analytical solution
Mathematics Subject Classification: 32B15, 34A34, 35A22, 35A24, 45A10

1. Introduction

Many engineering and scientific issues have been solved using fractional differential equations (FDEs). In the past two decades, fractional differential equations have garnered great attention due to their capacity to simulate numerous events in various scientific and engineering disciplines. Models based on fractional differential equations can depict various physical applications in science and engineering [1–3]. These models are precious for a wide range of physical problems. These equations are represented by fractional linear and non-linear PDEs, and fractional differential equations must be solved [4–6]. Most non-linear FDEs need approximate and numerical solutions since they cannot be solved precisely [7–9]. Variational iteration method [10], Adomian decomposition method [11], homotopy analysis method [12], homotopy perturbation method [13], tanh-coth method [14], spectral collocation method [15], Mittag-Leffler function method [16], exp function method [17] and differential quadrature method [18], are a few of the more recent analytical techniques for non-linear problems [19–21].

The Gardner equation [22] is an amalgamation of KdV and modified KdV equations, and it is generated to demonstrate the description of solitary inner waves in shallow water. The Gardner equation is frequently utilized in many areas of physics, including quantum area theories, plasma physics and fluid physics [23, 24]. It also discusses many wave phenomena in the plasma and solid states [25]. In the current research, we recognize the fractional Gardner (FG) equation of the form [26]:

$$D_\eta^\alpha u(\omega, \eta) + 6\left(u - \lambda^2 u^2\right) u_\omega + u_{\omega\omega\omega} = 0,\ 0 < \alpha \leq 1,$$

where λ is real constant, here, $u(\omega, \eta)$ is the wave term with scaling variable spaces (ω) and time (η), the functions uu_ω and $u^2 u_\omega$ are symbolizes the non-linear wave steepening and $u_{\omega\omega\omega}$ defines the wave dispersive effect.

The Cahn-Hilliard equation, first presented in 1958 by Cahn and Hilliard [27], serves as an example of the phase separation of a binary alloy under critical temperature. This equation is a key component of several intriguing physical processes, including spinodal decomposition, phase separation, and phase ordering dynamics [28, 29]. In this framework, the following fractional Cahn-Hilliard (FCH) equation is taken into consideration [26, 30] :

$$D_\eta^\alpha u(\omega, \eta) - u_\omega - 6uu_\omega^2 - \left(3u^2 - 1\right) u_{\omega\omega} + u_{\omega\omega\omega\omega} = 0,\ 0 < \alpha \leq 1.$$

Several novels and cutting-edge approaches to studying non-linear differential systems with fractional order have been developed over the previous thirty years, concurrently developing new computing techniques and symbolic programming. In the pre-computer age, most complex phenomena, such as solitons, chaos, singular formation, asymptotic characteristics, etc., remained unnoticed or, at best, weakly projected. This revolution in understanding has been sparked by analytical methods, new mathematical theories and computing techniques that let us investigate non-linear complex events. Many techniques have been used, including the F-expansion method [31], the q-Homotopy analysis method [32], the reduced differential transform method [33], the generalized Kudryashov method [34], the sub equation method [35], the Adomian decomposition method, the homotopy analysis method [36], variational iteration method [37], and improved (G/G)-expansion method [38].

The Jordanian mathematician, Omar Abu Arqub created the residual power series method in 2013 as a technique for quickly calculating the coefficients of the power series solutions for 1st and 2nd-order fuzzy differential equations [39]. Without perturbation, linearization, or discretization, the residual power series method provides a powerful and straightforward power series solution for highly linear and non-linear equations. The residual power series method has been used to solve an increasing variety of non-linear ordinary and partial differential equations of various sorts, orders, and classes during the past several years. It has been used to make solitary pattern results for non-linear dispersive fractional partial differential equations and to predict them [40], to solve the non-linear singular highly differential equation known as the generalized Lane-Emden equation [41], to solve higher-order ordinary differential equations numerically [42], to approximate solve the fractional non-linear KdV-Burger equations, to predict and represent The RPSM differs from several other analytical and numerical approaches in some crucial ways [43]. First, there is no requirement for a recursion connection or for the RPSM to compare the coefficients of the related terms. Second, by reducing the associated residual error, the RPSM offers a straightforward method to guarantee the convergence of the series solution. Thirdly, the RPSM doesn't suffer from computational rounding mistakes and doesn't use a lot of time or memory. Fourth, the approach may be used immediately to the provided issue by selecting an acceptable starting guess approximation since the residual power series method does not need any converting when transitionary from low-order to higher-order and from simple linearity to complicated nonlinearity [44–46].

This article uses the Laplace residual power series technique to achieve the definitive solution of the fractional-order non-linear Cahn-Hilliard and Gardner equations. The Laplace transformation efficiently integrates the RPSM for the renewability algorithmic technique. The fractional Caputo derivative explains quantitative categorizations of the Gardner and Cahn-Hilliard equations. The offered methodology is well demonstrated in modeling and calculation investigation.

2. Preliminaries

Definition 2.1. The fractional Caputo derivative of a function $u(\omega,\eta)$ of order α is given as

$$ {}^{C}D_{\eta}^{\alpha}u(\omega,\eta) = J_{\eta}^{m-\alpha}u^{m}(\omega,\eta),\ m-1<\alpha\leq m,\ \eta>0, \tag{2.1}$$

where $m \in N$ and J_{η}^{α} is the fractional Riemann-Liouville integral of $u(\omega,\eta)$ of fractional-order α is define as

$$ J_{\eta}^{\alpha}u(\omega,\eta) = \frac{1}{\Gamma(\alpha)}\int_{0}^{\eta}(t-\omega)u(\omega,t)dt,\ \alpha>0, \tag{2.2}$$

assuming that the given integral exists.

Definition 2.2. Assume that the continuous piecewise function $u(\omega,\eta)$ is expressed as:

$$ u(\omega,\upsilon) = £_{\eta}[u(\omega,\eta)] = \int_{0}^{\infty}e^{-\upsilon\eta}u(\omega,\eta)d\eta,\ \upsilon>\alpha, \tag{2.3}$$

where the inverse Laplace transform is expressed as

$$ u(\omega,\eta) = £_{\upsilon}^{-1}[u](\omega,\upsilon)] = \int_{l-i\infty}^{l+i\infty}e^{\upsilon\eta}u(\omega,\upsilon)d\upsilon,\ l=Re(\upsilon)>0. \tag{2.4}$$

Lemma 2.1. *Suppose that $u(\omega,\eta)$ is a piecewise continuous term and of exponential-order ψ and $u(\omega,\upsilon) = £_{\eta}[u(\omega,\eta)]$, we get*

(1) $£_{\eta}[J_{\eta}^{\beta}u(\omega,\eta)] = \frac{u(\omega,\upsilon)}{\upsilon^{\beta}},\ \beta>0.$
(2) $£_{\eta}[D_{\eta}^{\psi}u(\omega,\eta)] = \upsilon^{\psi}u(\omega,\upsilon) - \sum_{k=0}^{m-1}\upsilon^{\psi-k-1}u^{k}(\omega,0),\ m-1<\psi\leq m.$
(3) $£_{\eta}[D_{\eta}^{n\psi}u(\omega,\eta)] = \upsilon^{n\psi}u(\omega,\upsilon) - \sum_{k=0}^{n-1}\upsilon^{(n-k)\psi-1}D_{\eta}^{k\psi}u(\omega,0),\ 0<\psi\leq 1.$

Proof. The proof are in [1–3, 47]. □

Theorem 2.1. *Let us assume that $u(\omega,\eta)$ is a continuous piecewise on $\mathbf{I}\times[0,\infty)$. Consider that $u(\omega,\upsilon) = £_{\eta}[u(\omega,\eta)]$ has fractional power series (FPS) representation:*

$$ u(\omega,\upsilon) = \sum_{i=0}^{\infty}\frac{f_{i}(\omega)}{\upsilon^{1+i\alpha}},\ 0<\zeta\leq 1,\ \omega\in\mathbf{I},\ \upsilon>\psi. \tag{2.5}$$

Then, $f_{i}(\omega) = D_{\eta}^{n\alpha}u(\omega,0)$.

Proof. For proof, see Ref. [47]. □

Remark 2.1. The inverse Laplace transform of the Eq (2.5) represented as:

$$u(\omega,\eta)=\sum_{i=0}^{\infty}\frac{D_{\eta}^{\psi}u(\omega,0)}{\Gamma(1+i\psi)}\eta^{i(\psi)},\ 0<\psi\leq 1,\ \eta\geq 0. \tag{2.6}$$

which is the same as Taylor's formula for fractions, that can be found in [48].

Theorem 2.2. *Suppose that $u(\omega,\eta)$ is piecewise continuous on $\mathbf{I}\times[0,\infty)$ and of order ψ. As shown in Theorem 2.1, $u(\omega,\upsilon)=\pounds_{\eta}[u(\omega,\eta)]$. Taylor's formula can be written in its new form. If $\left|\upsilon\pounds_{\eta}[D_{\eta}^{i\alpha+1}u(\omega,\eta)]\right|\leq M(\omega)$, on $\mathbf{I}\times(\psi,\gamma]$, where $0<\alpha\leq 1$, then $R_i(\omega,\upsilon)$ the rest of the new way of writing fractions. The following inequality is true about Taylor's formula in Theorem 2.1:*

$$|R_i(\omega,\upsilon)|\leq\frac{M(\omega)}{S^{1+(i+1)\alpha}},\ \omega\in\mathbf{I},\ \psi<\upsilon\leq\gamma. \tag{2.7}$$

Proof. Let us consider that $\pounds_{\eta}\left[D_{\eta}^{k\alpha}u(\omega,\eta)\right](\upsilon)$ on interval $\mathbf{I}\times(\psi,\gamma]$ for $k=0,1,2,3,\cdots,i+1$, suppose that

$$\left|\upsilon\pounds_{\eta}[D_{\eta}^{i\alpha+1}u(\omega,\eta)]\right|\leq M(\omega),\ \omega\in\mathbf{I},\ \psi<\upsilon\leq\gamma. \tag{2.8}$$

Using the definition of remainder, $R_i(\omega,\upsilon)=u(\omega,\upsilon)-\sum_{k=0}^{i}\frac{D_{\eta}^{k\alpha}u(\omega,0)}{\upsilon^{1+k\alpha}}$, we can obtain

$$\begin{aligned}S^{1+(i+1)\alpha}R_i(\omega,\upsilon)=&\upsilon^{1+(i+1)\alpha}u(\omega,\upsilon)-\sum_{k=0}^{i}\upsilon^{(i+1-k)\alpha}D_{\eta}^{k\alpha}u(\omega,0)\\=&\upsilon\left(\upsilon^{(i+1)\alpha}u(\omega,\upsilon)-\sum_{k=0}^{i}\upsilon^{(i+1-k)\alpha-1}D_{\eta}^{k\alpha}u(\omega,0)\right)\\=&\upsilon\pounds_{\eta}\left[D_{\eta}^{(n+1)\zeta}u(\omega,\eta)\right].\end{aligned} \tag{2.9}$$

From Eqs (2.8) and (2.9) that $|\upsilon^{1+(i+1)\alpha}R_i(\omega,\upsilon)|\leq M(\omega)$. Thus,

$$-M(\omega)\leq\upsilon^{1+(i+1)\alpha}R_i(\omega,\upsilon)\leq M(\omega),\ \omega\in\mathbf{I},\ \psi<s\leq\gamma. \tag{2.10}$$

The proof of Theorem 2.2 is completed. □

3. General implementation of Laplace residual power series method

$$D_{\eta}^{\alpha}u(\omega,\eta)-u_{\omega}+6uu_{\omega}^{2}-\left(3u^{2}-1\right)u_{\omega\omega}+u_{\omega\omega\omega\omega}=0,\ 0<\alpha\leq 1, \tag{3.1}$$

with the initial condition,

$$u(\omega,\eta)=f_0(\omega), \tag{3.2}$$

where a and c are free constants and D_{η}^{α} is the Caputo-fractional derivative. First, we use the Laplace transformation to Eq (3.1),

$$\pounds\left[D_{\eta}^{\alpha}u(\omega,\eta)\right]=-\pounds\Big[u_{\omega}+6uu_{\omega}^{2}-\left(3u^{2}-1\right)u_{\omega\omega}+u_{\omega\omega\omega\omega}\Big]. \tag{3.3}$$

By the fact that $\pounds\left[D_\eta^\alpha u(\omega,\eta)\right] = \upsilon^a \pounds[u(\omega,\eta)] - \upsilon^{a-1}u(\omega,0)$ and using the initial condition (3.2), we rewrite (3.3) as

$$U(\omega,\upsilon) = \frac{f_0(\omega)}{\upsilon} - \frac{b}{\upsilon^\alpha}\pounds\left[\left(\pounds^{-1}\pounds\left[U_\omega + 6UU_\omega^2 - \left(3U^2-1\right)U_{\omega\omega} + U_{\omega\omega\omega\omega}\right]\right)\right], \tag{3.4}$$

where $U(\omega,\upsilon) = \pounds[u(\omega,\eta)]$.

Second, we define the transform term $U(\omega,\upsilon)$ as the following expression:

$$U(\omega,\upsilon) = \sum_{n=0}^{\infty} \frac{f_\upsilon(\omega)}{\upsilon^{n\alpha+1}}. \tag{3.5}$$

The series form of kth-truncated of Eq (3.5):

$$U_k(\omega,\upsilon) = \sum_{n=0}^{k} \frac{f_\upsilon(\omega)}{\upsilon^{n\alpha+1}} = \frac{f_o(\omega)}{\upsilon} + \sum_{n=1}^{k} \frac{f_k(\omega)}{\upsilon^{n\alpha+1}}. \tag{3.6}$$

The laplace residual function to (3.5) is

$$\pounds Res_k(\omega,\upsilon) = U_k(\omega,\upsilon) - \frac{f_0(\omega)}{\upsilon} - \frac{b}{\upsilon^\alpha}\pounds\left[\left(\pounds^{-1}\pounds\left[U_\omega + 6UU_\omega^2 - \left(3U^2-1\right)U_{\omega\omega} + U_{\omega\omega\omega\omega}\right]\right)\right]. \tag{3.7}$$

Third, we use a few properties that come up in the standard RPSM to point out certain facts:
$\pounds\,\mathrm{Res}(\omega,\upsilon) = 0$ and $\lim_{k\to\infty} \pounds\,\mathrm{Res}\, u_k(\omega,\upsilon) = \pounds\,\mathrm{Res}(\omega,\upsilon)$ for each $\upsilon > 0$;
$\lim_{\upsilon\to\infty} u\pounds\,\mathrm{Res}(\omega,\upsilon) = 0 \Rightarrow \lim_{\upsilon\to\infty} u\pounds\,\mathrm{Res}(\omega,\upsilon) = 0$;
$\lim_{\upsilon\to\infty} u^{k\alpha+1}\pounds\,\mathrm{Res}(\omega,\upsilon) = \lim_{\upsilon\to\infty} u^{k\alpha+1}\pounds\,\mathrm{Res}_k(\omega,\upsilon) = 0, 0 < \alpha \le 1, k = 1,2,3,\ldots.$

So, to find the co-efficient functions $f_n(\omega)$, we solve the following scheme successively:

$$\lim_{\upsilon\to\infty}\left(u^{ka+1}\pounds Res_k(\omega,\upsilon)\right) = 0,\ 0 < \alpha \le 1,\ k = 1,2,3,\ldots.$$

Finally, we use the inverse Laplace to $U_k(\omega,\upsilon)$, to achieved the kth approximated supportive result $u_k(\omega,\eta)$.

Next, we investigate the efficiency of the suggested above technique by investigating a numerical problem of the Cahn-Hilliard and Gardner models.

Example 3.1. Consider fractional-order Cahn-Hilliard equation,

$$D_\eta^\alpha u(\omega,\eta) - u_\omega + 6uu_\omega^2 - \left(3u^2-1\right)u_{\omega\omega} + u_{\omega\omega\omega\omega} = 0,\ 0 < \alpha \le 1, \tag{3.8}$$

with initial condition,

$$u(\omega,0) = \tanh\left(\frac{\sqrt{2}}{2}\omega\right). \tag{3.9}$$

The exact result when $\alpha = 1$ is

$$u(\omega,\eta) = \tanh\left(\frac{\sqrt{2}}{2}(\omega+\eta)\right). \tag{3.10}$$

Applying laplace transform to (3.8) and using the initial condition (3.9), we get

$$\begin{aligned} U(\omega,\upsilon) = \frac{\tanh\left(\frac{\sqrt{2}}{2}\omega\right)}{\upsilon} - \frac{1}{\upsilon^{\alpha}}£_{\eta}\Big[£_{\eta}^{-1}\left(U(\omega,\upsilon)\right) - 6£_{\eta}^{-1}\left(U(\omega,\upsilon)\right)£_{\eta}^{-1}\left(U_{\omega}^{2}(\omega,\upsilon)\right) \\ - 3£_{\eta}^{-1}\left(U(\omega,\upsilon)\right)£_{\eta}^{-1}\left(U_{\omega\omega}(\omega,\upsilon)\right) + £_{\eta}^{-1}\left(U_{\omega\omega}(\omega,\upsilon)\right) + £_{\eta}^{-1}\left(U_{\omega\omega\omega\omega}(\omega,\upsilon)\right)\Big]. \end{aligned} \tag{3.11}$$

The k-th truncated term series of (3.20) is

$$U(\omega,\upsilon) = \frac{\tanh\left(\frac{\sqrt{2}}{2}\omega\right)}{\upsilon} + \sum_{n=1}^{k}\frac{f_n(\omega)}{\upsilon^{n\alpha+1}}, \tag{3.12}$$

and the Laplace residual k-th term is

$$\begin{aligned} £_{\eta}Res_k(\omega,\upsilon) = U(\omega,\upsilon) - \frac{\tanh\left(\frac{\sqrt{2}}{2}\omega\right)}{\upsilon} + \frac{1}{\upsilon^{\alpha}}£_{\eta}\Big[£_{\eta}^{-1}\left(U(\omega,\upsilon)\right) - 6£_{\eta}^{-1}\left(U(\omega,\upsilon)\right)£_{\eta}^{-1}\left(U_{\omega}^{2}(\omega,\upsilon)\right) \\ - 3£_{\eta}^{-1}\left(U(\omega,\upsilon)\right)£_{\eta}^{-1}\left(U_{\omega\omega}(\omega,\upsilon)\right) + £_{\eta}^{-1}\left(U_{\omega\omega}(\omega,\upsilon)\right) + £_{\eta}^{-1}\left(U_{\omega\omega\omega\omega}(\omega,\upsilon)\right)\Big]. \end{aligned} \tag{3.13}$$

Now, to determine $f_k(x)$, $k = 1, 2, 3, \cdots$, we put the kth-truncated series (3.12) into the kth-Laplace residual term (3.13), multiply the solution of equation by $\upsilon^{k\alpha+1}$, and then solve recursively the relation $\lim_{\upsilon\to\infty}[\upsilon^{k\alpha+1}Res_k(x,\upsilon)] = 0$, $k = 1, 2, 3, \cdots$ for f_k. Following are the first some components of the sequences $f_k(x)$:

$$\begin{aligned} f_1(\omega) &= \frac{\operatorname{sech}\left(\frac{\omega}{\sqrt{2}}\right)^2}{\sqrt{2}}, \\ f_2(\omega) &= -\operatorname{sech}\left(\frac{\omega}{\sqrt{2}}\right)^2\tanh\left(\frac{\omega}{\sqrt{2}}\right), \\ f_3(\omega) &= \frac{1}{8}\operatorname{sech}\left(\frac{\omega}{\sqrt{2}}\right)^6\left(-4\sqrt{2} + (264 - 96\cosh(\sqrt{2}\omega) + \sqrt{2}\sinh(2\sqrt{2}\omega))\tanh\left(\frac{\omega}{\sqrt{2}}\right)\right) \\ &\quad + \left(\frac{-21}{2}\operatorname{sech}\left(\frac{\omega}{\sqrt{2}}\right)^6\tanh\left(\frac{\omega}{\sqrt{2}}\right) + 12\operatorname{sech}\left(\frac{\omega}{\sqrt{2}}\right)^4\tanh\left(\frac{\omega}{\sqrt{2}}\right)^3\right)\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}, \\ &\ \ \vdots \end{aligned} \tag{3.14}$$

Putting the values of $f_n(\omega)$, $(n \geq 1)$ in Eq (3.12), we have

$$\begin{aligned} U(\omega,\upsilon) &= \frac{\tanh\left(\frac{\sqrt{2}}{2}\omega\right)}{\upsilon} + \frac{\operatorname{sech}\left(\frac{\omega}{\sqrt{2}}\right)^2}{\sqrt{2}}\frac{1}{\upsilon^{\alpha+1}} - \operatorname{sech}\left(\frac{\omega}{\sqrt{2}}\right)^2\tanh\left(\frac{\omega}{\sqrt{2}}\right)\frac{1}{\upsilon^{2\alpha+1}} \\ &\quad + \frac{1}{8}\operatorname{sech}\left(\frac{\omega}{\sqrt{2}}\right)^6\left(-4\sqrt{2} + (264 - 96\cosh(\sqrt{2}\omega) + \sqrt{2}\sinh(2\sqrt{2}\omega))\tanh\left(\frac{\omega}{\sqrt{2}}\right)\right) \\ &\quad + \left(\frac{-21}{2}\operatorname{sech}\left(\frac{\omega}{\sqrt{2}}\right)^6\tanh\left(\frac{\omega}{\sqrt{2}}\right) + 12\operatorname{sech}\left(\frac{\omega}{\sqrt{2}}\right)^4\tanh\left(\frac{\omega}{\sqrt{2}}\right)^3\right)\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}\frac{1}{\upsilon^{3\alpha+1}} \\ &\quad + \cdots. \end{aligned} \tag{3.15}$$

Applying inverse Laplace transform, we get

$$
\begin{aligned}
u(\omega,\eta) &= \tanh\left(\frac{\sqrt{2}}{2}\omega\right) + \frac{\operatorname{sech}\left(\frac{\omega}{\sqrt{2}}\right)^2}{\sqrt{2}}\frac{\eta^{\alpha}}{\Gamma(\alpha+1)} - \operatorname{sech}\left(\frac{\omega}{\sqrt{2}}\right)^2 \tanh\left(\frac{\omega}{\sqrt{2}}\right)\frac{\eta^{2\alpha}}{\Gamma(2\alpha+1)} \\
&+ \frac{1}{8}\operatorname{sech}\left(\frac{\omega}{\sqrt{2}}\right)^6\left(-4\sqrt{2} + (264 - 96\cosh(\sqrt{2}\omega) + \sqrt{2}\sinh(2\sqrt{2}\omega))\tanh\left(\frac{\omega}{\sqrt{2}}\right)\right) \\
&+ \left(\frac{-21}{2}\operatorname{sech}\left(\frac{\omega}{\sqrt{2}}\right)^6 \tanh\left(\frac{\omega}{\sqrt{2}}\right) + 12\operatorname{sech}\left(\frac{\omega}{\sqrt{2}}\right)^4 \tanh\left(\frac{\omega}{\sqrt{2}}\right)^3\right)\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}\frac{\eta^{3\alpha}}{\Gamma(3\alpha+1)} \\
&+ \cdots .
\end{aligned} \tag{3.16}
$$

Throughout this investigation, the method are being employed to assess the precise analytical solution of fractional-order Cahn-Hilliard equation. For various spatial and temporal parameters, the Caputo fractional derivative operators in facilitate appropriate numerical findings for the Cahn-Hilliard equation option revenue framework utilizing multiple orders. In Figure 1, actual and approximate solutions graph and second fractional order $\alpha = 0.8$ of Example 3.1 at $\alpha = 1$. In Figure 2, approximate result graph at $\alpha = 0.6, 0.4$ and Figure 3, the approximate result at various value of α of Example 3.1.

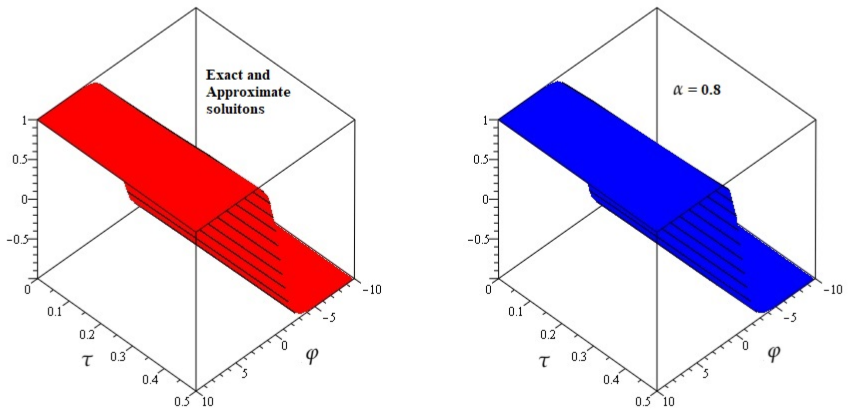

Figure 1. The exact and approximate results graph and 2nd fractional-order $\alpha = 0.8$ of Example 3.1 at $\alpha = 1$.

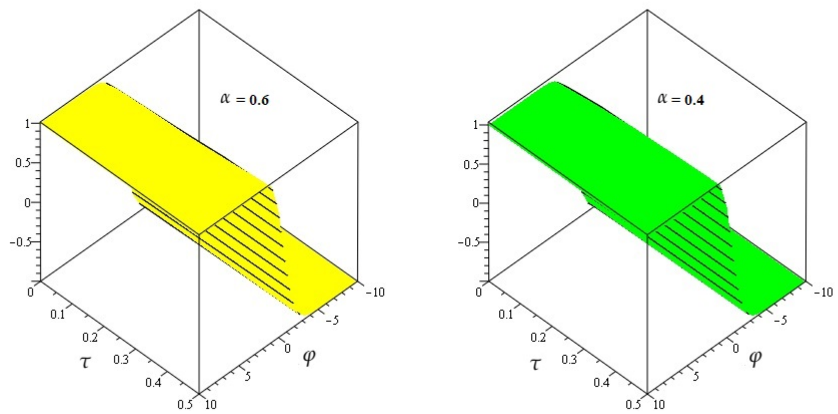

Figure 2. The approximate result graph at $\alpha = 0.6, 0.4$ for Example 3.1.

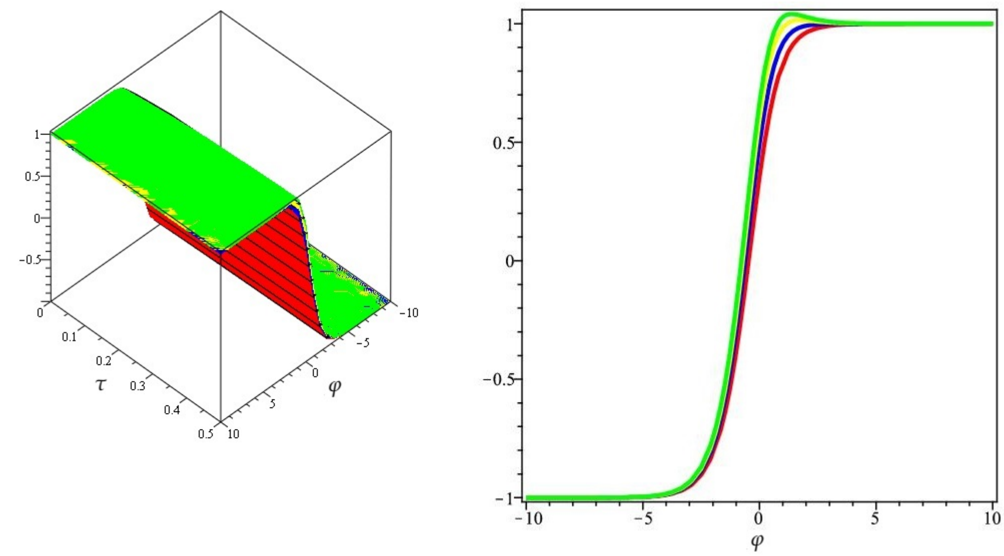

Figure 3. The approximate result at various value of α for Example 3.1.

Example 3.2. Consider the homogeneous fractional Gardner equation,

$$D_{\eta}^{\alpha}u(\omega,\eta)+6\left(u-\epsilon^{2}u^{2}\right)u_{\omega}+u_{\omega\omega\omega}=0,\ 0<\alpha\leq 1, \tag{3.17}$$

with the initial condition,

$$u(\omega,0)=\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{\omega}{2}\right). \tag{3.18}$$

The exact result when $\epsilon=1$, $\alpha=1$ is

$$u(\omega,\eta)=\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{\omega-\eta}{2}\right). \tag{3.19}$$

Applying Laplace transform to (3.17) and using the initial condition (3.18), we get

$$\begin{aligned}U(\omega,\upsilon)=&\left(\frac{1}{2}+\frac{1}{2}\tanh(\frac{\omega}{2})\right)\frac{1}{\upsilon}-\frac{1}{\upsilon^{\alpha}}£_{\eta}\Big[6\Big[£_{\eta}^{-1}\left[U(\omega,\upsilon)\right]£_{\eta}^{-1}\left[U_{\omega}(\omega,\upsilon)\right]\\&-\epsilon^{2}£_{\eta}^{-1}\left[U^{2}(\omega,\upsilon)\right]£_{\eta}^{-1}\left[U_{\omega}(\omega,\upsilon)\right]\Big]\Big]-\frac{1}{\upsilon^{\alpha}}£_{\eta}\left[£_{\eta}^{-1}\left[U_{\omega\omega\omega}(\omega,\upsilon)\right]\right].\end{aligned} \tag{3.20}$$

The k-th truncated term series of (3.20) is

$$U(\omega,\upsilon)=\left(\frac{1}{2}+\frac{1}{2}\tanh(\frac{\omega}{2})\right)\frac{1}{\upsilon}+\sum_{n=1}^{k}\frac{f_{n}(\omega)}{\upsilon^{n\alpha+1}}, \tag{3.21}$$

and the k-th Laplace residual function is

$$\begin{aligned}£_{\eta}Res_{k}(\omega,\upsilon)=&\ U(\omega,\upsilon)-\left(\frac{1}{2}+\frac{1}{2}\tanh(\frac{\omega}{2})\right)\frac{1}{\upsilon}+\frac{1}{\upsilon^{\alpha}}£_{\eta}\Big[6\Big[£_{\eta}^{-1}\left[U_{k}(\omega,\upsilon)\right]£_{\eta}^{-1}\left[U_{k\omega}(\omega,\upsilon)\right]\\&-\epsilon^{2}£_{\eta}^{-1}\left[U_{k}^{2}(\omega,\upsilon)\right]£_{\eta}^{-1}\left[U_{k\omega}(\omega,\upsilon)\right]\Big]\Big]+\frac{1}{\upsilon^{\alpha}}£_{\eta}\left[£_{\eta}^{-1}\left[U_{k\omega\omega\omega}(\omega,\upsilon)\right]\right].\end{aligned} \tag{3.22}$$

Now, to determine $f_k(x)$, $k = 1, 2, 3, \cdots$, we put the kth-truncate series (3.21) into the residual term of kth-Laplace (3.22), multiply the result equation by $\upsilon^{k\alpha+1}$, and then solve recursively the relations $\lim_{s\to\infty}[\upsilon^{k\alpha+1} Res_k(x,\upsilon)] = 0$, $k = 1, 2, 3, \cdots$ for f_k. First few components of the sequence $f_k(x)$,

$$\begin{aligned}
f_1(\omega) &= \frac{1}{8}\operatorname{sech}\left(\frac{\omega}{4}\right)^4\left[-1 + (-4 + 3\epsilon^3)\cosh(\omega) + 3(-1+\epsilon^2)\sinh(\omega)\right],\\
f_2(\omega) &= \frac{-1}{64}\operatorname{sech}\left(\frac{\omega}{4}\right)^7\Big[-24(-1+\epsilon^2)\cosh\left(\frac{\omega}{2}\right) - 6(22 - 37\epsilon^2 + 15\epsilon^4)\cosh\left(\frac{3\omega}{2}\right) + 24\cosh\left(\frac{5\omega}{2}\right)\\
&\quad - 42\epsilon^2\cosh\left(\frac{5\omega}{2}\right) + 18\epsilon^4\cosh\left(\frac{5\omega}{2}\right) + 206\sinh\left(\frac{\omega}{2}\right) - 204\epsilon^2\sinh\left(\frac{\omega}{2}\right) - 129\sinh\left(\frac{3\omega}{2}\right)\\
&\quad + 222\epsilon^2\sinh\left(\frac{3\omega}{2}\right) - 90\epsilon^4\sinh\left(\frac{3\omega}{2}\right) + 25\sinh\left(\frac{5\omega}{2}\right) - 42\epsilon^2\sinh\left(\frac{5\omega}{2}\right) + 18\epsilon^4\sinh\left(\frac{5\omega}{2}\right)\Big],\\
&\vdots
\end{aligned} \tag{3.23}$$

Putting the values of $f_n(x)$, $(n \geq 1)$ in Eq (3.21), we have

$$\begin{aligned}
U(\omega,\upsilon) &= \left(\frac{1}{2} + \frac{1}{2}\tanh(\frac{\omega}{2})\right)\frac{1}{\upsilon} + \frac{1}{8}\operatorname{sech}\left(\frac{\omega}{4}\right)^4\left[-1 + (-4 + 3\epsilon^3)\cosh(\omega) + 3(-1+\epsilon^2)\sinh(\omega)\right]\frac{1}{\upsilon^{\alpha+1}}\\
&\quad - \frac{1}{64}\operatorname{sech}\left(\frac{\omega}{4}\right)^7\Big[-24(-1+\epsilon^2)\cosh\left(\frac{\omega}{2}\right) - 6(22 - 37\epsilon^2 + 15\epsilon^4)\cosh\left(\frac{3\omega}{2}\right) + 24\cosh\left(\frac{5\omega}{2}\right)\\
&\quad - 42\epsilon^2\cosh\left(\frac{5\omega}{2}\right) + 18\epsilon^4\cosh\left(\frac{5\omega}{2}\right) + 206\sinh\left(\frac{\omega}{2}\right) - 204\epsilon^2\sinh\left(\frac{\omega}{2}\right) - 129\sinh\left(\frac{3\omega}{2}\right)\\
&\quad + 222\epsilon^2\sinh\left(\frac{3\omega}{2}\right) - 90\epsilon^4\sinh\left(\frac{3\omega}{2}\right) + 25\sinh\left(\frac{5\omega}{2}\right) - 42\epsilon^2\sinh\left(\frac{5\omega}{2}\right)\\
&\quad + 18\epsilon^4\sinh\left(\frac{5\omega}{2}\right)\Big]\frac{1}{\upsilon^{2\alpha+1}} + \cdots.
\end{aligned} \tag{3.24}$$

Applying inverse Laplace transform, we get

$$\begin{aligned}
u(\omega,\eta) &= \left(\frac{1}{2} + \frac{1}{2}\tanh(\frac{\omega}{2})\right) + \frac{1}{8}\operatorname{sech}\left(\frac{\omega}{4}\right)^4\left[-1 + (-4 + 3\epsilon^3)\cosh(\omega) + 3(-1+\epsilon^2)\sinh(\omega)\right]\frac{\eta^{\alpha}}{\Gamma(\alpha+1)}\\
&\quad - \frac{1}{64}\operatorname{sech}\left(\frac{\omega}{4}\right)^7\Big[-24(-1+\epsilon^2)\cosh\left(\frac{\omega}{2}\right) - 6(22 - 37\epsilon^2 + 15\epsilon^4)\cosh\left(\frac{3\omega}{2}\right) + 24\cosh\left(\frac{5\omega}{2}\right)\\
&\quad - 42\epsilon^2\cosh\left(\frac{5\omega}{2}\right) + 18\epsilon^4\cosh\left(\frac{5\omega}{2}\right) + 206\sinh\left(\frac{\omega}{2}\right) - 204\epsilon^2\sinh\left(\frac{\omega}{2}\right) - 129\sinh\left(\frac{3\omega}{2}\right)\\
&\quad + 222\epsilon^2\sinh\left(\frac{3\omega}{2}\right) - 90\epsilon^4\sinh\left(\frac{3\omega}{2}\right) + 25\sinh\left(\frac{5\omega}{2}\right) - 42\epsilon^2\sinh\left(\frac{5\omega}{2}\right)\\
&\quad + 18\epsilon^4\sinh\left(\frac{5\omega}{2}\right)\Big]\frac{\eta^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots.
\end{aligned} \tag{3.25}$$

Throughout this investigation, the method are being employed to assess the precise analytical solution of fractional-order Gardner equation. For various spatial and temporal parameters, the Caputo fractional derivative operators in facilitate appropriate numerical findings for the Cahn-Hilliard equation option revenue framework utilizing multiple orders. In Figure 4, actual and approximate solutions graph and second fractional order $\alpha = 0.8$ of Example 3.2 at $\alpha = 1$. In Figure 5, approximate result graph at $\alpha = 0.6, 0.4$ and Figure 6, the approximate result at various value of α of Example 3.2.

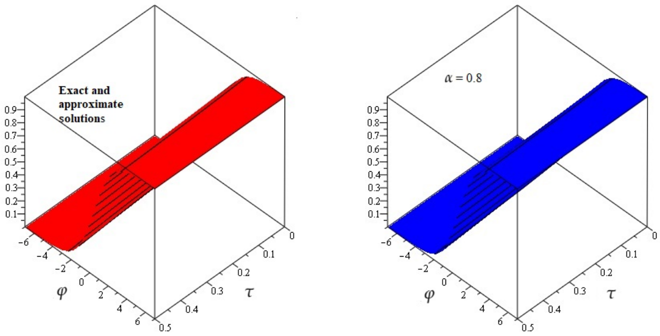

Figure 4. The actual and approximate results graph and second fractional order at $\alpha = 0.8$ of Example 3.2.

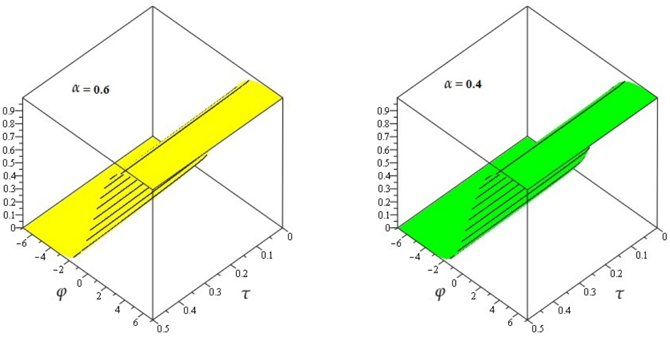

Figure 5. The approximate result at $\alpha = 0.6, 0.4$ of Example 3.2.

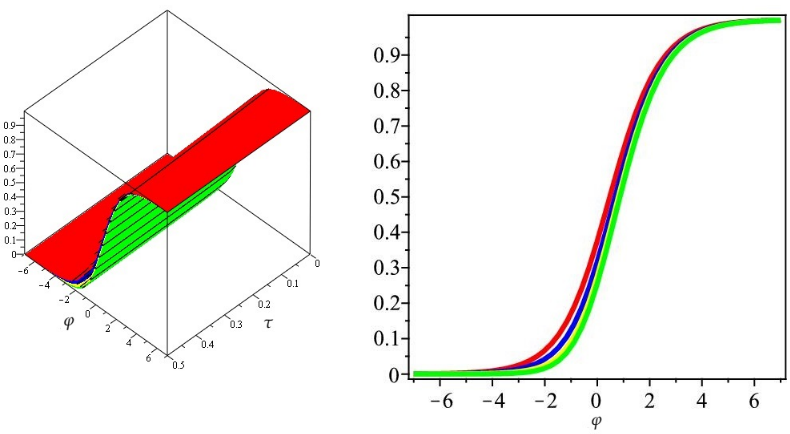

Figure 6. The approximate result at various value of α for Example 3.2.

4. Conclusions

In this article, significant nonlinear fractional Cahn-Hilliard and Gardner equations are solved utilizing a combination of the Laplace transformation and the residual power series. This study demonstrated that the suggested method, Laplace residual power series, is a straightforward and effective analytical technique for constructing exact and approximation solutions for partial differential equations with suitable initial conditions. The aforementioned method provided us with solutions in the Laplace transform space via a straightforward method for obtaining the expansion series constants with the aid of the limit idea at infinity. With fewer series terms, the approximate solutions are achieved. The proposed technique is used to solve two separate physical models, and its capacity to handle fractional nonlinear equations with high precision and straightforward computing processes has been demonstrated.

Conflict of interest

The authors declare no conflict of interest.

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