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Research article

On common fixed point results in bicomplex valued metric spaces with application

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Abstract: Metric fixed-point theory has become an essential tool in computer science, communication engineering and complex systems to validate the processes and algorithms by using functional equations and iterative procedures. The aim of this article is to obtain common fixed point results in a bicomplex valued metric space for rational contractions involving control functions of two variables. Our theorems generalize some famous results from literature. We supply an example to show the originality of our main result. As an application, we develop common fixed point results for rational contractions involving control functions of one variable in the context of bicomplex valued metric space.

Keywords: bicomplex valued metric space; common fixed point; control functions; rational expressions
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1. Introduction

Metric fixed-point theory has newly emerging applications to study the internet topology [1] and modelling the cyberspace as a digital ecosystem [2]. Moreover, new researches in fixed-point theory determine the significance to find the solution of real-world problems. A routing problem, for example, can be solved using functional equations and iterative procedures. The capacitated vehicle routing problem (CVRP) [3] outlines a method for determining the best plan to meet the demand of a globally dispersed network of clients while distributing cohesive products from a pickup point that used a large number of (the same) automobiles with a specific adaptive capacity. Meanwhile, fixed point theory is used as a problem-solving tool in communication engineering. Other real-world applications include the solution of chemical equations, genetics, algorithm testing, and control theory. Such results offer delightful circumstances in the study of mathematical analysis to approximating the solutions of linear and nonlinear differential and integral equations [4]. Because the theory of fixed-point is an odd synthesis of analysis [5, 6] and geometry [7–10]. Therefore, it has emerged as a powerful and crucial instrument for the investigation of nonlinear problems [11–14]. More recently, Işık and collaborators have discussed such results by using rational [15] as well as generalized Wardowski type contractive multi-valued mappings [16] and also investigated the common solutions to integral and functional equations [17, 18]. The aim of this article is to obtain common fixed point results in a bicomplex valued metric space. Therefore, we first define the basic preliminaries involving bicomplex numbers and review further developments related to them in the following paragraphs.

The emergence of complex numbers was established in the 17^{th} century by Sir Carl Fredrich Gauss, but his work was not on record. Later, in the year 1840 Augustin Louis Cauchy started doing analysis of complex numbers, who is known to be an effective founder of complex analysis. The theory of complex numbers has its source in the fact that the solution of the quadratic equation $ax^2 + bx + c = 0$ was not worthwhile for $b^2 - 4ac < 0$, in the set of real numbers. Under this background, Euler was the first mathematician who presented the symbol i, for $\sqrt{-1}$ with the property, $i^2 = -1$.

On the other hand, the beginning of bicomplex numbers was set up by Segre [19] which provides a commutative substitute to the skew field of quaternions. These numbers generalize complex numbers more precisely to quaternions. We refer readers to [20] for a more in-depth examination of bicomplex numbers. In 2011, Azam et al. [21] gave the concept of a complex valued metric space (CVMS) as a special case of cone metric space. Since the concept to introduce complex valued metric spaces is designed to define rational expressions that cannot be defined in cone metric spaces and therefore several results of fixed point theory cannot be proved to cone metric spaces, so complex valued metric space form a special class of cone metric space. Actually, the definition of a cone metric space banks on the underlying Banach space which is not a division ring. However, we can study generalizations of many results of fixed point theory involving divisions in complex valued metric spaces. Moreover, this idea is also used to define complex valued Banach spaces [22] which offer a lot of scope for further investigation. In 2017, Choi et al. [23] combined the concepts of bicomplex numbers and CVMS and introduced the notion of bicomplex valued metric spaces (bi CVMS) and established common fixed point results for weakly compatible mappings. Later on, Jebril et al. [24], utilized this notion of newly introduced space and obtained common fixed point results under rational contractions for a pair of mappings in the background of bi CVMS. More specifically, CVMS [25, 26] and bi CVMS [27, 28] has been remained a focus point of recent and past researches. By taking motivation from these facts, we establish some common fixed point theorems in bi CVMS for rational contractions involving control functions of two variables. As an application, we investigate the solutions of integral equations.

2. Preliminaries

We represent $\mathbb{C}_0$, $\mathbb{C}_1$ and $\mathbb{C}_2$ as the set of real numbers, complex numbers and bicomplex numbers respectively. Segre [19] defined the notion of bicomplex number as follows:

$$\varrho = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2$$

where $a_1, a_2, a_3, a_4 \in \mathbb{C}_0$, and the independent units i_1, i_2 are such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_{1,}$ and $\mathbb{C}_2$ is defined as

$$\mathbb{C}_2 = \{\varrho : \varrho = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 : a_1, a_2, a_3, a_4 \in \mathbb{C}_0\}$$

that is

$$\mathbb{C}_2 = \{\varrho : \varrho = z_1 + i_2 z_2 : z_1, z_2 \in \mathbb{C}_1\}$$

where $z_1 = a_1 + a_2 i_1 \in \mathbb{C}_1$ and $z_2 = a_3 + a_4 i_1 \in \mathbb{C}_1$. If $\varrho = z_1 + i_2 z_2$ and $\hbar = \omega_1 + i_2 \omega_2$, then the sum is

$$\varrho \pm \hbar = (z_1 + i_2 z_2) \pm (\omega_1 + i_2 \omega_2) = (z_1 \pm \omega_1) + i_2 (z_2 \pm \omega_2)$$

and the product is

$$\varrho \cdot \hbar = (z_1 + i_2 z_2) \cdot (\omega_1 + i_2 \omega_2) = (z_1 \omega_1 - z_2 \omega_2) + i_2 (z_1 \omega_2 + z_2 \omega_1).$$

There are four idempotent members in $\mathbb{C}_2$, which are, $0, 1, e_1 = \frac{1+i_1 i_2}{2}$ and $e_2 = \frac{1-i_1 i_2}{2}$ out of which e_1 and e_2 are nontrivial such that $e_1 + e_2 = 1$ and $e_1 e_2 = 0$. Every bicomplex number $z_1 + i_2 z_2$ can uniquely be demonstrated as the mixture of $e_1$and e_2, namely

$$\varrho = z_1 + i_2 z_2 = (z_1 - i_1 z_2) e_1 + (z_1 + i_1 z_2) e_2.$$

This characterization of ϱ is familiar as the idempotent characterization of ϱ and the complex coefficients $\varrho_1 = (z_1 - i_1 z_2)$ and $\varrho_2 = (z_1 + i_1 z_2)$ are called as idempotent components of ϱ.

An element $\varrho = z_1 + i_2 z_2 \in \mathbb{C}_2$ is called invertible if there exists $\hbar \in \mathbb{C}_2$ such that $\varrho\hbar = 1$ and $\hbar$ is called the inverse (multiplicative) of ϱ. Therefore ϱ is called the inverse of $\hbar$.

An element $\varrho = z_1 + i_2 z_2 \in \mathbb{C}_2$ is nonsingular iff $\left|z_1^2 + z_2^2\right| \neq 0$ and singular iff $\left|z_1^2 + z_2^2\right| = 0$. The inverse of ϱ is defined as

$$\varrho^{-1} = \hbar = \frac{z_1 - i_2 z_2}{z_1^2 + z_2^2}.$$

Zero is the at most member in $\mathbb{C}_0$ that does not possess a multiplicative inverse and in $\mathbb{C}_1$, $0 = 0 + i0$ is the at most member that does not possess a multiplicative inverse. We represent the set of singular members of $\mathbb{C}_0$ and $\mathbb{C}_1$ by $\aleph_0$ and $\aleph_1$ in this order. There are many members in $\mathbb{C}_2$ that do not have multiplicative inverse. We represents this set by $\aleph_2$ and evidently $\aleph_0 = \aleph_1 \subset \aleph_2$.

A bicomplex number $\varrho = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 \in \mathbb{C}_2$ is said to be degenerated if the matrix

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}_{2\times 2}$$

is degenerated. In this way ϱ^{-1} exists and it is degenerated too and $\|\cdot\| : \mathbb{C}_2 \to \mathbb{C}_0^+$ is defined as

$$\begin{aligned}\|\varrho\| &= \|z_1 + i_2 z_2\| = \left\{|z_1|^2 + |z_2|^2\right\}^{\frac{1}{2}} \\ &= \left[\frac{|(z_1 - i_1 z_2)|^2 + |(z_1 + i_1 z_2)|^2}{2}\right]^{\frac{1}{2}} \\ &= \left(a_1^2 + a_2^2 + a_3^2 + a_4^2\right)^{\frac{1}{2}},\end{aligned}$$

where $\varrho = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 = z_1 + i_2 z_2 \in \mathbb{C}_2$.

The space $\mathbb{C}_2$ with respect to the norm given above is a Banach space. If $\varrho, \hbar \in \mathbb{C}_2$, then

$$\|\varrho\hbar\| \leq \sqrt{2}\,\|\varrho\|\,\|\hbar\|$$

holds instead of

$$\|\varrho\hbar\| \leq \|\varrho\|\,\|\hbar\|.$$

Therefore, $\mathbb{C}_2$ is not a Banach algebra. Let $\varrho = z_1 + i_2 z_2$, $\hbar = \omega_1 + i_2\omega_2 \in \mathbb{C}_2$, then we define

$$\varrho \preceq_{i_2} \hbar \Leftrightarrow \operatorname{Re}(z_1) \preceq \operatorname{Re}(\omega_1) \text{ and } \operatorname{Im}(z_2) \preceq \operatorname{Im}(\omega_2).$$

It implies

$$\varrho \preceq_{i_2} \hbar,$$

if one of these assertions hold:

$$\begin{aligned}\text{(i) } (z_1) &= \omega_1,\ z_2 \prec \omega_2, \\ \text{(ii) } z_1 &\prec \omega_1,\ z_2 = \omega_2, \\ \text{(iii) } z_1 &\prec \omega_1,\ z_2 \prec \omega_2, \\ \text{(iv) } z_1 &= \omega_1,\ z_2 = \omega_2.\end{aligned}$$

Specifically, $\varrho \precnsim_{i_2} \hbar$ if $\varrho \preceq_{i_2} \hbar$ and $\varrho \neq \hbar$, that is, one of (i), (ii) and (iii) holds. Also $\varrho \prec_{i_2} \hbar$ if only condition (iii) is satisfied. For $\varrho, \hbar \in \mathbb{C}_2$, we can prove the followings:

(i) $\varrho \preceq_{i_2} \hbar \Longrightarrow \|\varrho\| \leq \|\hbar\|$,

(ii) $\|\varrho + \hbar\| \leq \|\varrho\| + \|\hbar\|$,

(iii) $\|a\varrho\| \leq a\,\|\hbar\|$, where a is a non negative real number,

(iv) $\|\varrho\hbar\| \leq \sqrt{2}\,\|\varrho\|\,\|\hbar\|$,

(v) $\left\|\varrho^{-1}\right\| = \|\varrho\|^{-1}$,

(vi) $\left\|\frac{\varrho}{\hbar}\right\| = \frac{\|\varrho\|}{\|\hbar\|}$, if $\hbar$ is a degenerated bicomplex number.

Azam et al. [21] gave the conception of CVMS in this way:

Definition 1. ([21]) Let $\mathfrak{L} \neq \emptyset$, $\preceq$ is a partial order on $\mathbb{C}$ and $\varsigma : \mathfrak{L} \times \mathfrak{L} \to \mathbb{C}_1$ be a mapping satisfying

(i) $0 \preceq \varsigma(\varrho, \hbar)$, for all $\varrho, \hbar \in \mathfrak{L}$ and $\varsigma(\varrho, \hbar) = 0$ if and only if $\varrho = \hbar$;

(ii) $\varsigma(\varrho, \hbar) = \varsigma(\hbar, \varrho)$ for all $\varrho, \hbar \in \mathfrak{L}$;

(iii) $\varsigma(\varrho, \hbar) \preceq \varsigma(\varrho, \nu) + \varsigma(\nu, \hbar)$, for all $\varrho, \hbar, \nu \in \mathfrak{L}$, then $(\mathfrak{L}, \varsigma)$ is a CVMS.

Choi et al. [23] defined the bi CVMS as follows:

Definition 2. ([23]) Let $\mathfrak{L} \neq \emptyset$, $\preceq_{i_2}$ is a partial order on $\mathbb{C}_2$ and $\varsigma : \mathfrak{L} \times \mathfrak{L} \to \mathbb{C}_2$ be a mapping satisfying

(i) $0 \preceq_{i_2} \varsigma(\varrho, \hbar)$, for all $\varrho, \hbar \in \mathfrak{L}$ and $\varsigma(\varrho, \hbar) = 0$ if and only if $\varrho = \hbar$;
(ii) $\varsigma(\varrho, \hbar) = \varsigma(\hbar, \varrho)$ for all $\varrho, \hbar \in \mathfrak{L}$;
(iii) $\varsigma(\varrho, \hbar) \preceq_{i_2} \varsigma(\varrho, \nu) + \varsigma(\nu, \hbar)$, for all $\varrho, \hbar, \nu \in \mathfrak{L}$,

then $(\mathfrak{L}, \varsigma)$ is a bi CVMS.

Example 1. ([29]) Let $\mathfrak{L} = \mathbb{C}_2$ and $\varrho, \hbar \in \mathfrak{L}$. Define $\varsigma : \mathfrak{L} \times \mathfrak{L} \to \mathbb{C}_2$ by

$$\varsigma(\varrho, \hbar) = |z_1 - \omega_1| + i_2 |z_2 - \omega_2|$$

where $\varrho = z_1 + i_2 z_2$ and $\hbar = \omega_1 + i_2 \omega_2 \in \mathbb{C}_2$. Then $(\mathfrak{L}, \varsigma)$ is a bi CVMS.

Lemma 1. *([29]) Let $(\mathfrak{L}, \varsigma)$ be a bi CVMS and let $\{\varrho_r\} \subseteq \mathfrak{L}$. Then $\{\varrho_r\}$ converges to ϱ if and only if $\|\varsigma(\varrho_r, \varrho)\| \to 0$ as $r \to \infty$.*

Lemma 2. *([29]) Let $(\mathfrak{L}, \varsigma)$ be a bi CVMS and let $\{\varrho_r\} \subseteq \mathfrak{L}$. Then $\{\varrho_r\}$ is a Cauchy sequence if and only if $\|\varsigma(\varrho_r, \varrho_{r+m})\| \to 0$ as $r \to \infty$, where $m \in \mathbb{N}$.*

3. Main result

We state and prove the following proposition which is required in the sequel.

Proposition 1. *Let $(\mathfrak{L}, \varsigma)$ be a bi CVMS and $\mathfrak{I}_1, \mathfrak{I}_2 : (\mathfrak{L}, \varsigma) \to (\mathfrak{L}, \varsigma)$. Let $\varrho_0 \in \mathfrak{L}$. Define the sequence $\{\varrho_r\}$ by*

$$\varrho_{2r+1} = \mathfrak{I}_1 \varrho_{2r} \text{ and } \varrho_{2r+2} = \mathfrak{I}_2 \varrho_{2r+1} \tag{3.1}$$

for all $r = 0, 1, 2, ...$

Assume that there exist $\rho : \mathfrak{L} \times \mathfrak{L} \to [0, 1)$ satisfying

$$\rho(\mathfrak{I}_2 \mathfrak{I}_1 \varrho, \hbar) \leq \rho(\varrho, \hbar) \text{ and } \rho(\varrho, \mathfrak{I}_1 \mathfrak{I}_2 \hbar) \leq \rho(\varrho, \hbar)$$

for all $\varrho, \hbar \in \mathfrak{L}$. Then

$$\rho(\varrho_{2r}, \hbar) \leq \rho(\varrho_0, \hbar) \text{ and } \rho(\varrho, \varrho_{2r+1}) \leq \rho(\varrho, \varrho_1)$$

for all $\varrho, \hbar \in \mathfrak{L}$ and $r = 0, 1, 2, ...$

Proof. Let $\varrho, \hbar \in \mathfrak{L}$ and $r = 0, 1, 2, ...$ Then we have

$$\begin{aligned} \rho(\varrho_{2r}, \hbar) &= \rho(\mathfrak{I}_2 \mathfrak{I}_1 \varrho_{2r-2}, \hbar) \leq \rho(\varrho_{2r-2}, \hbar) \\ &= \rho(\mathfrak{I}_2 \mathfrak{I}_1 \varrho_{2r-4}, \hbar) \leq \rho(\varrho_{2r-4}, \hbar) \\ &\leq \cdots \leq \rho(\varrho_0, \hbar). \end{aligned}$$

Similarly, we have

$$\begin{aligned}\rho(\varrho,\varrho_{2r+1}) &= \rho(\varrho,\mathfrak{I}_1\mathfrak{I}_2\varrho_{2r-1}) \le \rho(\varrho,\varrho_{2r-1})\\ &= \rho(\varrho,\mathfrak{I}_1\mathfrak{I}_2\varrho_{2r-3}) \le \rho(\varrho,\varrho_{2r-3})\\ &\le \cdots \le \rho(\varrho,\varrho_1).\end{aligned}$$

□

Lemma 3. *Let* $\rho,\kappa:\mathfrak{L}\times\mathfrak{L}\to[0,1)$ *and* $\varrho,\hbar\in\mathfrak{L}$. *If* $\mathfrak{I}_1,\mathfrak{I}_2:\mathfrak{L}\to\mathfrak{L}$ *satisfy*

$$\varsigma(\mathfrak{I}_1\varrho,\mathfrak{I}_2\mathfrak{I}_1\varrho) \preceq_{i_2} \rho(\varrho,\mathfrak{I}_1\varrho)\varsigma(\varrho,\mathfrak{I}_1\varrho)+\kappa(\varrho,\mathfrak{I}_1\varrho)\frac{\varsigma(\varrho,\mathfrak{I}_1\varrho)\varsigma(\mathfrak{I}_1\varrho,\mathfrak{I}_2\mathfrak{I}_1\varrho)}{1+\varsigma(\varrho,\mathfrak{I}_1\varrho)}$$

and

$$\varsigma(\mathfrak{I}_1\mathfrak{I}_2\hbar,\mathfrak{I}_2\hbar) \preceq_{i_2} \rho(\mathfrak{I}_2\hbar,\hbar)\varsigma(\mathfrak{I}_2\hbar,\hbar)+\kappa(\mathfrak{I}_2\hbar,\hbar)\frac{\varsigma(\mathfrak{I}_2\hbar,\mathfrak{I}_1\mathfrak{I}_2\hbar)\varsigma(\hbar,\mathfrak{I}_2\hbar)}{1+\varsigma(\mathfrak{I}_2\hbar,\hbar)}$$

then

$$\left\|\varsigma(\mathfrak{I}_1\varrho,\mathfrak{I}_2\mathfrak{I}_1\varrho)\right\| \le \rho(\varrho,\mathfrak{I}_1\varrho)\left\|\varsigma(\varrho,\mathfrak{I}_1\varrho)\right\| + \sqrt{2}\kappa(\varrho,\mathfrak{I}_1\varrho)\left\|\varsigma(\mathfrak{I}_1\varrho,\mathfrak{I}_2\mathfrak{I}_1\varrho)\right\|$$

and

$$\left\|\varsigma(\mathfrak{I}_1\mathfrak{I}_2\hbar,\mathfrak{I}_2\hbar)\right\| \le \rho(\mathfrak{I}_2\hbar,\hbar)\left\|\varsigma(\mathfrak{I}_2\hbar,\hbar)\right\| + \sqrt{2}\kappa(\mathfrak{I}_2\hbar,\hbar)\left\|\varsigma(\mathfrak{I}_2\hbar,\mathfrak{I}_1\mathfrak{I}_2\hbar)\right\|.$$

Proof. We can write

$$\begin{aligned}\left\|\varsigma(\mathfrak{I}_1\varrho,\mathfrak{I}_2\mathfrak{I}_1\varrho)\right\| &\le \left\|\rho(\varrho,\mathfrak{I}_1\varrho)\varsigma(\varrho,\mathfrak{I}_1\varrho)+\kappa(\varrho,\mathfrak{I}_1\varrho)\frac{\varsigma(\varrho,\mathfrak{I}_1\varrho)\varsigma(\mathfrak{I}_1\varrho,\mathfrak{I}_2\mathfrak{I}_1\varrho)}{1+\varsigma(\varrho,\mathfrak{I}_1\varrho)}\right\|\\ &\le \rho(\varrho,\mathfrak{I}_1\varrho)\left\|\varsigma(\varrho,\mathfrak{I}_1\varrho)\right\| + \sqrt{2}\kappa(\varrho,\mathfrak{I}_1\varrho)\left\|\frac{\varsigma(\varrho,\mathfrak{I}_1\varrho)}{1+\varsigma(\varrho,\mathfrak{I}_1\varrho)}\right\|\left\|\varsigma(\mathfrak{I}_1\varrho,\mathfrak{I}_2\mathfrak{I}_1\varrho)\right\|\\ &\le \rho(\varrho,\mathfrak{I}_1\varrho)\left\|\varsigma(\varrho,\mathfrak{I}_1\varrho)\right\| + \sqrt{2}\kappa(\varrho,\mathfrak{I}_1\varrho)\left\|\varsigma(\mathfrak{I}_1\varrho,\mathfrak{I}_2\mathfrak{I}_1\varrho)\right\|.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\left\|\varsigma(\mathfrak{I}_1\mathfrak{I}_2\hbar,\mathfrak{I}_2\hbar)\right\| &\le \left\|\rho(\mathfrak{I}_2\hbar,\hbar)\varsigma(\mathfrak{I}_2\hbar,\hbar)+\kappa(\mathfrak{I}_2\hbar,\hbar)\frac{\varsigma(\mathfrak{I}_2\hbar,\mathfrak{I}_1\mathfrak{I}_2\hbar)\varsigma(\hbar,\mathfrak{I}_2\hbar)}{1+\varsigma(\mathfrak{I}_2\hbar,\hbar)}\right\|\\ &\le \rho(\mathfrak{I}_2\hbar,\hbar)\left\|\varsigma(\mathfrak{I}_2\hbar,\hbar)\right\| + \sqrt{2}\kappa(\mathfrak{I}_2\hbar,\hbar)\left\|\frac{\varsigma(\hbar,\mathfrak{I}_2\hbar)}{1+\varsigma(\mathfrak{I}_2\hbar,\hbar)}\right\|\left\|\varsigma(\mathfrak{I}_2\hbar,\mathfrak{I}_1\mathfrak{I}_2\hbar)\right\|\\ &\le \rho(\mathfrak{I}_2\hbar,\hbar)\left\|\varsigma(\mathfrak{I}_2\hbar,\hbar)\right\| + \sqrt{2}\kappa(\mathfrak{I}_2\hbar,\hbar)\left\|\varsigma(\mathfrak{I}_2\hbar,\mathfrak{I}_1\mathfrak{I}_2\hbar)\right\|.\end{aligned}$$

□

Theorem 1. *Let* $(\mathfrak{L},\varsigma)$ *be a complete bi CVMS and* $\mathfrak{I}_1,\mathfrak{I}_2$*:* $\mathfrak{L}\to\mathfrak{L}$. *If there exist mappings* $\rho,\kappa,\varpi:\mathfrak{L}\times\mathfrak{L}\to[0,1)$ *such that for all* $\varrho,\hbar\in\mathfrak{L}$,

(a) $\rho(\mathfrak{I}_2\mathfrak{I}_1\varrho,\hbar)\le\rho(\varrho,\hbar)$ and $\rho(\varrho,\mathfrak{I}_1\mathfrak{I}_2\hbar)\le\rho(\varrho,\hbar)$,
$\kappa(\mathfrak{I}_2\mathfrak{I}_1\varrho,\hbar)\le\kappa(\varrho,\hbar)$ and $\kappa(\varrho,\mathfrak{I}_1\mathfrak{I}_2\hbar)\le\kappa(\varrho,\hbar)$,
$\varpi(\mathfrak{I}_2\mathfrak{I}_1\varrho,\hbar)\le\varpi(\varrho,\hbar)$ and $\varpi(\varrho,\mathfrak{I}_1\mathfrak{I}_2\hbar)\le\varpi(\varrho,\hbar)$,
(b) $\rho(\varrho,\hbar)+\sqrt{2}\kappa(\varrho,\hbar)+\sqrt{2}\varpi(\varrho,\hbar)<1$,

(c)

$$\varsigma(\Im_1\varrho, \Im_2\hbar) \preceq_{i_2} \rho(\varrho, \hbar)\varsigma(\varrho, \hbar) + \kappa(\varrho, \hbar)\frac{\varsigma(\varrho, \Im_1\varrho)\varsigma(\hbar, \Im_2\hbar)}{1+\varsigma(\varrho, \hbar)} + \varpi(\varrho, \hbar)\frac{\varsigma(\hbar, \Im_1\varrho)\varsigma(\varrho, \Im_2\hbar)}{1+\varsigma(\varrho, \hbar)}, \quad (3.2)$$

then $\Im_1$ and $\Im_2$ have a unique common fixed point.

Proof. Let $\varrho, \hbar \in \mathfrak{L}$. From (3.2), we have

$$\varsigma(\Im_1\varrho, \Im_2\Im_1\varrho) \preceq_{i_2} \rho(\varrho, \Im_1\varrho)\varsigma(\varrho, \Im_1\varrho) + \kappa(\varrho, \Im_1\varrho)\frac{\varsigma(\varrho, \Im_1\varrho)\varsigma(\Im_1\varrho, \Im_2\Im_1\varrho)}{1+\varsigma(\varrho, \Im_1\varrho)}$$

$$+\varpi(\varrho, \Im_1\varrho)\frac{\varsigma(\Im_1\varrho, \Im_1\varrho)\varsigma(\varrho, \Im_2\Im_1\varrho)}{1+\varsigma(\varrho, \Im_1\varrho)}.$$

By Lemma (3), we get

$$\left\|\varsigma(\Im_1\varrho, \Im_2\Im_1\varrho)\right\| \leq \rho(\varrho, \Im_1\varrho)\left\|\varsigma(\varrho, \Im_1\varrho)\right\| + \sqrt{2}\kappa(\varrho, \Im_1\varrho)\left\|\varsigma(\Im_1\varrho, \Im_2\Im_1\varrho)\right\|. \quad (3.3)$$

Similarly, we have

$$\varsigma(\Im_1\Im_2\hbar, \Im_2\hbar) \preceq_{i_2} \rho(\Im_2\hbar, \hbar)\varsigma(\Im_2\hbar, \hbar) + \kappa(\Im_2\hbar, \hbar)\frac{\varsigma(\Im_2\hbar, \Im_1\Im_2\hbar)\varsigma(\hbar, \Im_2\hbar)}{1+\varsigma(\Im_2\hbar, \hbar)}$$

$$+\varpi(\varrho, \hbar)\frac{\varsigma(\hbar, \Im_1\Im_2\hbar)\varsigma(\Im_2\hbar, \Im_2\hbar)}{1+\varsigma(\Im_2\hbar, \hbar)}$$

$$= \rho(\Im_2\hbar, \hbar)\varsigma(\Im_2\hbar, \hbar) + \kappa(\Im_2\hbar, \hbar)\frac{\varsigma(\Im_2\hbar, \Im_1\Im_2\hbar)\varsigma(\hbar, \Im_2\hbar)}{1+\varsigma(\Im_2\hbar, \hbar)}.$$

By Lemma (3), we get

$$\left\|\varsigma(\Im_1\Im_2\hbar, \Im_2\hbar)\right\| \leq \rho(\Im_2\hbar, \hbar)\left\|\varsigma(\Im_2\hbar, \hbar)\right\| + \sqrt{2}\kappa(\Im_2\hbar, \hbar)\left\|\varsigma(\Im_2\hbar, \Im_1\Im_2\hbar)\right\|. \quad (3.4)$$

Let $\varrho_0 \in \mathfrak{L}$ and the sequence $\{\varrho_r\}$ be defined by (3.1). From Proposition (1) and inequalities (3.3) and (3.4), we have

$$\begin{aligned}
\|\varsigma(\varrho_{2r+1}, \varrho_{2r})\| &= \left\|\varsigma(\Im_1\Im_2\varrho_{2r-1}, \Im_2\varrho_{2r-1})\right\| \\
&\leq \rho(\Im_2\varrho_{2r-1}, \varrho_{2r-1})\left\|\varsigma(\Im_2\varrho_{2r-1}, \varrho_{2r-1})\right\| \\
&\quad + \sqrt{2}\kappa(\Im_2\varrho_{2r-1}, \varrho_{2r-1})\left\|\varsigma(\Im_2\varrho_{2r-1}, \Im_1\Im_2\varrho_{2r-1})\right\| \\
&= \rho(\varrho_{2r}, \varrho_{2r-1})\|\varsigma(\varrho_{2r}, \varrho_{2r-1})\| + \sqrt{2}\kappa(\varrho_{2r}, \varrho_{2r-1})\|\varsigma(\varrho_{2r}, \varrho_{2r+1})\| \\
&\leq \rho(\varrho_0, \varrho_{2r-1})\|\varsigma(\varrho_{2r}, \varrho_{2r-1})\| + \sqrt{2}\kappa(\varrho_0, \varrho_{2r-1})\|\varsigma(\varrho_{2r}, \varrho_{2r+1})\| \\
&\leq \rho(\varrho_0, \varrho_1)\|\varsigma(\varrho_{2r}, \varrho_{2r-1})\| + \sqrt{2}\kappa(\varrho_0, \varrho_1)\|\varsigma(\varrho_{2r}, \varrho_{2r+1})\|
\end{aligned}$$

for all $r = 0, 1, 2, \ldots$ This implies that

$$\|\varsigma(\varrho_{2r+1}, \varrho_{2r})\| \leq \frac{\rho(\varrho_0, \varrho_1)}{1-\sqrt{2}\kappa(\varrho_0, \varrho_1)}\|\varsigma(\varrho_{2r}, \varrho_{2r-1})\|. \quad (3.5)$$

Similarly, we have

$$\begin{aligned}
\|\varsigma(\varrho_{2r+2},\varrho_{2r+1})\| &= \left\|\varsigma(\Im_2\Im_1\varrho_{2r},\Im_1\varrho_{2r})\right\| \\
&\leq \rho(\varrho_{2r},\Im_1\varrho_{2r})\left\|\varsigma(\varrho_{2r},\Im_1\varrho_{2r})\right\| \\
&\quad+\sqrt{2}\kappa(\varrho_{2r},\Im_1\varrho_{2r})\left\|\varsigma(\Im_1\varrho_{2r},\Im_2\Im_1\varrho_{2r})\right\| \\
&= \rho(\varrho_{2r},\varrho_{2r+1})\|\varsigma(\varrho_{2r},\varrho_{2r+1})\|+\sqrt{2}\kappa(\varrho_{2r},\varrho_{2r+1})\|\varsigma(\varrho_{2r+1},\varrho_{2r+2})\| \\
&\leq \rho(\varrho_0,\varrho_{2r+1})\|\varsigma(\varrho_{2r},\varrho_{2r+1})\|+\sqrt{2}\kappa(\varrho_0,\varrho_{2r+1})\|\varsigma(\varrho_{2r+1},\varrho_{2r+2})\| \\
&\leq \rho(\varrho_0,\varrho_1)\|\varsigma(\varrho_{2r},\varrho_{2r+1})\|+\sqrt{2}\kappa(\varrho_0,\varrho_1)\|\varsigma(\varrho_{2r+1},\varrho_{2r+2})\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|\varsigma(\varrho_{2r+2},\varrho_{2r+1})\| &\leq \frac{\rho(\varrho_0,\varrho_1)}{1-\sqrt{2}\kappa(\varrho_0,\varrho_1)}\|\varsigma(\varrho_{2r},\varrho_{2r+1})\| \\
&= \frac{\rho(\varrho_0,\varrho_1)}{1-\sqrt{2}\kappa(\varrho_0,\varrho_1)}\|\varsigma(\varrho_{2r+1},\varrho_{2r})\|. \qquad (3.6)
\end{aligned}$$

Let $\lambda=\frac{\rho(\varrho_0,\varrho_1)}{1-\sqrt{2}\kappa(\varrho_0,\varrho_1)}<1$. Then from (3.5) and (3.6), we have

$$\|\varsigma(\varrho_{r+1},\varrho_r)\|\leq\lambda\|\varsigma(\varrho_r,\varrho_{r-1})\|$$

for all $r\in\mathbb{N}$. Inductively, we can construct a sequence $\{\varrho_r\}$ in $\mathfrak{L}$ such that

$$\begin{aligned}
|\varsigma(\varrho_{r+1},\varrho_r)| &\leq \lambda|\varsigma(\varrho_r,\varrho_{r-1})| \\
|\varsigma(\varrho_{r+1},\varrho_r)| &\leq \lambda^2|\varsigma(\varrho_{r-1},\varrho_{r-2})| \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
|\varsigma(\varrho_{r+1},\varrho_r)| &\leq \lambda^r|\varsigma(\varrho_1,\varrho_0)|=\lambda^r|\varsigma(\varrho_0,\varrho_1)|
\end{aligned}$$

for all $r\in\mathbb{N}$. Now for $m>r$, we get

$$\begin{aligned}
\|\varsigma(\varrho_r,\varrho_m)\| &\leq \lambda^r\|\varsigma(\varrho_0,\varrho_1)\| \\
&\quad+\lambda^{r+1}\|\varsigma(\varrho_0,\varrho_1)\| \\
&\quad+\cdots+ \\
&\quad\lambda^{m-1}\|\varsigma(\varrho_0,\varrho_1)\| \\
&\leq \frac{\lambda^r}{1-\lambda}\|\varsigma(\varrho_0,\varrho_1)\|.
\end{aligned}$$

Now, by taking $r,m\to\infty$, we get

$$\|\varsigma(\varrho_r,\varrho_m)\|\to 0.$$

By Lemma 2, $\{\varrho_r\}$ is a Cauchy sequence. As $\mathfrak{L}$ is complete, so there exists $\varrho^*\in\mathfrak{L}$ such that $\varrho_r\to\varrho^*$ as $r\to\infty$. □

Now, we show that ϱ^* is a fixed point of $\Im_1$. From (3.2), we have

$$
\begin{aligned}
\varsigma\left(\varrho^{*},\Im_{1}\varrho^{*}\right) &\preceq_{i_2} \varsigma\left(\varrho^{*},\Im_{2}\varrho_{2r+1}\right)+\varsigma\left(\Im_{2}\varrho_{2r+1},\Im_{1}\varrho^{*}\right)\\
&=\varsigma\left(\varrho^{*},\Im_{2}\varrho_{2r+1}\right)+\varsigma\left(\Im_{1}\varrho^{*},\Im_{2}\varrho_{2r+1}\right)\\
&\preceq_{i_2}\left(\begin{array}{c}\varsigma\left(\varrho^{*},\varrho_{2r+2}\right)+\rho\left(\varrho^{*},\varrho_{2r+1}\right)\varsigma\left(\varrho^{*},\varrho_{2r+1}\right)\\ +\kappa\left(\varrho^{*},\varrho_{2r+1}\right)\frac{\varsigma\left(\varrho^{*},\Im_{1}\varrho^{*}\right)\varsigma\left(\varrho_{2r+1},\Im_{2}\varrho_{2r+1}\right)}{1+\varsigma\left(\varrho^{*},\varrho_{2r+1}\right)}\\ +\varpi\left(\varrho^{*},\varrho_{2r+1}\right)\frac{\varsigma\left(\varrho_{2r+1},\Im_{1}\varrho^{*}\right)\varsigma\left(\varrho^{*},\Im_{2}\varrho_{2r+1}\right)}{1+\varsigma\left(\varrho^{*},\varrho_{2r+1}\right)}\end{array}\right)\\
&\preceq_{i_2}\left(\begin{array}{c}\varsigma\left(\varrho^{*},\varrho_{2r+2}\right)+\rho\left(\varrho^{*},\varrho_{2r+1}\right)\varsigma\left(\varrho^{*},\varrho_{2r+1}\right)\\ +\kappa\left(\varrho^{*},\varrho_{2r+1}\right)\frac{\varsigma\left(\varrho^{*},\Im_{1}\varrho^{*}\right)\varsigma\left(\varrho_{2r+1},\varrho_{2r+2}\right)}{1+\varsigma\left(\varrho^{*},\varrho_{2r+1}\right)}\\ +\varpi\left(\varrho^{*},\varrho_{2r+1}\right)\frac{\varsigma\left(\varrho_{2r+1},\Im_{1}\varrho^{*}\right)\varsigma\left(\varrho^{*},\varrho_{2r+2}\right)}{1+\varsigma\left(\varrho^{*},\varrho_{2r+1}\right)}\end{array}\right).
\end{aligned}
$$

This implies that

$$
\left\|\varsigma\left(\varrho^{*},\Im_{1}\varrho^{*}\right)\right\|\leq\left(\begin{array}{c}\left\|\varsigma\left(\varrho^{*},\varrho_{2r+2}\right)\right\|+\rho\left(\varrho^{*},\varrho_{2r+1}\right)\left\|\varsigma\left(\varrho^{*},\varrho_{2r+1}\right)\right\|\\ +\sqrt{2}\kappa\left(\varrho^{*},\varrho_{2r+1}\right)\frac{\left\|\varsigma\left(\varrho^{*},\Im_{1}\varrho^{*}\right)\right\|\left\|\varsigma\left(\varrho_{2r+1},\varrho_{2r+2}\right)\right\|}{\left\|1+\varsigma\left(\varrho^{*},\varrho_{2r+1}\right)\right\|}\\ +\sqrt{2}\varpi\left(\varrho^{*},\varrho_{2r+1}\right)\frac{\left\|\varsigma\left(\varrho_{2r+1},\Im_{1}\varrho^{*}\right)\right\|\left\|\varsigma\left(\varrho^{*},\varrho_{2r+2}\right)\right\|}{\left\|1+\varsigma\left(\varrho^{*},\varrho_{2r+1}\right)\right\|}\end{array}\right).
$$

Letting $r\to\infty$, we have $\left\|\varsigma\left(\varrho^{*},\Im_{1}\varrho^{*}\right)\right\|=0$. Thus $\varrho^{*}=\Im_{1}\varrho^{*}$. Now we prove that ϱ^{*} is a fixed point of $\Im_{2}$. By (3.2), we have

$$
\begin{aligned}
\varsigma\left(\varrho^{*},\Im_{2}\varrho^{*}\right) &\preceq_{i_2}\left(\varsigma\left(\varrho^{*},\Im_{1}\varrho_{2r}\right)+\varsigma\left(\Im_{1}\varrho_{2r},\Im_{2}\varrho^{*}\right)\right)\\
&\preceq_{i_2}\left(\begin{array}{c}\varsigma\left(\varrho^{*},\Im_{1}\varrho_{2r}\right)+\rho\left(\varrho_{2r},\varrho^{*}\right)\varsigma\left(\varrho_{2r},\varrho^{*}\right)\\ +\kappa\left(\varrho_{2r},\varrho^{*}\right)\frac{\varsigma\left(\varrho_{2r},\Im_{1}\varrho_{2r}\right)\varsigma\left(\varrho^{*},\Im_{2}\varrho^{*}\right)}{1+\varsigma\left(\varrho_{2r},\varrho^{*}\right)}\\ +\varpi\left(\varrho_{2r},\varrho^{*}\right)\frac{\varsigma\left(\varrho^{*},\Im_{1}\varrho_{2r}\right)\varsigma\left(\varrho_{2r},\Im_{2}\varrho^{*}\right)}{1+\varsigma\left(\varrho_{2r},\varrho^{*}\right)}\end{array}\right)\\
&\preceq_{i_2}\left(\begin{array}{c}\varsigma\left(\varrho^{*},\varrho_{2r+1}\right)+\rho\left(\varrho_{2r},\varrho^{*}\right)\varsigma\left(\varrho_{2r},\varrho^{*}\right)\\ +\kappa\left(\varrho_{2r},\varrho^{*}\right)\frac{\varsigma\left(\varrho_{2r},\varrho_{2r+1}\right)\varsigma\left(\varrho^{*},\Im_{2}\varrho^{*}\right)}{1+\varsigma\left(\varrho_{2r},\varrho^{*}\right)}\\ +\varpi\left(\varrho_{2r},\varrho^{*}\right)\frac{\varsigma\left(\varrho^{*},\varrho_{2r+1}\right)\varsigma\left(\varrho_{2r},\Im_{2}\varrho^{*}\right)}{1+\varsigma\left(\varrho_{2r},\varrho^{*}\right)}\end{array}\right).
\end{aligned}
$$

This implies that

$$
\left\|\varsigma\left(\varrho^{*},\Im_{2}\varrho^{*}\right)\right\|\leq\left(\begin{array}{c}\left\|\varsigma\left(\varrho^{*},\varrho_{2r+1}\right)\right\|+\rho\left(\varrho_{2r},\varrho^{*}\right)\left\|\varsigma\left(\varrho_{2r},\varrho^{*}\right)\right\|\\ +\sqrt{2}\kappa\left(\varrho_{2r},\varrho^{*}\right)\frac{\left\|\varsigma\left(\varrho_{2r},\varrho_{2r+1}\right)\right\|\left\|\varsigma\left(\varrho^{*},\Im_{2}\varrho^{*}\right)\right\|}{\left\|1+\varsigma\left(\varrho_{2r},\varrho^{*}\right)\right\|}\\ +\sqrt{2}\varpi\left(\varrho_{2r},\varrho^{*}\right)\frac{\left\|\varsigma\left(\varrho^{*},\varrho_{2r+1}\right)\right\|\left\|\varsigma\left(\varrho_{2r},\Im_{2}\varrho^{*}\right)\right\|}{\left\|1+\varsigma\left(\varrho_{2r},\varrho^{*}\right)\right\|}\end{array}\right).
$$

Letting $r\to\infty$, we have $\left\|\varsigma\left(\varrho^{*},\Im_{2}\varrho^{*}\right)\right\|=0$. Thus $\varrho^{*}=\Im_{2}\varrho^{*}$. Thus ϱ^{*} is a common fixed point of $\Im_{1}$ and $\Im_{2}$. Now we prove that ϱ^{*} is unique. We suppose that

$$
\varrho^{/}=\Im_{1}\varrho^{/}=\Im_{2}\varrho^{/},
$$

but $\varrho^{*}\neq\varrho^{/}$. Now from (3.2), we have

$$\varsigma\left(\varrho^*,\varrho^/\right)=\varsigma\left(\Im_1\varrho^*,\Im_2\varrho^/\right)$$

$$\preceq_{i_2}\rho\left(\varrho^*,\varrho^/\right)\varsigma\left(\varrho^*,\varrho^/\right)+\kappa\left(\varrho^*,\varrho^/\right)\frac{\varsigma\left(\varrho^*,\Im\varrho^*\right)\varsigma\left(\varrho^/,\Im_2\varrho^/\right)}{1+\varsigma\left(\varrho^*,\varrho^/\right)}$$

$$+\varpi\left(\varrho^*,\varrho^/\right)\frac{\varsigma\left(\varrho^/,\Im_1\varrho^*\right)\varsigma\left(\varrho^*,\Im_2\varrho^/\right)}{1+\varsigma\left(\varrho^*,\varrho^/\right)}$$

$$=\rho\left(\varrho^*,\varrho^/\right)\varsigma\left(\varrho^*,\varrho^/\right)+\kappa\left(\varrho^*,\varrho^/\right)\frac{\varsigma\left(\varrho^*,\varrho^*\right)\varsigma\left(\varrho^/,\varrho^/\right)}{1+\varsigma\left(\varrho^*,\varrho^/\right)}$$

$$+\varpi\left(\varrho^*,\varrho^/\right)\frac{\varsigma\left(\varrho^/,\varrho^*\right)\varsigma\left(\varrho^*,\varrho^/\right)}{1+\varsigma\left(\varrho^*,\varrho^/\right)}.$$

This implies that

$$\begin{aligned}\left\|\varsigma\left(\varrho^*,\varrho^/\right)\right\| &\leq \rho\left(\varrho^*,\varrho^/\right)\left\|\varsigma\left(\varrho^*,\varrho^/\right)\right\| \\ &\quad+\sqrt{2}\varpi\left(\varrho^*,\varrho^/\right)\left\|\varsigma\left(\varrho^*,\varrho^/\right)\right\|\left\|\frac{\varsigma\left(\varrho^*,\varrho^/\right)}{1+\varsigma\left(\varrho^*,\varrho^/\right)}\right\| \\ &\leq \rho\left(\varrho^*,\varrho^/\right)\left\|\varsigma\left(\varrho^*,\varrho^/\right)\right\|+\sqrt{2}\varpi\left(\varrho^*,\varrho^/\right)\left\|\varsigma\left(\varrho^*,\varrho^/\right)\right\| \\ &= \left(\rho\left(\varrho^*,\varrho^/\right)+\sqrt{2}\varpi\left(\varrho^*,\varrho^/\right)\right)\left\|\varsigma\left(\varrho^*,\varrho^/\right)\right\|.\end{aligned}$$

As $\rho\left(\varrho^*,\varrho^/\right)+\sqrt{2}\varpi\left(\varrho^*,\varrho^/\right)<1$, we have

$$\left\|\varsigma\left(\varrho^*,\varrho^/\right)\right\|=0.$$

Thus $\varrho^*=\varrho^/$.

Corollary 1. *Let* $(\mathfrak{L},\varsigma)$ *be a complete bi CVMS and* $\Im_1,\Im_2:\mathfrak{L}\to\mathfrak{L}$. *If there exist mappings* $\rho,\kappa:\mathfrak{L}\times\mathfrak{L}\to[0,1)$ *such that*

(a) $\rho\left(\Im_2\Im_1\varrho,\hbar\right)\leq\rho\left(\varrho,\hbar\right)$ and $\rho\left(\varrho,\Im_1\Im_2\hbar\right)\leq\rho\left(\varrho,\hbar\right)$,
$\kappa\left(\Im_2\Im_1\varrho,\hbar\right)\leq\kappa\left(\varrho,\hbar\right)$ and $\kappa\left(\varrho,\Im_1\Im_2\hbar\right)\leq\kappa\left(\varrho,\hbar\right)$,
(b) $\rho\left(\varrho,\hbar\right)+\kappa\left(\varrho,\hbar\right)<1$,
(c) $\varsigma\left(\Im_1\varrho,\Im_2\hbar\right)\preceq_{i_2}\rho\left(\varrho,\hbar\right)\varsigma\left(\varrho,\hbar\right)+\kappa\left(\varrho,\hbar\right)\frac{\varsigma\left(\varrho,\Im_1\varrho\right)\varsigma\left(\hbar,\Im_2\hbar\right)}{1+\varsigma\left(\varrho,\hbar\right)}$,
for all $\varrho,\hbar\in\mathfrak{L}$, then $\Im_1$ and $\Im_2$ have a unique common fixed point.

Proof. Setting $\varpi:\mathfrak{L}\times\mathfrak{L}\to[0,1)$ by $\varpi\left(\varrho,\hbar\right)=0$ in Theorem 1. □

Corollary 2. *Let* $(\mathfrak{L},\varsigma)$ *be a complete bi CVMS and* $\Im_1,\Im_2:\mathfrak{L}\to\mathfrak{L}$. *If there exist mappings* $\rho,\varpi:\mathfrak{L}\times\mathfrak{L}\to[0,1)$ *such that for all* $\varrho,\hbar\in\mathfrak{L}$,

(a) $\rho(\Im_2\Im_1\varrho,\hbar) \leq \rho(\varrho,\hbar)$ and $\rho(\varrho,\Im_1\Im_2\hbar) \leq \rho(\varrho,\hbar)$,
$\varpi(\Im_2\Im_1\varrho,\hbar) \leq \varpi(\varrho,\hbar)$ and $\varpi(\varrho,\Im_1\Im_2\hbar) \leq \varpi(\varrho,\hbar)$,
(b) $\rho(\varrho,\hbar) + \varpi(\varrho,\hbar) < 1$,
(c) $\varsigma(\Im_1\varrho,\Im_2\hbar) \precsim_{i_2} \rho(\varrho,\hbar)\varsigma(\varrho,\hbar) + \varpi(\varrho,\hbar)\frac{\varsigma(\hbar,\Im_1\varrho)\varsigma(\varrho,\Im_2\hbar)}{1+\varsigma(\varrho,\hbar)}$,
then $\Im_1$ and $\Im_2$ have a unique common fixed point.

Proof. Setting $\kappa : \mathfrak{L}\times\mathfrak{L} \to [0,1)$ by $\kappa(\varrho,\hbar) = 0$ in Theorem 1. □

Corollary 3. *Let $(\mathfrak{L},\varsigma)$ be a complete bi CVMS and $\Im_1,\Im_2$: $\mathfrak{L}\to\mathfrak{L}$. If there exists mapping ρ : $\mathfrak{L}\times\mathfrak{L} \to [0,1)$ such that*

(a) $\rho(\Im_2\Im_1\varrho,\hbar) \leq \rho(\varrho,\hbar)$ and $\rho(\varrho,\Im_1\Im_2\hbar) \leq \rho(\varrho,\hbar)$,
(b) $\varsigma(\Im_1\varrho,\Im_2\hbar) \precsim_{i_2} \rho(\varrho,\hbar)\varsigma(\varrho,\hbar)$,
for all $\varrho,\hbar\in\mathfrak{L}$, then $\Im_1$ and $\Im_2$ have a unique common fixed point.

Proof. Setting $\kappa,\varpi : \mathfrak{L}\times\mathfrak{L} \to [0,1)$ by $\kappa(\varrho,\hbar) = \varpi(\varrho,\hbar) = 0$ in Theorem 1. □

Corollary 4. *Let $(\mathfrak{L},\varsigma)$ be a complete bi CVMS and $\Im : \mathfrak{L}\to\mathfrak{L}$. If there exist mappings ρ,κ,ϖ : $\mathfrak{L}\times\mathfrak{L} \to [0,1)$ such that*

(a) $\rho(\Im\varrho,\hbar) \leq \rho(\varrho,\hbar)$ and $\rho(\varrho,\Im\hbar) \leq \rho(\varrho,\hbar)$,
$\kappa(\Im\varrho,\hbar) \leq \kappa(\varrho,\hbar)$ and $\kappa(\varrho,\Im\hbar) \leq \kappa(\varrho,\hbar)$,
$\varpi(\Im\varrho,\hbar) \leq \varpi(\varrho,\hbar)$ and $\varpi(\varrho,\Im\hbar) \leq \varpi(\varrho,\hbar)$,
(b) $\rho(\varrho,\hbar) + \kappa(\varrho,\hbar) + \varpi(\varrho,\hbar) < 1$,
(c) $\varsigma(\Im\varrho,\Im\hbar) \precsim_{i_2} \rho(\varrho,\hbar)\varsigma(\varrho,\hbar) + \kappa(\varrho,\hbar)\frac{\varsigma(\varrho,\Im\varrho)\varsigma(\hbar,\Im\hbar)}{1+\varsigma(\varrho,\hbar)} + \varpi(\varrho,\hbar)\frac{\varsigma(\hbar,\Im\varrho)\varsigma(\varrho,\Im\hbar)}{1+\varsigma(\varrho,\hbar)}$,
for all $\varrho,\hbar\in\mathfrak{L}$, then $\Im$ has a unique fixed point.

Proof. Setting $\Im_1 = \Im_2 = \Im$ in Theorem 1. □

Example 2. Let $\mathfrak{L} = [0,1]$ and $\varsigma : \mathfrak{L}\times\mathfrak{L} \to \mathbb{C}$ defined by

$$\varsigma(\varrho,\hbar) = |\varrho-\hbar| + i_2|\varrho-\hbar|$$

for all $\varrho,\hbar\in\mathfrak{L}$. Then $(\mathfrak{L},\varsigma)$ is a complete bi CVMS. Define $\Im_1,\Im_2 : \mathfrak{L}\to\mathfrak{L}$ by

$$\Im_1\varrho = \frac{\varrho}{5} \text{ and } \Im_2\varrho = \frac{\varrho}{4}.$$

Consider

$$\rho,\kappa,\varpi : \mathfrak{L}\times\mathfrak{L} \to [0,1)$$

by

$$\rho(\varrho,\hbar) = \frac{\varrho}{3} + \frac{\hbar}{4}$$

and

$$\kappa(\varrho,\hbar) = \frac{\varrho^2\hbar^2}{30}$$

and

$$\varpi(\varrho,\hbar) = \frac{\varrho^2}{9} + \frac{\hbar^2}{16}.$$

Then evidently,

$$\rho(\varrho,\hbar)+\kappa(\varrho,\hbar)+\varpi(\varrho,\hbar)<1.$$

Now

$$\rho(\Im_2\Im_1\varrho,\hbar)=\rho\left(\Im_2(\frac{\varrho}{5}),\hbar\right)=\rho\left(\frac{\varrho}{20},\hbar\right)=\frac{\varrho}{60}+\frac{\hbar}{4}\leq\frac{\varrho}{3}+\frac{\hbar}{4}=\rho(\varrho,\hbar)$$

and

$$\rho(\varrho,\Im_1\Im_2\hbar)=\rho\left(\varrho,\Im_1(\frac{\hbar}{4})\right)=\rho\left(\varrho,\frac{\hbar}{20}\right)=\frac{\varrho}{3}+\frac{\hbar}{80}\leq\frac{\varrho}{3}+\frac{\hbar}{4}=\rho(\varrho,\hbar).$$

Also,

$$\kappa(\Im_2\Im_1\varrho,\hbar)=\kappa\left(\Im_2(\frac{\varrho}{5}),\hbar\right)=\kappa\left(\frac{\varrho}{20},\hbar\right)=\frac{\varrho^2\hbar^2}{12000}\leq\frac{\varrho^2\hbar^2}{30}=\kappa(\varrho,\hbar)$$

and

$$\kappa(\varrho,\Im_1\Im_2\hbar)=\kappa\left(\varrho,\Im_1(\frac{\hbar}{4})\right)=\kappa\left(\varrho,\frac{\hbar}{20}\right)=\frac{\varrho^2\hbar^2}{12000}\leq\frac{\varrho^2\hbar^2}{30}=\kappa(\varrho,\hbar)$$

and

$$\varpi(\Im_2\Im_1\varrho,\hbar)=\varpi\left(\Im_2(\frac{\varrho}{5}),\hbar\right)=\varpi\left(\frac{\varrho}{20},\hbar\right)=\frac{\varrho^2}{3600}+\frac{\hbar^2}{16}\leq\frac{\varrho^2}{9}+\frac{\hbar^2}{16}=\varpi(\varrho,\hbar)$$

and

$$\varpi(\varrho,\Im_1\Im_2\hbar)=\varpi\left(\varrho,\Im_1(\frac{\hbar}{4})\right)=\varpi\left(\varrho,\frac{\hbar}{20}\right)=\frac{\varrho^2}{9}+\frac{\hbar^2}{6400}\leq\frac{\varrho^2}{9}+\frac{\hbar^2}{16}=\varpi(\varrho,\hbar).$$

Now

$$\begin{aligned}\varsigma(\Im_1\varrho,\Im_2\hbar) &= \varsigma(\frac{\varrho}{5},\frac{\hbar}{4})=\left|\frac{\varrho}{5}-\frac{\hbar}{4}\right|+i_2\left|\frac{\varrho}{5}-\frac{\hbar}{4}\right| \\ &= \left|\frac{4\varrho-5\hbar}{20}\right|+i_2\left|\frac{4\varrho-5\hbar}{20}\right|\end{aligned}$$

$$\begin{aligned}&\preceq_{i_2}\left|\frac{4\varrho-4\hbar}{20}\right|+i_2\left|\frac{4\varrho-4\hbar}{20}\right| \\ &=\frac{1}{5}\left(|\varrho-\hbar|+i_2|\varrho-\hbar|\right) \\ &\preceq_{i_2}\frac{7}{12}\left(|\varrho-\hbar|+i_2|\varrho-\hbar|\right) \\ &\preceq_{i_2}\rho(\varrho,\hbar)\varsigma(\varrho,\hbar)+\kappa(\varrho,\hbar)\frac{\varsigma(\varrho,\Im_1\varrho)\varsigma(\hbar,\Im_2\hbar)}{1+\varsigma(\varrho,\hbar)} \\ &\quad+\varpi(\varrho,\hbar)\frac{\varsigma(\hbar,\Im_1\varrho)\varsigma(\varrho,\Im_2\hbar)}{1+\varsigma(\varrho,\hbar)}.\end{aligned}$$

Then it is very simple to prove that all the conditions of Theorem 1 are satisfied and 0 is a common fixed point of mappings $\Im_1$ and $\Im_2$.

Corollary 5. *Let* $(\mathfrak{L},\varsigma)$ *be a complete bi CVMS and let* $\Im:\mathfrak{L}\to\mathfrak{L}$. *If there exist* $\rho,\kappa,\varpi:\mathfrak{L}\times\mathfrak{L}\to[0,1)$ *such that*

(a) $\rho(\Im\varrho,\hbar) \leq \rho(\varrho,\hbar)$ and $\rho(\varrho,\Im\hbar) \leq \rho(\varrho,\hbar)$,
$\kappa(\Im\varrho,\hbar) \leq \kappa(\varrho,\hbar)$ and $\kappa(\varrho,\Im\hbar) \leq \kappa(\varrho,\hbar)$,
$\varpi(\Im\varrho,\hbar) \leq \varpi(\varrho,\hbar)$ and $\varpi(\varrho,\Im\hbar) \leq \varpi(\varrho,\hbar)$,
(b) $\rho(\varrho,\hbar) + \kappa(\varrho,\hbar) + \varpi(\varrho,\hbar) < 1$,

$$(c)\ \varsigma(\Im^n\varrho,\Im^n\hbar) \preceq_{i_2} \rho(\varrho,\hbar)\varsigma(\varrho,\hbar) + \kappa(\varrho,\hbar)\frac{\varsigma(\varrho,\Im^n\varrho)\varsigma(\hbar,\Im^n\hbar)}{1+\varsigma(\varrho,\hbar)} + \varpi(\varrho,\hbar)\frac{\varsigma(\hbar,\Im^n\varrho)\varsigma(\varrho,\Im^n\hbar)}{1+\varsigma(\varrho,\hbar)}, \tag{3.7}$$

for all $\varrho,\hbar \in \mathfrak{L}$, then $\Im$ has a unique fixed point.

Proof. From the Corollary (4), we have $\varrho \in \mathfrak{L}$ such that $\Im^n\varrho = \varrho$. Now from

$$\begin{aligned}
\varsigma(\Im\varrho,\varrho) &= \varsigma(\Im\Im^n\varrho,\Im^n\varrho) \\
&= \varsigma(\Im^n\Im\varrho,\Im^n\varrho) \preceq \rho(\Im\varrho,\varrho)\varsigma(\Im\varrho,\varrho) + \kappa(\Im\varrho,\varrho)\frac{\varsigma(\Im\varrho,\Im^n\Im\varrho)\varsigma(\varrho,\Im^n\varrho)}{1+\varsigma(\Im\varrho,\varrho)} \\
&\quad +\varpi(\Im\varrho,\varrho)\frac{\varsigma(\varrho,\Im^n\Im\varrho)\varsigma(\Im\varrho,\Im^n\varrho)}{1+\varsigma(\Im\varrho,\varrho)} \\
&\preceq_{i_2} \rho(\Im\varrho,\varrho)\varsigma(\Im\varrho,\varrho) + \kappa(\Im\varrho,\varrho)\frac{\varsigma(\Im\varrho,\Im\varrho)\varsigma(\varrho,\varrho)}{1+\varsigma(\Im\varrho,\varrho)} + \varpi(\Im\varrho,\varrho)\frac{\varsigma(\varrho,\Im\varrho)\varsigma(\Im\varrho,\varrho)}{1+\varsigma(\Im\varrho,\varrho)} \\
&= \rho(\Im\varrho,\varrho)\varsigma(\Im\varrho,\varrho) + \varpi(\Im\varrho,\varrho)\frac{\varsigma(\varrho,\Im\varrho)\varsigma(\Im\varrho,\varrho)}{1+\varsigma(\Im\varrho,\varrho)}
\end{aligned}$$

which implies that

$$\begin{aligned}
\left\|\varsigma(\Im\varrho,\varrho)\right\| &\leq \rho(\Im\varrho,\varrho)\left\|\varsigma(\Im\varrho,\varrho)\right\| + \varpi(\Im\varrho,\varrho)\left\|\varsigma(\varrho,\Im\varrho)\right\|\left\|\frac{\varsigma(\Im\varrho,\varrho)}{1+\varsigma(\Im\varrho,\varrho)}\right\| \\
&\leq \rho(\Im\varrho,\varrho)\left\|\varsigma(\Im\varrho,\varrho)\right\| + \varpi(\Im\varrho,\varrho)\left\|\varsigma(\varrho,\Im\varrho)\right\| \\
&= (\rho(\Im\varrho,\varrho) + \varpi(\Im\varrho,\varrho))\left\|\varsigma(\varrho,\Im\varrho)\right\|
\end{aligned}$$

which is possible only whenever $\left|\varsigma(\Im\varrho,\varrho)\right| = 0$. Thus $\Im\varrho = \varrho$. □

4. Deduced results

Corollary 6. *Let $(\mathfrak{L},\varsigma)$ be a complete bi CVMS and let $\Im_1,\Im_2 : \mathfrak{L} \to \mathfrak{L}$. If there exist $\rho,\kappa,\varpi : \mathfrak{L} \to [0,1)$ such that for all $\varrho,\hbar \in \mathfrak{L}$,*

(a) $\rho(\Im_2\Im_1\varrho) \leq \rho(\varrho)$,
$\kappa(\Im_2\Im_1\varrho) \leq \kappa(\varrho)$,
$\varpi(\Im_2\Im_1\varrho) \leq \varpi(\varrho)$,
(b) $\rho(\varrho) + \kappa(\varrho) + \varpi(\varrho) < 1$,
(c) $\varsigma(\Im_1\varrho,\Im_2\hbar) \preceq_{i_2} \rho(\varrho)\varsigma(\varrho,\hbar) + \kappa(\varrho)\frac{\varsigma(\varrho,\Im_1\varrho)\varsigma(\hbar,\Im_2\hbar)}{1+\varsigma(\varrho,\hbar)} + \varpi(\varrho)\frac{\varsigma(\hbar,\Im_1\varrho)\varsigma(\varrho,\Im_2\hbar)}{1+\varsigma(\varrho,\hbar)}$,

Corollary 7. *Let $(\mathfrak{L},\varsigma)$ be a complete bi CVMS and let $\Im_1,\Im_2 : \mathfrak{L} \to \mathfrak{L}$. If there exist $\rho,\kappa,\varpi : \mathfrak{L} \to [0,1)$ such that for all $\varrho,\hbar \in \mathfrak{L}$,*

(a) $\rho(\Im_2\Im_1\varrho) \leq \rho(\varrho)$,
$\kappa(\Im_2\Im_1\varrho) \leq \kappa(\varrho)$,
$\varpi(\Im_2\Im_1\varrho) \leq \varpi(\varrho)$,
(b) $\rho(\varrho) + \kappa(\varrho) + \varpi(\varrho) < 1$,
(c) $\varsigma(\Im_1\varrho, \Im_2\hbar) \preceq_{i_2} \rho(\varrho)\varsigma(\varrho,\hbar) + \kappa(\varrho)\frac{\varsigma(\varrho,\Im_1\varrho)\varsigma(\hbar,\Im_2\hbar)}{1+\varsigma(\varrho,\hbar)} + \varpi(\varrho)\frac{\varsigma(\hbar,\Im_1\varrho)\varsigma(\varrho,\Im_2\hbar)}{1+\varsigma(\varrho,\hbar)}$,
then $\Im_1$ and $\Im_2$ have a unique common fixed point.

Proof. Define $\rho,\kappa,\varpi : \mathfrak{L}\times\mathfrak{L} \to [0,1)$ by

$$\rho(\varrho,\hbar) = \rho(\varrho), \quad \kappa(\varrho,\hbar) = \kappa(\varrho) \quad \text{and} \quad \varpi(\varrho,\hbar) = \varpi(\varrho)$$

for all $\varrho,\hbar \in \mathfrak{L}$. Then for all $\varrho,\hbar \in \mathfrak{L}$, we have
(a) $\rho(\Im_2\Im_1\varrho,\hbar) = \rho(\Im_2\Im_1\varrho) \leq \rho(\varrho) = \rho(\varrho,\hbar)\, and \rho(\varrho,\Im_1\Im_2\hbar) = \rho(\varrho) = \rho(\varrho,\hbar)$,
$\kappa(\Im_2\Im_1\varrho,\hbar) = \kappa(\Im_2\Im_1\varrho) \leq \kappa(\varrho) = \kappa(\varrho,\hbar)\, and \kappa(\varrho,\Im_1\Im_2\hbar) = \kappa(\varrho) = \kappa(\varrho,\hbar)$,
$\varpi(\Im_2\Im_1\varrho,\hbar) = \varpi(\Im_2\Im_1\varrho) \leq \varpi(\varrho) = \varpi(\varrho,\hbar)$ and $\varpi(\varrho,\Im_1\Im_2\hbar) = \varpi(\varrho) = \varpi(\varrho,\hbar)$,
(b) $\rho(\varrho,\hbar) + \kappa(\varrho,\hbar) + \varpi(\varrho,\hbar) = \rho(\varrho) + \kappa(\varrho) + \varpi(\varrho) < 1$,
(c) $\varsigma(\Im_1\varrho, \Im_2\hbar) \preceq_{i_2} \rho(\varrho)\varsigma(\varrho,\hbar) + \kappa(\varrho)\frac{\varsigma(\varrho,\Im_1\varrho)\varsigma(\hbar,\Im_2\hbar)}{1+\varsigma(\varrho,\hbar)} + \varpi(\varrho)\frac{\varsigma(\hbar,\Im_1\varrho)\varsigma(\varrho,\Im_2\hbar)}{1+\varsigma(\varrho,\hbar)}$
$=\rho(\varrho,\hbar)\varsigma(\varrho,\hbar) + \kappa(\varrho,\hbar)\frac{\varsigma(\varrho,\Im_1\varrho)\varsigma(\hbar,\Im_2\hbar)}{1+\varsigma(\varrho,\hbar)} + \varpi(\varrho,\hbar)\frac{\varsigma(\hbar,\Im_1\varrho)\varsigma(\varrho,\Im_2\hbar)}{1+\varsigma(\varrho,\hbar)}$,
(d) $\lambda = \frac{\rho(\varrho_0,\varrho_1)}{1-\kappa(\varrho_0,\varrho_1)} = \frac{\rho(\varrho_0)}{1-\kappa(\varrho_0)} < 1$.
By Theorem 1, $\Im_1$ and $\Im_2$ have a unique common fixed point. □

Corollary 8. *Let $(\mathfrak{L},\varsigma)$ be a complete bi CVMS and let $\Im_1,\Im_2 : \mathfrak{L}\to\mathfrak{L}$. If there exist $\rho,\kappa,\varpi \in [0,1)$ with $\rho+\kappa+\varpi < 1$ such that*

$$\varsigma(\Im_1\varrho, \Im_2\hbar) \preceq_{i_2} \rho\varsigma(\varrho,\hbar) + \kappa\frac{\varsigma(\varrho,\Im_1\varrho)\varsigma(\hbar,\Im_2\hbar)}{1+\varsigma(\varrho,\hbar)} + \varpi\frac{\varsigma(\hbar,\Im_1\varrho)\varsigma(\varrho,\Im_2\hbar)}{1+\varsigma(\varrho,\hbar)},$$

for all $\varrho,\hbar \in \mathfrak{L}$, then $\Im_1$ and $\Im_2$ have a unique common fixed point.

Proof. Taking $\rho(\cdot) = \rho$, $\kappa(\cdot) = \kappa$ and $\varpi(\cdot) = \varpi$ in Corollary (7). □

Corollary 9. *Let $(\mathfrak{L},\varsigma)$ be a complete bi CVMS and let $\Im_1,\Im_2 : \mathfrak{L}\to\mathfrak{L}$. If there exist $\rho,\kappa \in [0,1)$ with $\rho+\kappa < 1$ such that*

$$\varsigma(\Im_1\varrho, \Im_2\hbar) \preceq_{i_2} \rho\varsigma(\varrho,\hbar) + \kappa\frac{\varsigma(\varrho,\Im_1\varrho)\varsigma(\hbar,\Im_2\hbar)}{1+\varsigma(\varrho,\hbar)}$$

for all $\varrho,\hbar \in \mathfrak{L}$, then $\Im_1$ and $\Im_2$ have a unique common fixed point.

5. Applications

Let $\mathfrak{L} = C([a,b],\mathbb{R})$, $(a > 0)$ where $C[a,b]$ denotes the set of all real continuous functions defined on the closed interval $[a,b]$ and $d : \mathfrak{L}\times\mathfrak{L} \to \mathbb{C}_2$ be defined in this way

$$d(\varrho,\hbar) = \max_{t\in[a,b]}(1+i)(|\varrho(t) - \hbar(t)|)$$

for all $\varrho, \hbar \in \mathfrak{L}$ and $t \in [a, b]$, where $|\cdot|$ is the usual real modulus. Then $(\mathfrak{L}, d)$ is complete bi CVMS. Consider the integral equations of Urysohn type

$$\varrho(t) = \int_a^b K_1(t, s, \varrho(s))ds + g(t), \tag{5.1}$$

$$\varrho(t) = \int_a^b K_2(t, s, \varrho(s))ds + g(t), \tag{5.2}$$

where $g : [a, b] \to \mathbb{R}$ and $K_1, K_2 : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are continuous for $t \in [a, b]$. We define partial order $\preceq_{i_2}$ in $\mathbb{C}_2$ as follows $\varrho(t) \preceq_{i_2} \hbar(t)$ if and only if $\varrho \le \hbar$.

Theorem 2. *Suppose the following condition*

$$|K_1(t, s, \varrho(s)) - K_2(t, s, \hbar(s))| \le \rho(\varrho, \hbar) |\varrho(s) - \hbar(s)|$$

holds, for all $\varrho, \hbar \in \mathfrak{L}$ with $\varrho \neq \hbar$ and for some control function $\rho : \mathfrak{L} \times \mathfrak{L} \to [0, 1)$, then the integral operators defined by (5.1) and (5.2) have a unique common solution.

Proof. Define continuous mappings $\mathfrak{I}_1, \mathfrak{I}_2 \colon \mathfrak{L} \to \mathfrak{L}$ by

$$\mathfrak{I}_1\varrho(t) = \frac{1}{b-a} \int_a^b K_1(t, s, \varrho(s))ds + g(t),$$

$$\mathfrak{I}_2\varrho(t) = \frac{1}{b-a} \int_a^b K_2(t, s, \varrho(s))ds + g(t),$$

for all $t \in [a, b]$. Consider

$$\begin{aligned} d(\mathfrak{I}_1\varrho, \mathfrak{I}_2\hbar) &= \max_{t\in[a,b]} (1 + i_2) \left|\mathfrak{I}_1\varrho(t) - \mathfrak{I}_2 h(t)\right| \\ &= \max_{t\in[a,b]} (1 + i_2) \left(\frac{1}{b-a} \left|\int_a^b K_1(t, s, \varrho(s))ds - \int_a^b K_2(t, s, h(s))ds\right|\right) \\ &\preceq_{i_2} \max_{t\in[a,b]} (1 + i_2) \left(\frac{1}{b-a} \int_a^b |K_1(t, s, \varrho(s)) - K_2(t, s, h(s))| \, ds\right) \\ &\preceq_{i_2} \max_{t\in[a,b]} (1 + i_2) \left(\frac{\rho(\varrho, \hbar)}{b-a} \int_a^b |\varrho(s) - \hbar(s)| \, ds\right). \end{aligned}$$

Thus

$$d(\mathfrak{I}_1\varrho, \mathfrak{I}_2\hbar) \preceq_{i_2} \rho(\varrho, \hbar) \, d(\varrho, \hbar).$$

Now with $\kappa, \varpi : \mathfrak{L} \times \mathfrak{L} \to [0, 1)$ defined by

$$\kappa(\varrho, \hbar) = \varpi(\varrho, \hbar) = 0$$

for every $\varrho, \hbar \in \mathfrak{L}$, all the assumptions of Theorem (1) are satisfied and the integral equations (5.1) and (5.2) have a unique common solution. □

6. Conclusions

In this article, we have utilized the notion of bicomplex valued metric space (bi CVMS) and obtained common fixed point results for rational contractions involving control functions of two variables. We have derived common fixed points and fixed points of single valued mappings for contractions involving control functions of one variable and constants. We anticipate that the obtained theorems in this article will establish new relationships for those who use bi CVMS. Still there are some open problems that can be addressed in future work. For example:
1) Can the notion of bi complex valued metric space be extended to hypercomplex valued metric space?
2) Can the results proved in this article be extended to multivalued mappings and fuzzy set valued mappings [30]?
3) Can differential and integral inclusions can be solved as applications of fixed point results for multivalued mappings in the setting of bi complex valued metric space?

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Conflict of interest

The authors declare that they have no conflicts of interest.

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