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Research article

Solving fractional partial differential equations via a new scheme

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Abstract: In this paper, we introduce a new technique, called the direct power series method to solve several types of time-fractional partial differential equations and systems, in terms of the Caputo derivative. We illustrate the method with a simple algorithm that can be used to solve different types of time-fractional partial problems. We introduce a new theorem to explain the required substitutions of the proposed method. In addition, convergence analysis conditions of the method are given. Furthermore, some different illustrative examples of time-fractional partial differential equations and systems are discussed to show the applicability and simplicity of the new approach.

Keywords: fractional differential equations; time fractional partial differential equations; analytical solution; power series
Mathematics Subject Classification: 35R11, 40G10

1. Introduction

Mathematical equations that contain two or more independent variables are called partial differential equations. They are found in various scientific applications, such as chemistry, physics, engineering and mathematics, which is why researchers have developed many techniques to solve such equations as homotopy perturbation method, variation iteration method, Adomian decomposition method and others [1–9].

Fractional calculus is a generalization of regular calculus, that calculates derivatives of functions of non-integer orders. Many definitions of fractional derivatives have been presented in the literature such as Riemann Louville, Caputo, conformable and others [10–15]. Furthermore, using each of these definitions can be viewed as a generalization of the normal calculus. For this reason, applying either of these definitions allows us to generalize our research to normal calculus, and since Caputo's fractional derivative is one of the most popular definitions, we establish our new results in terms of Caputo's fractional definition in this article. There are a number of different methods that mathematicians have used to solve ordinary and partial differential equations and systems in fractional calculus. One of these methods is the power series method, because it can solve such problems. On the other hand, there are too many methods that depend on the idea of power series, such as residual power series method [16–26], and some other techniques that combine the power series idea with transformations just as Laplace transform, ARA-transform, formable transform and others [27–37] to construct new ones. However, all these methods, introduce the solution in a convergent series form, but the difference lies in the level of difficulty during the applications.

The direct power series method (DPSM) used in this work was first introduced in [38], transformations are unnecessary, no limits or differentiations are required, this method only focuses on finding the nth coefficients of the series expansion of the analytic solution.

In DPSM we find the solution by computing a general term of the nth coefficients without going back to the hole power series every time we want to compute new coefficients. For this reason, DPSM helps mathematicians to find the approximate solutions better and faster. This method only takes two steps to write the general solution of some equations or systems that can be solved with other power series methods. At the end of the second step, we have a general form of the solution, expressed as an infinite series, and this makes it easier to find too many new coefficients of the series solution with computer programs. Furthermore, some different illustrative examples are presented in the fourth chapter and solved with the proposed method. We show that DPSM could be used to solve different types of problems and systems.

This paper is organized as follows, in the next section, we introduce some preparatory explanations on fractional operators, Section 3 introduces the methodology and the basic idea of DPSM and finally we consider some different examples on fractional partial differential equations and systems.

2. Basics about fractional operators and power series

In this section we introduce the definition of the fractional Caputo and some theorems about power series.

Definition 2.1. If $\psi(\chi,\tau)$ is a function of two variables and n is any natural number, then Caputo fractional partial derivative of order α with respect to τ is denoted and defined as

$$D_\tau^\alpha \psi(\chi,\tau)=\begin{cases}\frac{1}{\Gamma(n-\alpha)}\int_0^\tau (\tau-t)^{n-\alpha-1}\frac{\partial^n}{\partial\tau^n}\psi(\chi,\tau)dt, & n-1<\alpha<n,\\ \frac{\partial^n}{\partial\tau^n}\psi(\chi,\tau), & n=\alpha,\end{cases} \tag{2.1}$$

where $n\in\mathbb{N}$.

It is worth noting here that Caputo fractional partial derivative has the memory property. For more details and properties, see [39–41].

Lemma 2.1. [10,11]. Let $\varrho(\chi)$ be a real valued continuous function and $\alpha,\tau>0$. Then the following properties of Caputo's derivative are hold:

1) $D_\tau^\alpha \varrho(\chi)=0$.

2) $D_\tau^\alpha\left(\tau^\beta \varrho(\chi)\right)=\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}\tau^{\beta-\alpha}\varrho(\chi)$.

3) $D_\tau^{k\alpha}\left(\tau^{n\alpha}\varrho(\chi)\right)=\begin{cases}\frac{\Gamma(n\alpha+1)}{\Gamma((n-k)\alpha+1)}\tau^{(n-k)\alpha}\varrho(\chi), & n\geq k,\\ 0 & n<k.\end{cases}$

4) $D_\tau^{k\alpha}\left(\sum_{n=0}^{\infty}\frac{\varrho_n(\chi)(\gamma\tau)^{n\alpha}}{\Gamma(n\alpha+1)}\right)=\sum_{n=0}^{\infty}\frac{\gamma^{(n+k)\alpha}\varrho_{n+k}(\chi)\tau^{n\alpha}}{\Gamma(n\alpha+1)}$.

Definition 2.2. [22] For any α such that, $r-1<\alpha\leq r,\ r\in\mathbb{N}$ a power series of the form:

$$\sum_{n=0}^{\infty}\frac{\varrho_n(\chi)(\tau-\tau_0)^{n\alpha}}{\Gamma(n\alpha+1)}=\varrho_0(\chi)+\frac{\varrho_1(\chi)(\tau-\tau_0)^{\alpha}}{\Gamma(\alpha+1)}+\frac{\varrho_2(\chi)(\tau-\tau_0)^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots, \tag{2.2}$$

is called the multiple fractional power series about $\tau=\tau_0$, where τ is a variable and $\varrho_n(\chi)$, $\forall n=0,1,\cdots$ are functions of χ called the coefficients of the series.

Theorem 2.1. [22,23] Suppose that $\psi(\chi,\tau)$ has a fractional power series representation at $\tau=0$, of the form:

$$\psi(\chi,\tau)=\sum_{n=0}^{\infty}\frac{\varrho_n(\chi)\tau^{n\alpha}}{\Gamma(n\alpha+1)},\ \alpha>0,\chi\in I, \tag{2.3}$$

where $0\leq\tau<R$ and R is the radius of convergence. If $D_\tau^{n\alpha}\psi(\chi,\tau)$ is continuous on $(0,R)$, then, the coefficients $\varrho_n(\chi)$; $\forall n=0,1,\cdots$, of the power series (2.3) are given by

$$\varrho_n(\chi)=D_\tau^{n\alpha}\psi(\chi,0). \tag{2.4}$$

For the proof, see [22].

The convergence analysis of the presented power series are illustrated in the following theorem. We mention here, that these conditions are the required convergence conditions for DPSM [13].

Theorem 2.2. [35,36] Consider the fractional power series representation in (2.3), then we have the following cases:

a) If $\tau=0$, the series representation (2.3) is convergent and the radius of convergence is $R=0$.
b) If $\tau\geq 0$, the series representation (2.3) is convergent and the radius of convergence is $R=\infty$.
c) If $\tau\in[0,R]$, the series representation (2.3) is convergent for some positive real number R and is divergent for $\tau>R$, where R is the radius of convergence.

In the following arguments, we state some properties of the fractional power series (2.3):

1) The kth derivative of the fractional power series representation (2.3) is given by

$$D_\chi^k\psi(\chi,\tau)=\sum_{n=0}^{\infty}\frac{\varrho_n^{(k)}(\chi)\tau^{n\alpha}}{\Gamma(n\alpha+1)}, \tag{2.5}$$

and the coefficients of the equation $\varrho_n^{(m)}(\chi)$; $\forall n=0,1,\ldots$ are given by

$$\varrho_n^{(k)}(\chi)=D_\chi^k\left(D_\tau^{n\alpha}\psi(\chi,0)\right). \tag{2.6}$$

2) If we have $\{\psi_i(\chi,\tau)\}_{i=1}^m$ a sequence of functions of two variables, that has a power series expansion as follows:

$$\psi_i(\chi,\tau)=\sum_{n=0}^{\infty}\frac{\varrho_{i_n}(\chi)\tau^{n\alpha}}{\Gamma(n\alpha+1)}, \tag{2.7}$$

then the coefficients of $\frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)}$ of $\prod_{i=1}^{m}\psi_i(\chi,\tau)$ can be written by the following summation:

$$\prod_{i=1}^{m}\psi_i(\chi,\tau)=\sum_{\substack{j_1,j_2,\ldots j_m;\\ j_1+j_2+\cdots+j_m=n}}\frac{\varrho_{1j_1}(\chi)\varrho_{2j_2}(\chi)\ldots\varrho_{mj_m}(\chi)\Gamma(n\alpha+1)}{\Gamma(j_1\alpha+1)\Gamma(j_2\alpha+1)\ldots\Gamma(j_m\alpha+1)}\frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)}. \quad (2.8)$$

3. The methodology of DPSM

In this section we present the main idea of the DPSM for solving time fractional partial differential equations (TFPDEs). We illustrate the technique of using DPSM to solve TFPDEs.

3.1. Construction of DPSM solution of TFPDEs

Assume that $\psi(\chi,\tau)$ can be presented in the series representation (2.3). Consider the FPDE,

$$\mathcal{L}[\psi(\chi,\tau)]+\mathcal{N}[\psi(\chi,\tau)]=0, \quad (3.1)$$

where $\mathcal{L}$ and $\mathcal{N}$ denote linear and nonlinear operators respectively of fractional or integer orders derivatives. Using DPSM we can get the solution of some cases-that will be considered in Theorem 3.1, below-of Eq (3.1) in a series representation which means, to find the values of the coefficients $\varrho_n(\chi)$ of the series expansion (2.3). The Caputo fractional derivative affects the series formula and changes the summation, so we need to illustrate the following theorem that study the effects of the fractional derivatives on the power series representations.

Theorem 3.1. [32] Suppose that $\psi(\chi,\tau)$ and $\varphi(\chi,\tau)$ have fractional power series representations such as:

$$\psi(\chi,\tau)=\sum_{n=0}^{\infty}\frac{\varrho_n(\chi)\tau^{n\alpha}}{\Gamma(n\alpha+1)} \text{ and } \varphi(\chi,\tau)=\sum_{m=0}^{\infty}\frac{\rho_m(\chi)\tau^{m\alpha}}{\Gamma(m\alpha+1)}, \quad (3.2)$$

where $\psi(\chi,\tau)$ and $\varphi(\chi,\tau)$ are analytical functions, then we have $\varrho_n(\chi)$ and $\rho_m(\chi)$ are the coefficients of $\frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)}$ and $\frac{\tau^{m\alpha}}{\Gamma(m\alpha+1)}$ in $\psi(\chi,\tau)$ and $\varphi(\chi,\tau)$ respectively, $\forall m=0,1,\ldots,$ and $n=0,1,\cdots$. Then we have:

a) $\varrho_{n+k}(\chi)$ is the coefficient for $\frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)}$ in the series expansion of $D_\tau^{k\alpha}\psi(\chi,\tau)$ for any $k=0,1,\cdots$.

b) $\gamma^{(n+k)\alpha}\varrho_{n+k}(\chi)$ is the coefficient for $\frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)}$ in the series expansion of $D_\tau^{k\alpha}\psi(\chi,\gamma\tau)$ for any $k=0,1,\ldots,$ where $\gamma\in\mathbb{R}$.

c) $\sum_{i=0}^{n}\frac{\varrho_i(\chi)\rho_{n-i}(\chi)\Gamma(n\alpha+1)}{\Gamma(i\alpha+1)\Gamma((n-i)\alpha+1)}$ is the coefficient for $\frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)}$ in the series expansion of $\psi(\chi,\tau)\varphi(\chi,\tau)$.

d) $\sum_{i=0}^{n}\frac{\beta^{i\alpha}\gamma^{(n-i)\alpha}\varrho_i(\chi)\rho_{n-i}(\chi)\Gamma(n\alpha+1)}{\Gamma(i\alpha+1)\Gamma((n-i)\alpha+1)}$ is the coefficient for $\frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)}$ in the series expansion of $\psi(\chi,\beta\tau)\varphi(\chi,\gamma\tau)$, where β and $\gamma\in\mathbb{R}$.

e) $\sum_{i=0}^{n}\frac{\beta^{(i+k)\alpha}\gamma^{(n-i+m)\alpha}\varrho_{i+k}(\chi)\rho_{n-i+m}(\chi)\Gamma(n\alpha+1)}{\Gamma(i\alpha+1)\Gamma((n-i)\alpha+1)}$ is the coefficient for $\frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)}$ in the series expansion of $D_{\tau}^{k\alpha}\psi(\chi,\beta\tau)\ D_{\tau}^{s\alpha}\varphi(\chi,\gamma\tau)$, where $\beta,\ \gamma\in\mathbb{R}$ and $k, s=0,1,\ldots$.

Proof. Substituting the series expansion of $\psi(\chi,\tau)$ in $D_{\tau}^{k\alpha}\psi(\chi,\tau)$,

$$D_{\tau}^{k\alpha}\psi(\chi,\tau)=D_{\tau}^{k\alpha}\left(\sum_{n=0}^{\infty}\frac{\varrho_n(\chi)\tau^{n\alpha}}{\Gamma(n\alpha+1)}\right). \tag{3.3}$$

Using part (4) of Lemma 2.1, we get

$$D_{\tau}^{k\alpha}\psi(\chi,\tau)=\sum_{n=j}^{\infty}\frac{\varrho_n(\chi)\tau^{(n-k)\alpha}}{\Gamma((n-k)\alpha+1)}. \tag{3.4}$$

Thus, the series expansion can be written as

$$D_{\tau}^{k\alpha}\psi(\chi,\tau)=\sum_{n=0}^{\infty}\frac{\varrho_{n+k}(\chi)\tau^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{3.5}$$

Proof. Substituting the series expansion of $\psi(\chi,\gamma\tau)$ in $D_{\tau}^{k\alpha}\psi(\chi,\gamma\tau)$, we get

$$D_{\tau}^{k\alpha}\psi(\chi,\gamma\tau)=D_{\tau}^{k\alpha}\left(\sum_{n=0}^{\infty}\frac{\varrho_n(\chi)(\gamma\tau)^{n\alpha}}{\Gamma(n\alpha+1)}\right). \tag{3.6}$$

Using part (4) of Lemma 2.1, we get

$$D_{\tau}^{k\alpha}\psi(\chi,\gamma\tau)=\sum_{n=j}^{\infty}\frac{\gamma^{n\alpha}\varrho_n(\chi)\tau^{(n-k)\alpha}}{\Gamma((n-k)\alpha+1)}. \tag{3.7}$$

Thus, Eq (3.7), can be written as

$$D_{\tau}^{k\alpha}\psi(\chi,\gamma\tau)=\sum_{n=0}^{\infty}\frac{\gamma^{(n+k)\alpha}\varrho_{n+k}(\chi)\tau^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{3.8}$$

Proof. Multiplying the series expansion of $\psi(\chi,\tau)$ and $\varphi(\chi,\tau)$,

$$\psi(\chi,\tau)\varphi(\chi,\tau)=\sum_{n=0}^{\infty}\frac{\varrho_n(\chi)\tau^{n\alpha}}{\Gamma(n\alpha+1)}\sum_{m=0}^{\infty}\frac{\rho_m(\chi)\tau^{m\alpha}}{\Gamma(m\alpha+1)}. \tag{3.9}$$

Equation (3.9), can be simplified as

$$\psi(\chi,\tau)\varphi(\chi,\tau)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\varrho_n(\chi)\rho_m(\chi)\tau^{(n+m)\alpha}}{\Gamma(n\alpha+1)\Gamma(m\alpha+1)}, \tag{3.10}$$

which can be rewritten as

$$\psi(\chi,\tau)\varphi(\chi,\tau)=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\frac{\varrho_i(\chi)\rho_{n-i}(\chi)\Gamma(n\alpha+1)}{\Gamma(i\alpha+1)\Gamma((n-i)\alpha+1)}\right)\frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{3.11}$$

Proof. Multiplying the series expansion of $\psi(\chi,\beta\tau)$ and $\varphi(\chi,\gamma\tau)$,

$$\psi(\chi,\beta\tau)\varphi(\chi,\gamma\tau)=\left(\sum_{n=0}^{\infty}\frac{\varrho_n(\chi)(\beta\tau)^{n\alpha}}{\Gamma(n\alpha+1)}\right)\left(\sum_{m=0}^{\infty}\frac{\rho_m(\chi)(\gamma\tau)^{m\alpha}}{\Gamma(m\alpha+1)}\right). \tag{3.12}$$

Equation (3.12), can be written as

$$\psi(\chi,\beta\tau)\varphi(\chi,\gamma\tau)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\varrho_n(\chi)\rho_m(\chi)\beta^{n\alpha}\gamma^{m\alpha}\tau^{(n+m)\alpha}}{\Gamma(n\alpha+1)\Gamma(m\alpha+1)}, \tag{3.13}$$

which can be simplified as

$$\psi(\chi,\beta\tau)\varphi(\chi,\gamma\tau)=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\frac{\beta^{i\alpha}\gamma^{(n-i)\alpha}\varrho_i(\chi)\rho_{n-i}(\chi)\Gamma(n\alpha+1)}{\Gamma(i\alpha+1)\Gamma((n-i)\alpha+1)}\right)\frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{3.14}$$

Proof. Substituting the series expansion of $\psi(\chi,\gamma\tau)$ and $\varphi(\chi,\gamma\tau)$ in $D_\tau^{k\alpha}\psi(\chi,\beta\tau)\ D_\tau^{s\alpha}\varphi(\chi,\gamma\tau)$, we get:

$$D_\tau^{k\alpha}\psi(\chi,\beta\tau)\ D_\tau^{s\alpha}\varphi(\chi,\gamma\tau),=D_t^{j\alpha}\left(\sum_{i=0}^{\infty}\frac{\varrho_i(\chi)(\beta\tau)^{i\alpha}}{\Gamma(i\alpha+1)}\right)D_t^{m\alpha}\left(\sum_{j=0}^{\infty}\frac{\rho_j(\chi)(\gamma\tau)^{k\alpha}}{\Gamma(k\alpha+1)}\right). \tag{3.15}$$

Using part (2) of Lemma 2.1, Eq (3.15) can be written as

$$D_\tau^{k\alpha}\psi(\chi,\beta\tau)D_\tau^{s\alpha}\varphi(\chi,\gamma\tau)=\left(\sum_{n=0}^{\infty}\frac{\beta^{(n+k)\alpha}\varrho_{n+k}(\chi)\tau^{n\alpha}}{\Gamma(n\alpha+1)}\right)\left(\sum_{m=0}^{\infty}\frac{\gamma^{(m+s)\alpha}\rho_{m+s}(\chi)\tau^{m\alpha}}{\Gamma(m\alpha+1)}\right), \tag{3.16}$$

which can be simplified as

$$D_\tau^{k\alpha}\psi(\chi,\beta\tau)D_\tau^{s\alpha}\varphi(\chi,\gamma\tau)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\beta^{(n+k)\alpha}\gamma^{(m+s)\alpha}\varrho_{n+k}(\chi)\rho_{m+s}(\chi)\tau^{(n+m)\alpha}}{\Gamma(n\alpha+1)\Gamma(m\alpha+1)}. \tag{3.17}$$

Equation (3.17) can be written as

$$D_\tau^{k\alpha}\psi(\chi,\beta\tau)D_\tau^{s\alpha}\varphi(\chi,\gamma\tau)=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\frac{\beta^{(i+k)\alpha}\gamma^{(n-i+m)\alpha}\varrho_{i+k}(\chi)\rho_{n-i+m}(\chi)\Gamma(n\alpha+1)}{\Gamma(i\alpha+1)\Gamma((n-i)\alpha+1)}\right)\frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{3.18}$$

The proof is complete.

The main idea of DPSM depends on replacing each part of the target equation or system with its $\frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)}$ coefficients, in any equation that contains similar terms in Theorem 3.1, these replacements can be applied separately for each additive part, or if each part is multiplied by a real number. The main idea of the method is to do some replacements in the target equation and simplify the obtained series expansions in one series after simple computations, to get a general term of the coefficients in the series expansion (2.3) and hence, we get the analytic series solution of the equation by substituting $n=1,2,\ldots$ in the series form and so on.

3.2. Algorithm of DPSM for solving TFPDEs

Our goal in this section, is to explain the usage of DPSM in solving some TFPDEs and get numerical solutions for them. The method is basically, depends on assuming the series representation (2.3) of the solution and then find a general term of the series coefficients, that allows researchers to get better approximate solutions by getting many terms of the numerical solution they study.

The following steps illustrate the algorithm of DPSM in solving TFPDEs:

Step 3.1. Apply the replacements from Theorem 3.1 that is, replace each term of the target equation by its suitable similar coefficient $\varrho_n(\chi)$ of $\frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)}$.

Step 3.2. Simplify the obtained series representations from Step 3.1, and define a general form of the series solution by putting the higher index $\varrho_{n+k}(\chi)$ to the left-hand side and the rest of the coefficients into the right, to get a formula of the shape:

$$\varrho_{n+k}(\chi) = \rho(a_{n+k-1}, a_{n+k-2}, \dots, a_0). \tag{3.19}$$

Step 3.3. Substitute the values of n, recursively from $n = 1,2, \dots$ as much as you need terms of the series solution.

4. Numerical examples

In this section, some interesting examples on TFPDEs are solved, we clarify the steps of DPSM by solving fractional partial differential equations and system of fractional partial differential equations, each example is of different kind.

Example 4.1.[26] Consider the following temporal-fractional Burger equation of the form:

$$D_\tau^\alpha \psi(\chi,\tau) - \psi_{\chi\chi}(\chi,\tau) + \psi(\chi,\tau)\psi_\chi(\chi,\tau) = 0,\ \ 0 < \alpha \le 1, \tag{4.1}$$

subject to the initial condition,

$$\psi(\chi,0) = 2\chi. \tag{4.2}$$

Note that, when $\alpha = 1$ the exact solution of Eq (4.1) the integer case is

$$\psi(\chi,\tau) = \frac{2\chi}{1+2\tau}. \tag{4.3}$$

Solution 4.1. Applying the replacements of Theorem 3.1 on Eq (4.1), we get

$$D_\tau^\alpha \psi(\chi,\tau) \leftrightarrow \varrho_{n+1}(\chi),$$

$$\psi_{\chi\chi}(\chi,\tau) \leftrightarrow \varrho_n''(\chi),$$

and

$$\psi(\chi,\tau)\psi_\chi(\chi,\tau) \leftrightarrow \sum_{i=0}^{n} \frac{\varrho_i(\chi)\varrho_{n-i}'(\chi)\Gamma(n\alpha+1)}{\Gamma(i\alpha+1)\Gamma\big((n-i)\alpha+1\big)}.$$

Substituting the new terms from the replacements into Eq (4.1), we get

$$\varrho_{n+1}(\chi) - \varrho_n''(\chi) + \sum_{i=0}^{n} \frac{\varrho_i(\chi)\varrho_{n-i}'(\chi)\Gamma(n\alpha+1)}{\Gamma(i\alpha+1)\Gamma\big((n-i)\alpha+1\big)} = 0. \tag{4.4}$$

Then the Eq (4.1) can be expressed as

$$\varrho_{n+1}(\chi) = \varrho_n''(\chi) - \sum_{i=0}^{n} \frac{\varrho_i(\chi)\varrho_{n-i}'(\chi)\Gamma(n\alpha+1)}{\Gamma(i\alpha+1)\Gamma\big((n-i)\alpha+1\big)}. \tag{4.5}$$

From the initial condition in (4.2),
for $n = 0,\ \varrho_1(\chi) = -4\chi,$
for $n = 1,\ \varrho_2(\chi) = 16\chi,$
for $n = 2,\ \varrho_3(\chi) = -16\chi\left(4 + \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\right),$
for $n = 3,\ \varrho_4(\chi) = 64\chi\left(4 + \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} + \frac{2\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)}\right),$

for $n = 4$, $\varrho_5(\chi) = -256\chi\left(4 + \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} + \frac{2\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)} + \frac{\Gamma(4\alpha+1)}{\Gamma^2(2\alpha+1)}\right) - 128\chi\left(4 + \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\right)\frac{\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)\Gamma(\alpha+1)}$, which are the same results obtained in [26]. DPSM save a lot of time in calculating the coefficients and the steps here are written completely.

Thus, the solution of Eqs (4.1) and (4.2), can be expressed as

$$\begin{aligned}\psi(\chi,\tau) &= \varrho_0(\chi) + \frac{\tau^\alpha \varrho_1(\chi)}{\Gamma(1+\alpha)} + \frac{\tau^{2\alpha}\varrho_2(\chi)}{\Gamma(1+2\alpha)} + \frac{\tau^{3\alpha}\varrho_3(\chi)}{\Gamma(1+3\alpha)} + \frac{\tau^{4\alpha}\varrho_4(\chi)}{\Gamma(1+4\alpha)} + \frac{\tau^{5\alpha}\varrho_5(\chi)}{\Gamma(1+5\alpha)} \\ &= 2\chi - 4\chi\frac{\tau^\alpha}{\Gamma(1+\alpha)} + 16\chi\frac{\tau^{2\alpha}}{\Gamma(1+2\alpha)} - 16\chi\left(4 + \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\right)\frac{\tau^{3\alpha}}{\Gamma(1+3\alpha)} \\ &+ 64\chi\left(4 + \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} + \frac{2\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)}\right)\frac{\tau^{4\alpha}}{\Gamma(1+4\alpha)} \\ &+ \begin{pmatrix} -256\chi\left(4 + \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} + \frac{2\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)} + \frac{\Gamma(4\alpha+1)}{\Gamma^2(2\alpha+1)}\right) \\ -128\chi\left(4 + \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\right)\frac{\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)\Gamma(\alpha+1)} \end{pmatrix}\frac{\tau^{5\alpha}}{\Gamma(1+5\alpha)} + \cdots. \end{aligned} \tag{4.6}$$

Substituting $\alpha = 1$ we get

$$\begin{aligned}\psi(\chi,\tau) &= 2\chi - 4\tau\chi + 8\tau^2\chi - 16\tau^3\chi + 32\tau^4\chi - 64\tau^5\chi + 128\tau^6\chi - \cdots \\ &= 2\chi(1 - 2\tau + 4\tau^2 - 8\tau^3 + 16\tau^4 - 32\tau^5\chi + 64\tau^6\chi - \cdots),\end{aligned} \tag{4.7}$$

this result agrees, with the Maclaurin series of the exact solution of Eq (4.1) in the integer case, which is

$$\psi(\chi,\tau) = \frac{2\chi}{1+2\tau}. \tag{4.8}$$

The following figures illustrate some simulations of Example 4.1.

In Figure 1, we sketch the solution of Example 4.1 in 3D space with $\alpha = 1$. We compare the exact solution of the integer order case of Example 4.1 with the fifth approximate solution from DPSM and sketch the error in Figure 2. Figure 3 present the contour graphs of the solution with different values of α.

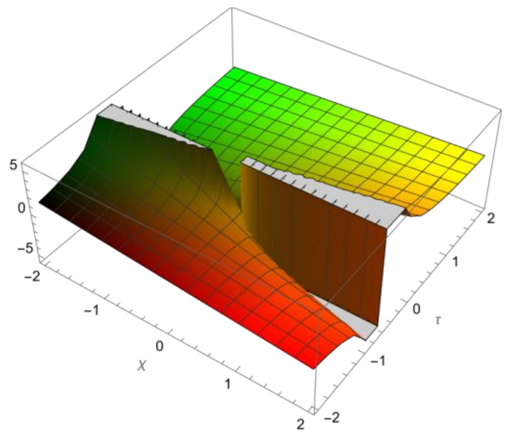
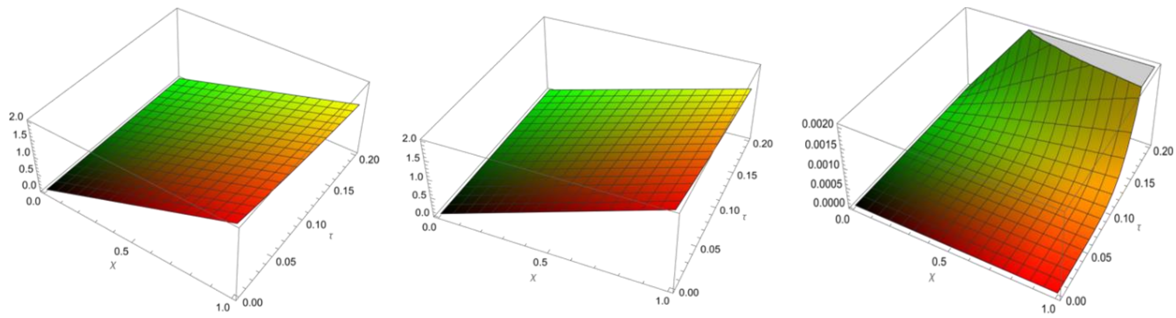
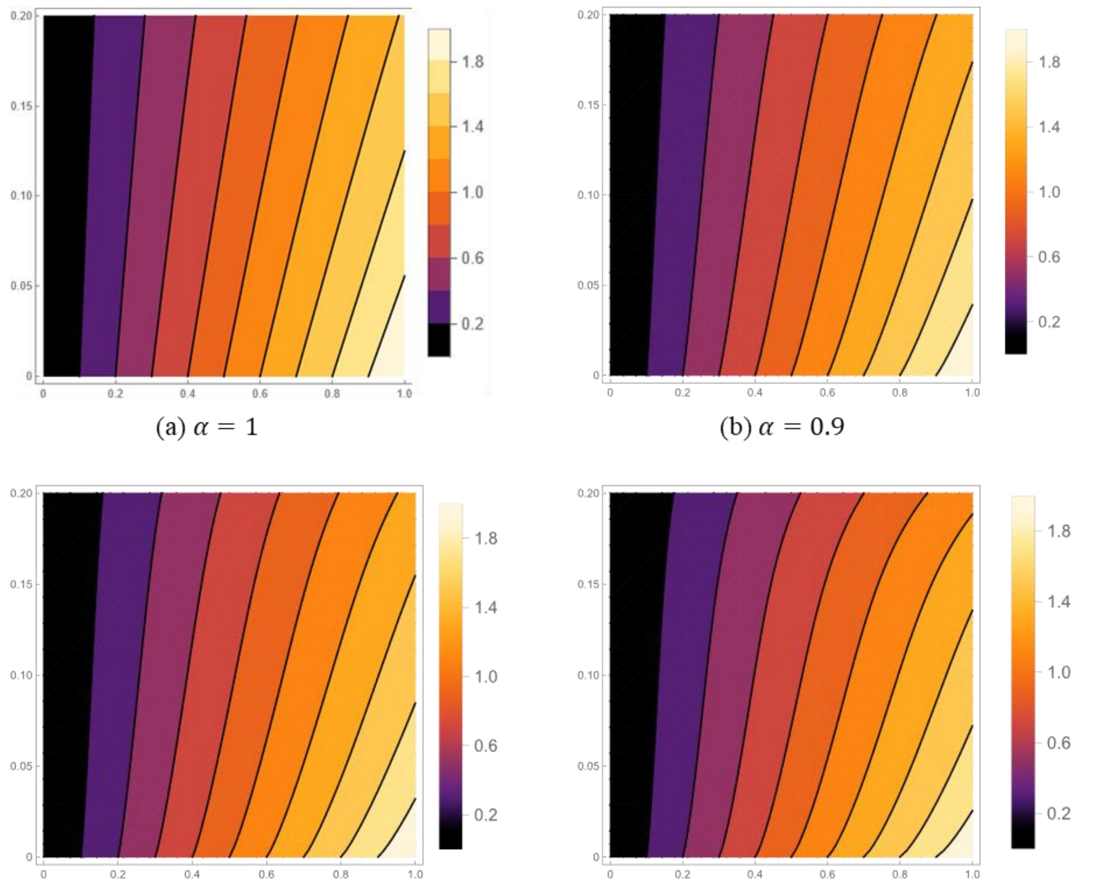


Figure 1. The 3D surface plot of the solution $\psi(\chi,\tau)$ for Example 4.1 with $\alpha = 1$, $-2 \le \chi \le 2$ and $-2 \le \tau \le 2$.



(a) $\psi(\chi,\tau)$ (b) $\psi_5(\chi,\tau)$ (c) $|\psi(\chi,\tau)-\psi_5(\chi,\tau)|$

Figure 2. The 3D surface plot of (a) exact solution $\psi(\chi,\tau)$, (b) the fifth approximation solution $\psi_5(\chi,\tau)$ and (c) the absolute error of the exact solution and approximation solution, with $\alpha=1$, $0\le\chi\le 1$ and $0\le\tau\le 0.2$.



(a) $\alpha=1$ (b) $\alpha=0.9$

(c) $\alpha=0.8$ (d) $\alpha=0.7$

Figure 3. The contour graph of the solution $\psi(\chi,\tau)$ of the fractional Burger equation at several values of α in Example 4.1.

Example 4.2. [28] Consider the following time-fractional Phi-4 equation of the form:

$$D_\tau^{2\alpha}\psi(\chi,\tau)=\psi_{\chi\chi}(\chi,\tau)-\mu^2\psi(\chi,\tau)-\lambda\psi^3(\chi,\tau), \tag{4.9}$$

with initial conditions,

$$\psi(\chi,0)=\sqrt{\frac{-\mu^2}{\lambda}}\tanh\left(\mu\chi\sqrt{\frac{1}{2(\nu^2-1)}}\right), \tag{4.10}$$

$$D_\tau^\alpha\psi(\chi,0)=-\mu\nu\sqrt{\frac{-\mu}{2\lambda(\nu^2-1)}}\operatorname{sech}^2\left(\mu\chi\sqrt{\frac{1}{2(\nu^2-1)}}\right), \tag{4.11}$$

where ν is the speed of the traveling wave $\alpha\epsilon(0,1)$. The exact solution for $\alpha=1$ is given by [28],

$$\psi(\chi,\tau)=\sqrt{\frac{-\mu^2}{\lambda}}\tanh\left(\mu(\chi-\nu\tau)\sqrt{\frac{1}{2(\nu^2-1)}}\right). \tag{4.12}$$

Solution 4.2. The solution by DPSM can be obtained replacing each part of Eq (4.9) with its suitable coefficient of $\frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)}$ as in Theorem 3.1 to get the following:

$$\varrho_{n+2}(\chi)=\varrho_n''(\chi)-\mu^2\varrho_n(\chi)-\lambda\sum_{\substack{i,j,k\\ i+j+k=n}}\frac{\varrho_i(\chi)\varrho_j(\chi)\varrho_k(\chi)\Gamma((i+j+k)\alpha+1)}{\Gamma(i\alpha+1)\Gamma(j\alpha+1)\Gamma(k\alpha+1)}, \tag{4.13}$$

from the initial conditions we have that $\varrho_0(\chi)=\psi(\chi,0)$ and $\varrho_1(\chi)=D_\tau^\alpha\psi(\chi,0)$.
For $n=0$,

$$\begin{aligned}\varrho_2(\chi)&=\varrho_0''(\chi)-\mu^2\varrho_0(\chi)-\lambda\varrho_0^3(\chi)\\&=\frac{\lambda\left(-\frac{\mu^2}{\lambda}\right)^{3/2}\nu^2\operatorname{Sech}^2\left(\frac{\mu\sqrt{\frac{1}{-1+\nu^2}}\chi}{\sqrt{2}}\right)\operatorname{Tanh}\left(\frac{\mu\sqrt{\frac{1}{-1+\nu^2}}\chi}{\sqrt{2}}\right)}{-1+\nu^2}.\end{aligned} \tag{4.14}$$

For $n=1$,

$$\begin{aligned}\varrho_3(\chi)&=\varrho_1''(\chi)-\mu^2\varrho_1(\chi)-3\lambda\varrho_1(\chi)\varrho_0^2(\chi)\\&=\frac{\mu^3\nu^3\left(-2+\cosh\left(\sqrt{2}\mu\sqrt{\frac{1}{-1+\nu^2}}\chi\right)\right)\operatorname{sech}^4\left(\frac{\mu\sqrt{\frac{1}{-1+\nu^2}}\chi}{\sqrt{2}}\right)\sqrt{-\frac{\mu}{\lambda(-1+\nu^2)}}}{\sqrt{2}(-1+\nu^2)}.\end{aligned} \tag{4.15}$$

For $n=2$,

$$\begin{aligned}\varrho_4(\chi)&=\varrho_2''(\chi)-\mu^2\,\varrho_2(\chi)-3\lambda\varrho_1^2(\chi)\varrho_0(\chi)-3\lambda\varrho_2(\chi)\varrho_0^2(\chi)\\&=\frac{1}{2(\nu^2-1)^2\,\lambda(\nu^2-1)}\lambda\mu\nu^2\left(-\frac{\mu^2}{\lambda}\right)^{3/2}\tanh\left(\frac{\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)\operatorname{sech}^4\left(\frac{\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)\\&\quad\left(2\mu\,\lambda(\nu^2-1)\left(\nu^2\cosh\left(\sqrt{2}\mu\sqrt{\frac{1}{\nu^2-1}}\chi\right)-2\nu^2-3\right)-3\lambda(\nu^2-1)^2\right).\end{aligned} \tag{4.16}$$

For $n=3$,

$$\begin{aligned}\varrho_5(\chi)&=\varrho_3''(\chi)-\mu^2\,\varrho_3(\chi)-\lambda\varrho_1^3(\chi)-6\lambda\varrho_0(\chi)\varrho_1(\chi)\varrho_2(\chi)-3\lambda\varrho_3(\chi)\varrho_0^3(\chi)\\&=\frac{1}{8\sqrt{2}(\nu^2-1)^2}\mu^3\nu^3\left(-\frac{\mu}{\lambda(\nu^2-1)}\right)^{3/2}\left[sech^7\left(\frac{\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)\mu\,\lambda(\nu^2-1)\,48\nu^2\sqrt{-\frac{\mu^2}{\lambda}}sinh\left(\frac{\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)\right.\\&\quad-21\nu^2\sqrt{-\frac{\mu^2}{\lambda}}sinh\left(\frac{3\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)+3\nu^2\sqrt{-\frac{\mu^2}{\lambda}}sinh\left(\frac{5\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)-48\sqrt{-\frac{\mu^2}{\lambda}}sinh\left(\frac{\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)\end{aligned}$$

$$+21\sqrt{-\frac{\mu^2}{\lambda}}\,sinh\left(\frac{3\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)-3\sqrt{-\frac{\mu^2}{\lambda}}\,sinh\left(\frac{5\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)+\nu^2\left(-cosh\left(\frac{5\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)\right)\Bigg)$$

$$+20(\nu^2+3)\,cosh\left(\frac{\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)-(11\nu^2+39)\,cosh\left(\frac{3\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)$$

$$+3\,cosh\left(\frac{5\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)+4\lambda(\nu^2-1)^2\,cosh\left(\frac{\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)\Bigg], \tag{4.17}$$

and then the 5th truncated series for $\psi(\chi,\tau)$ can be expressed as,

$$\psi_5(\chi,\tau)=\varrho_0(\chi)+\frac{\varrho_1(\chi)(\tau-\tau_0)}{\Gamma(\alpha+1)}+\sum_{i=2}^{5}\frac{\varrho_i(\chi)(t-t_0)^{i\alpha}}{\Gamma(i\alpha+1)}$$

$$=\sqrt{-\frac{\mu^2}{\lambda}}\,tanh\left(\frac{\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)-\frac{\mu\nu sech^2\left(\frac{\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)\sqrt{-\frac{\mu}{\lambda(\nu^2-1)}}(\tau-\tau_0)}{\sqrt{2}\Gamma(\alpha+1)}$$

$$+\frac{\lambda\left(-\frac{\mu^2}{\lambda}\right)^{3/2}\nu^2 sech^2\left(\frac{\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)tanh\left(\frac{\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)(\tau-\tau_0)^{2\alpha}}{(\nu^2-1)\,\Gamma(2\alpha+1)}$$

$$-\frac{\mu^3\nu^3\left(cosh\left(\sqrt{2}\mu\sqrt{\frac{1}{\nu^2-1}}\chi\right)-2\right)sech^4\left(\frac{\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)\sqrt{-\frac{\mu}{\lambda(\nu^2-1)}}(\tau-\tau_0)^{3\alpha}}{\sqrt{2}(\nu^2-1)\,\Gamma(3\alpha+1)}$$

$$+\frac{\lambda\mu\left(-\frac{\mu^2}{\lambda}\right)^{3/2}\nu^2 sech^4\left(\frac{\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)tanh\left(\frac{\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)\left(2\mu\left(cosh\left(\sqrt{2}\mu\sqrt{\frac{1}{\nu^2-1}}\chi\right)\nu^2-2\nu^2-3\right)\lambda(\nu^2-1)-3\lambda(\nu^2-1)^2\right)(\tau-\tau_0)^{4\alpha}}{2(\nu^2-1)^2\,\Gamma(4\alpha+1)\,\lambda(\nu^2-1)}$$

$$+\frac{1}{8\sqrt{2}(\nu^2-1)^2\,\Gamma(5\alpha+1)}\mu^3\nu^3 sech^7\left(\frac{\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)\left(-\frac{\mu}{\lambda(\nu^2-1)}\right)^{3/2}\Bigg[4\lambda\,cosh\left(\frac{\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)(\nu^2-1)^2$$

$$+\mu\Bigg[-cosh\left(\frac{5\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)\nu^2+48\sqrt{-\frac{\mu^2}{\lambda}}\,sinh\left(\frac{\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)\nu^2-21\sqrt{-\frac{\mu^2}{\lambda}}\,sinh\left(\frac{3\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)\nu^2$$

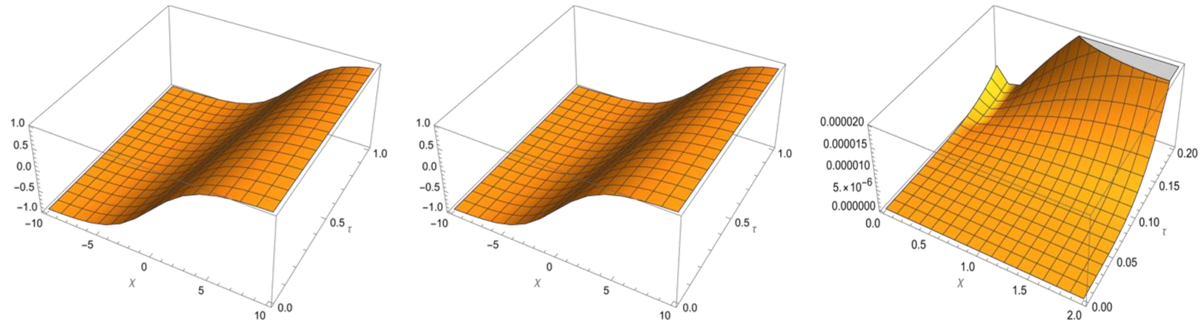
$$+3\sqrt{-\frac{\mu^2}{\lambda}}\,sinh\left(\frac{5\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)\nu^2+20(\nu^2+3)\,cosh\left(\frac{\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)-(11\nu^2+39)\,cosh\left(\frac{3\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)$$

$$+3\,cosh\left(\frac{5\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)-48\sqrt{-\frac{\mu^2}{\lambda}}\,sinh\left(\frac{\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)+21\sqrt{-\frac{\mu^2}{\lambda}}\,sinh\left(\frac{3\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)$$

$$-3\sqrt{-\frac{\mu^2}{\lambda}}\,sinh\left(\frac{5\mu\sqrt{\frac{1}{\nu^2-1}}\chi}{\sqrt{2}}\right)\Bigg]\lambda(\nu^2-1)\Bigg](\tau-\tau_0)^{5\alpha}. \tag{4.18}$$

We mention, that it's hard to find the fifth term by other analytical methods. By DPSM we can get many terms of the series solution by computer software.

In the following Figures 4 and 5, we sketch the exact solution of Example 4.2 in the integer case and compare it with approximate solution of fifth order in 2D and 3D. Also, we sketch the application solution in the 2D plane with different values of α to show how the approximation solution converges to the exact solution of integer order.



(a) $\psi(\chi,\tau)$ (b) $\psi_5(\chi,\tau)$ (c) $|\psi(\chi,\tau)-\psi_5(\chi,\tau)|$

Figure 4. The 3D surface plot of (a) exact solution $\psi(\chi,\tau)$, (b) the fifth approximation solution $\psi_5(\chi,\tau)$ and (c) the absolute error of the exact solution and approximation solution, for time-fractional Phi-4 at $\tau\in(0,1)$, $\chi\in(-10,10)$, $\alpha=1,\mu=1,\lambda=-1,\upsilon=3$.

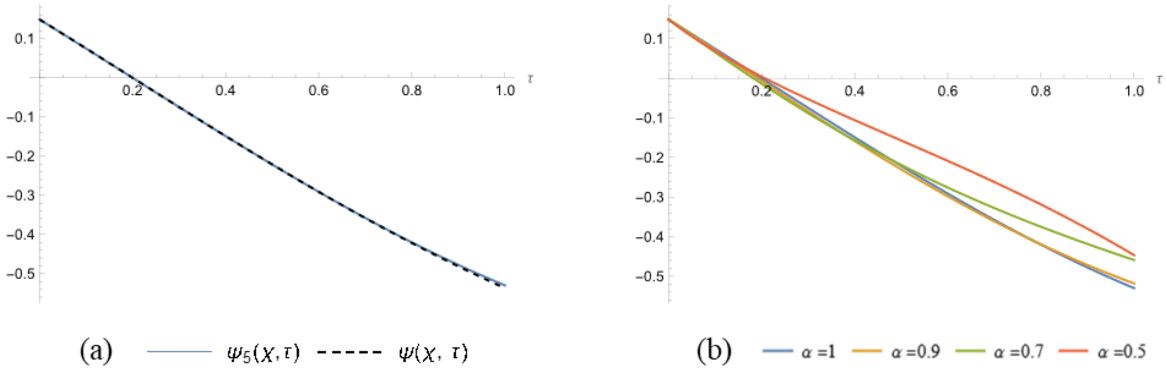


Figure 5. The 2D plot of (a) approximation solution $\psi_5(\chi,\tau)$ and exact solution $\psi(\chi,\tau)$ for time-fractional Phi-4 at $\tau\in(0,1)$, $\chi=0.6$, $\alpha=1,\mu=1,\lambda=-1,\upsilon=3$, (b) the DPSM solutions of different values of α in Example 4.2.

Example 4.3. [22] The following form is considered:

$$D_\tau^\alpha\psi(\chi,\tau)+\varphi_{\chi\chi}(\chi,\tau)+2\big(\psi^2(\chi,\tau)+\varphi^2(\chi,\tau)\big)\varphi(\chi,\tau)=0, \tag{4.19}$$

$$D_\tau^\alpha\varphi(\chi,\tau)-\psi_{\chi\chi}(\chi,\tau)-2\big(\psi^2(\chi,\tau)+\varphi^2(\chi,\tau)\big)\psi(\chi,\tau)=0, \tag{4.20}$$

with initial conditions,

$$\psi(\chi,0) = \cos(\chi), \tag{4.21}$$
$$\varphi(\chi,0) = \sin(\chi). \tag{4.22}$$

To find the general form of solutions by DPSM just do the replacements in Theorem 3.1 which could be directly written by,

$$\varrho_{n+1}(\chi) = -\rho_n''(\chi) - 2\sum_{\substack{i,j,k \\ i+j+k=n}} \frac{\left(\varrho_i(\chi)\varrho_j(\chi)\rho_k(\chi)+\rho_i(\chi)\rho_j(\chi)\rho_k(\chi)\right)\Gamma(n\alpha+1)}{\Gamma(i\alpha+1)\Gamma(j\alpha+1)\Gamma(k\alpha+1)}, \tag{4.23}$$

$$\rho_{n+1}(\chi) = \varrho_n''(\chi) + 2\sum_{\substack{i,j,k \\ i+j+k=n}} \frac{\left(\varrho_i(\chi)\varrho_j(\chi)\varrho_k(\chi)+\rho_i(\chi)\rho_j(\chi)\varrho_k(\chi)\right)\Gamma(n\alpha+1)}{\Gamma(i\alpha+1)\Gamma(j\alpha+1)\Gamma(k\alpha+1)}. \tag{4.24}$$

For $n = 0$,

$$\varrho_1(\chi) = -\delta\rho_0''(\chi) - \gamma\left(\varrho_0(\chi)\varrho_0(\chi)\rho_0(\chi) + \rho_0(\chi)\rho_0(\chi)\rho_0(\chi)\right) - \phi(\chi)\rho_0(\chi) = -\sin x,$$

$$\rho_1(\chi) = \delta\varrho_0''(\chi) + \gamma\left(\varrho_0(\chi)\varrho_0(\chi)\varrho_0(\chi) + \rho_0(\chi)\rho_0(\chi)\varrho_0(\chi)\right) + \phi(\chi)\varrho_0(\chi) = \cos x.$$

For $n = 1$,

$$\varrho_2(\chi) = -\cos x,$$
$$\rho_2(\chi) = -\sin x.$$

For $n = 2$,

$$\varrho_3(\chi) = \left(5 - 2\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}\right)\sin x,$$

$$\rho_3(\chi) = -\left(5 - 2\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}\right)\cos x.$$

For $n = 3$,

$$\varrho_4(\chi) = \left(5 - 2\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2} + \frac{4\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} - \frac{2\Gamma(1+3\alpha)}{\Gamma(1+\alpha)^3}\right)\cos x,$$
$$\rho_4(\chi) = \left(5 - 2\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2} + \frac{4\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} - \frac{2\Gamma(1+3\alpha)}{\Gamma(1+\alpha)^3}\right)\sin x,$$

which is the same general form of the solutions obtained in [22].

Example 4.4. [37] Consider the following time-fractional 3-dimentional Navier Stokes equation:

$$D_\tau^\alpha\psi_1(\chi,\varsigma,\zeta,\tau) + \psi_1(\chi,\varsigma,\zeta,\tau)\frac{\partial\psi_1(\chi,\varsigma,\zeta,\tau)}{\partial\chi} + \psi_2(\chi,\varsigma,\zeta,\tau)\frac{\partial\psi_1(\chi,\varsigma,\zeta,\tau)}{\partial\varsigma} + \psi_3(\chi,\varsigma,\zeta,\tau)\frac{\partial\psi_1(\chi,\varsigma,\zeta,\tau)}{\partial\zeta}$$

$$= \upsilon\left(\frac{\partial^2\psi_1(\chi,\varsigma,\zeta,\tau)}{\partial\chi^2} + \frac{\partial^2\psi_1(\chi,\varsigma,\zeta,\tau)}{\partial\varsigma^2} + \frac{\partial^2\psi_1(\chi,\varsigma,\zeta,\tau)}{\partial\zeta^2}\right), \tag{4.25}$$

$$D_\tau^\alpha \psi_2(\chi,\varsigma,\zeta,\tau) + \psi_1(\chi,\varsigma,\zeta,\tau)\frac{\partial \psi_2(\chi,\varsigma,\zeta,\tau)}{\partial \chi} + \psi_2(\chi,\varsigma,\zeta,\tau)\frac{\partial \psi_2(\chi,\varsigma,\zeta,\tau)}{\partial \varsigma} + \psi_3(\chi,\varsigma,\zeta,\tau)\frac{\partial \psi_2(\chi,\varsigma,\zeta,\tau)}{\partial \zeta}$$
$$= \upsilon\left(\frac{\partial^2 \psi_2(\chi,\varsigma,\zeta,\tau)}{\partial \chi^2} + \frac{\partial^2 \psi_2(\chi,\varsigma,\zeta,\tau)}{\partial \varsigma^2} + \frac{\partial^2 \psi_2(\chi,\varsigma,\zeta,\tau)}{\partial \zeta^2}\right), \tag{4.26}$$

$$D_\tau^\alpha \psi_3(\chi,\varsigma,\zeta,\tau) + \psi_1(\chi,\varsigma,\zeta,\tau)\frac{\partial \psi_3(\chi,\varsigma,\zeta,\tau)}{\partial \chi} + \psi_2(\chi,\varsigma,\zeta,\tau)\frac{\partial \psi_3(\chi,\varsigma,\zeta,\tau)}{\partial \varsigma} + \psi_3(\chi,\varsigma,\zeta,\tau)\frac{\partial \psi_3(\chi,\varsigma,\zeta,\tau)}{\partial \zeta}$$
$$= \upsilon\left(\frac{\partial^2 \psi_3(\chi,\varsigma,\zeta,\tau)}{\partial \chi^2} + \frac{\partial^2 \psi_3(\chi,\varsigma,\zeta,\tau)}{\partial \varsigma^2} + \frac{\partial^2 \psi_3(\chi,\varsigma,\zeta,\tau)}{\partial \zeta^2}\right), \tag{4.27}$$

where $\varphi \in \mathbb{R}$, and with the ICs,

$$\psi_1(\chi,\varsigma,\zeta,0) = -0.5\chi + \varsigma + \zeta, \tag{4.28}$$
$$\psi_2(\chi,\varsigma,\zeta,0) = \chi - 0.5\varsigma + \zeta, \tag{4.29}$$
$$\psi_3(\chi,\varsigma,\zeta,0) = \chi + \varsigma - 0.5\zeta. \tag{4.30}$$

The exact solution, when $\alpha = 1$, of the integer case is

$$\psi_1(\chi,\varsigma,\zeta,\tau) = \frac{-0.5\chi+\varsigma+\zeta-2.25\,\chi\tau}{1-2.25\,\tau^2},$$

$$\psi_2(\chi,\varsigma,\zeta,\tau) = \frac{\chi-0.5\varsigma+\zeta-2.25\varsigma\tau}{1-2.25\,\tau^2},$$

$$\psi_3(\chi,\varsigma,\zeta,\tau) = \frac{\chi+\varsigma-0.5\zeta-2.25\zeta\tau}{1-2.25\,\tau^2}.$$

Solution 4.3. The solution by DPSM can be obtained by replacing each part of Eqs (4.25)–(4.27) with its suitable coefficient of $\frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)}$ as in Theorem 3.1 to get the following relations:

$$\varrho_{n+1}(\chi,\varsigma,\zeta) = -\sum_{i=0}^{n} \frac{\left(\varrho_i(\chi,\varsigma,\zeta)\frac{\partial \varrho_{n-i}(\chi,\varsigma,\zeta)}{\partial \chi} + \rho_i(\chi,\varsigma,\zeta)\frac{\partial \varrho_{n-i}(\chi,\varsigma,\zeta)}{\partial \varsigma} + \sigma_i(\chi,\varsigma,\zeta)\frac{\partial \varrho_{n-i}(\chi,\varsigma,\zeta)}{\partial \zeta}\right)\Gamma(n\alpha+1)}{\Gamma(i\alpha+1)\Gamma((n-i)\alpha+1)}$$
$$+\upsilon\left(\frac{\partial^2 \varrho_n(\chi,\varsigma,\zeta)}{\partial \chi^2} + \frac{\partial^2 \varrho_n(\chi,\varsigma,\zeta)}{\partial \varsigma^2} + \frac{\partial^2 \varrho_n(\chi,\varsigma,\zeta)}{\partial \zeta^2}\right),$$

$$\rho_{n+1}(\chi,\varsigma,\zeta) = -\sum_{i=0}^{n} \frac{\left(\varrho_i(\chi,\varsigma,\zeta)\frac{\partial \rho_{n-i}(\chi,\varsigma,\zeta)}{\partial \chi} + \rho_i\frac{\partial \rho_{n-i}(\chi,\varsigma,\zeta)}{\varsigma} + \sigma_i\frac{\partial \rho_{n-i}(\chi,\varsigma,\zeta)}{\partial \zeta}\right)\Gamma(n\alpha+1)}{\Gamma(i\alpha+1)\Gamma((n-i)\alpha+1)}$$
$$+\upsilon\left(\frac{\partial^2 \rho_n(\chi,\varsigma,\zeta)}{\partial \chi^2} + \frac{\partial^2 \rho_n(\chi,\varsigma,\zeta)}{\partial \varsigma^2} + \frac{\partial^2 \rho_n(\chi,\varsigma,\zeta)}{\partial \zeta^2}\right),$$

$$\sigma_{n+1}(\chi,\varsigma,\zeta) = -\sum_{i=0}^{n} \frac{\left(\varrho_i(\chi,\varsigma,\zeta)\frac{\partial \sigma_{n-i}(\chi,\varsigma,\zeta)}{\partial \chi} + \sigma_i\frac{\partial \sigma_{n-i}(\chi,\varsigma,\zeta)}{\partial \varsigma} + \sigma_i\frac{\partial \sigma_{n-i}(\chi,\varsigma,\zeta)}{\partial \zeta}\right)\Gamma(n\alpha+1)}{\Gamma(i\alpha+1)\Gamma((n-i)\alpha+1)}$$
$$+\upsilon\left(\frac{\partial^2 \sigma_n(\chi,\varsigma,\zeta)}{\partial \chi^2} + \frac{\partial^2 \sigma_n(\chi,\varsigma,\zeta)}{\partial \varsigma^2} + \frac{\partial^2 \sigma_n}{\partial \zeta^2}\right).$$

For $n = 0$,

$$\varrho_1 = -\varrho_0 \frac{\partial \varrho_0}{\partial \chi} - \rho_0 \frac{\partial \varrho_0}{\partial \varsigma} - \sigma_0 \frac{\partial \varrho_0}{\partial \zeta} + \upsilon \left(\frac{\partial^2 \varrho_0}{\partial \chi^2} + \frac{\partial^2 \varrho_0}{\partial \varsigma^2} + \frac{\partial^2 \varrho_0}{\partial \zeta^2} \right)$$
$$= -(-0.5\chi + \varsigma + \zeta)(-0.5) - (\chi - 0.5\varsigma + \zeta)(1) - (\chi + \varsigma - 0.5\zeta)(1) = -2.25\chi,$$
$$\rho_1 = -\varrho_0 \frac{\partial \rho_0}{\partial \chi} - \rho_0 \frac{\partial \rho_0}{\partial \varsigma} - \sigma_0 \frac{\partial \rho_0}{\partial \zeta} + \upsilon \left(\frac{\partial^2 \rho_0}{\partial \chi^2} + \frac{\partial^2 \rho_0}{\partial \varsigma^2} + \frac{\partial^2 \rho_0}{\partial \zeta^2} \right)$$
$$= -(-0.5\chi + \varsigma + \zeta)(1) - (\chi - 0.5\varsigma + \zeta)(-0.5) - (\chi + \varsigma - 0.5\zeta)(1) = -2.25\varsigma,$$
$$\sigma_1 = -\varrho_0 \frac{\partial \sigma_0}{\partial \chi} - \rho_0 \frac{\partial \sigma_0}{\partial \varsigma} - \sigma_0 \frac{\partial \sigma_0}{\partial \zeta} + \upsilon \left(\frac{\partial^2 \sigma_0}{\partial \chi^2} + \frac{\partial^2 \sigma_0}{\partial \varsigma^2} + \frac{\partial^2 \sigma_0}{\partial \zeta^2} \right)$$
$$= -(-0.5\chi + \varsigma + \zeta)(1) - (\chi - 0.5\varsigma + \zeta)(1) - (\chi + \varsigma - 0.5\zeta)(-0.5) = -2.25\zeta.$$

For $n = 1$,

$$\varrho_2 = -\varrho_1 \frac{\partial \varrho_0}{\partial \chi} - \varrho_0 \frac{\partial \varrho_1}{\partial \chi} - \rho_1 \frac{\partial \varrho_0}{\partial \varsigma} - \rho_0 \frac{\partial \varrho_1}{\partial \varsigma} - \sigma_1 \frac{\partial \varrho_0}{\partial \zeta} - \sigma_0 \frac{\partial \varrho_1}{\partial \zeta} + \upsilon \left(\frac{\partial^2 \varrho_1}{\partial \chi^2} + \frac{\partial^2 \varrho_1}{\partial \varsigma^2} + \frac{\partial^2 \varrho_1}{\partial \zeta^2} \right)$$
$$= -(-2.25\chi)(-0.5) - (-0.5\chi + \varsigma + \zeta)(-2.25) - (-2.25\varsigma)(1) - (-2.25\zeta)(1)$$
$$= -2(0.5)(2.25)\chi + 2(2.25)\varsigma + 2(2.25)\zeta = 2(2.25)(-0.5\chi + \varsigma + \zeta) = 4.5\varrho_0,$$
$$\rho_2 = -\varrho_1 \frac{\partial \rho_0}{\partial \chi} - \varrho_0 \frac{\partial \rho_1}{\partial \chi} - \rho_1 \frac{\partial \rho_0}{\partial \varsigma} - \rho_0 \frac{\partial \rho_1}{\partial \varsigma} - \sigma_1 \frac{\partial \rho_0}{\partial \zeta} - \sigma_0 \frac{\partial \rho_1}{\partial \zeta} + \upsilon \left(\frac{\partial^2 \rho_1}{\partial \chi^2} + \frac{\partial^2 \rho_1}{\partial \varsigma^2} + \frac{\partial^2 \rho_1}{\partial \zeta^2} \right)$$
$$= 2(2.25)\chi - 2(0.5)(2.25)\varsigma + 2(2.25)\zeta = 2(2.25)(\chi - 0.5\varsigma + \zeta) = 4.5\rho_0,$$
$$\sigma_2 = -\varrho_1 \frac{\partial \sigma_0}{\partial \chi} - \varrho_0 \frac{\partial \sigma_1}{\partial \chi} - \rho_1 \frac{\partial \sigma_0}{\partial \varsigma} - \rho_0 \frac{\partial \sigma_1}{\partial \varsigma} - \sigma_1 \frac{\partial \sigma_0}{\partial \zeta} - \sigma_0 \frac{\partial \sigma_1}{\partial \zeta} + \upsilon \left(\frac{\partial^2 \sigma_1}{\partial \chi^2} + \frac{\partial^2 \sigma_1}{\partial \varsigma^2} + \frac{\partial^2 \sigma_1}{\partial \zeta^2} \right)$$
$$= 2(2.25)\chi + 2(2.25)\varsigma - 2(0.5)(2.25)\zeta = 2(2.25)(\chi + \varsigma - 0.5\zeta) = 4.5\sigma_0,$$

For $n = 2$,

$$\varrho_3 = -\left(20.25 + 5.0625 \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\right)\chi,$$
$$\rho_3 = -\left(20.25 + 5.0625 \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\right)\varsigma,$$
$$\sigma_3 = -\left(20.25 + 5.0625 \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\right)\zeta.$$

For $n = 3$,

$$\varrho_4 = \left(40.5 + 10.125 \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} + \frac{20.25\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)}\right)(-0.5\chi + \varsigma + \zeta),$$
$$\rho_4 = \left(40.5 + 10.125 \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} + \frac{20.25\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)}\right)(\chi - 0.5\varsigma + \zeta),$$
$$\sigma_4 = \left(40.5 + 10.125 \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} + \frac{20.25\,\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)}\right)(\chi + \varsigma - 0.5\zeta).$$

So, the solution of systems (4.25)–(4.27) and (4.28)–(4.30) has the following series form:

$$\psi_1(\chi, \varsigma, \zeta, \tau) = -0.5\chi + \varsigma + \zeta - \frac{2.25}{\Gamma(1+\alpha)}\chi\,\tau^{\alpha}$$
$$+ \frac{2(2.25)}{\Gamma(1+2\alpha)}(-0.5\chi + \varsigma + \zeta)\tau^{2\alpha} - \frac{(2.25)^2}{\Gamma(1+3\alpha)}\left(4 + \frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}\right)\chi\,\tau^{3\alpha}$$

$$+\frac{(2.25)^2}{\Gamma(1+4\alpha)}\left(8+\frac{2\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}+\frac{4\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\right)(-0.5\chi+\varsigma+\zeta)\tau^{4\alpha}+\cdots, \quad (4.31)$$

$$\psi_2(\chi,\varsigma,\zeta,\tau)=\chi-0.5\varsigma+\zeta-\frac{2.25}{\Gamma(1+\alpha)}\varsigma\,\tau^{\alpha}$$
$$+\frac{2(2.25)}{\Gamma(1+2\alpha)}(\chi-0.5\varsigma+\zeta)\tau^{2\alpha}-\frac{(2.25)^2}{\Gamma(1+3\alpha)}\left(4+\frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}\right)\varsigma\,\tau^{3\alpha}$$
$$+\frac{(2.25)^2}{\Gamma(1+4\alpha)}\left(8+\frac{2\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}+\frac{4\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\right)(\chi-0.5\varsigma+\zeta)\tau^{4\alpha}+\cdots, \quad (4.32)$$

$$\psi_3(\chi,\varsigma,\zeta,\tau)=\chi+\varsigma-0.5\zeta-\frac{2.25}{\Gamma(1+\alpha)}\zeta\,\tau^{\alpha}$$
$$+\frac{2(2.25)}{\Gamma(1+2\alpha)}(\chi+\varsigma-0.5\zeta)\tau^{2\alpha}-\frac{(2.25)^2}{\Gamma(1+3\alpha)}\left(4+\frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}\right)\zeta\,\tau^{3\alpha}$$
$$+\frac{(2.25)^2}{\Gamma(1+4\alpha)}\left(8+\frac{2\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}+\frac{4\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\right)(\chi+\varsigma-0.5\zeta)\tau^{4\alpha}+\cdots. \quad (4.33)$$

The following figure (Figure 6) illustrates the graph of the exact solution $(\alpha=1)$ of Example 4.3 and the 10^{th} approximation solutions.

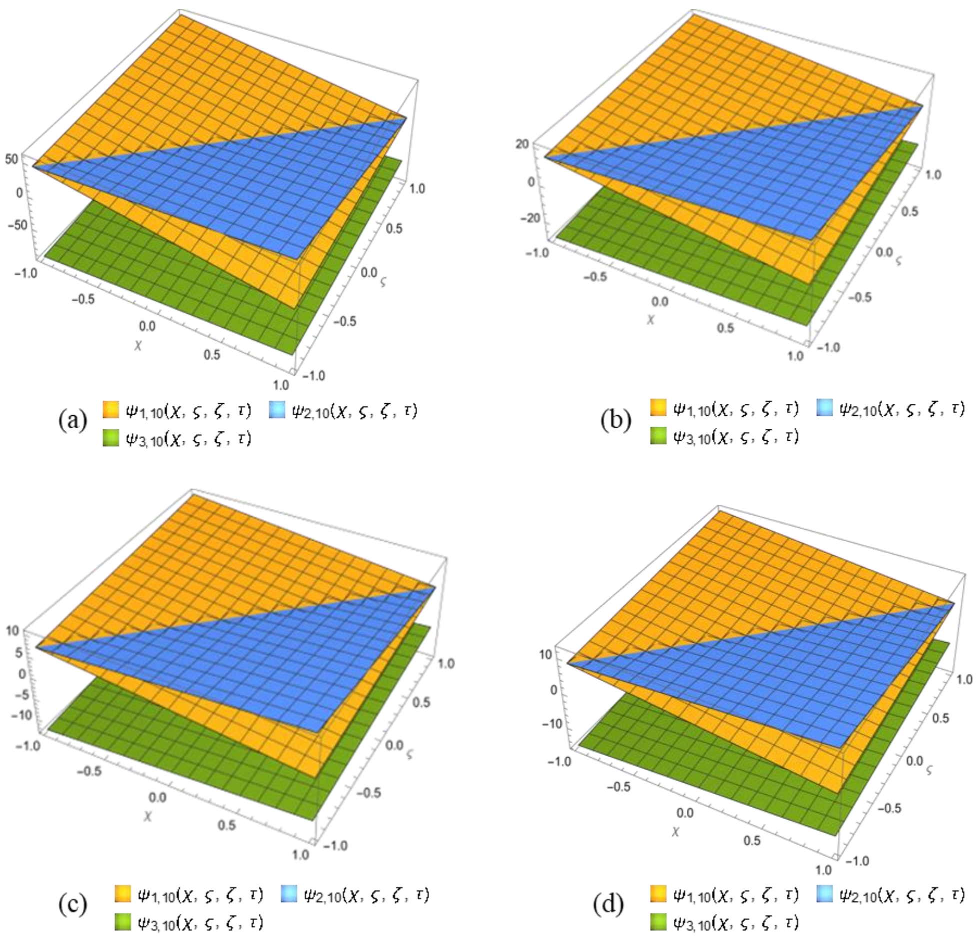


Figure 6. The 3D surface plot of the 10^{th} approximate solutions of ψ_1, ψ_2, and ψ_3 at various values of α and $\tau=0.5$ & $\zeta=3$ for the problem in Example 4.3. (a) $\alpha=0.6$, (b) $\alpha=0.8$, (c) $\alpha=1$, (d) $\alpha=1$ (exact solutions).

5. Conclusions

In several types of science like mathematics, physics, engineering statics, etc., there are massive numbers of equations and systems that need solutions. Mathematicians created and developed many analytical and numerical methods to find the solution accurately or approximately. Power series methods like residual power series, Laplace residual power series, and many others give exact solutions or sometimes provide approximate solutions that converge to the exact one. However, these methods require a lot of steps each time, especially for solving nonlinear equations and systems. In this paper, a new technique has been presented for the first time called the DPSM for solving TFPDEs. In the following, we mention some remarks on the new method

1) DPSM introduces an analytical series solution depending on the idea of power series representation.
2) The method is simple and applicable in presenting series solutions.
3) DPSM can obtain many terms of the series solution.
4) The proposed method, need no discretization, transformation differentiation or taking limit.
5) All numerical results obtained in this article by Mathematica 3.12.

This method is a simple method among all analytical methods. It depends on the idea of the power series expansion for solving time-fractional differential equations or systems. In the future, we attend to expand the replacements and solve integral equations.

Conflict of interest

The authors declare no conflicts of interest.

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