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Research article

Variational approach for a Steklov problem involving nonstandard growth conditions

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Abstract: The aim of this paper is to study the multiplicity of solutions for a nonlocal $p(x)$-Kirchhoff type problem with Steklov boundary value in variable exponent Sobolev spaces. We prove the existence of at least three solutions and a nontrivial weak solution of the problem, using the Ricceri's three critical points theorem together with Mountain Pass theorem.

Keywords: variational methods; $p(x)$-Kirchhoff type equation; Steklov boundary value; Ricceri's critical points theorem; weak solution
Mathematics Subject Classification: 35J60, 47J30, 35A15

1. Introduction

In this paper, we investigate the following $p(x)-$Kirchhoff type problem

$$\begin{cases} M\left(A\left(x,\nabla u\right)\right)\ \operatorname{div}\left(a\left(x,\nabla u\right)\right)=|u|^{p(x)-2}\,u, & in\ \ \Omega, \\ a\left(x,\nabla u\right)\frac{\partial u}{\partial \nu}=\lambda f\left(x,u\right), & on\ \ \partial\Omega, \end{cases} \tag{\textbf{P}}$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 2)$ is a smooth bounded domain, λ is a positive parameter, p is continuous function on $\overline{\Omega}$ with $p^- := \inf\limits_{x\in\overline{\Omega}} p\left(x\right)$, $\operatorname{div}\left(a\left(x,\nabla u\right)\right)$ is a $p\left(x\right)-$Laplace type operator and $a(x,\xi):\overline{\Omega}\times\mathbb{R}^N\to\mathbb{R}$ is the continuous derivative with respect to ξ of the mapping $A:\overline{\Omega}\times\mathbb{R}^N\to\mathbb{R}$, $A=A(x,\xi)$, i.e. $a(x,\xi)=\nabla_\xi A(x,\xi)$. Furthermore, $f:\partial\Omega\times\mathbb{R}\to\mathbb{R}$ satisfies Carathéodory function and $M:(0,\infty)\to(0,\infty)$ is a continuous function.

Problem (**P**) is generalization of a model, the so-called Kirchhoff equation, introduced by Kirchhoff [22]. Kirchhoff established a model given by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.1}$$

where $0 \leq x \leq L$, $t \geq 0$, u is the lateral deflection, ρ is the mass density, h is the cross-sectional area, L is the length, E is the Young's modulus and P_0 is the initial axial tension. This equation is an extension of the classical D'Alambert's wave equation by considering the effects of the changes in the length of the strings during the vibrations.

Recently, equations with nonstandard growth condition have started to attract more attention due to their various physical applications. In fact, there are applications concerning image restoration [11], elastic mechanics [38], the image restoration or the motion of the so called electrorheological fluids [30], stationary thermo-rheological viscous flows of non-Newtonian fluids [3] and the mathematical description of the processes filtration of an idea barotropic gas through a porous medium [4].

As in the study of differential and partial differential equations, the investigate of Kirchhoff type equations under different boundary conditions has initially been extended to the case involving the $p-$growth conditions, and then the equations involving the $p(x)-$growth conditions. Especially, researchers have studied extensively the existence, multiplicity, uniqueness, nontrivial weak solution and regularity of solutions for various Kirchhoff type equations [5, 8, 12–14, 16, 19, 26, 34, 36]. For example, Zhang and all in [36] studied the existence of nontrivial solutions and many solutions for a nonlocal $p(x)-$Kirchhoff problem with a p^+-superlinear subcritical Caratheodory reaction term, which does not satisfy the Ambrosetti–Rabinowitz condition using Mountain Pass theorem and Fountain theorem. Cao and all in [10] established the existence of nontrivial solutions for $p(x)-$Laplacian equations without any growth and Ambrosetti-Rabinowitz conditions. In [31], the author proved the existence of positive solutions using the Nehari manifold approach in $W_0^{1,p(x)}(\Omega)$.

The Steklov problems involving $p(x)-$Laplacian have been worked by some of the authors [1, 7, 15, 32, 35]. Especially, the authors have studied the problems of type (**P**) when $M(t) = 1$. For instance, in [37], by applying the Ricceri's three critical points theorem, the authors investigated the existence of at least three solutions to the following elliptic problem:

$$\begin{cases} \operatorname{div}(a(x, \nabla u)) + |u|^{p(x)-2} u = \lambda f(x, u), & in \;\; x \in \Omega, \\ |\nabla u|^{p(x)-2} \upsilon = \mu g(x, u), & on \;\; x \in \partial\Omega, \end{cases}$$

where $\mu, \lambda \in [0, \infty)$, $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain of smooth boundary $\partial\Omega$, υ is the outward unit normal vector on $\partial\Omega$, $p(x) \in C(\Omega)$ is the variable exponent and $f : \Omega\times\mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega\times\mathbb{R} \to \mathbb{R}$ are two Carathéodory functions.

In [25], using Ricceri's variational principle and mountain pass theorem, they showed we prove in a different cases the existenceand multiplicity of a-harmonic solutions for the following elliptic problem:

$$\begin{cases} -\operatorname{div}(a(x, \nabla u)) = 0, & in \;\; x \in \Omega, \\ |\nabla u|^{p(x)-2} \upsilon = f(x, u), & on \;\; x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain of smooth boundary $\partial\Omega$ and υ is the outward unit normal vector on $\partial\Omega$. $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ and $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ satisfy appropriate conditions. In [24], the authors studied the problem (1.2) for case $f(x, u) = \lambda|u|^{m(x)-2}u$, where the functions $m(x) \in L^\infty(\partial\Omega)$.

If $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the appropriate conditions, then the authors proved infinitely many positive eigenvalue sequences via the Ljusternik–Schnirelmann principle and a new variational technique. Now, we also mention some new paper that are related to our work. In [21], the authors concerned with a nontrivial weak solution under appropriate conditions a weighted Steklov problem involving the $p(x)$ −Laplacian operator in Sobolev spaces with variable exponents by variational method and Ekeland's principle. Ourraoui in [28] proved some results on the existence and uniqueness of solutions concerned a class of elliptic problems involving $p(x)$ −Laplacian with Steklov boundary condition. In [6], the author investigated the existence and multiplicity of solutions for Steklov problem with non-standard growth condition without using the Ambrosetti-Rabinowitz type condition. In [2], the authors obtained the existence and multiplicity of solutions for the nonlinear Steklov boundary value problem, using Mountain Pass, Fountain and Ricceri three critical points theorems for $M(t) = 1$ in problem (**P**). In [20], the authors established the existence of infinitely many solutions for perturbed nonlocal problems with variable exponent and nonhomogeneous Neumann conditions using variational methods and critical point theory with $m_1 \leq M(t) \leq m_2$, where for all $s > 0$ and m_1 and m_2 are positive constants and $A(x, \nabla u) = \int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + a(x) |u|^{p(x)} \right) dx$ for problem (**P**). Chammen and all in [9] studied the existence and the multiplicity of solutions is obtained by using variational methods, and mountain pass lemma combined with Ekeland variational principle for a class of Steklov Neumann boundary value problems involving $p(x)$−Laplacian operator when $M(t) = 1$ for problem (**P**).

Inspired by the papers above mentioned, we studied the Steklov problem involving the $p(x)$ −Kirchhoff type operator. The present article is composed of three sections. In the second part, we introduce necessary notations, fundamental hypothesis and the variable exponent Lebesgue-Sobolev spaces on which we work. In the third part, after giving some basic results that will be useful for the proof of our principal theorems, we give the main theorems and their proofs.

2. Preliminaries and main results

In order to discuss problem (**P**), we review some basic properties about the variable exponent Lebesgue- Sobolev spaces ($L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$), we refer to [18, 23, 32, 34].

Set

$$C_+\left(\overline{\Omega}\right) = \left\{p : \ p \in C\left(\overline{\Omega}\right), \ p(x) > 1, \ \text{ for all } x \in \overline{\Omega}\right\}.$$

Denote $1 < p^- := \inf_{x \in \overline{\Omega}} p(x) \leq p(x) \leq p^+ := \sup_{x \in \overline{\Omega}} p(x) < \infty$ for all $p(x) \in C_+\left(\overline{\Omega}\right)$.

We define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{ u \,|u : \Omega \to \mathbb{R} \text{ is a measurable and } \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{(p(x),\Omega)} = \inf \left\{ \iota > 0 : \int_\Omega \left| \frac{u(x)}{\iota} \right|^{p(x)} dx \leq 1 \right\}.$$

Moreover, we can define $C_+(\partial\Omega)$ and p^-, p^+ for any $p(x) \in C(\partial\Omega)$, and denote

$$L^{p(x)}(\partial\Omega) = \left\{ u \,|u : \partial\Omega \to \mathbb{R} \text{ is a measureable and } \int_{\partial\Omega} |u(x)|^{p(x)} d\sigma < \infty \right\},$$

endowed with the norm

$$|u|_{L^{p(x)}(\partial\Omega)} = |u|_{(p(x),\partial\Omega)} = \inf\left\{\varrho > 0 : \int_{\partial\Omega} \left|\frac{u(x)}{\varrho}\right|^{p(x)} d\sigma \leq 1\right\}.$$

where $d\sigma$ is the measure on the boundary. Moreover, if $p_1(x)$ and $p_2(x)$ are two functions in $C_+\left(\overline{\Omega}\right)$ such that $p_1(x) \leq p_2(x)$ almost everywhere $x \in \Omega$, then there exists a continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$, and ıf $L^{p'(x)}(\Omega)$ denotes the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p'(x)} + \frac{1}{p(x)} = 1$, then we write Hölder-Type inequality

$$\int_\Omega |uv|\, dx \leq (\frac{1}{p^-} + \frac{1}{(p^-)'})\, |u|_{p(x)}\, |v|_{p'(x)}, \tag{2.1}$$

for any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$.

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : \ |\nabla u| \in L^{p(x)}(\Omega)\right\},$$

with the norm

$$\|u\|_{1,p(x)} : = \inf\left\{\zeta > 0 : \int_\Omega \left(\left|\frac{\nabla u(x)}{\zeta}\right|^{p(x)} + \left|\frac{u(x)}{\zeta}\right|^{p(x)}\right) dx \leq 1\right\},$$

or

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}, \text{ for all } u \in W^{1,p(x)}(\Omega).$$

The space $W_0^{1,p(x)}(\Omega)$ is denoted by the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm $\|u\|_{1,p(x)}$. We can define an equivalent norm

$$\|u\| = |\nabla u|_{p(x)}, \text{ for all } u \in W_0^{1,p(x)}(\Omega).$$

If $p^- > 1$ and $p^+ < \infty$, the $L^{p(x)}(\Omega)$, $L^{p(x)}(\partial\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces. An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the modular of the $L^{p(x)}(\Omega)$ space, which is the mapping $\psi(u) : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\psi(u) := \int_\Omega |u(x)|^{p(x)}\, dx, \ \forall u \in L^{p(x)}(\Omega).$$

Proposition 2.1 [18, 23]. *For all $u, u_n \in L^{p(x)}(\Omega)$ ($n = 1, 2, ...$) and $p^+ < \infty$, the following properties hold true:*

(i) $|u|_{p(x)} > 1\ (= 1, < 1) \iff \psi(u) > 1\ (= 1, < 1)$,
(ii) $\min\left(|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}\right) \leq \psi(u) \leq \max\left(|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}\right)$,
(iii) $|u_n - u|_{p(x)} \to 0\ (\to \infty) \Leftrightarrow \psi(u_n - u) \to 0\ (\to \infty)$.

Proposition 2.2 [15]. *Let $\varphi(u) = \int_{\partial\Omega} |u(x)|^{p(x)}\, d\sigma$. For all $u, u_n \in L^{p(x)}(\partial\Omega)$ ($n = 1, 2, ...$), we have*

(i) $|u|_{L^{p(x)}(\partial\Omega)} > 1 \Rightarrow |u|^{p^-}_{L^{p(x)}(\partial\Omega)} \le \varphi(u) \le |u|^{p^+}_{L^{p(x)}(\partial\Omega)}$,
(ii) $|u|_{L^{p(x)}(\partial\Omega)} < 1 \Rightarrow |u|^{p^+}_{L^{p(x)}(\partial\Omega)} \le \varphi(u) \le |u|^{p^-}_{L^{p(x)}(\partial\Omega)}$,
(iii) $|u_n - u|_{p(x)} \to 0 \ (\to \infty) \Leftrightarrow \varphi(u_n - u) \to 0 \ (\to \infty)$.

Proposition 2.3 [17]. *Let $p(x)$ and $q(x)$ be measurable functions such that $1 \le p(x)q(x) \le \infty$ and $p(x) \in L^\infty(\Omega)$ for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$, $u \ne 0$. Then*

$$\min\left(|u|^{p^-}_{p(x)q(x)}, |u|^{p^+}_{p(x)q(x)}\right) \le \left||u|^{p(x)}\right|_{q(x)} \le \max\left(|u|^{p^-}_{p(x)q(x)}, |u|^{p^+}_{p(x)q(x)}\right).$$

In particular, if $p(x) = p$ is constant, then we have

$$||u|^p|_{q(x)} = |u|^p_{pq(x)}.$$

Proposition 2.4 [17, 23, 32].

(i) if $q(x) \in C_+\left(\overline{\Omega}\right)$ and $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous, where

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } N > p(x), \\ \infty, & \text{if } N \le p(x), \end{cases}$$

(ii) if $q(x) \in C_+(\partial\Omega)$ and $q(x) < p^\partial(x)$ for all $x \in \partial\Omega$, then the trace embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$ is compact and continuous, where

$$p^\partial(x) := \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & \text{if } N > p(x), \\ \infty, & \text{if } N \le p(x), \end{cases}$$

(iii) Poincaré inequality ; i.e.there is a positive constant $C > 0$ such that

$$|u|_{p(x)} \le C\|u\|, \textit{for all } u \in W_0^{1,p(x)}(\Omega).$$

Remark 2.5. If $N < p^- \le p(x)$ for any $x \in \overline{\Omega}$, by Theorem 2.2 in [18] and Remark 1 in [29], we deduce that $W^{1,p(x)}(\Omega)$ is continuously embedded in $W^{1,p^-}(\Omega)$. Since $N < p^-$, it follows that $W^{1,p(x)}(\Omega)$ is compactly embedded in $C\left(\overline{\Omega}\right)$. Defining $\|u\|_{C(\overline{\Omega})} = \sup_{x\in\overline{\Omega}} |u(x)|$, we find that there exists a positive constant $c_* > 0$ such that

$$\|u\|_{C(\overline{\Omega})} \le c_* \|u\|_{1,p(x)}, \ \textit{for all } u \in W^{1,p(x)}(\Omega).$$

Theorem 2.6 (Mountain-Pass Geometry) [33]. Let X be a Banach spaces and $J_\lambda \in C^1(X, \mathbb{R})$ satisfies Palais-Smale condition. Assume that $J_\lambda(0) = 0$, and there exist two positive real numbers η and r such that

(i) There exist two positive real numbers η and r such that $J_\lambda(u) \ge r > 0$ with $\|u\| = \eta$,
(ii) There exists $u_1 \in X$ such that $\|u_1\| > \eta$ and $J_\lambda(u_1) < 0$.

Put

$$G = \{\phi \in C([0,1], X) : \phi(0) = 0, \phi(1) = u_1\}.$$

Set $\beta = \inf\{\max J_\lambda(\phi([0,1])) : \phi \in G\}$. Then $\beta \geq r$ and β is a critical value of J_λ.

Theorem 2.7 [8]. Let X be a separable and reflexive real Banach space, $\Phi : X \to \mathbb{R}$ a continuous Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ a continuous Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that,

(i) $\lim\limits_{\|u\|\to\infty} J_\lambda(u) = \lim\limits_{\|u\|\to\infty} (\Phi(u) + \lambda\Psi(u)) = \infty$ for all $\lambda > 0$,

(ii) There are $r \in \mathbb{R}$ and $u_0, u_1 \in X$ such that $\Phi(u_0) < r < \Phi(u_1)$,

(iii) $\inf_{u\in\Phi^{-1}((-\infty,r])} \Psi(u) > \frac{(\Phi(u_1)-r)\Psi(u_0)+(r-\Phi(u_0))\Psi(u_1)}{\Phi(u_1)-\Phi(u_0)}$.

Then there exist an open interval $\Theta \subset (0,\infty)$ and a positive real number ρ such that for each $\lambda \in \Theta$ the equation

$$\Phi'(u) + \lambda\Psi'(u) = 0$$

has at least three solutions in X whose norms are lees than ρ.

Throughout this paper, we consider the following assumptions:

($\mathbf{M}_1$) $M : (0,\infty) \to (0,\infty)$ is a continuous function such that

$$m_1 s^{\alpha-1} \leq M(s) \leq m_2 s^{\alpha-1}, \forall\ s > 0$$

where m_1, m_2 and α are real numbers such that $0 < m_1 \leq m_2$ and $\alpha > 1$.

(**A1**) There exists a constant $c_0 > 0$ such that satisfies the following growth condition

$$|a(x,\xi)| \leq c_0(1 + |\xi|^{p(x)-1}),\ \text{for all } x \in \overline{\Omega} \text{ and } \xi \in \mathbb{R}^N$$

(**A2**) The following inequalities hold

$$|\xi|^{p(x)} \leq a(x,\xi)\,\xi \leq p(x)\,A(x,\xi),\ \text{for all } x \in \overline{\Omega} \text{ and } \xi \in \mathbb{R}^N.$$

(**A3**) $A(x,0) = 0$, for all $x \in \overline{\Omega}$.

(**A4**) A is $p(x)$-uniformly convex: There exists a constant $k_0 > 0$ such that

$$A(x, \frac{u+\upsilon}{2}) \leq \frac{1}{2}A(x,u) + \frac{1}{2}A(x,\upsilon) - k_0\,|u-\upsilon|^{p(x)},\ \text{for all } x \in \overline{\Omega} \text{ and } u,\upsilon \in \mathbb{R}^N.$$

Our main results in this paper are the proofs of the following theorems, which are based on the Mountain Pass Theorem [33] and the Ricceri Theorem [8] . Let X denote the variable exponent Sobolev space $W_0^{1,p(x)}(\Omega)$.

Theorem 2.8. *Suppose that* $(\mathbf{M}_1), (\mathbf{A1}) - (\mathbf{A5})$, $p^+ < \alpha p^-$ *and* f *satisfies the following conditions hold*

($\mathbf{f}_1$) *There exits c_1 is a positive constant such that*

$$|f(x,t)| \leq c_1 \left(1 + |t|^{m(x)-1}\right), \forall (x,t) \in \partial\Omega \times \mathbb{R},$$

where $m(x) \in C_+(\partial\Omega)$ *such that* $p^+ < m^- := \inf_{x\in\partial\Omega} m(x) \leq m(x) \leq m^+ := \sup_{x\in\partial\Omega} m(x) < p^\partial(x)$

($\mathbf{f}_2$) $f(x,t) = o\left(|t|^{\alpha p^+ -1}\right)$ *as* $t \to 0$, *for* $x \in \partial\Omega$ *and* $\alpha p^+ < m^-$

(**AR**) *Ambrosetti-Rabinowitz's Condition holds, i.e., there exists* $M > 0$ *and* $\theta > \frac{m_2 \alpha p^+}{m_1}$ *such that*

$$0 < \theta F(x,t) \leq f(x,t)t, \ \ |t| \geq M, \text{ for all } x \in \partial\Omega.$$

Then there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, problem (**P**) has a nontrivial weak solution in X .

Theorem 2.9. Assume that (M_1), (**A1**) – (**A5**) and f satisfies the following conditions hold

($\mathbf{f}_3$) $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory condition and

$$|f(x,t)| \leq n(x) + k\,|t|^{\beta(x)-1}, \text{for all } (x,t) \in \partial\Omega \times \mathbb{R},$$

where $k \geq 0$ is a constant, $n(x) \in L^{\frac{\beta(x)}{\beta(x)-1}}(\partial\Omega)$, and $\beta(x) \in C_+(\partial\Omega)$ such that

$$1 < \beta^- := \inf_{x\in\Omega} \beta(x) \leq \beta(x) \leq \beta^+ := \sup_{x\in\overline{\Omega}} \beta(x) < p^- \text{and } N < p^-.$$

($\mathbf{f}_4$) If $|t| \in (0,1)$, then $F(x,t) < 0$ and $t \in (t_0, \infty)$ for $t_0 > 1$, then $F(x,t) > \varpi > 0$.

Then there exist an open interval $\Theta \subset (0,\infty)$ and a constant $\rho > 0$ such that for any $\lambda \in \Theta$, problem (P) has at least three weak solutions in X whose norms are less than ρ.

3. Proofs of main results

Definition 3.1. We say that $u \in X$ is a weak solution of the boundary value problem (**P**) if and only if

$$\begin{aligned} & M\left(\int_\Omega A(x,\nabla u)\right)\int_\Omega a(x,\nabla u)\nabla v + \int_\Omega |u|^{p(x)-2}\, uv dx \\ = \; & \lambda \int_{\partial\Omega} f(x,u) v d\sigma \end{aligned}$$

for any $v \in X$. We define the functionals ; $\Phi, \Psi : X \to \mathbb{R}$

$$\Phi(u) = \widehat{M}(\Lambda(u)) + \int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx$$

$$\Psi(u) = -\int_{\partial\Omega} F(x,u)\, d\sigma, \ u \in X$$

where $\widehat{M}(t)$, $\Lambda(u)$ and $F(x,t)$ are denoted by

$$\widehat{M}(t) = \int_0^t M(k)\, dk, \ \Lambda(u) = \int_\Omega A(x,\nabla u)\, dx \ \text{ and } F(x,t) = \int_0^t f(x,k)\, dk,$$

for all $t > 0$ and $(x, k) \in \partial\Omega \times \mathbb{R}$.

Proposition 3.2 [15]. *Let* $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying $(\mathbf{f}_1)$. For each $u \in X$, set $\varkappa(u) = \int_{\partial\Omega} F(x, u)\, d\sigma$. Then $\varkappa(u) \in C^1(X, \mathbb{R})$ *and*

$$\left\langle \varkappa'(u), \upsilon \right\rangle = \int_{\partial\Omega} f(x, u)\, \upsilon d\sigma,$$

for all $\upsilon \in X$. Moreover, the operator $\varkappa : X \to X^*$ is compact.

Lemma 3.3 [27].

(i) A verifies the growth condition $|A(x, \xi)| \leq c_0(|\xi| + |\xi|^{p(x)})$, for all $x \in \Omega$ and $\xi \in \mathbb{R}^N$;
(ii) A is $p(x)-$ homogeneous, $A(x, z\xi) \leq A(x, \xi)\, z^{p(x)}$, for all $z \geq 1$, $\xi \in \mathbb{R}^N$ *and* $x \in \Omega$.

Lemma 3.4 [27].

(i) The functional Λ is well-defined on X,
(ii) The functional Λ is of class $C^1(X, \mathbb{R})$ and

$$\langle \Lambda'(u), v \rangle = \int_\Omega a(x, \nabla u) .\nabla v dx, \text{ for all } u, v \in X,$$

(iii) The functional Λ is weakly lower semi-continuous on X,
(iv) For all $u, \upsilon \in X$

$$\Lambda(\frac{u + \upsilon}{2}) \leq \frac{1}{2}\Lambda(u) + \frac{1}{2}\Lambda(\upsilon) - k_0 \|u - \upsilon\|^{p^-},$$

(v) For all $u, \upsilon \in X$

$$\Lambda(u) - \Lambda(\upsilon) \geq \langle \Lambda'(\upsilon), u - \upsilon \rangle,$$

(vi) J_λ is weakly lower semi-continuous on X.

Then energy functional associated to the problem (**P**) is $J_\lambda(u) = \Phi(u) + \lambda\Psi(u)$. Furthermore, from Proposition 3.2, Lemma 3.3, Lemma 3.4, $(\mathbf{f}_1)$ and $(\mathbf{M}_1)$, it is easy to see that the functional $\Phi, \Psi \in C^1(X, \mathbb{R})$ and the derivate of J_λ is the mapping $J'_\lambda : X \to \mathbb{R}$. Then, we have

$$\langle J'_\lambda(u), \upsilon \rangle =$$

$$M\left(\int_\Omega A(x, \nabla u)\right) \int_\Omega a(x, \nabla u)\, \nabla \upsilon dx + \int_\Omega |u|^{p(x)-2}\, u\upsilon dx - \lambda \int_{\partial\Omega} f(x, u)\, v d\sigma,$$

for any $u, \upsilon \in X$ and we can infer that critical points of functional J_λ are the weak solutions for problem (**P**).

Lemma 3.5. Suppose $(\mathbf{M}_1), (\mathbf{f}_1), (\mathbf{A2}), (\mathbf{AR})$ and $\alpha p^- > p^+$ hold. Then, the functional J_λ satisfies Palais-Smale (PS) condition for any $\lambda \in (0, +\infty)$.

Proof. Let assume that there exists a sequence $\{u_n\} \subset X$ such that

$$|J_\lambda(u_n)| \leq C \text{ and } J'_\lambda(u_n) \to 0 \text{ as } n \to \infty. \tag{3.1}$$

Firstly, we prove that $\{u_n\}$ is bounded in X. Arguing by contradiction and passing to a subsequence, we have $\|u_n\| \to \infty$ as $n \to \infty$. From $(\mathbf{M}_1)$,$(\mathbf{AR})$, (3.1) and considering $\|u_n\| > 1$, for n large enough, we get

$$\begin{aligned} C + \|u_n\| &\geq J_\lambda(u_n) - \frac{1}{\theta}\langle J'_\lambda(u_n), u_n\rangle \\ &\geq \frac{m_1}{\alpha}\left(\int_\Omega A(x,\nabla u_n)\,dx\right)^\alpha - \frac{m_2 p^+}{\theta}\left(\int_\Omega A(x,\nabla u_n)\,dx\right)^{\alpha-1}\int_\Omega A(x,\nabla u_n)\,dx \\ &\quad -\frac{1}{p^+}\int_\Omega |u_n|^{p(x)}\,dx + \frac{1}{p^-}\int_\Omega |u_n|^{p(x)}\,dx - \lambda\left(\int_{\partial\Omega}\left(\frac{1}{\theta}f(x,u_n)\,u_n - F(x,u_n)\right)d\sigma\right) \\ &\geq \left(\frac{m_1}{\alpha} - \frac{m_2 p^+}{\theta}\right)\left(\int_\Omega A(x,\nabla u_n)\,dx\right)^\alpha - \frac{1}{p^+}\|u_n\|^{p^+}. \end{aligned}$$

From (**A3**) and Proposition 2.2 (ii), we have

$$C + \|u_n\| \geq \left(\frac{m_1}{\alpha} - \frac{m_2 p^+}{\theta}\right)\frac{1}{(p^+)^\alpha}\|u_n\|^{\alpha p^-} - \frac{1}{p^+}\|u_n\|^{p^+}$$

If this last inequality is divided by $\|u_n\|^{\alpha p^-}$ and pass to the limit as $n \to \infty$, we obtain a contradiction with the condition $(\mathbf{AR})$. So, $\alpha p^- > p^+$, $\{u_n\}$ is bounded in X. Thus, we may extract a subsequence $\{u_n\} \subset X$ and $u \in X$ such that $u_n \rightharpoonup u$ in X.

Next, we will show that $u_n \to u$ in X. Taking into account relation (3.1), we obtain that $\left\langle J'_\lambda(u_n), u_n - u\right\rangle \to 0$. That is,

$$\langle J'_\lambda(u_n), u_n - u\rangle = \tag{3.2}$$

$$M\left(\int_\Omega A(x,\nabla u_n)\,dx\right)\int_\Omega a(x,\nabla u_n)(\nabla u_n - \nabla u)\,dx$$

$$-\int_{\partial\Omega} f(x,u_n)(u_n - u)\,d\sigma \to 0$$

Moreover, using $(\mathbf{f}_1)$ and the inequality (2.1), we deduce that

$$\begin{aligned} \left|\int_{\partial\Omega} f(x,u_n)(u_n-u)\,d\sigma\right| &\leq \left|\int_{\partial\Omega}\left(c_1 + c_1|u_n|^{m(x)-1}\right)(u_n - u)\;d\sigma\right| \\ &\leq c_1\int_{\partial\Omega}|(u_n - u)|\;d\sigma \\ &\quad + c_2\left||u_n|^{m(x)-1}\right|_{L^{m'(x)}}|u_n - u|_{L^{m(x)}(\partial\Omega)} \end{aligned}$$

where $c_2 > 0$ is a constant. On the other hand, from Proposition 2.3, if we consider the compact embedding $X \hookrightarrow L^{m(x)}(\partial\Omega)$, that is, $|u_n - u|_{L^{m(x)}(\partial\Omega)} \to 0$ as $n \to \infty$, we obtain

$$\int_{\partial\Omega} f(x, u_n)(u_n - u)\, d\sigma \to 0. \tag{3.3}$$

So, we use (3.3) in the above inequality (3.2), we have

$$M\left(\int_\Omega A(x, \nabla u_n)\, dx\right)\int_\Omega a(x, \nabla u_n)(\nabla u_n - \nabla u)\, dx \to 0.$$

Moreover, from ($\mathbf{M}_1$), we conclude that

$$\int_\Omega a(x, \nabla u_n)(\nabla u_n - \nabla u)\, dx \to 0.$$

that is, $\lim_{n\to\infty} \langle \Lambda'(u_n), u_n - u\rangle = 0$. From Lemma 3.4 (v), we write

$$0 = \lim_{n\to\infty} \langle \Lambda'(u_n), u - u_n\rangle \leq \lim_{n\to\infty} (\Lambda(u) - \Lambda(u_n)) = \Lambda(u) - \lim_{n\to\infty}\Lambda(u_n)$$

or

$$\lim_{n\to\infty}\Lambda(u_n) \leq \Lambda(u). \tag{3.4}$$

Thus, from Lemma 3.4 (iii) and the above inequality (3.4), we have

$$\lim_{n\to\infty}\Lambda(u_n) = \Lambda(u). \tag{3.5}$$

Now, we assume by contradiction that $\{u_n\}$ does not converge strongly to u in X. Then, there exists $\xi > 0$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\left\|u_{n_k} - u\right\| \geq \epsilon$. Moreover, by Lemma 3.4 (iv), we get

$$\frac{1}{2}\Lambda(u) + \frac{1}{2}\Lambda(u_{n_k}) - \Lambda(\frac{u_{n_k} + u}{2}) \geq k_0 \left\|u_{n_k} - u\right\|^{p^-} \geq k_0\epsilon^{p^-}. \tag{3.6}$$

Letting $k \to \infty$ in the inequality (3.6) and and using the inequalities (3.5), we have

$$\limsup_{n\to\infty}\Lambda(\frac{u_{n_k} + u}{2}) \leq \Lambda(u) - k_0\epsilon^{p^-}.$$

Moreover, we have $\{\frac{u_{n_k} + u}{2}\}$ converges weakly to u in X. Using Lemma 3.4 (iii), we obtain

$$\Lambda(u) \leq \liminf_{n\to\infty}\Lambda(\frac{u_{n_k} + u}{2}),$$

which is a contradiction. Therefore, it follows that $\{u_n\}$ converges strongly to u in X. The proof of Lemma 3.5 is complete.

Lemma 3.6. Suppose ($\mathbf{M}_1$), ($\mathbf{f}_1$), ($\mathbf{f}_2$), (**A3**)and (**AR**) hold. Then the following statements hold;

(i) There exist two positive real numbers η and r such that $J_\lambda(u) \geq r > 0$, $u \in X$ with $\|u\| = \eta$.
(ii) There exists $u \in X$ such that $\|u\| > \eta$, $J_\lambda(u) < 0$.

Proof. (i) Let $\|u\| < 1$. From $(\mathbf{f}_1)$ and $(\mathbf{f}_2)$, we obtain

$$F(x,t) \leq \varepsilon |t|^{\alpha p^+} + c_\varepsilon |t|^{m(x)}, \forall (x,t) \in \partial\Omega \times \mathbb{R}. \tag{3.7}$$

On the other hand, using the continuous embeddings $X \hookrightarrow L^{p(x)}(\Omega)$, $X \hookrightarrow L^{\alpha p^+}(\partial\Omega)$ and $X \hookrightarrow L^{m(x)}(\partial\Omega)$, there exists positive constants c_3, c_4 and c_5 such that

$$|u|_{L^{p(x)}(\Omega)} \leq c_3 \|u\|, \ |u|_{L^{m(x)}(\partial\Omega)} \leq c_4 \|u\|$$

$$\text{and } |u|_{L^{\alpha p^+}(\partial\Omega)} \leq c_5 \|u\|, \ \forall\, u \in X. \tag{3.8}$$

Therefore, by Proposition 2.1, Proposition 2.2, $(\mathbf{M}_1)$, the inequalities (3.7) and (3.8), we obtain

$$\begin{aligned} J_\lambda(u) &\geq \frac{m_1}{\alpha (p^+)^\alpha} \|u\|^{\alpha p^+} + \int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx - \lambda \int_{\partial\Omega} \left(\varepsilon |t|^{\alpha p^+} + c_\varepsilon |t|^{m(x)}\right) d\sigma \\ &\geq \frac{m_1}{\alpha (p^+)^\alpha} \|u\|^{\alpha p^+} + \frac{c_3^{p^+}}{p^+} \|u\|^{p^-} - \lambda \varepsilon c_4^{\alpha p^+} \|u\|^{\alpha p^+} - \lambda c_\varepsilon c_5^{m^-} \|u\|^{m^-}, \end{aligned}$$

for $\|u\|$ small enought. Let $\varepsilon > 0$ be small enough such that $0 < \lambda \varepsilon c_4^{\alpha p^+} \leq \left(\frac{m_1}{2\alpha(p^+)^\alpha} + \frac{c_3^{p^+}}{p^+}\right)$, we have

$$\begin{aligned} J_\lambda(u) &\geq \frac{m_1}{2\alpha (p^+)^\alpha} \|u\|^{\alpha p^+} - \lambda c_\varepsilon c_5^{m^-} \|u\|^{m^-} \\ &\geq \|u\|^{\alpha p^+} \left(\frac{m_1}{2\alpha (p^+)^\alpha} - \lambda c_\varepsilon c_5^{m^-} \|u\|^{m^- - \alpha p^+}\right) \end{aligned}$$

As $\alpha p^+ < m^-$, the functional $g : [0,1] \to R$ defined by

$$g(t) = \frac{m_1}{2\alpha (p^+)^\alpha} - \lambda c_\varepsilon c_5^{m^-} t^{m^- - \alpha p^+}$$

is positive on neighborhood of the origin. It follows that there exist two positive real numbers η *and* r such that $J_\lambda(u) \geq r > 0$, $u \in X$ with $\|u\| = \eta \in (0,1)$.

(ii) From $(\mathbf{AR})$, we obtain that there exist positive constant c_6 such that

$$F(x,t) \geq c_6 |t|^\theta, \quad |t| \geq t_*, \text{ for all } (x,t) \in \partial\Omega \times \mathbb{R}. \tag{3.9}$$

On the other hand, from $(\mathbf{M}_1)$ and $t > 1$, we have

$$\widehat{M}(t) \leq \frac{m_2}{\alpha} t^\alpha \leq \frac{m_2}{\alpha} t^{\frac{m_2}{m_1} p^+ \alpha}. \tag{3.10}$$

Moreover, we use the inequalities (3.9) and (3.10) together with Lemma 3.3, we obtain

$$J_\lambda(t\phi) = \widehat{M}\left(\int_\Omega A(x, \nabla t\phi)\, dx\right) + \int_\Omega \frac{1}{p(x)} |t\phi|^{p(x)} dx - \lambda \int_{\partial\Omega} F(x, t\phi) d\sigma$$

$$\leq \frac{m_2}{\alpha \left(p^-\right)^{\frac{m_2}{m_1}\alpha}} t^{\frac{m_2}{m_1}\alpha p^+} \int_\Omega A\left(x, \nabla \phi\right) dx + \frac{t^{p^+}}{p^-} \int_\Omega |\phi|^{p(x)} dx - \lambda c_6 t^\theta \int_{\partial\Omega} |\phi|^\theta \, d\sigma.$$

From (**AR**), we show that $J_\lambda \left(t\phi\right) \to -\infty$ as $t \to +\infty$. Then, we can take $u_1 = t\phi$ such that $\|u_1\| > \eta$ and $J_\lambda \left(u_1\right) < 0$. The proof of Lemma 3.6 is complete.

Proof of Theorem 2.8. By (**A3**), we obtain $J_\lambda \left(0\right) = 0$. If we also take into account Lemma 3.5 and Lemma 3.6, J_λ satisfies the Mountain Pass theorem [33]. Hence, J_λ has at least one nontrivial critical point, i.e., problem (**P**) has a nontrivial weak solution.

Proof of Theorem 2.9. To prove this theorem, it is sufficient to show that the conditions of Theorem 2.7 are satisfied. Since X is continuous embedding $L^{m^+}\left(\partial\Omega\right)$ from Proposition 2.4, there exist positive constant c_7 such that

$$|u|_{L^{m^+}(\partial\Omega)} \leq c_7 \|u\| \tag{3.11}$$

Also, consider the case $\|u\| > 1$ for $u \in X$. Then, using (**M**$_1$) and Proposition 2.1, we get that

$$\begin{aligned}
\Phi\left(u\right) &= \widehat{M}\left(\int_\Omega A\left(x, \nabla u\right) dx\right) + \int_\Omega \frac{1}{p\left(x\right)} |u|^{p(x)} dx \\
&\geq \frac{m_1}{\alpha \left(p^+\right)^\alpha} \|u\|^{\alpha p^-} + \frac{1}{p^+} \|u\|^{p^-} \\
&\geq \frac{m_1}{\alpha \left(p^+\right)^\alpha} \|u\|^{\alpha p^-}
\end{aligned} \tag{3.12}$$

If we use the inequalities (2.1) and (3.11) together with the conditional (**f**$_3$), we can write

$$\begin{aligned}
\lambda\Psi\left(u\right) &= -\lambda \int_{\partial\Omega} F\left(x, u\right) d\sigma = -\lambda \int_{\partial\Omega} \left(\int_0^{u(x)} f\left(x, t\right) dt\right) d\sigma \\
&\geq -\lambda \int_{\partial\Omega} \left(n\left(x\right) |u\left(x\right)| + \frac{k}{\beta(x)} |u\left(x\right)|^{\beta(x)}\right) d\sigma \\
&\geq -\lambda c_8 \|n\left(x\right)\|_{\frac{\beta(x)}{\beta(x)-1}, \partial\Omega} \|u\left(x\right)\|_{\beta(x), \partial\Omega} - \frac{\lambda c_9}{\beta^-} \|u\|^{\beta^+},
\end{aligned} \tag{3.13}$$

where c_8 and c_9 are positive constants. Combining (3.12) and (3.13), we obtain

$$J_\lambda\left(u\right) \geq \frac{m_1}{\alpha \left(p^+\right)^\alpha} \|u\|^{p^-} - \lambda c_8 \|n\left(x\right)\|_{\frac{\beta(x)}{\beta(x)-1}, \partial\Omega} \|u\left(x\right)\|_{\beta(x), \partial\Omega} - \frac{\lambda c_9}{\beta^-} \|u\|^{\beta^+}$$

From $p^- > \beta^+$, it follows that

$$\lim_{\|u\| \to \infty} J_\lambda\left(u\right) = \lim_{\|u\| \to \infty} \left(\Phi\left(u\right) + \lambda\Psi\left(u\right)\right) = \infty.$$

So, condition (i) of Theorem 2.7 is satisfied.

Now, we show the condition (ii) of Theorem 2.7. By $f(x, t) = \frac{\partial F(x,t)}{\partial t}$ and from the condition (**f**$_3$), it is obtained that $F(x, t)$ is increasing for $t \in (t_0, \infty)$ and decreasing for $t \in (0, 1)$ uniformly for $x \in \partial\Omega$.

Also, $F(x,0)=0$ is obvious and $F(x,t)\to\infty$ when $t\to\infty$ because $F(x,t)\geq\varpi t$ uniformly on x. Then, there exists a real number $\gamma>t_0$ such that

$$F(x,t)\geq 0=F(x,0)\geq F(x,\tau),\forall u\in X,\ t>\gamma,\tau\in(0,1). \tag{3.14}$$

Let κ,δ be two real numbers such that $0<\kappa<\min\{1,c\}$ where c is a constant which satisfies

$$\begin{aligned}\|u\|_{C(\overline{\Omega})} &= \sup_{x\in\overline{\Omega}}|u(x)|, \\ \|u\|_{C(\overline{\Omega})} &\leq c\|u\| \text{ for all } u\in X.\end{aligned} \tag{3.15}$$

Since $N<p^-$,we have from Remark 2.5 that compact embedding from X to $C\left(\overline{\Omega}\right)$. If we choose $\delta>\gamma$ satisfying $\delta^{p^-}|\Omega|>1$ and using (3.14), we obtain

$$F(x,t)\leq F(x,0)=0,\ \text{for } t\in[0,\kappa].$$

Then, we have

$$\int_{\partial\Omega}\sup_{0<t<\kappa}F(x,t)d\sigma\leq\int_{\partial\Omega}F(x,0)d\sigma=0. \tag{3.16}$$

On the other hand, we can write from $\delta>\gamma$,

$$\int_{\partial\Omega}F(x,\delta)d\sigma>0$$

and

$$\frac{1}{c^{p^+}}.\frac{\kappa^{p^+}}{\delta^{p^-}}\int_{\partial\Omega}F(x,\delta)d\sigma>0. \tag{3.17}$$

So, combining (3.16) and (3.17), we show that

$$\int_{\partial\Omega}\sup_{0<t<\kappa}F(x,t)\leq 0<\frac{1}{c^{p^+}}.\frac{\kappa^{p^+}}{\delta^{p^-}}\int_{\partial\Omega}F(x,\delta)d\sigma. \tag{3.18}$$

Let $u_0,u_1\in X$, $u_0(x)=0$ and $u_1(x)=\delta$ for any $x\in\overline{\Omega}$. If we define $\mu=\frac{1}{\alpha(p^+)^\alpha}.\left(\frac{\kappa}{c}\right)^{p^+}$, then we have $\mu\in(0,1)$, $\Phi(u_0)=\Psi(u_0)=0$,

$$\begin{aligned}\Phi(u_1) &= \widehat{M}\left(\int_\Omega A(x,\nabla\delta)\,dx\right)+\int_\Omega\frac{1}{p(x)}\delta^{p(x)}dx \\ &\geq \frac{m_1}{\alpha(p^+)^\alpha}\|\delta\|^{\alpha p^-}+\frac{1}{p^+}\|\delta\|^{p^-} \\ &\geq \frac{1}{\alpha(p^+)^\alpha}.\delta^{p^-}|\Omega|>\frac{1}{\alpha(p^+)^\alpha}.1>\frac{1}{\alpha(p^+)^\alpha}.\left(\frac{\kappa}{c}\right)^{p^+}=\mu\end{aligned}$$

and from (3.18) we obtain

$$\Psi(u_1)=-\int_{\partial\Omega}F(x,u_1(x))\,d\sigma=-\int_{\partial\Omega}F(x,\delta)\,d\sigma<0.$$

Thus we deduce that $\Phi(u_0) < r < \Phi(u_1)$, that is, condition (ii) of Theorem 2.7 is satisfied.

Finally, we show the condition (iii) of Theorem 2.7. we have

$$\begin{aligned}\frac{(\Phi(u_1)-r)\Psi(u_0)+(r-\Phi(u_0))\Psi(u_1)}{\Phi(u_1)-\Phi(u_0)} &= -\frac{\mu\,\Psi(u_1)}{\Phi(u_1)}\\ &= \mu\frac{\int_{\partial\Omega}F(x,\delta)d\sigma}{\widehat{M}\left(\int_\Omega A(x,\nabla\delta)\,dx\right)+\int_\Omega\frac{1}{p(x)}\delta^{p(x)}dx} > 0.\end{aligned} \tag{3.19}$$

Let $u\in X$ such that $\Phi(u)\le\mu<1$. Since $\frac{1}{\alpha(p^+)^\alpha}\psi(u)\le\Phi(u)\le\mu$ for $u\in X$, we obtain

$$\psi(u)\le\alpha\left(p^+\right)^\alpha\mu=\left(\frac{\kappa}{c}\right)^{p^+}<1. \tag{3.20}$$

So, by Proposition 2.1 for $\|u\|<1$, we have

$$\frac{1}{\alpha\left(p^+\right)^\alpha}\|u\|^{p^+}\le\frac{1}{\alpha\left(p^+\right)^\alpha}\psi(u)\le\Phi(u)\le\mu. \tag{3.21}$$

On the other hand, from (3.15), (3.20) and (3.21) we show that

$$\|u\|_{C(\overline{\Omega})}=\sup_{x\in\overline{\Omega}}|u(x)|\le c\|u\|\le c\left(\alpha\left(p^+\right)^\alpha\mu\right)^{\frac{1}{p^+}}=\kappa, \tag{3.22}$$

for all $u\in X$ and $x\in\overline{\Omega}$ with $\Phi(u)\le\mu$.

From (3.22) ,we obtain

$$-\inf_{u\in\Phi^{-1}\left((-\infty,\mu]\right)}\Psi(u)=\sup_{u\in\Phi^{-1}\left((-\infty,\mu]\right)}-\Psi(u)=\int_{\partial\Omega}\sup_{0<t<\kappa}F(x,t)d\sigma\le 0.$$

Thus, by (3.19) we have

$$-\inf_{u\in\Phi^{-1}\left((-\infty,\mu]\right)}\Psi(u)<\mu\frac{\int_{\partial\Omega}F(x,\delta)d\sigma}{\widehat{M}\left(\int_\Omega A(x,\nabla\delta)\,dx\right)+\int_\Omega\frac{1}{p(x)}\delta^{p(x)}dx}$$

or

$$\inf_{u\in\Phi^{-1}\left((-\infty,\mu]\right)}\Psi(u)>\frac{(\Phi(u_1)-r)\Psi(u_0)+(r-\Phi(u_0))\Psi(u_1)}{\Phi(u_1)-\Phi(u_0)}.$$

So, condition (iii) of Theorem 2.7 is obtained. Since all conditions of Theorem 2.7 are verified, there exists an open interval $\Theta\subset(0,\infty)$ and a positive real number ρ such that for each $\lambda\in\Theta$ the equation

$$\Phi'(u)+\lambda\Psi'(u)=0,$$

has at least three solutions in X whose norms are lees than ρ. The proof of Theorem 2.9 is complete.

4. Conclusions

In this paper, we studied a non-linear elliptic equation involving $p(x)$-growth conditions and satisfying Steklov boundary condition on a bounded domain Ω. The study was carried out the existence of at least three solutions and a nontrivial weak solution of the problem under appropriate conditions. Our basic approach is to use Ricceri's three critical point theorem and Mountain Pass theorem together with the variational approach to investigate the existence of a multiplicity of solutions.

Conflict of interest

The authors declare no conflict of interest.

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