AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(3): 5338–5351.
DOI: 10.3934/math.2023268
Received: 11 September 2022
Revised: 18 November 2022
Accepted: 24 November 2022
Published: 15 December 2022

Research article

Some results for multivalued mappings in extended fuzzy b-metric spaces

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Abstract: In this paper, some fixed point results for multivalued contractions are established in setting G-complete extended fuzzy b-metric spaces. An example is furnished to demonstrate the validity of results. An application of integral type inclusion is given to authenticate the theorems. Our results extend and generalize many existing results in literature.

Keywords: fuzzy metric space; fuzzy b-metric space; extended fuzzy b-metric space; multivalued mapping
Mathematics Subject Classification: 47H10, 54H25

1. Introduction

In the early twentieth century functional analysis was quite new in research. The mathematician of that time were demonstrating various notions of convergence on various spaces. There was a dire need to simplify things and unify arguments. This need was fulfilled by Frechet in his PhD dissertation on functional analysis by introducing the notion of metric space. This new idea lead Banach [2] to prove the well known fixed point theorem in 1922. This theorem become a land mark for the researchers to establish a number of extensions of metric spaces.The idea of b-metric was originated from the works of Bourbaki [6] and Bakhtin [5]. Czerwik [8] introduced an axiom which was weaker than the triangular inequality and precisely defined a b-metric space with a view of generalizing the Banach contraction mapping theorem. Several fixed point theorems on the platform of b-metric space endowed with different contractions are proved by many scholars for instance, see [7, 8, 10, 25, 28].

The concept of fuzzy sets has been introduced by Zadeh [30] in 1965. This concept was used in topology and analysis by many authors. The idea of fuzzy sets is utilized by Michalek and

Kramosoil [17] in 1975 to establish an important notion of fuzzy metric space. In 1988, Grabiec [12] proved the well known Banach fixed point theorem in the setting of fuzzy metric spaces. In 1994, George and Veermani [9] modified the definition of fuzzy metric space given by Michalek and Kramosil [17] and defined the Hausdorff topology of fuzzy metric spaces, which have important applications [26] in quantum particle physics. After that many fixed point results have been established by many researchers in fuzzy metric spaces. For instance see [1, 11, 19, 22, 29]. In 2016, Nădăban [20] introduced the concept of fuzzy b-metric space. In 2017, Mehmood et al. [18] generalized the idea of extended b-metric space given by Kamran and Samreen [27] by introducing the idea of an extended fuzzy b-metric space and proved the Banach fixed point theorem in this new frame. In [21] the notion of Hausdorff fuzzy metric (HFM) on compact set is introduced. Recently Batul et al. [3] proved some results using multivalued mappings (MVM) in FBMS. In this article, motivated by the idea of EFBMS given in [18] we generalizes the results of the new article [3] for MVM in Hausdorff extended fuzzy b-metric space (HEFBMS). These results generalizes both the results of [3, 23].

2. Preliminaries

Recently, the concept of EFBMS is introduced in [18] as follows:

Definition 2.1. [18] Let W be a non empty set , $\theta\colon W \times W \to [1, \infty)$ and $*$ be a continuous t-norm. A mapping $M_\theta : W \times W \times [0, +\infty) \to [0, 1]$ is called an extended fuzzy b-metric on W if for all $\rho_1, \rho_2, \rho_3 \in W$, the following conditions hold:

$[Mb_\theta 1] : M_\theta(\rho_1, \rho_2, 0) = 0;$

$[Mb_\theta 2] : M_\theta(\rho_1, \rho_2, \vartheta) = 1,$ for all $\vartheta > 0$ if and only if $\rho_1 = \rho_2$;

$[Mb_\theta 3] : M_\theta(\rho_1, \rho_2, \vartheta) = M_\theta(\rho_2, \rho_1, \vartheta);$

$[Mb_\theta 4] : M_\theta(\rho_1, \rho_3, \theta(\rho_1, \rho_3)(\vartheta + \beta)) \geq M_\theta(\rho_1, \rho_2, \vartheta) * M_\theta(\rho_2, \rho_3, \beta)$ for all $\vartheta, \beta \geq 0$;

$[Mb_\theta 5] : M_\theta(\rho_1, \rho_2, .)\colon (0, +\infty) \to [0, 1]$ is left continuous, and $\lim\limits_{t\to+\infty} M_\theta(\rho_1, \rho_2, \vartheta) = 1.$

Then $(W, M_\theta, *)$ is an EFBMS.

Remark 2.1. By taking $\theta(\rho_1, \rho_3) = b$ we get the notion of FBMS defined in [16] and by taking $\theta(\rho_1, \rho_3) = 1$ the notion of FMS defined in [9] is obtained.

Example 2.1. [18] Let $W = \{1, 2, 3\}$ and define $d_b : W \times W \to \mathbb{R}$ by

$$d_b(\rho_1, \rho_2) = (\rho_1 - \rho_2)^2.$$

Then (W, d_b) is a b-metric space. Define a mapping $\theta\colon W \times W \to [1, +\infty)$ by

$$\theta(\rho_1, \rho_2) = 1 + \rho_1 + \rho_2.$$

Let $M_\theta\colon W \times W \times [0, \infty) \to [0, 1]$ be defined by

$$M_\theta(\rho_1, \rho_2, \vartheta) = \begin{cases} \dfrac{\vartheta}{\vartheta + d_b(\rho_1, \rho_2)} & \text{if } \vartheta > 0 \\ 0 & \text{if } \vartheta = 0. \end{cases}$$

Then $(W, M_\theta, \wedge)$ is an EFBMS with $\vartheta_1 * \vartheta_2 = \vartheta_1 \wedge \vartheta_2 = \min\{\vartheta_1, \vartheta_2\}$.

Following are the definitions of G-Cauchy sequence and completeness in [18].

Definition 2.2. [18] For an EFBMS $(W, M_\theta, *)$:

(1) Let $\{\rho_n\}$ in W be any sequence, then $\{\rho_n\}$ is called G-Cauchy if $\lim_{n\to+\infty} M_\theta(\rho_n, \rho_{n+q}, \vartheta) = 1$ for $\vartheta > 0$ and $q > 0$.
(2) If every G-Cauchy sequence is convergent in an EFBMS then EFBMS is called G-complete EFBMS.

Similarly, for an EFBMS $(W, M_\theta, *)$, a sequence $\{\rho_n\}$ in W is convergent if there exits $\rho \in W$ such that

$$\lim_{n\to+\infty} M_\theta(\rho_n, \rho, \vartheta) = 1 \text{ for all } \vartheta > 0.$$

Definition 2.3. [23] Let $(W, M, *)$ be FMS and Y be any non empty subset of $(W, M, *)$, the fuzzy distance $\mathcal{M}$ of an element $\sigma_1 \in W$ and the subset $Y \subset W$ is defined as

$$\mathcal{M}(\sigma_1, Y, \vartheta) = \sup\{M(\sigma_1, \sigma_2, \vartheta) \colon \sigma_2 \in Y\}.$$

Note that $\mathcal{M}(\sigma_1, Y, \vartheta) = \mathcal{M}(Y, \sigma_1, \vartheta)$.

Lemma 2.1. *[31] Suppose $X \in CB(W)$, then $\rho_1 \in X$ if and only if $\mathcal{M}(X, \rho_1, \vartheta) = 1$ for all $\vartheta > 0$.*

Definition 2.4. [23] Let $(W, M, *)$ be a FMS and $\hat{C}_0(W)$ be the collection of all nonempty compact subsets of W. By $\mathcal{H}_\mathcal{M}$ we mean a function on $\hat{C}_0(W) \times \hat{C}_0(W) \times (0, +\infty)$ defined by,

$$\mathcal{H}_\mathcal{M}(X, Y, \vartheta) = \min\{\inf_{\sigma_1\in X} \mathcal{F}(\sigma_1, Y, \vartheta), \inf_{\sigma_2\in Y} \mathcal{M}(X, \sigma_2, \vartheta)\}$$

for all $X, Y \in \hat{C}_0(W)$ and $\vartheta > 0$.

Lemma 2.2. *[14] Let $(W, M, *)$ be a FMS and $M(\rho_1, \rho_2, k\vartheta) \geq M(\rho_1, \rho_2, \vartheta)$ for all $\rho_1, \rho_2 \in W$, $k \in (0, 1)$ and $\vartheta > 0$ then $\rho_1 = \rho_2$.*

Lemma 2.3. *[23] Let $(W, M, *)$ be a FMS and $(\hat{C}_0, \mathcal{H}_\mathcal{M}, *)$ is a HFMS on $\hat{C}_0$. If for all $X, Y \in \hat{C}_0$, for each $\rho \in X$ and $\vartheta > 0$ there exist $\sigma_\rho \in Y$, such that $\mathcal{M}(\rho, Y, \vartheta) = M(\rho, \sigma_\rho, \vartheta)$ then*

$$\mathcal{H}_\mathcal{M}(X, Y, \vartheta) \leq M(\rho, \sigma_\rho, \vartheta).$$

The extention of Definition 2.3 of [23], in HEFBMS on $\hat{C}_0$ is given in the following definition.

Definition 2.5. Consider $(W, M_\theta, *)$ a EFBMS and define $\mathcal{H}_{\mathcal{M}_\theta}$ on $\hat{C}_0(W) \times \hat{C}_0(W) \times (0, \infty)$ by,

$$\mathcal{H}_{\mathcal{M}_\theta}(X, Y, \vartheta) = \min\{\inf_{\rho\in X} \mathcal{M}_\theta(\rho, Y, \vartheta), \inf_{\sigma\in Y} \mathcal{M}_\theta(X, \sigma, \vartheta)\}$$

for all $X, Y \in \hat{C}_0(W)$, $\vartheta > 0$, where $\mathcal{M}_\theta$ is defined in the same way as in Definition 2.3. That is,

$$\mathcal{M}_\theta(\rho, Y, \vartheta) = \sup\{M_\theta(\rho, \sigma, \vartheta) \colon \sigma \in Y\}.$$

3. Main results

In this section, we prove certain new fixed point results by using the idea of HFMS in HEFBMS. The extention of Lemmas 2.1–2.3 in the setting of EFBMS is as follows:

Lemma 3.1. *If $X \in CB(W)$, then $\rho \in X$ if and only if $\mathcal{M}_\theta(X, \rho, \vartheta) = 1 \ \forall \ \vartheta > 0$.*

Proof. Since

$$\mathcal{M}_\theta(X, \rho, \vartheta) = \sup\{M_\theta(\rho, \sigma, \vartheta) \colon \sigma \in X\} = 1,$$

there exists a sequence $\{\sigma_n\} \subset X$ such that

$$M_\theta(\rho, \sigma_n, \vartheta) > 1 - \frac{1}{n}.$$

Letting $n \to +\infty$, we get $\sigma_n \to \rho$. From $A \in CB(W)$, it follows that $\rho \in X$.

Conversely, if $\rho \in X$, we have

$$\mathcal{M}_\theta(X, \rho, \vartheta) = \sup\{M_\theta(\rho, \sigma, \vartheta) \colon \sigma \in X\} > M_\theta(\rho, \rho, \vartheta) = 1,$$

□

Again following [19], it follows from $Fb_\theta 5$.

Lemma 3.2. *In G-complete EFBMS $(W, M_\theta, *)$ if for $\rho, \sigma \in W$ and for $k \in (0, 1)$,*

$$M_\theta(\rho, \sigma, k\vartheta) \geq M_\theta\,(\rho, \sigma, \vartheta)$$

then $\rho = \sigma$.

Lemma 3.3. *Let $(W, M_\theta, *)$ be an EFBMS and $(\hat{C}_0, \mathcal{H}_{\mathcal{M}_\theta}, *)$ is a HEFBMS on $\hat{C}_0$. If for all $X, Y \in \hat{C}_0$, and each $\rho \in X$ there exists $\sigma_\rho \in Y$, satisfying $\mathcal{M}_\theta(\rho, Y, \vartheta) = M_\theta(\rho, \sigma_\rho, \vartheta)$, where $\vartheta > 0$ then*

$$\mathcal{H}_{\mathcal{M}_\theta}(X, Y, \vartheta) \leq M_\theta(\rho, \sigma_\rho, \vartheta).$$

Proof. If

$$\mathcal{H}_{\mathcal{M}_\theta}(X, Y, \vartheta) = \inf_{\rho \in X} \mathcal{M}_\theta(\rho, B, \vartheta),$$

then

$$\mathcal{H}_{\mathcal{F}_\theta}(X, Y, \vartheta) \leq \mathcal{M}_\theta(\rho, Y, \vartheta).$$

Since for each $\rho \in X$ there exists $\sigma_\rho \in Y$ satisfying

$$\mathcal{M}_\theta(\rho, Y, \vartheta) = M_\theta(\vartheta, \sigma_\rho, \vartheta).$$

Hence

$$\mathcal{H}_{\mathcal{M}_\theta}(X, Y, \vartheta) \leq M_\theta(\rho, \sigma_\rho, \vartheta).$$

Now if

$$\mathcal{H}_{\mathcal{M}_\theta}(X, Y, \vartheta) = \inf_{\sigma \in Y} \mathcal{M}_\theta(X, \sigma, \vartheta) \leq \inf_{\rho \in X} \mathcal{M}_\theta(\rho, Y, \vartheta) \leq \mathcal{M}_\theta(\rho, Y, \vartheta) = M_\theta(\rho, \sigma_\rho, \vartheta).$$

This implies

$$\mathcal{H}_{\mathcal{M}_\theta}(X, Y, \vartheta) \leq M_\theta(\rho, \sigma_\rho, \vartheta)$$

for some $\sigma_\rho \in Y$. Hence in both cases result is proved. □

Theorem 3.1. *Let $(W, M_\theta, *)$ be G-complete EFBMS with $\theta(\rho,\sigma) \geqslant 1$ and $\mathcal{H}_{\mathcal{M}_\theta}$ be a HEFBMS. Let $\Phi\colon W \to \hat{C}_0(W)$ be a multivalued mapping satisfying*

$$\mathcal{H}_{\mathcal{M}_\theta}(\Phi\rho, \Phi\sigma, k\vartheta) \geq M_\theta(\rho,\sigma,\vartheta) \tag{3.1}$$

$\forall\, \rho,\sigma \in W,\ k\theta(\rho,\sigma) < 1$. *Then Φ has a fixed point.*

Proof. We choose a sequence $\{c_i\}$ in W, for $c_0 \in W$ as follows: Let $c_1 \in W$ such that $c_1 \in \Phi c_0$ with the help of Lemma 3.3 we can choose $c_2 \in \Phi c_1$ such that

$$M_\theta(c_1, c_2, \vartheta) \geqslant \mathcal{H}_{\mathcal{M}_\theta}(\Phi c_0, \Phi c_1, \vartheta) \text{ for all } \vartheta > 0.$$

By induction we have $c_{i+1} \in \Phi c_i$ satisfying

$$M_\theta(c_i, c_{i+1}, \vartheta) \geqslant \mathcal{H}_{\mathcal{M}_\theta}(\Phi c_{i-1}, \Phi c_i, \vartheta) \text{ for all } i\ \in \mathbb{N}.$$

Now using (3.1) and Lemma 3.3 we can write

$$\begin{aligned}
M_\theta(c_i, c_{i+1}, \vartheta) &\geq \mathcal{H}_{\mathcal{M}_\theta}(\Phi c_{i-1}, \Phi c_i, \vartheta) \geq M_\theta\left(c_{i-1}, c_i, \frac{\vartheta}{k}\right)\\
&\geq \mathcal{H}_{\mathcal{M}_\theta}\left(\Phi c_{i-2}, \Phi c_{i-1}, \frac{\vartheta}{k}\right) \geq M_\theta\left(c_{i-2}, c_{i-1}, \frac{\vartheta}{k^2}\right)\\
&\ \vdots\\
&\geq \mathcal{H}_{\mathcal{M}_\theta}\left(\Phi c_0, \Phi c_1, \frac{\vartheta}{k^{i-1}}\right) \geq M_\theta\left(c_0, c_1, \frac{\vartheta}{k^i}\right).
\end{aligned} \tag{3.2}$$

For any $q \in \mathbb{N}$, writing $q(\frac{\vartheta}{q}) = \dfrac{\vartheta}{q} + \dfrac{\vartheta}{q} + \ldots + \dfrac{\vartheta}{q}$ and using $[Mb_\theta 4]$ repeatedly,

$$\begin{aligned}
M_\theta(c_i, c_{i+q}, \vartheta) \geq{}& M_\theta\left(c_i, c_{i+1}, \frac{\vartheta}{q\theta(c_i, c_{i+q})}\right) * M_\theta\left(c_{i+1}, c_{i+2}, \frac{\vartheta}{q\theta(c_i, c_{i+q})\theta(c_{i+1}, c_{i+q})}\right)\\
&* M_\theta\left(c_{i+2}, c_{i+3}, \frac{\vartheta}{q\theta(c_i, c_{i+q})\theta(c_{i+1}, c_{i+q})\theta(x_{n+2}, x_{n+q})}\right) * \ldots\\
&* M_\theta\left(c_{i+q-1}, c_{i+q}, \frac{\vartheta}{q\theta(c_i, c_{i+q})\theta(c_{i+1}, c_{i+q})\theta(c_{i+2}, c_{i+q})\ldots\theta(c_{i+q-1}, c_{i+q})}\right).
\end{aligned}$$

Using (3.2) and $[Mb_\theta 5]$, we get

$$\begin{aligned}
M_\theta(c_i, c_{i+q}, \vartheta) \geq{}& M_\theta\left(c_0, c_1, \frac{\vartheta}{q\theta(c_i, c_{i+q})k^i}\right) * M_\theta\left(c_0, c_1, \frac{\vartheta}{q\theta(c_i, c_{i+q})\theta(c_{i+1}, c_{i+q})k^{i+1}}\right)\\
&* M_\theta\left(c_0, c_1, \frac{\vartheta}{q\theta(c_i, c_{i+q})\theta(c_{i+1}, c_{i+q})\theta(c_{i+2}, c_{i+q})k^{n+3}}\right) * \ldots\\
&* M_\theta\left(c_0, c_1, \frac{\vartheta}{q\theta(c_i, c_{i+q})\theta(c_{i+1}, c_{i+q})\theta(c_{i+2}, c_{i+q})\ldots\theta(c_{i+q-1}, c_{i+q})k^{n+q}}\right).
\end{aligned}$$

Since $\theta(c_i, c_{i+q})k < 1$ for all $i, q \in \mathbb{N}$, taking limit as $i \to +\infty$, we get

$$\lim_{i\to+\infty} M_\theta(c_i, c_{i+q}, \vartheta) = 1 * 1 * \ldots * 1 = 1.$$

Hence $\{c_i\}$ is G-Cauchy sequence. As W is G-complete so there exists $z \in W$ such that $\{c_i\}$ converges to z. To prove that z is a fixed point of Φ we proceed as follows:

$$\begin{aligned}
M_\theta(z, \Phi z, \vartheta) &\geq M_\theta\left(z, c_{i+1}, \frac{\vartheta}{2\theta(z, \Phi z)}\right) * M_\theta\left(c_{i+1}, \Phi z, \frac{\vartheta}{2\theta(z, \Phi z)}\right) \\
&\geq M_\theta\left(z, c_{i+1}, \frac{\vartheta}{2\theta(z, \Phi z)}\right) * \mathcal{H}_{\mathcal{M}_\theta}\left(\Phi c_i, \Phi z, \frac{\vartheta}{2\theta(z, \Phi z)}\right) \\
&\geq M_\theta\left(z, c_{i+1}, \frac{\vartheta}{2\theta(z, \Phi z)}\right) * M_\theta\left(c_i, z, \frac{\vartheta}{2\theta(z, \Phi z)k}\right) \\
&\longrightarrow 1 \quad \text{as} \quad i \to +\infty.
\end{aligned}$$

By Lemma 3.1 $z \in \Phi z$.

This implies that z is fixed point of Φ. □

Example 3.1. Let $W = [0, 1]$ and $M_\theta(\rho, \sigma, \vartheta) = \dfrac{\vartheta}{\vartheta + (\rho - \sigma)^2}$. Then $(W, M_\theta, *)$ is a G-complete EFBMS with

$$\theta(\rho, \sigma) = 1 + \rho + \sigma.$$

Let $\Phi\colon W \to \hat{C}_0(W)$ be a mapping defined by

$$\Phi(\rho) = \begin{cases} \{0\} & \text{if } \rho = 0, \\ \{0, \frac{\sqrt{k}\rho}{n}\} & \text{otherwise,} \end{cases}$$

where $k \in (0, 1)$ and $n \geqslant 2$. For $\rho = \sigma$, we have

$$\mathcal{H}_{\mathcal{M}_\theta}(\Phi\rho, \Phi\sigma, k\vartheta) = 1 = M_\theta(\rho, \sigma, \vartheta).$$

For $\rho \neq \sigma$, we have the following cases:

For $\rho = 0$ and $\sigma \in (0, 1]$, we have

$$\begin{aligned}
&\mathcal{H}_{\mathcal{M}_\theta}(\Phi(0), \Phi(\sigma), k\vartheta) \\
&= \min\Big\{\inf_{a\in\Phi(0)} \mathcal{M}_\theta(a, \Phi(\sigma), k\vartheta), \inf_{b\in\Phi(\sigma)} \mathcal{M}_\theta(\Phi(0), b, k\vartheta)\Big\} \\
&= \min\left\{\inf_{a\in\Phi(0)} \mathcal{M}_\theta\left(a, \left\{0, \frac{\sqrt{k}\sigma}{n}\right\}, k\vartheta\right), \inf_{b\in\Phi(\sigma)} \mathcal{M}_\theta(\{0\}, b, k\vartheta)\right\} \\
&= \min\left\{\inf\left\{\mathcal{M}_\theta\left(0, \{0, \frac{\sqrt{k}\sigma}{n}\}, k\vartheta\right)\right\}, \inf\left\{\mathcal{M}_\theta(\{0\}, 0, kt), \mathcal{M}_\theta\left(\{0\}, \frac{\sqrt{k}\sigma}{n}, k\vartheta\right)\right\}\right\} \\
&= \min\left\{\inf\left\{\sup\left\{\mathcal{M}_\theta(0, 0, k\vartheta), \mathcal{M}_\theta(0, \frac{\sqrt{k}\sigma}{n}, k\vartheta)\right\}\right\}, \inf\left\{\mathcal{M}_\theta(0, 0, k\vartheta), \mathcal{M}_\theta(0, \frac{\sqrt{k}y}{n}, k\vartheta)\right\}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \min\left\{\inf\left\{\sup\left\{1, \frac{\vartheta}{\vartheta + \frac{\sigma^2}{n^2}}\right\}\right\}, \inf\left\{1, \frac{\vartheta}{\vartheta + \frac{\sigma^2}{n^2}}\right\}\right\} \\
&= \min\left\{\inf\{1\}, \frac{\vartheta}{\vartheta + \frac{\sigma^2}{n^2}}\right\} \\
&= \min\left\{1, \frac{\vartheta}{\vartheta + \frac{\sigma^2}{n^2}}\right\} \\
&= \frac{\vartheta}{\vartheta + \frac{\sigma^2}{n^2}}.
\end{aligned}
$$

It follows that

$$\mathcal{H}_{\mathcal{M}_\theta}(\Phi(0), \Phi(\sigma), k\vartheta) > M_\theta(0, \sigma, \vartheta) = \frac{\vartheta}{\vartheta + \sigma^2}.$$

For ρ and $\sigma \in (0, 1]$, after some simple calculation, we get:

$$
\begin{aligned}
\mathcal{H}_{\mathcal{M}_\theta}(\Phi(\rho), \Phi(\sigma), k\vartheta) &= \min\left\{\sup\left\{\frac{\vartheta}{\vartheta + \frac{\rho^2}{n^2}}, \frac{\vartheta}{\vartheta + \frac{(\rho-\sigma)^2}{n^2}}\right\}, \sup\left\{\frac{\vartheta}{\vartheta + \frac{\sigma^2}{n^2}}, \frac{\vartheta}{\vartheta + \frac{(\rho-\sigma)^2}{n^2}}\right\}\right\} \\
&\geq \frac{\vartheta}{\vartheta + \frac{(\rho-\sigma)^2}{n^2}} > \frac{\vartheta}{\vartheta + (\rho - \sigma)^2} = M_\theta(\rho, \sigma, \vartheta).
\end{aligned}
$$

Thus for all cases, we have

$$\mathcal{H}_{\mathcal{M}_\theta}(\Phi\rho, \Phi\sigma, k\vartheta) \geq M_\theta(\rho, \sigma, \vartheta).$$

Hence 0 is a fixed point of Φ.

Following results follows from Theorem 3.1.

Remark 3.1. (i) Taking $\theta(\rho, \sigma) = b$ in the above Thoerem, we get the Theorem 3.2 of [3].
(ii) Taking $\theta(\rho, \sigma) = 1$ in the above Theorem, the same result follows for FMS.

Theorem 3.2. *Let $(W, M_\theta, *)$ be a G-complete EFBMS with $\theta(\rho, \sigma) \geqslant 1$ and $\mathcal{H}_{\mathcal{M}_\theta}$ be a HEFBMS. Let $\Phi\colon W \to \hat{C}_0(W)$ be a multivalued mapping satisfying*

$$\mathcal{H}_{\mathcal{M}_\theta}(\Phi\rho, \Phi\sigma, k\vartheta) \geq \min\left\{\frac{\mathcal{M}_\theta(\sigma, \Phi\sigma, \vartheta)\,[1 + \mathcal{M}_\theta(\rho, \Phi\rho, \vartheta)]}{1 + M_\theta(\rho, \sigma, \vartheta)}, M_\theta(\rho, \sigma, \vartheta)\right\} \tag{3.3}$$

for all $\rho, \sigma \in W$, $k\theta(\rho, \sigma) < 1$. Then Φ has a fixed point.

Proof. Proceeding as in Theorem 3.1 we have

$$M_\theta(c_1, c_2, \vartheta) \geqslant \mathcal{H}_{\mathcal{M}_\theta}(\Phi c_0, \Phi c_1, \vartheta) \quad \text{for all } \vartheta > 0.$$

By induction, we have $c_{i+1} \in \Phi c_i$ satisfying

$$M_\theta(c_i, c_{i+1}, \vartheta) \geqslant \mathcal{H}_{\mathcal{M}_\theta}(\Phi c_{i-1}, S c_i, \vartheta) \quad \text{for all } i \in \mathbb{N}.$$

Now using (3.3) and Lemma 3.3 we can write

$$\begin{aligned}
M_\theta(c_i, c_{i+1}, \vartheta) &\geq \mathcal{H}_{\mathcal{M}_\theta}(\Phi c_{i-1}, \Phi c_i, \vartheta) \\
&\geq \min\left\{\frac{\mathcal{M}_\theta\left(c_i, \Phi c_i, \frac{\vartheta}{k}\right)\left[1 + \mathcal{M}_\theta\left(c_{i-1}, \Phi c_{i-1}, \frac{\vartheta}{k}\right)\right]}{1 + M_\theta\left(c_{i-1}, c_i, \frac{\vartheta}{k}\right)}, M_\theta\left(c_{i-1}, c_i, \frac{\vartheta}{k}\right)\right\} \\
&\geq \min\left\{\frac{M_\theta\left(c_i, c_{i+1}, \frac{\vartheta}{k}\right)\left[1 + M_\theta\left(c_{i-1}, c_i, \frac{\vartheta}{k}\right)\right]}{1 + M_\theta\left(c_{i-1}, c_i, \frac{\vartheta}{k}\right)}, M_\theta\left(c_{i-1}, c_i, \frac{\vartheta}{k}\right)\right\} \\
&\geq \min\left\{M_\theta\left(c_i, c_{i+1}, \frac{\vartheta}{k}\right), M_\theta\left(c_{i-1}, c_i, \frac{\vartheta}{k}\right)\right\}. \qquad (3.4)
\end{aligned}$$

If

$$\min\left\{M_\theta\left(c_i, c_{i+1}, \frac{\vartheta}{k}\right), M_\theta\left(c_{i-1}, c_i, \frac{\vartheta}{k}\right)\right\} = M_\theta\left(c_i, c_{i+1}, \frac{\vartheta}{k}\right).$$

Then (3.4) implies

$$M_\theta(c_i, c_{i+1}, \vartheta) \geq M_\theta\left(c_i, c_{i+1}, \frac{\vartheta}{k}\right).$$

Then Lemma 3.2 yield the proof and if

$$\min\left\{M_\theta\left(c_i, c_{i+1}, \frac{\vartheta}{k}\right), M_\theta\left(c_{i-1}, c_i, \frac{\vartheta}{k}\right)\right\} = M_\theta\left(c_{i-1}, c_i, \frac{\vartheta}{k}\right).$$

Then from (3.4) we have

$$M_\theta(c_i, c_{i+1}, \vartheta) \geq M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k}) \geqslant \ldots \geqslant M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k^i}).$$

One can complete the proof as in Theorem 3.1. □

Remark 3.2. By taking $\theta(\rho, \sigma) = b$ in Theorem 3.2, the Theorem 3.2 of [3] is obtained and by taking $\theta(\rho, \sigma) = 1$ in Theorem 3.2, the main result of [23] is obtained.

Theorem 3.3. *Let $(W, M_\theta, *)$ be a G-complete EFBMS with $\theta(\rho, \sigma) \geqslant 1$ and $\mathcal{H}_{\mathcal{M}_\theta}$ be a HEFBMS. Let $\Phi\colon W \to \hat{C}_0(W)$ be a multivalued mapping satisfying*

$$\mathcal{H}_{\mathcal{M}_\theta}(\Phi\rho, \Phi\sigma, k\vartheta) \geq \min\left\{\frac{\mathcal{M}_\theta(\sigma, \Phi\sigma, \vartheta)\left[1 + \mathcal{M}_\theta(\rho, \Phi\rho, \vartheta) + \mathcal{M}_\theta(\sigma, \Phi\rho, \vartheta)\right]}{2 + M_\theta(\rho, \sigma, \vartheta)}, M_\theta(\rho, \sigma, \vartheta)\right\} \qquad (3.5)$$

for all $\rho, \sigma \in W$, $k\theta(\rho, \sigma) < 1$. Then Φ has a fixed point.

Proof. Proceeding as in Theorem 3.1 we can write

$$M_\theta(c_1, c_2, \vartheta) \geqslant \mathcal{H}_{\mathcal{M}_\theta}(\Phi c_0, \Phi c_1, \vartheta) \quad \text{for all } \vartheta > 0.$$

By induction, we have $c_{i+1} \in \Phi c_i$ satisfying

$$M_\theta(c_i, c_{i+1}, \vartheta) \geqslant \mathcal{H}_{\mathcal{M}_\theta}(\Phi c_{i-1}, \Phi c_i, \vartheta) \quad \text{for all } i \in \mathbb{N}.$$

Using (3.5) and Lemma 3.3, we have

$$
\begin{aligned}
M_\theta(c_i, c_{i+1}, \vartheta) &\geq \mathcal{H}_{\mathcal{M}_\theta}(\Phi c_{i-1}, \Phi c_i, \vartheta) \\
&\geq \min\left\{\frac{\mathcal{M}_\theta(c_i, \Phi c_i, \frac{\vartheta}{k})\left[1 + \mathcal{M}_\theta(c_{i-1}, \Phi c_{i-1}, \frac{\vartheta}{k}) + \mathcal{M}_\theta(c_i, \Phi c_{i-1}, \frac{\vartheta}{k})\right]}{2 + M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k})}, M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k})\right\} \\
&\geq \min\left\{\frac{M_\theta(c_i, c_{i+1}, \frac{\vartheta}{k})\left[1 + M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k}) + M_\theta(c_i, c_i, \frac{\vartheta}{k})\right]}{2 + M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k})}, M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k})\right\} \\
&\geq \min\left\{\frac{M_\theta(c_i, c_{i+1}, \frac{\vartheta}{k})\left[1 + M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k}) + 1\right]}{2 + M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k})}, M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k})\right\} \\
&\geq \min\left\{\frac{M_\theta(c_i, c_{i+1}, \frac{\vartheta}{k})\left[2 + M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k})\right]}{2 + M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k})}, M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k})\right\} \\
&\geq \min\left\{M_\theta(c_i, c_{i+1}, \frac{\vartheta}{k}), M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k})\right\}. \qquad (3.6)
\end{aligned}
$$

If

$$\min\left\{M_\theta(c_i, c_{i+1}, \frac{\vartheta}{k}), M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k})\right\} = M_\theta(c_i, c_{i+1}, \frac{\vartheta}{k}).$$

Then (3.6) implies

$$M_\theta(c_i, c_{i+1}, \vartheta) \geq M_\theta(c_i, c_{i+1}, \frac{\vartheta}{k}).$$

Then it is trivial by Lemma 3.2.

If

$$\min\left\{M_\theta(c_i, c_{i+1}, \frac{\vartheta}{k}), M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k})\right\} = M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k}).$$

Then from (3.6) we have

$$M_\theta(c_i, c_{i+1}, \vartheta) \geq M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k}) \geqslant \ldots \geqslant M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k^i}).$$

Now one can complete the proof by using Theorem 3.1. □

Remark 3.3. Taking $\theta(\rho, \sigma) = b$ in Theorem 3.3, we get the Theorem 3.3 of [3] by taking $\theta(\rho, \sigma) = 1$, the same result follows for FMS.

Theorem 3.4. *Let $(W, M_\theta, *)$ be a G-complete EFBMS with $\theta(\rho, \sigma) \geqslant 1$ and $\mathcal{H}_{\mathcal{M}_\theta}$ be a HEFBMS. Let $\Phi\colon W \to \hat{C}_0(W)$ be a multivalued mapping satisfying*

$$
\begin{aligned}
\mathcal{H}_{\mathcal{M}_\theta}(\Phi\rho, \Phi\sigma, k\vartheta) \geq \min\Bigg\{&\frac{\mathcal{M}_\theta(\rho, \Phi\rho, \vartheta)\left[1 + \mathcal{M}_\theta(\sigma, \Phi\sigma, \vartheta)\right]}{1 + \mathcal{M}_\theta(\Phi\rho, \Phi\sigma, \vartheta)}, \frac{\mathcal{M}_\theta(\sigma, \Phi\sigma, \vartheta)\left[1 + \mathcal{M}_\theta(\rho, \Phi\rho, \vartheta)\right]}{1 + M_\theta(\rho, \sigma, \vartheta)}, \\
&\frac{\mathcal{M}_\theta(\rho, \Phi\rho, \vartheta)\left[2 + \mathcal{M}_\theta(\rho, \Phi\sigma, \vartheta)\right]}{1 + M_\theta(\rho, \Phi\sigma, \vartheta) + \mathcal{M}_\theta(\sigma, \Phi\rho, \vartheta)}, M_\theta(\rho, \sigma, \vartheta)\Bigg\} \qquad (3.7)
\end{aligned}
$$

for all $\rho, \sigma \in W$, $k\theta(\rho, \sigma) < 1$. Then Φ has a fixed point.

Proof. Proceeding as in Theorem 3.1, we get

$$M_\theta(c_1, c_2, \vartheta) \geqslant \mathcal{H}_{\mathcal{M}_\theta}(\Phi c_0, \Phi c_1, \vartheta) \quad \text{for all } \vartheta > 0.$$

By induction we have $c_{i+1} \in \Phi c_i$ satisfying

$$M_\theta(c_i, c_{i+1}, \vartheta) \geqslant \mathcal{H}_{\mathcal{M}_\theta}(\Phi c_{i-1}, \Phi c_i, \vartheta) \quad \text{for all } i \in \mathbb{N}.$$

Now by using (3.7) and Lemma 3.3, we can write

$$\begin{aligned}
M_\theta(c_i, c_{i+1}, \vartheta) &\geq \mathcal{H}_{\mathcal{M}_\theta}(\Phi c_{i-1}, \Phi c_i, \vartheta)\\
&\geq \min\left\{ \frac{\mathcal{M}_\theta(c_{i-1}, \Phi c_{i-1}, \frac{\vartheta}{k})\left[1 + \mathcal{M}_\theta(c_i, \Phi_{c_i}, \frac{\vartheta}{k})\right]}{1 + \mathcal{M}_\theta(\Phi c_{i-1}, \Phi c_i, \frac{\vartheta}{k})}, \frac{\mathcal{M}_\theta(c_i, \Phi c_i, \frac{\vartheta}{k})\left[1 + \mathcal{M}_\theta(c_{i-1}, \Phi c_{i-1}, \frac{\vartheta}{k})\right]}{1 + M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k})}, \right.\\
&\qquad \left. \frac{\mathcal{M}_\theta(c_{i-1}, \Phi c_{i-1}, \frac{\vartheta}{k})\left[2 + \mathcal{M}_\theta(c_{i-1}, \Phi c_i, \frac{\vartheta}{k})\right]}{1 + \mathcal{M}_\theta(c_{i-1}, \Phi c_i, \frac{\vartheta}{k}) + \mathcal{M}_\theta(c_i, \Phi c_{i-1}, \frac{\vartheta}{k})}, M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k}) \right\}\\
&\geq \min\left\{ \frac{M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k})\left[1 + M_\theta(c_i, c_{i+1}, \frac{\vartheta}{k})\right]}{1 + M_\theta(c_i, c_{i+1}, \frac{\vartheta}{k})}, \frac{M_\theta(c_i, c_{i+1}, \frac{\vartheta}{k})\left[1 + M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k})\right]}{1 + M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k})}, \right.\\
&\qquad \left. \frac{M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k})\left[2 + M_\theta(c_{i-1}, c_{i+1}, \frac{\vartheta}{k})\right]}{1 + M_\theta(c_{i-1}, c_{i+1}, \frac{\vartheta}{k}) + M_\theta(c_i, c_i, \frac{\vartheta}{k})}, M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k}) \right\},
\end{aligned}$$

$$M_\theta(c_i, c_{i+1}, \vartheta) \geq \min\left\{ M_\theta(c_i, c_{i+1}, \frac{\vartheta}{k}), M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k}) \right\}. \tag{3.8}$$

If

$$\min\left\{ M_\theta(c_i, c_{i+1}, \frac{\vartheta}{k}), M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k}) \right\} = M_\theta(c_i, c_{i+1}, \frac{\vartheta}{k}).$$

Then (3.8) implies

$$M_\theta(c_i, c_{i+1}, \vartheta) \geq M_\theta(c_i, c_{i+1}, \frac{\vartheta}{k}).$$

The case is trivial by Lemma 3.2 and if

$$\min\left\{ M_\theta(c_i, c_{i+1}, \frac{\vartheta}{k}), M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k}) \right\} = M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k}).$$

Then from (3.6) we have

$$M_\theta(c_i, c_{i+1}, \vartheta) \geq M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k}) \geqslant \ldots \geqslant M_\theta(c_{i-1}, c_i, \frac{\vartheta}{k^i}).$$

The proof then follows by Theorem 3.1. □

Remark 3.4. Taking $\theta(\rho, \sigma) = b$ in Theorem 3.4, the Theorem 3.4 of [3] is obtained and by taking $\theta(\rho, \sigma) = 1$ in Theorem 3.4, the same result follows in FMS.

4. Applications

Nonlinear integral equations in abstract spaces arise in different fields of physical sciences, engineering, biology, and applied mathematics [4, 15]. The theory of nonlinear integral equations in abstract spaces is a fast growing field with important applications to a number of areas of analysis as well as other branches of science [13].

As an application of our main fixed point result Theorem 3.1, Volterra-Type integral inclusion has been studied for the existence of the solution.

Let $W = C([0,1],\mathbb{R})$ and define the EFBMS on W with $\theta(\rho,\varrho) = 1+\rho+\varrho$ by

$$M_\theta(\rho,\varrho,\vartheta) = e^{-\frac{\sup\limits_{u\in[0,1]}|\rho(u)-\varrho(u)|^2}{\vartheta}}$$

for all $\vartheta > 0$ and $\rho,\varrho \in W$. Then $(W, M_\theta, *)$ is a G-complete EFBMS with t norm $a*b = ab$ for all $a, b \in [0,1]$. Consider

$$\rho(u) \in \int_0^u G(u,v,\rho(v))dv + \Theta(u) \quad \text{for all } \ u,v \in [0,1], \tag{4.1}$$

where $\Theta\colon [0,1] \to \mathbb{R}_+$ and $G\colon [0,1]\times[0,1]\times\mathbb{R} \to \mathbb{R}$ are continuous functions.

Define a multivalued operator $\Phi : W \to \hat{C}_0(W)$ by

$$\Phi(\rho(u)) = \left\{ w \in W : w \in \int_0^u \Psi(u,v,\rho(v))dv + \Theta(u), \quad u \in [0,1] \right\}.$$

The following theorem proves the existence of a solution of the integral inclusion (4.1).

Theorem 4.1. *Let $\Phi : W \to \hat{C}_0(W)$ be the multivalued integral operator given by*

$$\Phi(\rho(u)) = \left\{ w \in W : w \in \int_0^u \Psi(u,v,\rho(v))dv + \Theta(u), \quad u \in [0,1] \right\}.$$

Suppose the following hold:

(1) $\Psi\colon [0,1]\times[0,1]\times\mathbb{R} \to P_{cv}(\mathbb{R})$ is such that $\Psi(u,v,\rho(v))$ is lower semi-continuous in $[0,1]\times[0,1]$.
(2) For all $u, v \in [0,1]$, $\Theta(u,v) \in W$, we have

$$|\Psi(u,v,\rho(v)) - \Psi(u,v,\varrho(v))|^2 \le \Theta^2(u,v)|\rho(v)-\varrho(v)|^2.$$

(3) For $0 < k < 1$ we have

$$\sup_{u\in[0,1]} \int_0^u \Theta^2(u,v)dv \le k.$$

Then (4.1) has the solution in W.

Proof. For $\Psi\colon [0,1]\times[0,1]\times\mathbb{R} \to P_{cv}(\mathbb{R})$, by Michael's selection theorem there exists an operator $\Psi_i\colon [0,1]\times[0,1]\times\mathbb{R} \to \mathbb{R}$ such that $\Psi_i(u,v,\varrho(v)) \in \Psi(u,v,\varrho(v))$ for all $u, v \in [0,1]$.

It follows that

$$\rho(u) \in \int_0^u \Psi_i(u, v, \rho(v))dv + \Theta(u) \in \Phi(\rho(u))$$

hence $\Phi(\rho(u)) \neq \emptyset$ and closed. Moreover, since $\Theta(u)$ and Ψ are continuous are bounded. This means that $\Phi(\rho(u))$ is bounded and $\Phi(\rho(u)) \in \hat{C}_0(W)$ Let $q, r \in W$ there exist $q(u) \in \Phi(\rho(u))$ and $r(u) \in \Phi(\varrho(u))$ such that

$$q(\rho(u)) = \left\{w \in W : w \in \int_0^u \Psi_i(u, v, \rho(v))dv + \Theta(u), \quad v \in [0, 1]\right\}$$

and

$$r(\varrho(u)) = \left\{w \in W : w \in \int_0^u \Psi_i(u, v, \varrho(v))dv + \Theta(v), \quad v \in [0, 1]\right\}.$$

It follows from assumption (4.1) that

$$|\Psi_i(u, v, \rho(v)) - \Psi_i(u, v, \varrho(v))|^2 \leq \Theta^2(u, v)|\rho(v) - \varrho(v)|^2.$$

Now,

$$\begin{aligned}
e^{-\frac{\sup_{u\in[0,1]} |q(\rho(u) - r(\varrho(u))|^2}{k\vartheta}} &\geq e^{-\frac{\sup_{u\in[0,1]} \int_0^u |\Psi_i(u, v, \rho(v)) - \Psi_i(u, v, \varrho(v))|^2 dv}{k\vartheta}} \\
&\geq e^{-\frac{\sup_{u\in[0,1]} \int_0^u \Theta^2(u, v)|\rho(v) - \varrho(v)|^2 dv}{k\vartheta}} \\
&\geq e^{-\frac{|\rho(v) - \varrho(v)|^2 \sup_{u\in[0,1]} \int_0^u \Theta^2(u, v)dv}{k\vartheta}} \\
&\geq e^{-\frac{k|\rho(v) - \varrho(v)|^2}{k\vartheta}} \\
&= e^{-\frac{|\rho(v) - \varrho(v)|^2}{\vartheta}} \\
&\geq e^{-\frac{\sup_{v\in[0,1]} |\rho(v) - \varrho(v)|^2}{\vartheta}} \\
&= M_\theta(u, v, \vartheta).
\end{aligned}$$

So, we have

$$M_\theta(q, r, k\vartheta) \geq M_\theta(u, v, \vartheta).$$

By replacing the roll of u and v, we have

$$\mathcal{H}_{\mathcal{M}_\theta}(\Phi\rho, \Phi\varrho, k\vartheta) \geq M_\theta(\rho, \varrho, \vartheta).$$

Hence, Φ has a fixed point in W, which satisfies the integral inclusion (4.1). □

5. Conclusions

In this article fixed point results in the setting of Hausdorff extended fuzzy b-metric spaces have been estabilshed. The main results are validated by an example. Theorem 3.1 generalizes the result of [3]. These results extend the theory of fixed points for multivalued mappings in a more general class of extended fuzzy b-metric spaces. For instance, some fixed point results can be obtained by taking $\theta(\sigma, \rho) = b$ (corresponding to G-complete FBMSs). An application for the existence of a solution for a Volterra type integral inclusion is also presented.

Conflict of interest

The authors declare that they have no conflicts of interest.

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