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Research article

Numerical investigation of fractional-order wave-like equation

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Abstract: The two approaches to solving nonlinear Caputo time-fractional wave-like equations with variable coefficients are examined in this study. The Homotopy perturbation transform method and the Yang transform decomposition method are the names of these two techniques. Three separate numerical examples are provided to demonstrate the effectiveness and precision of the suggested methods. The results were acquired to demonstrate the effectiveness and power of the two approaches, providing estimates with better precision and closed form solutions. The solutions to these kinds of equations can be found using the suggested methods as infinite series, and when these series are in closed form, they provide the exact solution. The suggested techniques have been demonstrated to be effective and efficient in their application. Three numerical examples are used to examine the methods accuracy and effectiveness.

Keywords: Homotopy perturbation method; Adomian decomposition method; Yang transform; nonlinear time-fractional wave-like equations; Caputo operator
Mathematics Subject Classification: 32B15, 34A34, 35A22, 35A24, 45A10

1. Introduction

Today, fractional calculus (FC) is widely acknowledged as a vital technique for describing real life events [1–5]. The presence of fractional formulations is discussed in a range of contexts, despite the fact that scholars view FC as a useful tool in systematic investigation [6, 7]. In addition to being consistent with the current stage, the dynamical properties of fractional differential equation systems typically provide a sufficient justification for earlier stages [8, 9]. Due to the novelty of the diversification process, which can result in a modest adjustment leading to significant output variability, the conversion of integer-order Differential equation (DE)-regulated systems to fractional DE-regulated structures should be accurate. Fractional calculus has received a lot of attention, and various applications in the fields of science have been presented [10–12]. Fractional differential equations are widely used in domains including signal processing, system recognition, reaction diffusion processes, control systems using dynamical systems, random walk models, and neural networks to simulate real-world phenomena [13,14]. FC provides us with a useful tool for characterizing memory and hereditary characteristics of various systems.

Differential equations with fractional derivatives offer effective methods for identifying memory and heredity features that classical systems typically ignore. Fractional-order derivative modelling is helpful for studying dynamical systems and accurately describes how real-world systems behave. In applied mathematics and physics, fractional differential equations are frequently employed in interpreting and modeling many realistic issues, including diffusion, fluid mechanics, chemistry, viscoelasticity, damping laws, electrical circuits, mathematical biology, relaxation processes, and so on [15–17]. Finding the numerical solution to fractional differential equations, linear and nonlinear, is now receiving much more attention from mathematicians. Some of the methods are Homotopy Analysis Method [18], Differential Transform Method [19], and Adomian Decomposition Method [20]. Many researchers have suggested innovative integral transform techniques to discover the analytical solution of linear and nonlinear FDEs. Some of them are Sumudu [21], Elzaki [22], Laplace [23], Mahgoub [24] and Natural [25]. For solving the nonlinear system of FDEs, the Adomian decomposition method was combined with Sumudu transform method [26], with Elzaki transform method [27,28], with Laplace transform method [29], with Mahgoub transform method [30] and with Natural transform method [31,32] and so on [33–35].

This study aims to solve the nonlinear Caputo time-fractional wave-like equation with variable coefficients by combining two effective methods: The Yang transform decomposition method (YTDM), which combines the Yang transform method and the Adomian decomposition method, and the Homotopy perturbation transform method (HPTM), which combines the Yang transform method and the Homotopy perturbation method. Xiao-Jun Yang presented the Yang transform, which can solve various differential equations with constant coefficients. The Adomian decomposition technique (ADM) [36, 37] is a well-known systematic method for solving deterministic or stochastic operator equations, such as ordinary, partial, integral, and integro-differential equations. The ADM is a sophisticated technique that provides fast algorithms for analytical solutions and numeric simulation in engineering and applied sciences. On the other hand, He [38,39] presented the homotopy perturbation approach in 1998. This method sees the solution as the sum of an infinite series, which usually converges quickly to accurate solutions. This approach has been used for a wide range of mathematical problems. The nonlinear Caputo time-fractional wave-like equations with variable coefficients are presented as follows:

$$
\begin{aligned}
D_{\varsigma}^{\wp}\mathbb{U} = &\sum_{i,j=1}^{n} B_{1ij}(\chi,\varsigma,\mathbb{U})\frac{\partial^{k+m}}{\partial \mathbf{y}_i^k \partial \mathbf{y}_j^m} B_{2ij}(\mathbb{U}_{\mathbf{y}i},\mathbb{U}_{\mathbf{y}j}) \\
&+ \sum_{i=1}^{n} C_{1i}(\chi,\varsigma,\mathbb{U})\frac{\partial^p}{\partial \mathbf{y}_i^p} C_{2i}(\mathbb{U}_{\mathbf{y}i}) + D(\chi,\varsigma,\mathbb{U}) + E(\chi,\varsigma),
\end{aligned}
\tag{1.1}
$$

with initial values

$$
\mathbb{U}(\chi,0) = a_0(\chi) \quad \text{and} \quad \mathbb{U}_{\varsigma}(\chi,0) = a_1(\chi), \tag{1.2}
$$

where $D_{\varsigma}^{\wp}$ is (Caputo) fractional operator having order $\wp$ and $1 < \wp \leq 2, \mathbb{U} = \mathbb{U}(\chi, \varsigma), \chi = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n) \in \mathbb{R}^n$, $\varsigma \geq 0$, B_{1ij}, $C_{1i}i$, $j \in 1, 2, \ldots, n$ are nonlinear functions of χ, ς and $\mathbb{U}$, B_{2ij}, $C_{2i}i$, $j \in 1, 2, \ldots, n$, are nonlinear functions of derivatives of $\mathbb{U}$ with respect to $\mathbf{y}_i$ and $\mathbf{y}_j i$, $j \in 1, 2, \ldots, n$, respectively. In addition, D, E are nonlinear functions and k, m, p are integers.

It should be noted that (1.1) simplifies to the standard wave-like equations with variable coefficients when $\wp = 2$. Numerous applied sciences, including nonlinear hydrodynamics, mathematical physics, physics, plasma physics, astrophysics, human movement sciences, and engineering biophysics, all highly depend on these kinds of equations. These equations explain the development of microscopic particles moving erratically when submerged in fluids, variations in laser light intensity, and velocity distributions of fluid particles in turbulent flows [40].

The rest of the paper is organized as follows: Introduction is included in Section 1. The fundamental definitions of FC, the Yang transform, and its properties are given in Section 2. The notion of HPTM is presented in Section 3, whereas the idea of YTDM is presented in Section 4. Its application to the nonlinear time-fractional wave-like equations is demonstrated in Section 5. We summarise the conclusion in Section 6.

2. Preliminaries

This part clearly explains certain key facts about the fractional derivatives and Yang transform along with their properties.

Definition 2.1. [1] The Caputo fractional derivative is given as

$$D_{\varsigma}^{\wp}\mathbb{U}(\mathbf{y}, \varsigma) = \frac{1}{\Gamma(k-\wp)} \int_0^{\varsigma} (\varsigma - \wp)^{k-\wp-1} \mathbb{U}^{(k)}(\mathbf{y}, \wp) d\wp, \tag{2.1}$$

where $k - 1 < \wp \leq k$ and $k \in \mathbb{N}$.

Definition 2.2. [41] Yang transform of a function $\mathbb{U}(\varsigma)$ is given as:

$$Y\{\mathbb{U}(\varsigma)\} = M(u) = \int_0^{\infty} e^{\frac{-\varsigma}{u}} \mathbb{U}(\varsigma) d\varsigma, \quad \varsigma > 0, u \in (-\varsigma_1, \varsigma_2), \tag{2.2}$$

with Yang inverse transform as

$$Y^{-1}\{M(u)\} = \mathbb{U}(\varsigma). \tag{2.3}$$

Definition 2.3. [41] Yang transform of a function with derivative of nth order as

$$Y\{\mathbb{U}^n(\varsigma)\} = \frac{M(u)}{u^n} - \sum_{k=0}^{n-1} \frac{\mathbb{U}^k(0)}{u^{n-k-1}}, \tag{2.4}$$

for all $n \in \mathbb{N}$.

Definition 2.4. [41] Yang transform of the function with derivative having fractional-order as

$$Y\{\mathbb{U}^{\wp}(\varsigma)\} = \frac{M(u)}{u^{\wp}} - \sum_{k=0}^{n-1} \frac{\mathbb{U}^k(0)}{u^{\wp-(k+1)}}, \quad 0 < \wp \leq n. \tag{2.5}$$

3. Construction of Homotopy perturbation transform method for solving FDEs

To illustrate the process of solution of the HPTM, we take FDEs of the form to show the general implementation of the proposed method as stated in [42, 43]

$$D_{\varsigma}^{\wp}\mathbb{U}(\mathbf{y},\varsigma) = \mathcal{P}_1[\mathbf{y}]\mathbb{U}(\mathbf{y},\varsigma) + \mathcal{Q}_1[\mathbf{y}]\mathbb{U}(\mathbf{y},\varsigma), \quad 1 < \wp \leq 2, \tag{3.1}$$

having initial values

$$\mathbb{U}(\mathbf{y},0) = \xi(\mathbf{y}), \quad \frac{\partial}{\partial\varsigma}\mathbb{U}(\mathbf{y},0) = \zeta(\mathbf{y}), \tag{3.2}$$

where $D_{\varsigma}^{\wp} = \frac{\partial^{\wp}}{\partial\varsigma^{\wp}}$ is (Caputo) fractional operator, $\mathcal{P}_1[\mathbf{y}]$, $\mathcal{Q}_1[\mathbf{y}]$ are operators, linear and nonlinear, respectively.

Proceeds the YT and using (2.5), we have

$$Y\left\{D_{\varsigma}^{\wp}\mathbb{U}(\mathbf{y},\varsigma)\right\} = Y\left\{\mathcal{P}_1[\mathbf{y}]\mathbb{U}(\mathbf{y},\varsigma) + \mathcal{Q}_1[\mathbf{y}]\mathbb{U}(\mathbf{y},\varsigma)\right\},$$

that is

$$\frac{1}{u^{\wp}}\left\{M(u) - u\mathbb{U}(0) - u^2\mathbb{U}'(0)\right\} = Y\left\{\mathcal{P}_1[\mathbf{y}]\mathbb{U}(\mathbf{y},\varsigma) + \mathcal{Q}_1[\mathbf{y}]\mathbb{U}(\mathbf{y},\varsigma)\right\}.$$

After that, we get

$$M(\mathbb{U}) = u\mathbb{U}(0) + u^2\mathbb{U}'(0) + u^{\wp}Y\{\mathcal{P}_1[\mathbf{y}]\mathbb{U}(\mathbf{y},\varsigma) + \mathcal{Q}_1[\mathbf{y}]\mathbb{U}(\mathbf{y},\varsigma)\}.$$

Let us take Yang inverse transform on both sides, we have

$$\mathbb{U}(\mathbf{y},\varsigma) = \mathbb{U}(0) + \mathbb{U}'(0) + Y^{-1}\left\{u^{\wp}Y\left\{\mathcal{P}_1[\mathbf{y}]\mathbb{U}(\mathbf{y},\varsigma) + \mathcal{Q}_1[\mathbf{y}]\mathbb{U}(\mathbf{y},\varsigma)\right\}\right\}. \tag{3.3}$$

Applying HPM

$$\mathbb{U}(\mathbf{y},\varsigma) = \sum_{k=0}^{\infty}\epsilon^k\mathbb{U}_k(\mathbf{y},\varsigma), \tag{3.4}$$

with homotopy parameter $\epsilon \in [0,1]$.

Ultimately, the nonlinear factors are discarded as:

$$\mathcal{Q}_1[\mathbf{y}]\mathbb{U}(\mathbf{y},\varsigma) = \sum_{k=0}^{\infty}\epsilon^k H_n(\mathbb{U}), \tag{3.5}$$

where $H_k(\mathbb{U})$ is He‘s polynomials

$$H_n(\mathbb{U}_0,\mathbb{U}_1,\ldots,\mathbb{U}_n) = \frac{1}{\Gamma(n+1)}D_{\epsilon}^k\left[\mathcal{Q}_1\left(\sum_{k=0}^{\infty}\epsilon^i\mathbb{U}_i\right)\right]_{\epsilon=0},$$

with $D_{\epsilon}^k = \frac{\partial^k}{\partial\epsilon^k}$.

Utilizing (3.4) and (3.5) in (3.3), we have

$$\sum_{k=0}^{\infty}\epsilon^k\mathbb{U}_k(\mathbf{y},\varsigma) = \mathbb{U}(0) + \mathbb{U}'(0) + \epsilon\left(Y^{-1}\left\{u^{\wp}Y\left\{\mathcal{P}_1\sum_{k=0}^{\infty}\epsilon^k\mathbb{U}_k(\mathbf{y},\varsigma) + \sum_{k=0}^{\infty}\epsilon^k H_k(\mathbb{U})\right\}\right\}\right). \tag{3.6}$$

Equating the ϵ coefficients, we get

$$\begin{aligned}
\epsilon^0 &: \mathbb{U}_0(\mathbf{y},\varsigma) = \mathbb{U}(0) + \mathbb{U}^{'}(0),\\
\epsilon^1 &: \mathbb{U}_1(\mathbf{y},\varsigma) = Y^{-1}\left\{u^{\wp} Y\{\mathcal{P}_1[\mathbf{y}]\mathbb{U}_0(\mathbf{y},\varsigma) + H_0(\mathbb{U})\}\right\},\\
\epsilon^2 &: \mathbb{U}_2(\mathbf{y},\varsigma) = Y^{-1}\left\{u^{\wp} Y\{\mathcal{P}_1[\mathbf{y}]\mathbb{U}_1(\mathbf{y},\varsigma) + H_1(\mathbb{U})\}\right\},\\
&\vdots\\
\epsilon^k &: \mathbb{U}_k(\mathbf{y},\varsigma) = Y^{-1}\left\{u^{\wp} Y\{\mathcal{P}_1[\mathbf{y}]\mathbb{U}_{k-1}(\mathbf{y},\varsigma) + H_{k-1}(\mathbb{U})\}\right\},
\end{aligned} \tag{3.7}$$

where $k \in \mathbb{N}$.

At last, the analytical solution is approximated by the series,

$$\mathbb{U}(\mathbf{y},\varsigma) = \lim_{M\to\infty} \sum_{k=1}^{M} \mathbb{U}_k(\mathbf{y},\varsigma). \tag{3.8}$$

4. Construction of Yang transform decomposition method for solving FDEs

To illustrate the process of solution of the YTDM, we take FDEs of the form to show the general implementation of the proposed method as stated in [42, 43]

$$D_{\varsigma}^{\wp}\mathbb{U}(\mathbf{y},\varsigma) = \mathcal{P}_1(\mathbf{y},\varsigma) + \mathcal{Q}_1(\mathbf{y},\varsigma), \quad 1 < \wp \le 2, \tag{4.1}$$

having initial values

$$\mathbb{U}(\mathbf{y},0) = \xi(\mathbf{y}) \text{ and } \frac{\partial}{\partial\varsigma}\mathbb{U}(\mathbf{y},0) = \zeta(\mathbf{y}), \tag{4.2}$$

where $D_{\varsigma}^{\wp} = \frac{\partial^{\wp}}{\partial\varsigma^{\wp}}$ is (Caputo) fractional operator, $\mathcal{P}_1, \mathcal{Q}_1$ are operators, linear and nonlinear, respectively.

Proceeds the YT and using (2.5), we have

$$Y\left\{D_{\varsigma}^{\wp}\mathbb{U}(\mathbf{y},\varsigma)\right\} = Y\left\{\mathcal{P}_1(\mathbf{y},\varsigma) + \mathcal{Q}_1(\mathbf{y},\varsigma)\right\},$$

that is

$$\frac{1}{u^{\wp}}\left\{M(u) - u\mathbb{U}(0) - u^2\mathbb{U}^{'}(0)\right\} = Y\{\mathcal{P}_1(\mathbf{y},\varsigma) + \mathcal{Q}_1(\mathbf{y},\varsigma)\}.$$

After that, we get

$$M(\mathbb{U}) = u\mathbb{U}(0) + u^2\mathbb{U}^{'}(0) + u^{\wp} Y\left\{\mathcal{P}_1(\mathbf{y},\varsigma) + \mathcal{Q}_1(\mathbf{y},\varsigma)\right\}.$$

Let us take Yang inverse transform on both sides, we have

$$\mathbb{U}(\mathbf{y},\varsigma) = \mathbb{U}(0) + \mathbb{U}^{'}(0) + Y^{-1}\left\{u^{\wp} Y\left\{\mathcal{P}_1(\mathbf{y},\varsigma) + \mathcal{Q}_1(\mathbf{y},\varsigma)\right\}\right\}. \tag{4.3}$$

Now, we assume an infinite series solution as

$$\mathbb{U}(\mathbf{y},\varsigma) = \sum_{m=0}^{\infty} \mathbb{U}_m(\mathbf{y},\varsigma), \tag{4.4}$$

and the nonlinear terms $\mathcal{Q}_1$ are calculated by operate the formula

$$\mathcal{Q}_1(\mathbf{y},\varsigma) = \sum_{m=0}^{\infty} \mathcal{A}_m, \tag{4.5}$$

where

$$\mathcal{A}_m = \frac{1}{m!}\left[\frac{\partial^m}{\partial \ell^m}\left\{\mathcal{Q}_1\left(\sum_{k=0}^{\infty}\ell^k \mathbf{y}_k, \sum_{k=0}^{\infty}\ell^k \varsigma_k\right)\right\}\right]_{\ell=0}.$$

Utilizing (4.4) and (4.5) into (4.3), we have

$$\sum_{m=0}^{\infty}\mathbb{U}_m(\mathbf{y},\varsigma) = \mathbb{U}(0) + \mathbb{U}^{'}(0) + Y^{-1}\left\{u^{\wp}Y\left\{\mathcal{P}_1\left(\sum_{m=0}^{\infty}\mathbf{y}_m, \sum_{m=0}^{\infty}\varsigma_m\right) + \sum_{m=0}^{\infty}\mathcal{A}_m\right\}\right\}. \quad (4.6)$$

By equating both sides, we have

$$\mathbb{U}_0(\mathbf{y},\varsigma) = \mathbb{U}(0) + \varsigma\mathbb{U}^{'}(0),$$
$$\mathbb{U}_1(\mathbf{y},\varsigma) = Y^{-1}\left\{u^{\wp}Y^{+}\{\mathcal{P}_1(\mathbf{y}_0,\varsigma_0) + \mathcal{A}_0\}\right\}.$$

Continuing in this manner to get the general recursive relation,

$$\mathbb{U}_{m+1}(\mathbf{y},\varsigma) = Y^{-1}\left\{u^{\wp}Y^{+}\{\mathcal{P}_1(\mathbf{y}_m,\varsigma_m) + \mathcal{A}_m\}\right\}. \quad (4.7)$$

5. Examples

In this section, we present some Illustrative examples.

Example 1. Consider the 2-dimensional nonlinear time-fractional wave-like equation with variable coefficients:

$$\frac{\partial^{\wp}\mathbb{U}(\mathbf{y},\mathbf{z},\varsigma)}{\partial\varsigma^{\wp}} = \frac{\partial^2}{\partial\mathbf{y}\partial\mathbf{z}}\left(\mathbb{U}_{\mathbf{yy}}\mathbb{U}_{\mathbf{zz}}\right) - \frac{\partial^2}{\partial\mathbf{y}\partial\mathbf{z}}\left(\mathbf{yz}\mathbb{U}_{\mathbf{y}}\mathbb{U}_{\mathbf{z}}\right) - \mathbb{U}, \quad \varsigma > 0, 1 < \wp \leq 2, \quad (5.1)$$

having initial values

$$\mathbb{U}(\mathbf{y},\mathbf{z},0) = e^{\mathbf{yz}} \text{ and } \frac{\partial}{\partial\varsigma}\mathbb{U}(\mathbf{y},\mathbf{z},0) = e^{\mathbf{yz}}, \quad (\mathbf{y},\mathbf{z}) \in \mathbb{R}^2. \quad (5.2)$$

Proceeds the YT, we have

$$Y\left\{\frac{\partial^{\wp}\mathbb{U}}{\partial\varsigma^{\wp}}\right\} = Y\left\{\frac{\partial^2}{\partial\mathbf{y}\partial\mathbf{z}}\left(\mathbb{U}_{\mathbf{yy}}\mathbb{U}_{\mathbf{zz}}\right) - \frac{\partial^2}{\partial\mathbf{y}\partial\mathbf{z}}\left(\mathbf{yz}\mathbb{U}_{\mathbf{y}}\mathbb{U}_{\mathbf{z}}\right) - \mathbb{U}\right\},$$

and using (2.5), we have

$$\frac{1}{u^{\wp}}\left\{M(u) - u\mathbb{U}(0) - u^2\mathbb{U}^{'}(0)\right\} = Y\left\{\frac{\partial^2}{\partial\mathbf{y}\partial\mathbf{z}}\left(\mathbb{U}_{\mathbf{yy}}\mathbb{U}_{\mathbf{zz}}\right) - \frac{\partial^2}{\partial\mathbf{y}\partial\mathbf{z}}\left(\mathbf{yz}\mathbb{U}_{\mathbf{y}}\mathbb{U}_{\mathbf{z}}\right) - \mathbb{U}\right\},$$

that is

$$M(u) = u\mathbb{U}(0) + u^2\mathbb{U}^{'}(0) + u^{\wp}Y\left\{\frac{\partial^2}{\partial\mathbf{y}\partial\mathbf{z}}\left(\mathbb{U}_{\mathbf{yy}}\mathbb{U}_{\mathbf{zz}}\right) - \frac{\partial^2}{\partial\mathbf{y}\partial\mathbf{z}}\left(\mathbf{yz}\mathbb{U}_{\mathbf{y}}\mathbb{U}_{\mathbf{z}}\right) - \mathbb{U}\right\}.$$

Let us take Yang inverse transform on both sides, we have

$$\mathbb{U}(\mathbf{y},\mathbf{z},\varsigma) = Y^{-1}\{u\mathbb{U}(0) + u^2\mathbb{U}^{'}(0)\} + Y^{-1}\left\{u^{\wp}Y\left\{\frac{\partial^2}{\partial\mathbf{y}\partial\mathbf{z}}\left(\mathbb{U}_{\mathbf{yy}}\mathbb{U}_{\mathbf{zz}}\right) - \frac{\partial^2}{\partial\mathbf{y}\partial\mathbf{z}}\left(\mathbf{yz}\mathbb{U}_{\mathbf{y}}\mathbb{U}_{\mathbf{z}}\right) - \mathbb{U}\right\}\right\},$$

that is

$$\mathbb{U}(\mathbf{y},\mathbf{z},\varsigma) = (e^{\mathbf{yz}} + e^{\mathbf{yz}}\varsigma) + Y^{-1}\left\{u^{\wp} Y\left\{\frac{\partial^2}{\partial\mathbf{y}\partial\mathbf{z}}\left(\mathbb{U}_{\mathbf{yy}}\mathbb{U}_{\mathbf{zz}}\right) - \frac{\partial^2}{\partial\mathbf{y}\partial\mathbf{z}}\left(\mathbf{yz}\mathbb{U}_{\mathbf{y}}\mathbb{U}_{\mathbf{z}}\right) - \mathbb{U}\right\}\right\}.$$

Applying HPM

$$\sum_{k=0}^{\infty}\epsilon^k\mathbb{U}_k(\mathbf{y},\mathbf{z},\varsigma) = (e^{\mathbf{yz}} + e^{\mathbf{yz}}\varsigma) + \epsilon\left(Y^{-1}\left\{u^{\wp} Y\left\{\left(\sum_{k=0}^{\infty}\epsilon^k H_k^1(\mathbb{U})\right) - \left(\sum_{k=0}^{\infty}\epsilon^k H_k^2(\mathbb{U})\right) + \left(\sum_{k=0}^{\infty}\epsilon^k\mathbb{U}_k(\mathbf{y},\varsigma)\right)\right\}\right\}\right).$$

Ultimately, the nonlinear factors by He's polynomial $H_k(\mathbb{U})$ are given as:

$$\sum_{k=0}^{\infty}\epsilon^k H_k^1(\mathbb{U}) = \frac{\partial^2}{\partial\mathbf{y}\partial\mathbf{z}}\left(\mathbb{U}_{\mathbf{yy}}\mathbb{U}_{\mathbf{zz}}\right),$$

and

$$\sum_{k=0}^{\infty}\epsilon^k H_k^2(\mathbb{U}) = \frac{\partial^2}{\partial\mathbf{y}\partial\mathbf{z}}\left(\mathbf{yz}\mathbb{U}_{\mathbf{y}}\mathbb{U}_{\mathbf{z}}\right).$$

Few components are calculated as:

$$\begin{aligned}
H_0^1(\mathbb{U}) &= \mathbb{U}_{0\mathbf{yy}}\mathbb{U}_{0\mathbf{zz}},\\
H_1^1(\mathbb{U}) &= \mathbb{U}_{0\mathbf{yy}}\mathbb{U}_{1\mathbf{zz}} + \mathbb{U}_{1\mathbf{yy}}\mathbb{U}_{0\mathbf{zz}},\\
&\vdots\\
H_0^2(\mathbb{U}) &= \mathbf{yz}\mathbb{U}_{0\mathbf{y}}\mathbb{U}_{0\mathbf{z}},\\
H_1^2(\mathbb{U}) &= \mathbf{yz}\mathbb{U}_{0\mathbf{y}}\mathbb{U}_{1\mathbf{z}} + \mathbf{yz}\mathbb{U}_{1\mathbf{y}}\mathbb{U}_{0\mathbf{z}}.
\end{aligned}$$

Equating the ϵ coefficients, we get

$$\begin{aligned}
\epsilon^0 &: \mathbb{U}_0(\mathbf{y},\mathbf{z},\varsigma) = (e^{\mathbf{yz}} + e^{\mathbf{yz}}\varsigma),\\
\epsilon^1 &: \mathbb{U}_1(\mathbf{y},\mathbf{z},\varsigma) = -\left(\frac{\varsigma^{\wp}}{\Gamma(\wp+1)} + \frac{\varsigma^{\wp+1}}{\Gamma(\wp+2)}\right)e^{\mathbf{yz}},\\
\epsilon^2 &: \mathbb{U}_2(\mathbf{y},\mathbf{z},\varsigma) = \left(\frac{\varsigma^{2\wp}}{\Gamma(2\wp+1)} + \frac{\varsigma^{2\wp+1}}{\Gamma(2\wp+2)}\right)e^{\mathbf{yz}},\\
&\vdots
\end{aligned}$$

At last, the analytical solution is approximated by the series as:

$$\mathbb{U}(\mathbf{y},\mathbf{z},\varsigma) = \mathbb{U}_0(\mathbf{y},\mathbf{z},\varsigma) + \mathbb{U}_1(\mathbf{y},\mathbf{z},\varsigma) + \mathbb{U}_2(\mathbf{y},\mathbf{z},\varsigma) + \cdots,$$

that is

$$\mathbb{U}(\mathbf{y},\mathbf{z},\varsigma) = \left(1 + \varsigma - \frac{\varsigma^{\wp}}{\Gamma(\wp+1)} - \frac{\varsigma^{\wp+1}}{\Gamma(\wp+2)} + \frac{\varsigma^{2\wp}}{\Gamma(2\wp+1)} + \frac{\varsigma^{2\wp+1}}{\Gamma(2\wp+2)} + \cdots\right)e^{\mathbf{yz}}.$$

Application of the YTDM

Proceeds the YT, we have

$$Y\left\{\frac{\partial^{\wp}\mathbb{U}}{\partial\varsigma^{\wp}}\right\} = Y\left\{\frac{\partial^2}{\partial\mathbf{y}\partial\mathbf{z}}\left(\mathbb{U}_{\mathrm{yy}}\mathbb{U}_{\mathbf{zz}}\right) - \frac{\partial^2}{\partial\mathbf{y}\partial\mathbf{z}}\left(\mathbf{yz}\mathbb{U}_{\mathrm{y}}\mathbb{U}_{\mathrm{z}}\right) - \mathbb{U}\right\},$$

and using (2.5), we have

$$\frac{1}{u^{\wp}}\left\{M(u) - u\mathbb{U}(0) - u^2\mathbb{U}'(0)\right\} = Y\left\{\frac{\partial^2}{\partial\mathbf{y}\partial\mathbf{z}}\left(\mathbb{U}_{\mathrm{yy}}\mathbb{U}_{\mathbf{zz}}\right) - \frac{\partial^2}{\partial\mathbf{y}\partial\mathbf{z}}\left(\mathbf{yz}\mathbb{U}_{\mathrm{y}}\mathbb{U}_{\mathrm{z}}\right) - \mathbb{U}\right\},$$

equivalently

$$M(u) = u\mathbb{U}(0) + u^2\mathbb{U}'(0) + u^{\wp}Y\left\{\frac{\partial^2}{\partial\mathbf{y}\partial\mathbf{z}}\left(\mathbb{U}_{\mathrm{yy}}\mathbb{U}_{\mathbf{zz}}\right) - \frac{\partial^2}{\partial\mathbf{y}\partial\mathbf{z}}\left(\mathbf{yz}\mathbb{U}_{\mathrm{y}}\mathbb{U}_{\mathrm{z}}\right) - \mathbb{U}\right\}.$$

Let us take Yang inverse transform on both sides, we have

$$\begin{aligned}\mathbb{U}(\mathbf{y},\mathbf{z},\varsigma) &= Y^{-1}\left\{u\mathbb{U}(0) + u^2\mathbb{U}'(0)\right\} + Y^{-1}\left\{u^{\wp}Y\left\{\frac{\partial^2}{\partial\mathbf{y}\partial\mathbf{z}}\left(\mathbb{U}_{\mathrm{yy}}\mathbb{U}_{\mathbf{zz}}\right) - \frac{\partial^2}{\partial\mathbf{y}\partial\mathbf{z}}\left(\mathbf{yz}\mathbb{U}_{\mathrm{y}}\mathbb{U}_{\mathrm{z}}\right) - \mathbb{U}\right\}\right\}\\ &= (e^{\mathbf{yz}} + e^{\mathbf{yz}}\varsigma) + Y^{-1}\left\{u^{\wp}Y\left\{\frac{\partial^2}{\partial\mathbf{y}\partial\mathbf{z}}\left(\mathbb{U}_{\mathrm{yy}}\mathbb{U}_{\mathbf{zz}}\right) - \frac{\partial^2}{\partial\mathbf{y}\partial\mathbf{z}}\left(\mathbf{yz}\mathbb{U}_{\mathrm{y}}\mathbb{U}_{\mathrm{z}}\right) - \mathbb{U}\right\}\right\}.\end{aligned}$$

Now, we assume an infinite series solution as:

$$\mathbb{U}(\mathbf{y},\mathbf{z},\varsigma) = \sum_{m=0}^{\infty}\mathbb{U}_m(\mathbf{y},\mathbf{z},\varsigma),$$

where

$$\frac{\partial^2}{\partial\mathbf{y}\partial\mathbf{z}}\left(\mathbb{U}_{\mathrm{yy}}\mathbb{U}_{\mathbf{zz}}\right) = \sum_{m=0}^{\infty}\mathcal{A}_m$$

and

$$\frac{\partial^2}{\partial\mathbf{y}\partial\mathbf{z}}\left(\mathbf{yz}\mathbb{U}_{\mathrm{y}}\mathbb{U}_{\mathrm{z}}\right) = \sum_{m=0}^{\infty}\mathcal{B}_m$$

are the Adomian polynomials that represents the nonlinear terms and

$$\begin{aligned}\sum_{m=0}^{\infty}\mathbb{U}_m(\mathbf{y},\mathbf{z},\varsigma) &= \mathbb{U}(\mathbf{y},\mathbf{z},0) + Y^{-1}\left\{u^{\wp}Y\left\{\sum_{m=0}^{\infty}\mathcal{A}_m - \sum_{m=0}^{\infty}\mathcal{B}_m - \mathbb{U}\right\}\right\}\\ &= (e^{\mathbf{yz}} + e^{\mathbf{yz}}\varsigma) + Y^{-1}\left\{u^{\wp}Y\left\{\sum_{m=0}^{\infty}\mathcal{A}_m - \sum_{m=0}^{\infty}\mathcal{B}_m - \mathbb{U}\right\}\right\}.\end{aligned}$$

The first few nonlinear terms are as follows:

$$\mathcal{A}_0 = \mathbb{U}_{0\mathbf{yy}}\mathbb{U}_{0\mathbf{zz}},$$

$$\mathcal{A}_1 = \mathbf{yz}\mathbb{U}_{0\mathbf{y}}\mathbb{U}_{0\mathbf{z}},$$
$$\vdots$$
$$\mathcal{B}_0 = \mathbb{U}_{0\mathbf{yy}}\mathbb{U}_{1\mathbf{zz}} + \mathbb{U}_{1\mathbf{yy}}\mathbb{U}_{0\mathbf{zz}},$$
$$\mathcal{B}_1 = \mathbf{yz}\mathbb{U}_{0\mathbf{y}}\mathbb{U}_{1\mathbf{z}} - \mathbf{yz}\mathbb{U}_{1\mathbf{y}}\mathbb{U}_{0\mathbf{z}}.$$

By equating both sides, we have

$$\mathbb{U}_0(\mathbf{y}, \mathbf{z}, \varsigma) = (e^{\mathbf{yz}} + e^{\mathbf{yz}}\varsigma).$$

For $m = 0$, we have

$$\mathbb{U}_1(\mathbf{y}, \mathbf{z}, \varsigma) = -\left(\frac{\varsigma^{\wp}}{\Gamma(\wp + 1)} + \frac{\varsigma^{\wp+1}}{\Gamma(\wp + 2)}\right)e^{\mathbf{yz}},$$

and for $m = 1$, we have

$$\mathbb{U}_2(\mathbf{y}, \mathbf{z}, \varsigma) = \left(\frac{\varsigma^{2\wp}}{\Gamma(2\wp + 1)} + \frac{\varsigma^{2\wp+1}}{\Gamma(2\wp + 2)}\right)e^{\mathbf{yz}}.$$

The YTDM solution is

$$\begin{aligned}\mathbb{U}(\mathbf{y}, \mathbf{z}, \varsigma) &= \sum_{m=0}^{\infty} \mathbb{U}_m(\mathbf{y}, \mathbf{z}, \varsigma) = \mathbb{U}_0(\mathbf{y}, \mathbf{z}, \varsigma) + \mathbb{U}_1(\mathbf{y}, \mathbf{z}, \varsigma) + \mathbb{U}_2(\mathbf{y}, \mathbf{z}, \varsigma) + \cdots \\ &= \left(1 + \varsigma - \frac{\varsigma^{\wp}}{\Gamma(\wp + 1)} - \frac{\varsigma^{\wp+1}}{\Gamma(\wp + 2)} + \frac{\varsigma^{2\wp}}{\Gamma(2\wp + 1)} + \frac{\varsigma^{2\wp+1}}{\Gamma(2\wp + 2)} + \cdots\right)e^{\mathbf{yz}}.\end{aligned}$$

In the special case $\wp = 2$, we get exact solution as

$$\mathbb{U}(\mathbf{y}, \mathbf{z}, \varsigma) = (\cos\varsigma + \sin\varsigma)e^{\mathbf{yz}}. \tag{5.3}$$

The graphs in Figures 1(a) and 1(b) show the behavior of the exact and proposed methods solution in Caputo manner at $\wp = 1$. Figure 1(c) shows our methods solution at different fractional-orders of $\wp = 2, 1.9, 1.8, 1.7$, and $-1 \leq \mathbf{y}, \mathbf{z} \leq 1$ for Example 1 and Figure 1(d), respectively, at $\varsigma = 0.1$ and $-1 \leq \mathbf{y}, \mathbf{z} \leq 1$. In Table 1, we computed the absolute errors of the Shehu variational iteration method (SVIM) and suggested methods that confirm that our solution converges quickly compared to SVIM. The graphical representation shows that the exact solution and proposed methods solution are in good agreement.

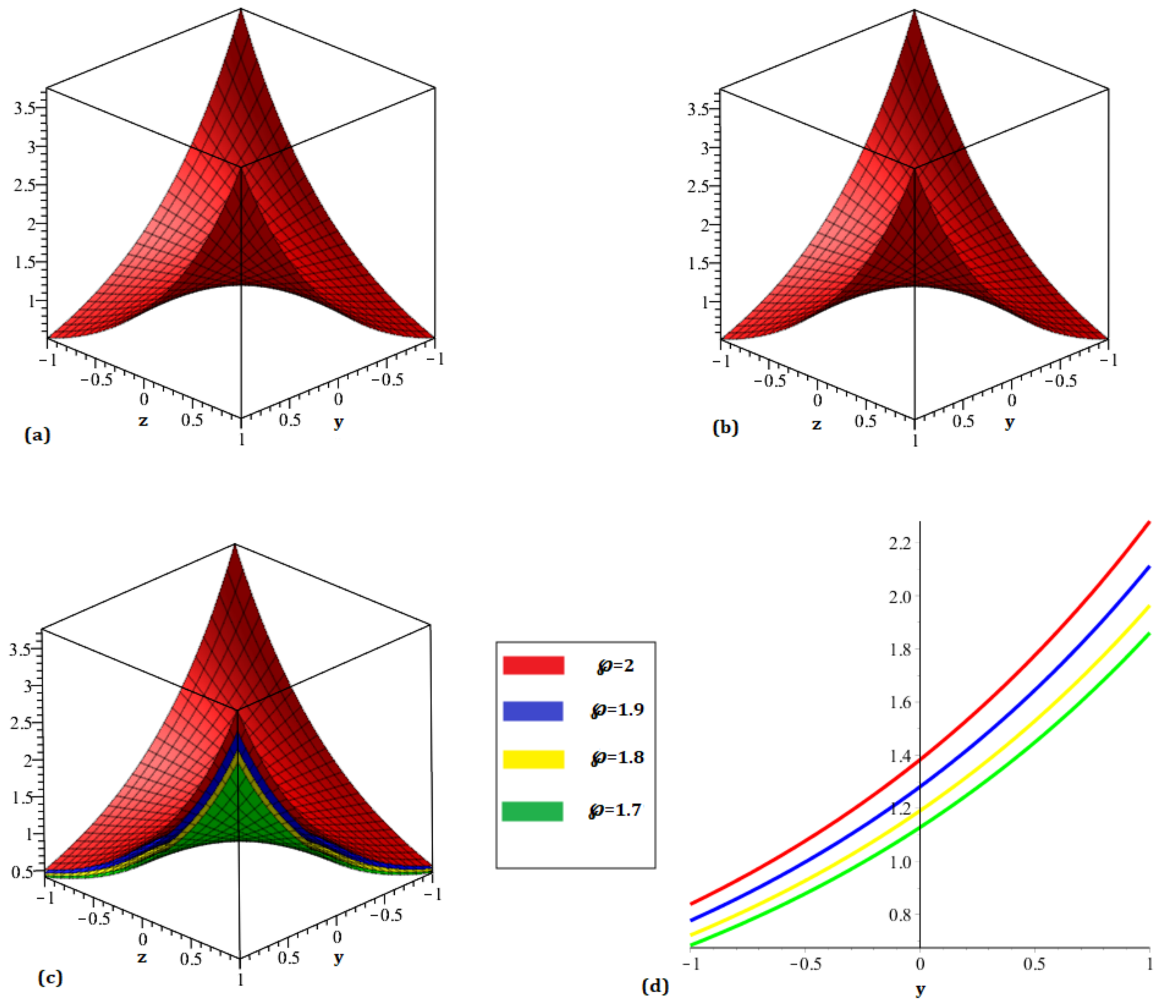

Figure 1. The nature of $\mathbb{U}(\mathbf{y}, \mathbf{z}, \varsigma)$ in terms of $\mathbf{y}, \mathbf{z}$ and ς at various values of $\wp$ for Example 1.

Table 1. Comparison of the Shehu variational iteration method (SVIM) and suggested solutions absolute errors at different orders of $\wp$ for Example 1.

$\varsigma/\mathbf{y}, \mathbf{z}$	$SVIM(\wp = 0.5)$	Proposed methods ($\wp = 0.5$)	$SVIM(\wp = 0.7)$	Proposed methods ($\wp = 0.7$)
0.1	3.2196 E-13	2.5111E-14	4.0929E-13	0.01010032E-14
0.3	2.1569 E-09	2.32229E-11	2.7420E-09	0.02040261E-11
0.5	1.3095 E-07	2.0335E-09	1.6647E-07	0.03090871E-09
0.7	1.9680 E-06	1.1448E-07	2.5019E-06	0.04162043E-07
0.9	1.4947 E-05	2.05640176E-06	1.9001E-05	0.05253943E-06

Example 2. Consider the following nonlinear time-fractional wave-like equation with variable coefficients:

$$\frac{\partial^{\wp}\mathbb{U}(\mathbf{y},\varsigma)}{\partial\varsigma^{\wp}} = \mathbb{U}^2\frac{\partial^2}{\partial\mathbf{y}^2}\left(\mathbb{U}_{\mathrm{y}}\mathbb{U}_{\mathrm{yy}}\mathbb{U}_{\mathrm{yyy}}\right) + \mathbb{U}_{\mathrm{y}}^2\frac{\partial^2}{\partial\mathbf{y}^2}\left(\mathbb{U}_{\mathrm{yy}}^3\right) - 18\mathbb{U}^5 + \mathbb{U} \quad \varsigma > 0, 1 < \wp \leq 2, \tag{5.4}$$

having initial values

$$\mathbb{U}(\mathbf{y},0) = e^{\mathbf{y}} \text{ and } \frac{\partial}{\partial\varsigma}\mathbb{U}(\mathbf{y},0) = e^{\mathbf{y}}. \tag{5.5}$$

Proceeds the YT, we have

$$Y\left\{\frac{\partial^{\wp}\mathbb{U}}{\partial\varsigma^{\wp}}\right\} = Y\left\{\mathbb{U}^2\frac{\partial^2}{\partial\mathbf{y}^2}(\mathbb{U}_{\mathrm{y}}\mathbb{U}_{\mathrm{yy}}\mathbb{U}_{\mathrm{yyy}}) + \mathbb{U}_{\mathrm{y}}^2\frac{\partial^2}{\partial\mathbf{y}^2}(\mathbb{U}_{\mathrm{yy}}^3) - 18\mathbb{U}^5 + \mathbb{U}\right\},$$

and using (2.5), we have

$$\frac{1}{u^{\wp}}\{M(u) - u\mathbb{U}(0) - u^2\mathbb{U}'(0)\} = Y\left\{\mathbb{U}^2\frac{\partial^2}{\partial\mathbf{y}^2}\left(\mathbb{U}_{\mathrm{y}}\mathbb{U}_{\mathrm{yy}}\mathbb{U}_{\mathrm{yyy}}\right) + \mathbb{U}_{\mathrm{y}}^2\frac{\partial^2}{\partial\mathbf{y}^2}\left(\mathbb{U}_{\mathrm{yy}}^3\right) - 18\mathbb{U}^5 + \mathbb{U}\right\},$$

that is

$$M(u) = u\mathbb{U}(0) + u^2\mathbb{U}'(0) + u^{\wp}Y\left\{\mathbb{U}^2\frac{\partial^2}{\partial\mathbf{y}^2}\left(\mathbb{U}_{\mathrm{y}}\mathbb{U}_{\mathrm{yy}}\mathbb{U}_{\mathrm{yyy}}\right) + \mathbb{U}_{\mathrm{y}}^2\frac{\partial^2}{\partial\mathbf{y}^2}\left(\mathbb{U}_{\mathrm{yy}}^3\right) - 18\mathbb{U}^5 + \mathbb{U}\right\}.$$

Let us take Yang inverse transform on both sides, we have

$$\begin{aligned}\mathbb{U}(\mathbf{y},\varsigma) &= Y^{-1}\left\{u\mathbb{U}(0) + u^2\mathbb{U}'(0)\right\} + Y^{-1}\left\{u^{\wp}Y\left\{\mathbb{U}^2\frac{\partial^2}{\partial\mathbf{y}^2}\left(\mathbb{U}_{\mathrm{y}}\mathbb{U}_{\mathrm{yy}}\mathbb{U}_{\mathrm{yyy}}\right) + \mathbb{U}_{\mathrm{y}}^2\frac{\partial^2}{\partial\mathbf{y}^2}\left(\mathbb{U}_{\mathrm{yy}}^3\right) - 18\mathbb{U}^5 + \mathbb{U}\right\}\right\} \\ &= (e^{\mathbf{yz}} + e^{\mathbf{yz}}\varsigma) + Y^{-1}\left\{u^{\wp}Y\left\{\mathbb{U}^2\frac{\partial^2}{\partial\mathbf{y}^2}\left(\mathbb{U}_{\mathrm{y}}\mathbb{U}_{\mathrm{yy}}\mathbb{U}_{\mathrm{yyy}}\right) + \mathbb{U}_{\mathrm{y}}^2\frac{\partial^2}{\partial\mathbf{y}^2}\left(\mathbb{U}_{\mathrm{yy}}^3\right) - 18\mathbb{U}^5 + \mathbb{U}\right\}\right\}.\end{aligned}$$

By applying HPM, we have

$$\begin{aligned}\sum_{k=0}^{\infty}\epsilon^k\mathbb{U}_k(\mathbf{y},\varsigma) = (e^{\mathbf{y}} + e^{\mathbf{y}}\varsigma) + \epsilon\Bigg(Y^{-1}\Bigg\{u^{\wp}Y\Bigg\{\left(\sum_{k=0}^{\infty}\epsilon^k H_k^1(\mathbb{U})\right) + \left(\sum_{k=0}^{\infty}\epsilon^k H_k^2(\mathbb{U})\right) \\ -18\left(\sum_{k=0}^{\infty}\epsilon^k H_k^3(\mathbb{U})\right) + \left(\sum_{k=0}^{\infty}\epsilon^k\mathbb{U}_k(\mathbf{y},\varsigma)\right)\Bigg\}\Bigg\}\Bigg).\end{aligned}$$

Ultimately, the nonlinear factors by He's polynomial $H_k(\mathbb{U})$ are given as:

$$\begin{aligned}\sum_{k=0}^{\infty}\epsilon^k H_k^1(\mathbb{U}) &= \mathbb{U}^2\frac{\partial^2}{\partial\mathbf{y}^2}\left(\mathbb{U}_{\mathrm{y}}\mathbb{U}_{\mathrm{yy}}\mathbb{U}_{\mathrm{yyy}}\right), \\ \sum_{k=0}^{\infty}\epsilon^k H_k^2(\mathbb{U}) &= \mathbb{U}_{\mathrm{y}}^2\frac{\partial^2}{\partial\mathbf{y}^2}\left(\mathbb{U}_{\mathrm{yy}}^3\right), \\ \sum_{k=0}^{\infty}\epsilon^k H_k^3(\mathbb{U}) &= \mathbb{U}^5.\end{aligned}$$

Few components are calculated as:

$$
\begin{aligned}
H_0^1(\mathbb{U}) &= \mathbb{U}_0^2\frac{\partial^2}{\partial \mathbf{y}^2}\left(\mathbb{U}_{0\mathbf{y}}\mathbb{U}_{0\mathbf{yy}}\mathbb{U}_{0\mathbf{yyy}}\right),\\
H_1^1(\mathbb{U}) &= 2\mathbb{U}_0\mathbb{U}_1\frac{\partial^2}{\partial \mathbf{y}^2}\left(\mathbb{U}_{0\mathbf{y}}\mathbb{U}_{0\mathbf{yy}}\mathbb{U}_{0\mathbf{yyy}}\right)+\mathbb{U}_0^2\frac{\partial^2}{\partial \mathbf{y}^2}\left(\mathbb{U}_{1\mathbf{y}}\mathbb{U}_{0\mathbf{yy}}\mathbb{U}_{0\mathbf{yyy}}+\mathbb{U}_{0\mathbf{y}}\mathbb{U}_{1\mathbf{yy}}\mathbb{U}_{0\mathbf{yyy}}+\mathbb{U}_{0\mathbf{y}}\mathbb{U}_{0\mathbf{yy}}\mathbb{U}_{1\mathbf{yyy}}\right),\\
&\vdots\\
H_0^2(\mathbb{U}) &= \mathbb{U}_{0\mathbf{y}}^2\frac{\partial^2}{\partial \mathbf{y}^2}\left(\mathbb{U}_{0\mathbf{yy}}^3\right),\\
H_1^2(\mathbb{U}) &= 2\mathbb{U}_{0\mathbf{y}}\mathbb{U}_{1\mathbf{y}}\frac{\partial^2}{\partial \mathbf{y}^2}\left(\mathbb{U}_{0\mathbf{yy}}^3\right)+3\mathbb{U}_{0\mathbf{y}}^2\frac{\partial^2}{\partial \mathbf{y}^2}\left(\mathbb{U}_{0\mathbf{yy}}^2\mathbb{U}_{1\mathbf{yy}}\right),\\
&\vdots\\
H_0^3(\mathbb{U}) &= \mathbb{U}_0^5,\\
H_1^3(\mathbb{U}) &= 5\mathbb{U}_0^4\mathbb{U}_1.
\end{aligned}
$$

Equating the ϵ coefficients, we get

$$
\begin{aligned}
\epsilon^0 &: \mathbb{U}_0(\mathbf{y},\varsigma) = (e^{\mathbf{y}}+e^{\mathbf{y}}\varsigma),\\
\epsilon^1 &: \mathbb{U}_1(\mathbf{y},\varsigma) = \left(\frac{\varsigma^{\wp}}{\Gamma(\wp+1)}+\frac{\varsigma^{\wp+1}}{\Gamma(\wp+2)}\right)e^{\mathbf{y}},\\
\epsilon^2 &: \mathbb{U}_2(\mathbf{y},\varsigma) = \left(\frac{\varsigma^{2\wp}}{\Gamma(2\wp+1)}+\frac{\varsigma^{2\wp+1}}{\Gamma(2\wp+2)}\right)e^{\mathbf{y}},\\
&\vdots
\end{aligned}
$$

At last, the analytical solution is approximated by the series as:

$$
\begin{aligned}
\mathbb{U}(\mathbf{y},\varsigma) &= \mathbb{U}_0(\mathbf{y},\varsigma)+\mathbb{U}_1(\mathbf{y},\varsigma)+\mathbb{U}_2(\mathbf{y},\varsigma)+\cdots\\
&= \left(1+\varsigma+\frac{\varsigma^{\wp}}{\Gamma(\wp+1)}+\frac{\varsigma^{\wp+1}}{\Gamma(\wp+2)}+\frac{\varsigma^{2\wp}}{\Gamma(2\wp+1)}+\frac{\varsigma^{2\wp+1}}{\Gamma(2\wp+2)}+\cdots\right)e^{\mathbf{y}}.
\end{aligned}
$$

Application of the YTDM

Proceeds the YT, we have

$$
Y\left\{\frac{\partial^{\wp}\mathbb{U}}{\partial\varsigma^{\wp}}\right\} = Y\left\{\mathbb{U}^2\frac{\partial^2}{\partial \mathbf{y}^2}\left(\mathbb{U}_{\mathbf{y}}\mathbb{U}_{\mathbf{yy}}\mathbb{U}_{\mathbf{yyy}}\right)+\mathbb{U}_{\mathbf{y}}^2\frac{\partial^2}{\partial \mathbf{y}^2}\left(\mathbb{U}_{\mathbf{yy}}^3\right)-18\mathbb{U}^5+\mathbb{U}\right\},
$$

and using (2.5), we have

$$
\frac{1}{u^{\wp}}\left\{M(u)-u\mathbb{U}(0)-u^2\mathbb{U}'(0)\right\} = Y\left\{\mathbb{U}^2\frac{\partial^2}{\partial \mathbf{y}^2}\left(\mathbb{U}_{\mathbf{y}}\mathbb{U}_{\mathbf{yy}}\mathbb{U}_{\mathbf{yyy}}\right)+\mathbb{U}_{\mathbf{y}}^2\frac{\partial^2}{\partial \mathbf{y}^2}\left(\mathbb{U}_{\mathbf{yy}}^3\right)-18\mathbb{U}^5+\mathbb{U}\right\},
$$

that is

$$
M(u) = u\mathbb{U}(0)+u^2\mathbb{U}'(0)+u^{\wp}Y\left\{\mathbb{U}^2\frac{\partial^2}{\partial \mathbf{y}^2}\left(\mathbb{U}_{\mathbf{y}}\mathbb{U}_{\mathbf{yy}}\mathbb{U}_{\mathbf{yyy}}\right)+\mathbb{U}_{\mathbf{y}}^2\frac{\partial^2}{\partial \mathbf{y}^2}\left(\mathbb{U}_{\mathbf{yy}}^3\right)-18\mathbb{U}^5+\mathbb{U}\right\}.
$$

Let us take Yang inverse transform on both sides, we have

$$\begin{aligned}\mathbb{U}(\mathbf{y},\mathbf{z},\varsigma) &= Y^{-1}\left\{u\mathbb{U}(0)+u^{2}\mathbb{U}^{'}(0)\right\}+Y^{-1}\left\{u^{\wp}Y\left\{\mathbb{U}^{2}\frac{\partial^{2}}{\partial\mathbf{y}^{2}}\left(\mathbb{U}_{\mathbf{y}}\mathbb{U}_{\mathbf{yy}}\mathbb{U}_{\mathbf{yyy}}\right)+\mathbb{U}_{\mathbf{y}}^{2}\frac{\partial^{2}}{\partial\mathbf{y}^{2}}\left(\mathbb{U}_{\mathbf{yy}}^{3}\right)-18\mathbb{U}^{5}+\mathbb{U}\right\}\right\}\\
&= (e^{\mathbf{y}}+e^{\mathbf{y}}\varsigma)+Y^{-1}\left\{u^{\wp}Y\left\{\mathbb{U}^{2}\frac{\partial^{2}}{\partial\mathbf{y}^{2}}\left(\mathbb{U}_{\mathbf{y}}\mathbb{U}_{\mathbf{yy}}\mathbb{U}_{\mathbf{yyy}}\right)+\mathbb{U}_{\mathbf{y}}^{2}\frac{\partial^{2}}{\partial\mathbf{y}^{2}}\left(\mathbb{U}_{\mathbf{yy}}^{3}\right)-18\mathbb{U}^{5}+\mathbb{U}\right\}\right\}.\end{aligned}$$

Now, we assume an infinite series solution as:

$$\mathbb{U}(\mathbf{y},\mathbf{z},\varsigma)=\sum_{m=0}^{\infty}\mathbb{U}_{m}(\mathbf{y},\mathbf{z},\varsigma), \tag{5.6}$$

where

$$\mathbb{U}^{2}\frac{\partial^{2}}{\partial\mathbf{y}^{2}}\left(\mathbb{U}_{\mathbf{y}}\mathbb{U}_{\mathbf{yy}}\mathbb{U}_{\mathbf{yyy}}\right)=\sum_{m=0}^{\infty}\mathcal{A}_{m},$$
$$\mathbb{U}_{\mathbf{y}}^{2}\frac{\partial^{2}}{\partial\mathbf{y}^{2}}\left(\mathbb{U}_{\mathbf{yy}}^{3}\right)=\sum_{m=0}^{\infty}\mathcal{B}_{m}$$

and

$$\mathbb{U}^{5}=\sum_{m=0}^{\infty}\mathcal{C}_{m}$$

are the Adomian polynomials that represents the nonlinear terms and

$$\begin{aligned}\sum_{m=0}^{\infty}\mathbb{U}_{m}(\mathbf{y},\mathbf{z},\varsigma) &= \mathbb{U}(\mathbf{y},\mathbf{z},0)+Y^{-1}\left\{u^{\wp}Y\left\{\sum_{m=0}^{\infty}\mathcal{A}_{m}+\sum_{m=0}^{\infty}\mathcal{B}_{m}-18\sum_{m=0}^{\infty}\mathcal{C}_{m}+\mathbb{U}\right\}\right\}\\
&= (e^{\mathbf{y}}+e^{\mathbf{y}}\varsigma)+Y^{-1}\left\{u^{\wp}Y\left\{\sum_{m=0}^{\infty}\mathcal{A}_{m}+\sum_{m=0}^{\infty}\mathcal{B}_{m}-18\sum_{m=0}^{\infty}\mathcal{C}_{m}+\mathbb{U}\right\}\right\}.\end{aligned}$$

The first few nonlinear terms are as follows:

$$\begin{aligned}\mathcal{A}_{0} &= \mathbb{U}_{0}^{2}\frac{\partial^{2}}{\partial\mathbf{y}^{2}}\left(\mathbb{U}_{0\mathbf{y}}\mathbb{U}_{0\mathbf{yy}}\mathbb{U}_{0\mathbf{yyy}}\right),\\
\mathcal{A}_{1} &= 2\mathbb{U}_{0}\mathbb{U}_{1}\frac{\partial^{2}}{\partial\mathbf{y}^{2}}\left(\mathbb{U}_{0\mathbf{y}}\mathbb{U}_{0\mathbf{yy}}\mathbb{U}_{0\mathbf{yyy}}\right)+\mathbb{U}_{0}^{2}\frac{\partial^{2}}{\partial\mathbf{y}^{2}}\left(\mathbb{U}_{1\mathbf{y}}\mathbb{U}_{0\mathbf{yy}}\mathbb{U}_{0\mathbf{yyy}}+\mathbb{U}_{0\mathbf{y}}\mathbb{U}_{1\mathbf{yy}}\mathbb{U}_{0\mathbf{yyy}}+\mathbb{U}_{0\mathbf{y}}\mathbb{U}_{0\mathbf{yy}}\mathbb{U}_{1\mathbf{yyy}}\right),\\
&\vdots\\
\mathcal{B}_{0} &= \mathbb{U}_{0\mathbf{y}}^{2}\frac{\partial^{2}}{\partial\mathbf{y}^{2}}\left(\mathbb{U}_{0\mathbf{yy}}^{3}\right),\\
\mathcal{B}_{1} &= 2\mathbb{U}_{0\mathbf{y}}\mathbb{U}_{1\mathbf{y}}\frac{\partial^{2}}{\partial\mathbf{y}^{2}}\left(\mathbb{U}_{0\mathbf{yy}}^{3}\right)+3\mathbb{U}_{0\mathbf{y}}^{2}\frac{\partial^{2}}{\partial\mathbf{y}^{2}}\left(\mathbb{U}_{0\mathbf{yy}}^{2}\mathbb{U}_{1\mathbf{yy}}\right),\\
&\vdots\\
\mathcal{C}_{0} &= \mathbb{U}_{0}^{5},\\
\mathcal{C}_{1} &= 5\mathbb{U}_{0}^{4}\mathbb{U}_{1}.\end{aligned}$$

By equating both sides, we have

$$\mathbb{U}_0(\mathbf{y}, \varsigma) = (e^{\mathbf{y}} + e^{\mathbf{y}}\varsigma).$$

For $m = 0$, we have

$$\mathbb{U}_1(\mathbf{y}, \varsigma) = \left(\frac{\varsigma^{\wp}}{\Gamma(\wp + 1)} + \frac{\varsigma^{\wp+1}}{\Gamma(\wp + 2)}\right) e^{\mathbf{y}},$$

and for $m = 1$, we have

$$\mathbb{U}_2(\mathbf{y}, \varsigma) = \left(\frac{\varsigma^{2\wp}}{\Gamma(2\wp + 1)} + \frac{\varsigma^{2\wp+1}}{\Gamma(2\wp + 2)}\right) e^{\mathbf{y}}.$$

The YTDM solution is

$$\begin{aligned} \mathbb{U}(\mathbf{y}, \varsigma) = \sum_{m=0}^{\infty} \mathbb{U}_m(\mathbf{y}, \varsigma) &= \mathbb{U}_0(\mathbf{y}, \varsigma) + \mathbb{U}_1(\mathbf{y}, \varsigma) + \mathbb{U}_2(\mathbf{y}, \varsigma) + \cdots \\ &= \left(1 + \varsigma + \frac{\varsigma^{\wp}}{\Gamma(\wp + 1)} + \frac{\varsigma^{\wp+1}}{\Gamma(\wp + 2)} + \frac{\varsigma^{2\wp}}{\Gamma(2\wp + 1)} + \frac{\varsigma^{2\wp+1}}{\Gamma(2\wp + 2)} + \cdots\right) e^{\mathbf{y}}. \end{aligned}$$

In the special case $\wp = 2$, we get exact solution as:

$$\mathbb{U}(\mathbf{y}, \varsigma) = e^{\mathbf{y}+\varsigma}. \tag{5.7}$$

The graphs in Figures 2(a) and 2(b) show the behavior of the exact and proposed methods solution in Caputo manner at $\wp = 1$. Figure 2(c) shows our methods solution at different fractional-orders of $\wp = 2, 1.9, 1.8, 1.7$, and $-1 \leq \mathbf{y}, \varsigma \leq 1$ for Example 2 and Figure 2(d), respectively, at $\varsigma = 0.5$ and $-1 \leq \mathbf{y} \leq 1$. The graphical representation shows that the exact solution and proposed methods solution are in good agreement.

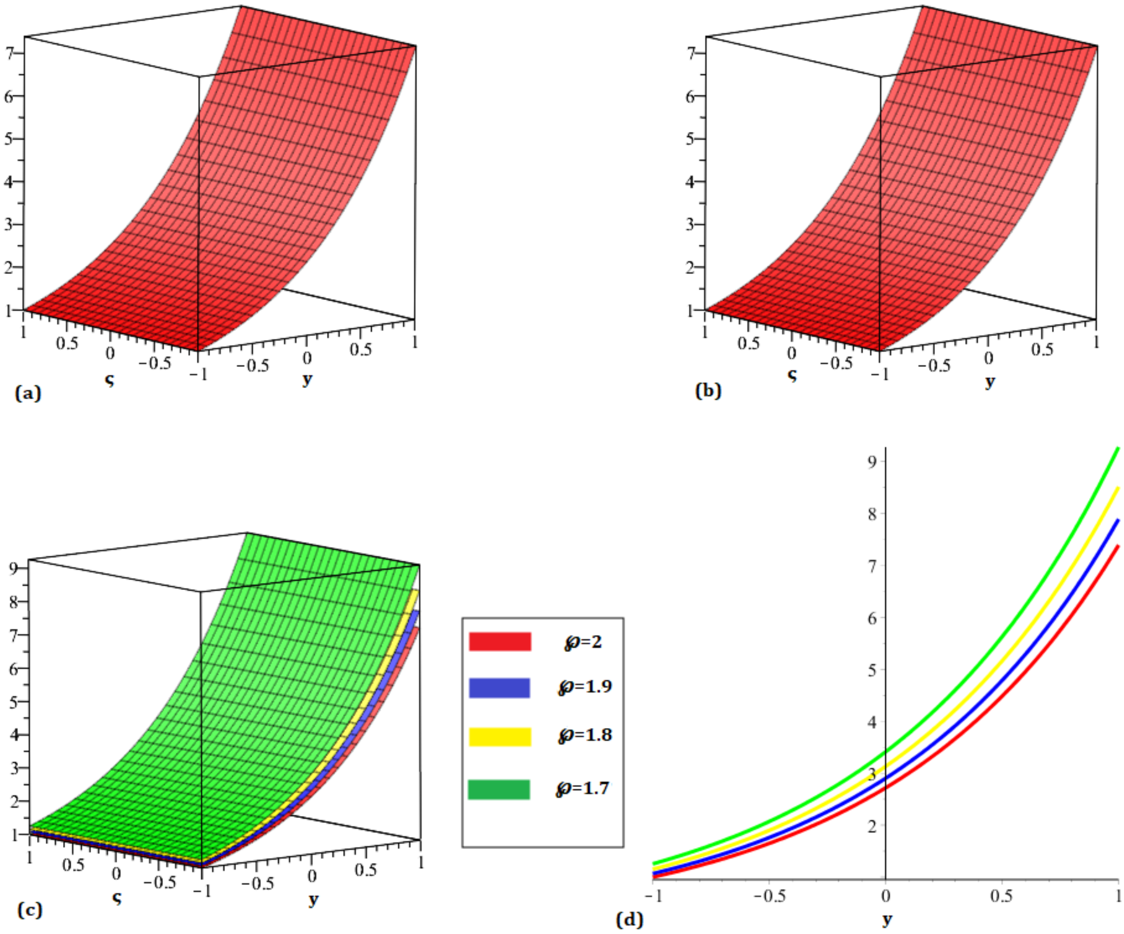

Figure 2. The nature of $\mathbb{U}(\mathbf{y}, \varsigma)$ in terms of $\mathbf{y}$ and ς at various values of $\wp$ for Example 2.

Example 3. Consider the following nonlinear time-fractional wave-like equation with variable coefficients:

$$\frac{\partial^{\wp}\mathbb{U}(\mathbf{y},\varsigma)}{\partial\varsigma^{\wp}} = \mathbf{y}^2\frac{\partial}{\partial\mathbf{y}}\left(\mathbb{U}_{\mathbf{y}}\mathbb{U}_{\mathbf{yy}}\right) - \mathbf{y}^2\left(\mathbb{U}_{\mathbf{yy}}\right)^2 - \mathbb{U}, \quad \varsigma > 0, 1 < \wp \leq 2, \tag{5.8}$$

having initial values

$$\mathbb{U}(\mathbf{y},0) = 0 \text{ and } \frac{\partial}{\partial\varsigma}\mathbb{U}(\mathbf{y},0) = \mathbf{y}^2. \tag{5.9}$$

Proceeds the YT, we have

$$Y\left\{\frac{\partial^{\wp}\mathbb{U}}{\partial\varsigma^{\wp}}\right\} = Y\left\{\mathbf{y}^2\frac{\partial}{\partial\mathbf{y}}(\mathbb{U}_{\mathbf{y}}\mathbb{U}_{\mathbf{yy}}) - \mathbf{y}^2(\mathbb{U}_{\mathbf{yy}})^2 - \mathbb{U}\right\},$$

and using (2.5), we have

$$\frac{1}{u^{\wp}}\left\{M(u) - u\mathbb{U}(0) - u^2\mathbb{U}'(0)\right\} = Y\left\{\mathbf{y}^2\frac{\partial}{\partial\mathbf{y}}\left(\mathbb{U}_{\mathbf{y}}\mathbb{U}_{\mathbf{yy}}\right) - \mathbf{y}^2\left(\mathbb{U}_{\mathbf{yy}}\right)^2 - \mathbb{U}\right\},$$

that is

$$M(u) = u\mathbb{U}(0) + u^2\mathbb{U}'(0) + u^{\wp}Y\left\{\mathbf{y}^2\frac{\partial}{\partial\mathbf{y}}\left(\mathbb{U}_{\mathbf{y}}\mathbb{U}_{\mathbf{yy}}\right) - \mathbf{y}^2\left(\mathbb{U}_{\mathbf{yy}}\right)^2 - \mathbb{U}\right\}.$$

Let us take Yang inverse transform on both sides, we have

$$\begin{aligned}\mathbb{U}(\mathbf{y},\varsigma) &= Y^{-1}\left\{u\mathbb{U}(0)+u^2\mathbb{U}^{'}(0)\right\}+Y^{-1}\left\{u^{\wp}Y\left\{\mathbf{y}^2\frac{\partial}{\partial\mathbf{y}}\left(\mathbb{U}_{\mathbf{y}}\mathbb{U}_{\mathbf{yy}}\right)-\mathbf{y}^2\left(\mathbb{U}_{\mathbf{yy}}\right)^2-\mathbb{U}\right\}\right\}\\ &= (e^{\mathbf{y}}\varsigma)+Y^{-1}\left\{u^{\wp}Y\left\{\mathbf{y}^2\frac{\partial}{\partial\mathbf{y}}\left(\mathbb{U}_{\mathbf{y}}\mathbb{U}_{\mathbf{yy}}\right)-\mathbf{y}^2\left(\mathbb{U}_{\mathbf{yy}}\right)^2-\mathbb{U}\right\}\right\}.\end{aligned}$$

By applying HPM, we obtain

$$\sum_{k=0}^{\infty}\epsilon^k\mathbb{U}_k(\mathbf{y},\varsigma)=e^{\mathbf{y}}\varsigma+\epsilon\left(Y^{-1}\left\{u^{\wp}Y\left\{\left(\sum_{k=0}^{\infty}\epsilon^kH_k^1(\mathbb{U})\right)-\left(\sum_{k=0}^{\infty}\epsilon^kH_k^2(\mathbb{U})\right)-\left(\sum_{k=0}^{\infty}\epsilon^k\mathbb{U}_k(\mathbf{y},\varsigma)\right)\right\}\right\}\right).$$

Ultimately, the nonlinear factors by He's polynomial $H_k(\mathbb{U})$ are given as

$$\begin{aligned}\sum_{k=0}^{\infty}\epsilon^kH_k^1(\mathbb{U}) &= \mathbf{y}^2\frac{\partial}{\partial\mathbf{y}}\left(\mathbb{U}_{\mathbf{y}}\mathbb{U}_{\mathbf{yy}}\right),\\ \sum_{k=0}^{\infty}\epsilon^kH_k^2(\mathbb{U}) &= \mathbf{y}^2\left(\mathbb{U}_{\mathbf{yy}}\right)^2.\end{aligned}$$

Few components are calculated as:

$$\begin{aligned}H_0^1(\mathbb{U}) &= \mathbb{U}_{0\mathbf{y}}\mathbb{U}_{0\mathbf{yy}},\\ H_1^1(\mathbb{U}) &= \mathbb{U}_{0\mathbf{y}}\mathbb{U}_{1\mathbf{yy}}+\mathbb{U}_{1\mathbf{y}}\mathbb{U}_{0\mathbf{yy}},\\ H_2^1(\mathbb{U}) &= \mathbb{U}_{0\mathbf{y}}\mathbb{U}_{2\mathbf{yy}}+\mathbb{U}_{1\mathbf{y}}\mathbb{U}_{1\mathbf{yy}}+\mathbb{U}_{2\mathbf{y}}\mathbb{U}_{0\mathbf{yy}},\\ &\vdots\\ H_0^2(\mathbb{U}) &= \mathbb{U}_{0\mathbf{yy}}^2,\\ H_1^2(\mathbb{U}) &= 2\mathbb{U}_{0\mathbf{yy}}\mathbb{U}_{1\mathbf{yy}},\\ H_2^2(\mathbb{U}) &= 2\mathbb{U}_{0\mathbf{yy}}\mathbb{U}_{2\mathbf{yy}}+\mathbb{U}_{1\mathbf{yy}}^2.\end{aligned}$$

Equating the ϵ coefficients, we get

$$\begin{aligned}\epsilon^0 &: \mathbb{U}_0(\mathbf{y},\varsigma)=\mathbf{y}^2\varsigma,\\ \epsilon^1 &: \mathbb{U}_1(\mathbf{y},\varsigma)=-\frac{\varsigma^{\wp+1}}{\Gamma(\wp+2)}\mathbf{y}^2,\\ \epsilon^2 &: \mathbb{U}_2(\mathbf{y},\varsigma)=\frac{\varsigma^{2\wp+1}}{\Gamma(2\wp+2)}\mathbf{y}^2,\\ \epsilon^3 &: \mathbb{U}_3(\mathbf{y},\varsigma)=-\frac{\varsigma^{3\wp+1}}{\Gamma(3\wp+2)}\mathbf{y}^2,\\ &\vdots\end{aligned}$$

At last, the analytical solution is approximated by the series as:

$$\begin{aligned}\mathbb{U}(\mathbf{y},\varsigma) &= \mathbb{U}_0(\mathbf{y},\varsigma)+\mathbb{U}_1(\mathbf{y},\varsigma)+\mathbb{U}_2(\mathbf{y},\varsigma)+\cdots\\ &= \mathbf{y}^2\left(\varsigma-\frac{\varsigma^{\wp+1}}{\Gamma(\wp+2)}+\frac{\varsigma^{2\wp+1}}{\Gamma(2\wp+2)}-\frac{\varsigma^{3\wp+1}}{\Gamma(3\wp+2)}+\cdots\right).\end{aligned}$$

Application of the YTDM

Proceeds the YT, we have

$$Y\left\{\frac{\partial^{\wp}\mathbb{U}}{\partial\varsigma^{\wp}}\right\} = Y\left\{\mathbf{y}^2\frac{\partial}{\partial\mathbf{y}}\left(\mathbb{U}_{\mathbf{y}}\mathbb{U}_{\mathbf{yy}}\right) - \mathbf{y}^2\left(\mathbb{U}_{\mathbf{yy}}\right)^2 - \mathbb{U}\right\},$$

and using the Def.(2.5), we have

$$\frac{1}{u^{\wp}}\left\{M(u) - u\mathbb{U}(0) - u^2\mathbb{U}'(0)\right\} = Y\left\{\mathbf{y}^2\frac{\partial}{\partial\mathbf{y}}\left(\mathbb{U}_{\mathbf{y}}\mathbb{U}_{\mathbf{yy}}\right) - \mathbf{y}^2\left(\mathbb{U}_{\mathbf{yy}}\right)^2 - \mathbb{U}\right\},$$

that is

$$M(u) = u\mathbb{U}(0) + u^2\mathbb{U}'(0) + u^{\wp}Y\left\{\mathbf{y}^2\frac{\partial}{\partial\mathbf{y}}\left(\mathbb{U}_{\mathbf{y}}\mathbb{U}_{\mathbf{yy}}\right) - \mathbf{y}^2\left(\mathbb{U}_{\mathbf{yy}}\right)^2 - \mathbb{U}\right\}.$$

Let us take Yang inverse transform on both sides, we have

$$\begin{aligned}\mathbb{U}(\mathbf{y},\varsigma) &= Y^{-1}\left\{u\mathbb{U}(0) + u^2\mathbb{U}'(0)]\right\} + Y^{-1}\left\{u^{\wp}Y\left\{\mathbf{y}^2\frac{\partial}{\partial\mathbf{y}}\left(\mathbb{U}_{\mathbf{y}}\mathbb{U}_{\mathbf{yy}}\right) - \mathbf{y}^2\left(\mathbb{U}_{\mathbf{yy}}\right)^2 - \mathbb{U}\right\}\right\},\\ &= \mathbf{y}^2\varsigma + Y^{-1}\left\{u^{\wp}Y\left\{\mathbf{y}^2\frac{\partial}{\partial\mathbf{y}}(\mathbb{U}_{\mathbf{y}}\mathbb{U}_{\mathbf{yy}}) - \mathbf{y}^2(\mathbb{U}_{\mathbf{yy}})^2 - \mathbb{U}\right\}\right\}.\end{aligned}$$

Now, we assume an infinite series solution as

$$\mathbb{U}(\mathbf{y},\varsigma) = \sum_{m=0}^{\infty}\mathbb{U}_m(\mathbf{y},\varsigma),$$

where

$$\mathbb{U}_{\mathbf{y}}\mathbb{U}_{\mathbf{yy}} = \sum_{m=0}^{\infty}\mathcal{A}_m$$

and

$$\left(\mathbb{U}_{\mathbf{yy}}\right)^2 = \sum_{m=0}^{\infty}\mathcal{B}_m$$

are the Adomian polynomials that represents the nonlinear terms and

$$\begin{aligned}\sum_{m=0}^{\infty}\mathbb{U}_m(\mathbf{y},\varsigma) &= \mathbb{U}(\mathbf{y},\mathbf{z},0) + Y^{-1}\left\{u^{\wp}Y\left\{\sum_{m=0}^{\infty}\mathcal{A}_m - \sum_{m=0}^{\infty}\mathcal{B}_m - \mathbb{U}\right\}\right\}\\ &= \mathbf{y}^2\varsigma + Y^{-1}\left\{u^{\wp}Y\left\{\sum_{m=0}^{\infty}\mathcal{A}_m - \sum_{m=0}^{\infty}\mathcal{B}_m - \mathbb{U}\right\}\right\}.\end{aligned}$$

The first few nonlinear terms are as follows:

$$\begin{aligned}\mathcal{A}_0 &= \mathbb{U}_{0\mathbf{y}}\mathbb{U}_{0\mathbf{yy}},\\ \mathcal{A}_1 &= \mathbb{U}_{0\mathbf{y}}\mathbb{U}_{1\mathbf{yy}} + \mathbb{U}_{1\mathbf{y}}\mathbb{U}_{0\mathbf{yy}},\\ \mathcal{A}_2 &= \mathbb{U}_{0\mathbf{y}}\mathbb{U}_{2\mathbf{yy}} + \mathbb{U}_{1\mathbf{y}}\mathbb{U}_{1\mathbf{yy}} + \mathbb{U}_{2\mathbf{y}}\mathbb{U}_{0\mathbf{yy}},\end{aligned}$$

$$\begin{aligned}
&\vdots\\
\mathcal{B}_0 &= \mathbb{U}_{0\mathbf{yy}}^2,\\
\mathcal{B}_1 &= 2\mathbb{U}_{0\mathbf{yy}}\mathbb{U}_{1\mathbf{yy}},\\
\mathcal{B}_2 &= 2\mathbb{U}_{0\mathbf{yy}}\mathbb{U}_{2\mathbf{yy}} + \mathbb{U}_{1\mathbf{yy}}^2.
\end{aligned}$$

By equating both sides, we have

$$\mathbb{U}_0(\mathbf{y},\varsigma) = \mathbf{y}^2\varsigma.$$

For $m = 0$, we have

$$\mathbb{U}_1(\mathbf{y},\varsigma) = -\frac{\varsigma^{\wp+1}}{\Gamma(\wp+2)}\mathbf{y}^2,$$

for $m = 1$, we have

$$\mathbb{U}_2(\mathbf{y},\varsigma) = \frac{\varsigma^{2\wp+1}}{\Gamma(2\wp+2)}\mathbf{y}^2,$$

and for $m = 2$, we have

$$\mathbb{U}_3(\mathbf{y},\varsigma) = -\frac{\varsigma^{3\wp+1}}{\Gamma(3\wp+2)}\mathbf{y}^2.$$

The YTDM solution is

$$\begin{aligned}
\mathbb{U}(\mathbf{y},\varsigma) &= \sum_{m=0}^{\infty}\mathbb{U}_m(\mathbf{y},\varsigma) = \mathbb{U}_0(\mathbf{y},\varsigma) + \mathbb{U}_1(\mathbf{y},\varsigma) + \mathbb{U}_2(\mathbf{y},\varsigma) + \mathbb{U}_3(\mathbf{y},\varsigma) + \cdots\\
&= \mathbf{y}^2\left(\varsigma - \frac{\varsigma^{\wp+1}}{\Gamma(\wp+2)} + \frac{\varsigma^{2\wp+1}}{\Gamma(2\wp+2)} - \frac{\varsigma^{3\wp+1}}{\Gamma(3\wp+2)} + \cdots\right).
\end{aligned}$$

In the special case $\wp = 2$, we get exact solution as

$$\mathbb{U}(\mathbf{y},\varsigma) = \mathbf{y}^2\sin\varsigma. \tag{5.10}$$

The graphs in Figures 3(a) and 3(b) show the behavior of the exact and proposed methods solution in Caputo manner at $\wp = 1$. Figure 3(c) shows our methods solution at different fractional-orders of $\wp = 2, 1.9, 1.8, 1.7$, and $0 \le \mathbf{y},\varsigma \le 3$ for Example 3 and Figure 3(d), respectively, at $\varsigma = 0.5$ and $0 \le \mathbf{y} \le 3$. The analysis reveals that the fractional-order solutions are strongly convergent to the integer-order solution. The convergence of the solution can be observed in the 2-D and 3-D graphs, respectively.

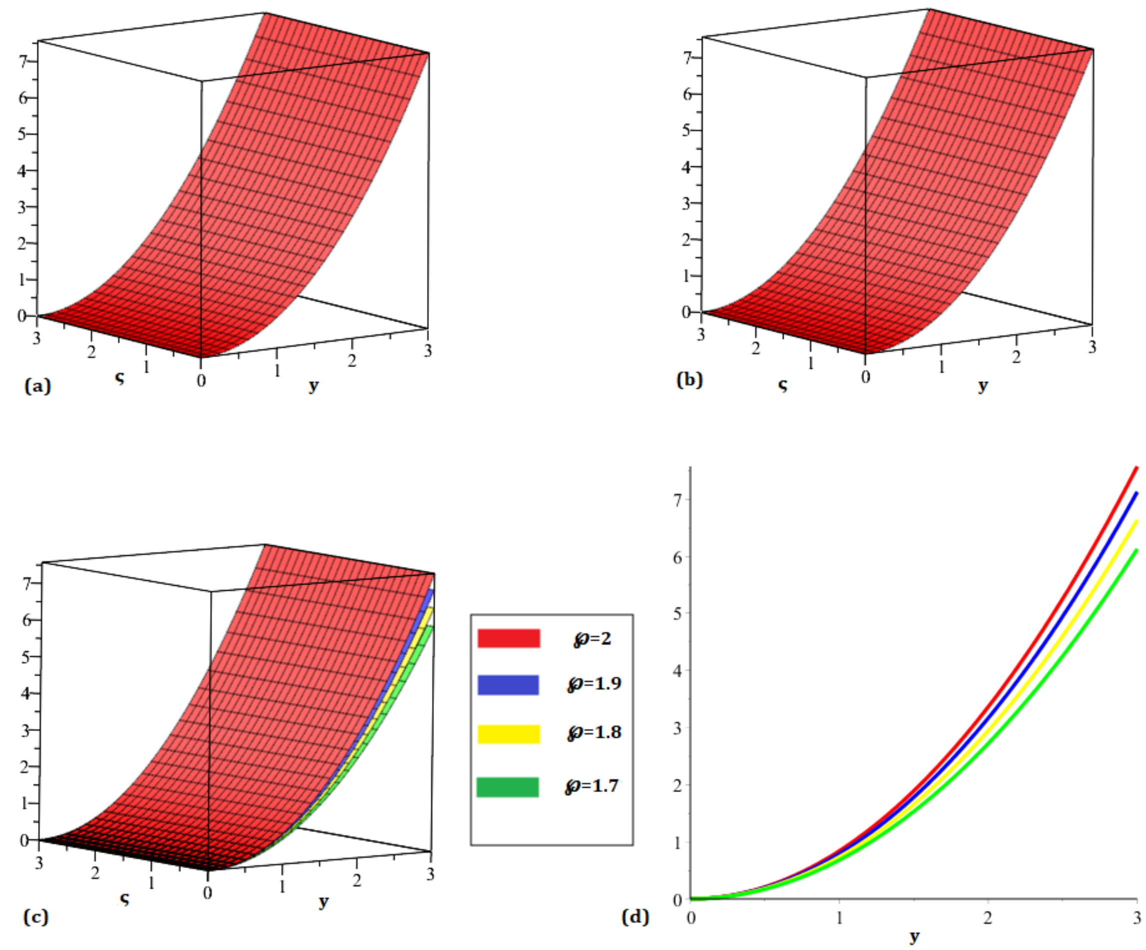

Figure 3. The nature of $\mathbb{U}(\mathbf{y}, \varsigma)$ in terms of $\mathbf{y}$ and ς at various values of $\wp$ for Example 3.

6. Conclusions

This article compares two approaches to tackling nonlinear Caputo time-fractional wave-like problems. Three numerical examples are used to examine the accuracy and effectiveness of the suggested method. This research used the Yang transform decomposition method and the Homotopy perturbation transform method to give a new representation of exact solutions for nonlinear time-fractional wave-like equations with variable coefficients. In three numerical cases, the techniques were used. In the numerical cases, our methods provided us with the results as infinite series, and when this series is in closed form, it provides the exact results to the related equations. In comparison to other analytical and numerical techniques, the current methods have proven to be an effective and simple procedure. Furthermore, the proposed methods needed fewer calculations and can thus be used to solve other fractional-order problems.

Conflict of interest

The authors declare no conflicts of interest.

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