AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(3): 5266–5280.
DOI: 10.3934/math.2023264
Received: 21 September 2022
Revised: 05 November 2022
Accepted: 17 November 2022
Published: 14 December 2022

Research article

Combination of Laplace transform and residual power series techniques of special fractional-order non-linear partial differential equations

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Abstract: This paper investigates fractional-order partial differential equations analytically by applying a modified technique called the Laplace residual power series method. The analytical solution was utilized to test the accuracy and precision of the proposed methodologies and shown by tables and graphs. The solution is a convergent series established on Taylor's new form. When determining the series coefficients like RPSM, the fractional derivatives must be calculated every time. We only need to perform a few computations to obtain the coefficients because LRPSM only requires the concept of an infinite limit. The advantage of this method is that it does not require Adomian polynomials or he's polynomials to solve nonlinear problems. As a result, the method's reduced computation size is a strength. The outcome we got supports the idea that the suggested method is the best one for handling any non-linear models that appear in technology and science.

Keywords: residual power series; fractional-order partial differential equations; Laplace transform; Caputo operator
Mathematics Subject Classification: 33B15, 34A34, 35A20, 35A22, 44A10

1. Introduction

Due to its well-established applications in various scientific and technical fields, fractional calculus has gained prominence during the last three decades. Many pioneers have shown that when adjusted by integer-order models, fractional-order models may accurately represent complex events [1, 2]. The Caputo fractional derivatives are nonlocal in contrast to the integer-order derivatives, which are local in nature [3]. In other words, the integer-order derivative may be used to analyze changes in the area around a point, but the Caputo fractional derivative can be used to analyze changes in the whole interval. Senior mathematicians including Riemann [4], Caputo [5], Podlubny [6], Ross [7], Liouville [8], Miller and others, collaborated to create the fundamental foundation for fractional order integrals and derivatives. The theory of fractional-order calculus has been related to real-world projects, and it has been applied to chaos theory [9], signal processing [10], electrodynamics [11], human diseases [12, 13], and other areas [14–16].

Due to the numerous applications of fractional differential equations in engineering and science such as electrodynamics [17], chaos ideas [18], accounting [19], continuum and fluid mechanics [20], digital signal [21] and biological population designs [22] fractional differential equations are now more widely known. For such issues to be resolved, efficient tools are needed [23–25]. Because of this, we will attempt to apply an efficient analytical technique to solve nonlinear arbitrary order differential equations in this article. Many strategies in collaboration fields may be delightfully and even more accurately analyzed using fractional differential equations. Various strategies have been developed in this regard, some of them are as follows, such as the fractional Reduced differential transformation technique [26], Adomian decomposition technique [27], the fractional Variational iteration technique [28], Elzaki decomposition technique [29, 30], iterative transformation technique [31], the fractional natural decomposition method (FNDM) [32], and the fractional homotopy perturbation method [33].

The power series solution is used to solve some classes of the differential and integral equations of fractional or non-fractional order, and it is based on assuming that the solution of the equation can be expanded as a power series. RPS is an easy and fast technique for determining the coefficients of the power series solution. The Jordanian mathematician Omar Abu Arqub created the residual power series method in 2013, as a technique for quickly calculating the coefficients of the power series solutions for 1st and 2nd-order fuzzy differential equations [34]. Without perturbation, linearization, or discretization, the residual power series method provides a powerful and straightforward power series solution for highly linear and nonlinear equations [35–38]. The residual power series method has been used to solve an increasing variety of nonlinear ordinary and partial differential equations of various sorts, orders, and classes during the past several years. It has been used to make nonlinear fractional dispersive partial differential equation have solitary pattern results and to predict them [39], to solve the highly nonlinear singular differential equation known as the generalized Lane-Emden equation [40], to solve higher-order ordinary differential equations numerically [41], to approximate solve the fractional nonlinear KdV-Burger equations, to predict and represent the RPSM differs from several other analytical and numerical approaches in some crucial ways [42]. First, there is no requirement for a recursion connection or for the RPSM to compare the coefficients of the related terms. Second, by reducing the associated residual error, the RPSM offers a straightforward method to guarantee the convergence of the series solution. Thirdly, the RPSM doesn't suffer from computational rounding mistakes and doesn't use a lot of time or memory. Fourth, the approach may be used immediately to the provided issue by selecting an acceptable starting guess approximation since the residual power series method does not need any converting when transitionary from low-order to higher-order and from simple linearity to complicated nonlinearity [43–45]. The process of solving linear differential equations using the LT method consists of three steps. The first step depends on transforming the original differential equation into a new space, called the Laplace space. In the second step, the new equation is solved algebraically in the Laplace space. In the last step, the solution in the second step is transformed back into the original space, resulting in the solution of the given problem.

In this article, we apply the Laplace residual power series method to achieve the definitive solution of the fractional-order nonlinear partial differential equations. The Laplace transformation efficiently integrates the residual power series method for the renewability algorithmic technique. This proposed technique produces interpretive findings in the sense of a convergent series. The Caputo fractional derivative operator explains quantitative categorizations of the partial differential equations. The offered methodology is well demonstrated in modelling and enumeration investigations. The exact-analytical findings are a valuable way to analyze the problematic dynamics of systems, notably for computational fractional partial differential equations.

2. Preliminaries

Definition 2.1. The fractional Caputo derivative of a function $u(\zeta, t)$ of order α is given as [46]

$$^{C}D_t^{\alpha} u(\zeta, t) = J_t^{m-\alpha} u^m(\zeta, t), \; m-1 < \alpha \leq m, \; t > 0, \tag{2.1}$$

where $m \in N$ and J_t^{α} is the fractional integral Riemann-Liouville (RL) of $u(\zeta, t)$ of order α is given as

$$J_t^{\sigma} u(\zeta, t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\varphi, \tau) d\tau \tag{2.2}$$

Definition 2.2. The Laplace transformation (LT) of $u(\zeta, t)$ is given as [46]

$$u(\zeta, s) = \mathcal{L}_t[u(\zeta, t)] = \int_0^{\infty} e^{-st} u(\zeta, t) dt, \; s > \alpha, \tag{2.3}$$

where the Laplace transform inverse is defined as

$$u(\zeta, t) = \mathcal{L}_t^{-1}[u(\zeta, s)] = \int_{l-i\infty}^{l+i\infty} e^{st} u(\zeta, s) ds, \; l = Re(s) > l_0. \tag{2.4}$$

Lemma 2.1. *Suppose that $u(\zeta, t)$ is piecewise continue term and $U(\zeta, s) = \mathcal{L}_t[u(\zeta, t)]$, we get*

(1) $\mathcal{L}_t[J_t^{\alpha} u(\zeta, t)] = \frac{U(\zeta, s)}{s^{\alpha}}, \; \alpha > 0.$
(2) $\mathcal{L}_t[D_t^{\alpha} u(\zeta, t)] = s^{\sigma} U(\zeta, s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} u^k(\zeta, 0), \; m-1 < \alpha \leq m.$
(3) $\mathcal{L}_t[D_t^{n\alpha} u(\zeta, t)] = s^{n\alpha} U(\zeta, s) - \sum_{k=0}^{n-1} s^{(n-k)\alpha-1} D_t^{k\alpha} u(\zeta, 0), \; 0 < \alpha \leq 1.$

Proof. For proof see Refs. [46]. □

Theorem 2.1. *Let $u(\zeta, t)$ be a piecewise continuous function on $I \times [0, \infty)$ with exponential order ζ. Assume that the fractional expansion of the function $U(\zeta, s) = \mathcal{L}_t[u(\zeta, t)]$ is as follows:*

$$U(\zeta, s) = \sum_{n=0}^{\infty} \frac{f_n(\zeta)}{s^{1+n\alpha}}, \; 0 < \alpha \leq 1, \; \zeta \in I, \; s > \zeta. \tag{2.5}$$

Then, $f_n(\zeta) = D_t^{n\sigma} u(\zeta, 0)$.

Proof. For proof see Refs. [46]. □

Remark 2.1. The inverse Laplace transform of the Eq (2.5) is represented as [46]

$$u(\zeta,t)=\sum_{i=0}^{\infty}\frac{D_t^{\alpha}u(\zeta,0)}{\Gamma(1+i\alpha)}t^{i(\zeta)},\ 0<\zeta\le 1,\ t\ge 0. \tag{2.6}$$

3. Road map of the proposed method

Consider the fractional order partial differential equation,

$$D_t^{\alpha}U(\zeta,t)+\frac{\partial^3U(\zeta,t)}{\partial t\partial\zeta^2}-\frac{\partial^4U(\zeta,t)}{\partial t^2\partial\zeta^2}+\frac{\partial^4U(\zeta,t)}{\partial\zeta^4}+a\left(\frac{\partial^2U(\zeta,t)}{\partial\zeta^2}\right)^2-b\left(\frac{\partial^2U(\zeta,t)}{\partial t^2}\right)^3+cU(\zeta,t)=0. \tag{3.1}$$

Applying LT of Eq (3.1), we get

$$\begin{aligned}U(\zeta,s)+\frac{f_0(\zeta,s)}{s}+\frac{1}{s^{\alpha}}\Bigg[&-\frac{\partial^3U(\zeta,s)}{\partial t\partial\zeta^2}-\frac{\partial^4U(\zeta,s)}{\partial t^2\partial\zeta^2}+\frac{\partial^4U(\zeta,s)}{\partial\zeta^4}\\&+a\mathcal{L}\left(\mathcal{L}_t^{-1}\left(\frac{\partial^2U(\zeta,s)}{\partial\zeta^2}\right)\right)^2-b\mathcal{L}\left(\mathcal{L}_t^{-1}\left(\frac{\partial^2U(\zeta,s)}{\partial t^2}\right)\right)^3+cU(\zeta,s)\Bigg]=0.\end{aligned} \tag{3.2}$$

Suppose that the result of Eq (3.2), we get

$$U(\zeta,s)=\sum_{n=0}^{\infty}\frac{f_n(\zeta,s)}{s^{n\alpha+1}}. \tag{3.3}$$

The k^{th}-truncated term series are

$$U(\zeta,s)=\frac{f_0(\zeta,s)}{s}+\sum_{n=1}^{k}\frac{f_n(\zeta,s)}{s^{n\alpha+1}},\ k=1,2,3,4\cdots. \tag{3.4}$$

Residual Laplace function (RLF) is given as

$$\begin{aligned}\mathcal{L}_tRes_u(\zeta,s)=U(\zeta,s)+\frac{f_0(\zeta,s)}{s}+\frac{1}{s^{\alpha}}\Bigg[&\frac{\partial^3U(\zeta,s)}{\partial t\partial\zeta^2}-\frac{\partial^4U(\zeta,s)}{\partial t^2\partial\zeta^2}+\frac{\partial^4U(\zeta,s)}{\partial\zeta^4}\\&+a\mathcal{L}\left(\mathcal{L}_t^{-1}\left(\frac{\partial^2U(\zeta,s)}{\partial\zeta^2}\right)\right)^2-b\mathcal{L}\left(\mathcal{L}_t^{-1}\left(\frac{\partial^2U(\zeta,s)}{\partial t^2}\right)\right)^3+cU(\zeta,s)\Bigg].\end{aligned} \tag{3.5}$$

And the k^{th}-LRFs as

$$\begin{aligned}\mathcal{L}_tRes_k(\zeta,s)=U_k(\zeta,s)+\frac{f_0(\zeta,s)}{s}+\frac{1}{s^{\sigma}}\Bigg[&\frac{\partial^3U_k(\zeta,s)}{\partial t\partial\zeta^2}-\frac{\partial^4U_k(\zeta,s)}{\partial t^2\partial\zeta^2}+\frac{\partial^4U(\zeta,s)}{\partial\zeta^4}\\&+a\mathcal{L}\left(\mathcal{L}_t^{-1}\left(\frac{\partial^2U_k(\zeta,s)}{\partial\zeta^2}\right)\right)^2-b\mathcal{L}\left(\mathcal{L}_t^{-1}\left(\frac{\partial^2U_k(\zeta,s)}{\partial t^2}\right)\right)^3+cU_k(\zeta,s)\Bigg].\end{aligned} \tag{3.6}$$

To illustrate a few facts, the following LRPSM features are provided:

(1) $\mathcal{L}_t Res(\zeta, s) = 0$ and $\lim_{j\to\infty} \mathcal{L}_t Res_k(\zeta, s) = \mathcal{L}_t Res_u(\zeta, s)$ for each $s > 0$.
(2) $\lim_{s\to\infty} s\mathcal{L}_t Res_u(\zeta, s) = 0 \Rightarrow \lim_{s\to\infty} s\mathcal{L}_t Res_{u,k}(\zeta, s) = 0$.
(3) $\lim_{s\to\infty} s^{k\alpha+1}\mathcal{L}_t Res_{u,k}(\zeta, s) = \lim_{s\to\infty} s^{k\alpha+1}\mathcal{L}_t Res_{u,k}(\zeta, s) = 0,\ 0 < \alpha \leq 1,\ k = 1, 2, 3, \cdots$.

To calculate the coefficients using $f_n(\zeta, s)$, $g_n(\zeta, s)$, $h_n(\zeta, s)$ and $l_n(\zeta, s)$, the following system is recursively solved:

$$\lim_{s\to\infty} s^{k\alpha+1}\mathcal{L}_t Res_{u,k}(\alpha, s) = 0,\ k = 1, 2, \cdots. \tag{3.7}$$

In finally inverse Laplace transform to Eq (3.4), to get the k^{th} analytical result of $u_k(\zeta, t)$.

4. Applications

Example 4.1. Consider the fractional partial differential equations [47],

$$\begin{aligned} &D_t^{\alpha} u(\zeta, t) - \frac{\partial^3 u(\zeta, t)}{\partial t \partial \zeta^2} - \frac{\partial^4 u(\zeta, t)}{\partial t^2 \partial \zeta^2} + \frac{\partial^4 u(\zeta, t)}{\partial \zeta^4} + \frac{1}{9}\left(\frac{\partial^2 u(\zeta, t)}{\partial \zeta^2}\right)^2 \\ &- \frac{1}{216}\left(\frac{\partial^2 u(\zeta, t)}{\partial t^2}\right)^3 + 16u(\zeta, t) = 0, \quad \text{where } 2 < \alpha \leq 3, \end{aligned} \tag{4.1}$$

with the following IC's:

$$u(x, 0) = -\zeta^4,\ \frac{\partial}{\partial t}u(\zeta, 0) = 0,\ \frac{\partial^2}{\partial t^2}u(\zeta, 0) = 0. \tag{4.2}$$

Using Laplace transform to Eq (4.1), we get

$$\begin{aligned} &U(\zeta, s) + \frac{\zeta^4}{s} + \frac{1}{s^{\alpha}}\Bigg[- \frac{\partial^3 U(\zeta, s)}{\partial t \partial \zeta^2} - \frac{\partial^4 U(\zeta, s)}{\partial t^2 \partial \zeta^2} + \frac{\partial^4 U(\zeta, s)}{\partial \zeta^4} \\ &+ \frac{1}{9}\mathcal{L}\left(\mathcal{L}_t^{-1}\left(\frac{\partial^2 U(\zeta, s)}{\partial \zeta^2}\right)\right)^2 - \frac{1}{216}\mathcal{L}\left(\mathcal{L}_t^{-1}\left(\frac{\partial^2 U(\zeta, s)}{\partial t^2}\right)\right)^3 + 16U(\zeta, s)\Bigg] = 0, \end{aligned} \tag{4.3}$$

and so the k^{th}-truncated term series are

$$\zeta u(\zeta, s) = \frac{-\zeta^4}{s} + \sum_{n=1}^{k} \frac{f_n(\zeta, s)}{s^{n\alpha+1}},\ k = 1, 2, 3, 4 \cdots. \tag{4.4}$$

Residual Laplace function is given as

$$\begin{aligned} \mathcal{L}_t Res_u(\zeta, s) = &\ U(\zeta, s) + \frac{\zeta^4}{s} + \frac{1}{s^{\alpha}}\Bigg[\frac{\partial^3 U(\zeta, s)}{\partial t \partial \zeta^2} - \frac{\partial^4 U(\zeta, s)}{\partial t^2 \partial \zeta^2} + \frac{\partial^4 U(\zeta, s)}{\partial \zeta^4} \\ &+ \frac{1}{9}\mathcal{L}\left(\mathcal{L}_t^{-1}\left(\frac{\partial^2 U(\zeta, s)}{\partial \zeta^2}\right)\right)^2 - \frac{1}{216}\mathcal{L}\left(\mathcal{L}_t^{-1}\left(\frac{\partial^2 U(\zeta, s)}{\partial t^2}\right)\right)^3 + 16U(\zeta, s)\Bigg], \end{aligned} \tag{4.5}$$

and the k^{th}-LRFs as:

$$\begin{aligned} \mathcal{L}_t Res_k(\zeta, s) = &\ U_k(\zeta, s) + \frac{\zeta^4}{s} + \frac{1}{s^{\sigma}}\Bigg[\frac{\partial^3 U_k(\zeta, s)}{\partial t \partial \zeta^2} - \frac{\partial^4 U_k(\zeta, s)}{\partial t^2 \partial \zeta^2} + \frac{\partial^4 U(\zeta, s)}{\partial \zeta^4} \\ &+ \frac{1}{9}\mathcal{L}\left(\mathcal{L}_t^{-1}\left(\frac{\partial^2 U_k(\zeta, s)}{\partial \zeta^2}\right)\right)^2 - \frac{1}{216}\mathcal{L}\left(\mathcal{L}_t^{-1}\left(\frac{\partial^2 U_k(\zeta, s)}{\partial t^2}\right)\right)^3 + 16U_k(\zeta, s)\Bigg]. \end{aligned} \tag{4.6}$$

Table 1. Comparison of the exact and proposed technique solution and various fractional-orders α and $t = 0.25$ for Example 4.1.

ζ	$\alpha = 2.5$	$\alpha = 2.7$	$\alpha = 2.9$	$\alpha = 3$	HPM [47]	Exact
0	0.0222397	0.0111683	0.00553658	0.00388069	0.00388069	0.0038812
0.2	0.0206397	0.00956834	0.00393658	0.00228069	0.00228069	0.0022812
0.4	-0.00336028	-0.0144317	-0.0200634	-0.0217193	-0.0217193	-0.0217188
0.6	-0.10736	-0.118432	-0.124063	-0.125719	-0.125719	-0.125719
0.8	-0.38736	-0.398432	-0.404063	-0.405719	-0.405719	-0.405719
1.0	-0.97776	-0.988832	-0.994463	-0.996119	-0.996119	-0.996119

Now, we calculate $f_k(\zeta, s)$, $k = 1, 2, 3, \cdots$, substituting the k^{th}-truncate series of Eq (4.4) into the k^{th} residual Laplace term Eq (4.6), multiply the solution equation by $s^{k\alpha+1}$, and then solve recursively the link $\lim_{s\to\infty}(s^{k\alpha+1}\mathcal{L}_t Res_{u,k}(\zeta, s)) = 0$, $k = 1, 2, 3, \cdots$. Following are the first some term:

$$f_1(\zeta, s) = 24, \;\; f_2(\zeta, s) = -384, \;\; f_3(\zeta, s) = 6144. \tag{4.7}$$

Putting the value of $f_k(\zeta, s)$, $k = 1, 2, 3, \cdots$, in Eq (4.4), we get

$$U(\zeta, s) = -\frac{\zeta^4}{s} + \frac{24}{s^{\alpha+1}} - \frac{384}{s^{2\alpha+1}} + \frac{6144}{s^{3\alpha+1}} + \cdots . \tag{4.8}$$

Using inverse LT, we get

$$u(\zeta, t) = -\zeta^4 + \frac{24t^{\alpha}}{\Gamma(\alpha+1)} - \frac{384t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{6144t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots , \tag{4.9}$$

and the exact solution are

$$u = -\zeta^4 + 4t^3. \tag{4.10}$$

In Figure 1, the exact and LRPSM solutions for $u(\zeta, t)$ at $\alpha = 3$ at ζ and $t = 0.3$ of Example 4.1. In Figure 2, analytical solution for $u(\zeta, t)$ at different value of $\alpha = 2.8$ and 2.6 at ζ and $t = 0.3$. In Figure 3, analytical solution for $u(\zeta, t)$ at various value of α at $t = 0.3$ of Example 4.1.

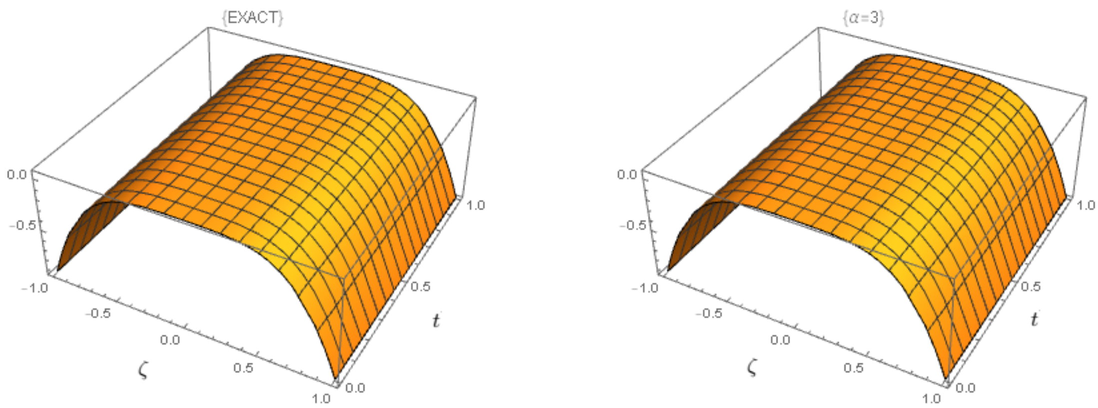

Figure 1. The actual and LRPSM results for $u(\zeta, t)$ at $\alpha = 3$ at ζ and $t = 0.3$.

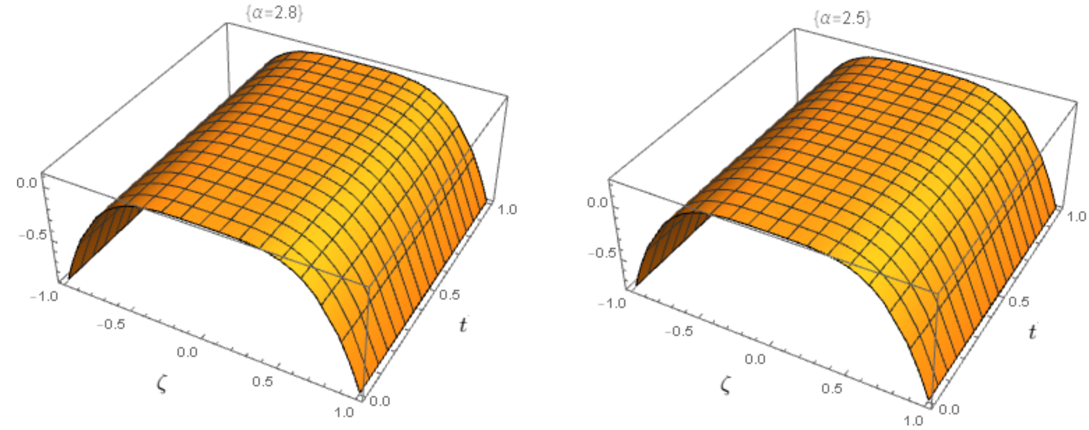

Figure 2. Analytical solution for $u(\zeta, t)$ at different value of $\alpha = 2.8$ and 2.6 at ζ and $t = 0.3$.

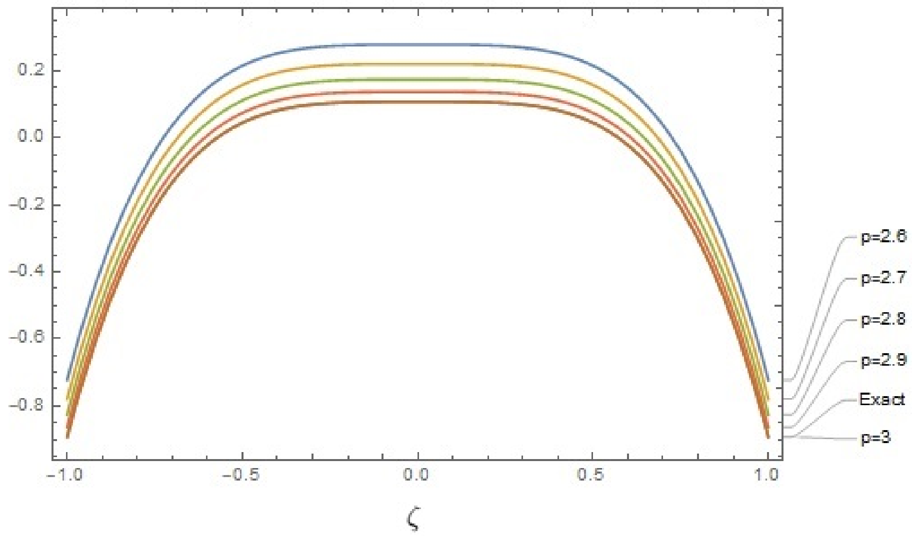

Figure 3. Analytical solution for $u(\zeta, t)$ at various value of α at $t = 0.3$.

Example 4.2. Consider the fractional partial differential equations [47]:

$$\begin{aligned} &D_t^{\alpha} U(\zeta, t) - \frac{\partial^3 U(\zeta, t)}{\partial t \partial \zeta^2} - \frac{\partial^4 U(\zeta, t)}{\partial t^2 \partial \zeta^2} + \frac{\partial^4 U(\zeta, t)}{\partial \zeta^4} + \left(\frac{\partial^2 U(\zeta, t)}{\partial \zeta^2}\right)^2 \\ &- \left(\frac{\partial^2 U(\zeta, t)}{\partial t^2}\right)^2 + 2U^2(\zeta, t) = 0, \quad \text{where } \ 2 < \alpha \leq 3, \end{aligned} \tag{4.11}$$

with the following IC's:

$$U(\zeta, 0) = e^{\zeta}, \ \frac{\partial}{\partial t} U(\zeta, 0) = e^{\zeta}, \ \frac{\partial^2}{\partial t^2} U(\zeta, 0) = e^{\zeta}. \tag{4.12}$$

Using Laplace transform to Eq (4.11), we get

$$U(\zeta,s)-\frac{e^{\zeta}}{s}-\frac{e^{\zeta}}{s^2}-\frac{e^{\zeta}}{s^3}+\frac{1}{s^{\alpha}}\Bigg[-\frac{\partial^3U(\zeta,s)}{\partial t\partial\zeta^2}-\frac{\partial^4U(\zeta,s)}{\partial t^2\partial\zeta^2}+\frac{\partial^4U(\zeta,s)}{\partial\zeta^4}$$
$$+\mathcal{L}_t\left(\mathcal{L}_t^{-1}\left(\frac{\partial^2U(\zeta,s)}{\partial\zeta^2}\right)\right)^2-\mathcal{L}_t\left(\mathcal{L}_t^{-1}\left(\frac{\partial^2U(\zeta,s)}{\partial t^2}\right)\right)^2+2\mathcal{L}_t\left(\mathcal{L}_t^{-1}\left(U(\zeta,s)\right)\right)^2\Bigg]=0. \tag{4.13}$$

Table 2. Comparison of the exact and proposed technique solution and various fractional-orders α and $t = 0.099$ for Example 4.2.

ζ	$\alpha = 2.5$	$\alpha = 2.7$	$\alpha = 2.9$	$\alpha = 3$	Exact
0	1.08911	1.09052	1.09122	1.09142	1.09199
0.2	1.32968	1.33169	1.33268	1.33297	1.33376
0.4	1.62325	1.62612	1.62754	1.62795	1.62905
0.6	1.98139	1.98553	1.98759	1.98818	1.98973
0.8	2.41823	2.42422	2.42721	2.42807	2.43026
1	2.95086	2.95959	2.96394	2.96519	2.96833

Residual Laplace function is given as

$$\mathcal{L}_tRes_u(\zeta,s)=U(\zeta,s)-\frac{e^{\zeta}}{s}-\frac{e^{\zeta}}{s^2}-\frac{e^{\zeta}}{s^3}+\frac{1}{s^{\alpha}}\Bigg[\frac{\partial^3U(\zeta,s)}{\partial t\partial\zeta^2}-\frac{\partial^4U(\zeta,s)}{\partial t^2\partial\zeta^2}+\frac{\partial^4U(\zeta,s)}{\partial\zeta^4}$$
$$+\mathcal{L}_t\left(\mathcal{L}_t^{-1}\left(\frac{\partial^2U(\zeta,s)}{\partial\zeta^2}\right)\right)^2-\mathcal{L}_t\left(\mathcal{L}_t^{-1}\left(\frac{\partial^2U(\zeta,s)}{\partial t^2}\right)\right)^2+2\mathcal{L}_t\left(\mathcal{L}_t^{-1}\left(U(\zeta,s)\right)\right)^2\Bigg], \tag{4.14}$$

and so the k^{th}-truncated term series are

$$u(\zeta,s)=\frac{e^{\zeta}}{s}+\frac{e^{\zeta}}{s^2}+\frac{e^{\zeta}}{s^3}+\sum_{n=1}^{k}\frac{f_n(\zeta,s)}{s^{n\alpha+1}},\ k=1,2,3,4\cdots, \tag{4.15}$$

and the k^{th}-LRFs as:

$$\mathcal{L}_tRes_k(\zeta,s)=U_k(\zeta,s)-\frac{e^{\zeta}}{s}-\frac{e^{\zeta}}{s^2}-\frac{e^{\zeta}}{s^3}+\frac{1}{s^{\sigma}}\Bigg[\frac{\partial^3U_k(\zeta,s)}{\partial t\partial\zeta^2}-\frac{\partial^4U_k(\zeta,s)}{\partial t^2\partial\zeta^2}+\frac{\partial^4U_k(\zeta,s)}{\partial\zeta^4}$$
$$+\mathcal{L}_t\left(\mathcal{L}_t^{-1}\left(\frac{\partial^2U_k(\zeta,s)}{\partial\zeta^2}\right)\right)^2-\mathcal{L}_t\left(\mathcal{L}_t^{-1}\left(\frac{\partial^2U_k(\zeta,s)}{\partial t^2}\right)\right)^2+2\mathcal{L}_t\left(\mathcal{L}_t^{-1}\left(U_k(\zeta,s)\right)\right)^2\Bigg]. \tag{4.16}$$

Now, we calculate $f_k(\zeta,s)$, $k=1,2,3,\cdots$, substituting the k^{th}-truncate series of Eq (4.15) into the k^{th} residual Laplace term Eq (4.16), multiply the solution equation by $s^{k\alpha+1}$, and then solve recursively the link $\lim_{s\to\infty}(s^{k\alpha+1}\mathcal{L}_tRes_{u,k}(\zeta,s))=0$, $k=1,2,3,\cdots$. Following are the first some term:

$$\begin{aligned}f_1(\zeta,s)&=-(e^{\zeta}+3e^{2\zeta}),\\ f_2(\zeta,s)&=e^{\zeta}+54e^{2\zeta}+36e^{3\zeta},\\ f_3(\zeta,s)&=-(e^{\zeta}+870e^{2\zeta}+3564e^{3\zeta}+792e^{4\zeta}).\end{aligned} \tag{4.17}$$

Putting the value of $f_k(\zeta, s)$, $k = 1, 2, 3, \cdots$, in Eq (4.15), we get

$$U(\zeta, s) = \frac{e^{\zeta}}{s} + \frac{e^{\zeta}}{s^2} + \frac{e^{\zeta}}{s^3} - \frac{e^{\zeta} + 3e^{2\zeta}}{s^{\alpha+1}} - \frac{e^{\zeta} + 54e^{2\zeta} + 36e^{3\zeta}}{s^{2\alpha+1}} - \frac{e^{\zeta} + 870e^{2\zeta} + 3564e^{3\zeta} + 792e^{4\zeta}}{s^{3\alpha+1}} + \cdots . \tag{4.18}$$

Using inverse LT, we get

$$u(\zeta, t) = e^{\zeta} + e^{\zeta}t + \frac{e^{\zeta}t}{2} - \frac{(e^{\zeta} + 3e^{2\zeta})t^{\alpha}}{\Gamma(\alpha+1)} + \frac{(e^{\zeta} + 54e^{2\zeta} + 36e^{3\zeta})t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{(e^{\zeta} + 870e^{2\zeta} + 3564e^{3\zeta} + 792e^{4\zeta})t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots , \tag{4.19}$$

and the exact solution are

$$u = e^{\zeta+t}. \tag{4.20}$$

In Figure 4, the exact and LRPSM solutions for $u(\zeta, t)$ at $\alpha = 3$ at ζ and $t = 0.3$ of Example 4.2. In Figure 5, LRPSM solutions for $u(\zeta, t)$ at $\alpha = 2.5$ and $\alpha = 2.8$ and $t = 0.3$ of Example 4.2.

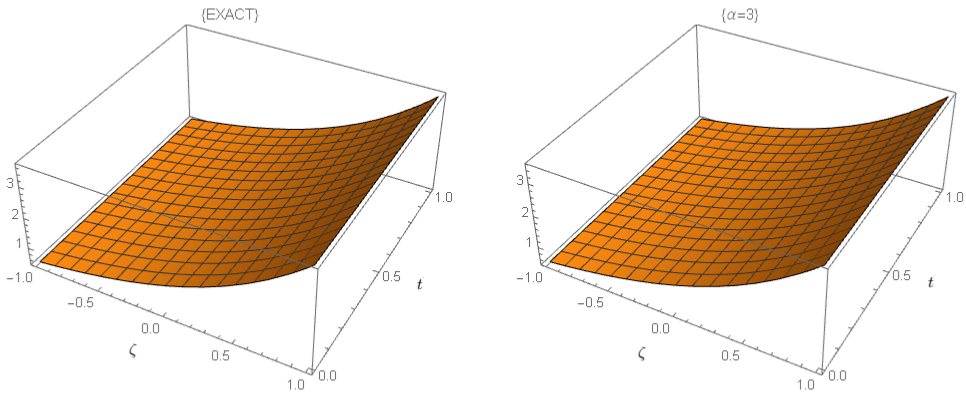

Figure 4. Exact and LRPSM solutions for $u(\zeta, t)$ at $\alpha = 3$ at ζ and $t = 0.3$.

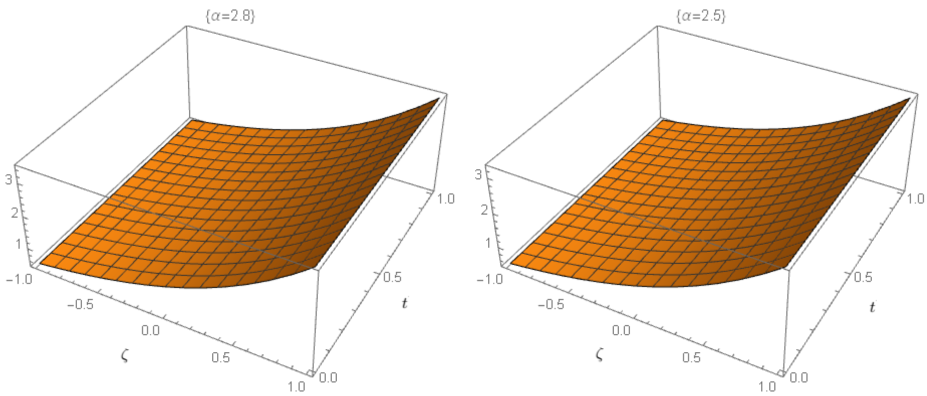

Figure 5. LRPSM solutions for $u(\zeta, t)$ at $\alpha = 2.5$ and $\alpha = 2.8$ and $t = 0.3$.

Example 4.3. Consider the fractional partial differential equations [47]:

$$\begin{aligned}&D_t^{\alpha}u(\zeta,t)-\frac{\partial^3 u(\zeta,t)}{\partial t\partial\zeta^2}-\frac{\partial^4 u(\zeta,t)}{\partial t^2\partial x^2}+\frac{\partial^4 u(\zeta,t)}{\partial\zeta^4}\\&-\left(\frac{\partial^2 u(\zeta,t)}{\partial t^2}\right)\left(\frac{\partial u(\zeta,t)}{\partial\zeta}\right)-u(\zeta,t)(\frac{\partial u(\zeta,t)}{\partial t})=0,\quad\text{where }\ 2<\alpha\le 3,\end{aligned}\tag{4.21}$$

with the following IC's:

$$U(\zeta,0)=\cos\zeta,\ \frac{\partial}{\partial t}U(\zeta,0)=-\sin\zeta,\ \frac{\partial^2}{\partial t^2}U(\zeta,0)=-\cos\zeta.\tag{4.22}$$

Table 3. Comparison of the exact and proposed technique solution and various fractional-orders α and $t=0.22$ for Example 4.3.

ζ	$\alpha=2.5$	$\alpha=2.7$	$\alpha=2.9$	$\alpha=3$	Exact
0	0.	0.97178	0.973463	0.974025	0.975897
0.2	-0.04370738	0.908702	0.910351	0.910903	0.913089
0.4	-0.085672	0.809397	0.810946	0.811465	0.813878
0.6	-0.124221	0.677823	0.679212	0.679677	0.682221
0.8	-0.157818	0.519227	0.5204	0.520792	0.523366
1	-0.185124	0.339931	0.34084	0.341145	0.343646

Using Laplace transform to Eq (4.21), we get

$$\begin{aligned}&U(\zeta,s)-\frac{\cos\zeta}{s}+\frac{\sin\zeta}{s^2}+\frac{\cos\zeta}{s^3}+\frac{1}{s^{\sigma}}\Big[-\frac{\partial^3 U(\zeta,s)}{\partial t\partial\zeta^2}-\frac{\partial^4 U(\zeta,s)}{\partial t^2\partial\zeta^2}+\frac{\partial^4 U(\zeta,s)}{\partial\zeta^4}\\&+\mathcal{L}\left(\mathcal{L}_t^{-1}\left(\frac{\partial^2 u(\zeta,s)}{\partial t^2}\right)\mathcal{L}_t^{-1}\left(\frac{\partial u(\zeta,s)}{\partial\zeta}\right)\right)-\mathcal{L}\left(\mathcal{L}_t^{-1}\left(U(\zeta,s)\right)\mathcal{L}_t^{-1}\left(\frac{\partial U(\zeta,s)}{\partial t}\right)\right)\Big]=0.\end{aligned}\tag{4.23}$$

Residual Laplace function is given as

$$\begin{aligned}\mathcal{L}_tRes_u(\zeta,s)=U(\zeta,s)&-\frac{\cos\zeta}{s}+\frac{\sin\zeta}{s^2}+\frac{\cos\zeta}{s^3}+\frac{1}{s^{\sigma}}\Big[-\frac{\partial^3 U(\zeta,s)}{\partial t\partial\zeta^2}-\frac{\partial^4 U(\zeta,s)}{\partial t^2\partial\zeta^2}+\frac{\partial^4 U(\zeta,s)}{\partial\zeta^4}\\&+\mathcal{L}\left(\mathcal{L}_t^{-1}\left(\frac{\partial^2 u(\zeta,s)}{\partial t^2}\right)\mathcal{L}_t^{-1}\left(\frac{\partial u(\zeta,s)}{\partial\zeta}\right)\right)-\mathcal{L}\left(\mathcal{L}_t^{-1}\left(U(\zeta,s)\right)\mathcal{L}_t^{-1}\left(\frac{\partial U(\zeta,s)}{\partial t}\right)\right)\Big],\end{aligned}\tag{4.24}$$

and so the k^{th}-truncated term series are

$$u(\zeta,s)=\frac{\cos\zeta}{s}+\frac{-\sin\zeta}{s^2}+\frac{-\cos\zeta}{s^3}+\sum_{n=1}^{k}\frac{f_n(\zeta,s)}{s^{n\alpha+1}},\ k=1,2,3,4\cdots,\tag{4.25}$$

and the k^{th}-LRFs as:

$$\begin{aligned}\mathcal{L}_tRes_k(\zeta,s)=U_k(\zeta,s)&-\frac{\cos\zeta}{s}+\frac{\sin\zeta}{s^2}+\frac{\cos\zeta}{s^3}+\frac{1}{s^{\alpha}}\Big[-\frac{\partial^3 U_k(\zeta,s)}{\partial t\partial\zeta^2}-\frac{\partial^4 U_k(\zeta,s)}{\partial t^2\partial\zeta^2}\\&+\frac{\partial^4 U_k(\zeta,s)}{\partial\zeta^4}+\mathcal{L}_t\Big(\mathcal{L}_t^{-1}\Big(\frac{\partial^2 U_k(\zeta,s)}{\partial t^2}\Big)\mathcal{L}_t^{-1}\Big(\frac{\partial U_k(\zeta,s)}{\partial\zeta}\Big)\Big)\\&-\mathcal{L}_t\Big(\mathcal{L}_t^{-1}\Big(U_k(\zeta,s)\Big)\mathcal{L}_t^{-1}\Big(\frac{\partial U_k(\zeta,s)}{\partial t}\Big)\Big)\Big].\end{aligned}\tag{4.26}$$

Now, we calculate $f_k(\zeta, s)$, $k = 1, 2, 3, \cdots$, substituting the k^{th}-truncate series of Eq (4.25) into the k^{th} residual Laplace term Eq (4.26), multiply the solution equation by $s^{k\alpha+1}$, and then solve recursively the link $\lim_{s\to\infty}(s^{k\alpha+1}\mathcal{L}_t Res_{u,k}(\zeta, s)) = 0$, $k = 1, 2, 3, \cdots$. Following are the first some term:

$$f_1(\zeta, s) = -\cos\zeta,\ f_2(\zeta, s) = \cos\zeta,\ f_3(\zeta, s) = -\cos\zeta. \tag{4.27}$$

Putting the value of $f_k(x, s)$, $k = 1, 2, 3, \cdots$, in Eq (4.25), we get

$$U(\zeta, s) = \frac{\cos\zeta}{s} - \frac{\sin\zeta}{s^2} - \frac{\cos\zeta}{s^3} - \frac{\cos\zeta}{s^{\alpha+1}} + \frac{\cos\zeta}{s^{2\alpha+1}} - \frac{\cos\zeta}{s^{3\alpha+1}} + \cdots. \tag{4.28}$$

Using inverse LT, we get

$$u(\zeta, t) = \cos\zeta - t\sin\zeta - \frac{t^2\cos\zeta}{2} - \frac{t^\alpha\cos\zeta}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}\cos\zeta}{\Gamma(2\alpha+1)} - \frac{t^{3\alpha}\cos\zeta}{\Gamma(3\alpha+1)} + \cdots, \tag{4.29}$$

and the exact solution are

$$u = \cos(\zeta + t). \tag{4.30}$$

In Figure 6, exact and LRPSM solutions for $u(\zeta, t)$ at $\alpha = 3$ and $t = 0.3$ of Example 4.3. Figure 7, LRPSM solutions for $u(\zeta, t)$ at $\alpha = 2.5$, $\alpha = 2.8$, and $t = 0.3$.

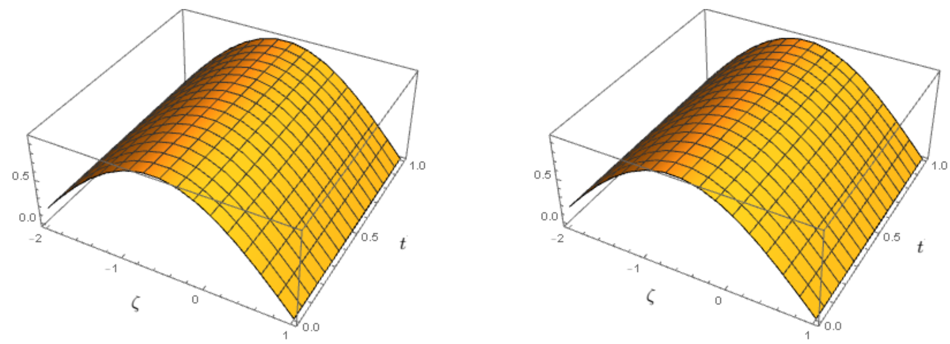

Figure 6. Exact and LRPSM solutions for $u(\zeta, t)$ at $\alpha = 3$ at and $t = 0.3$.

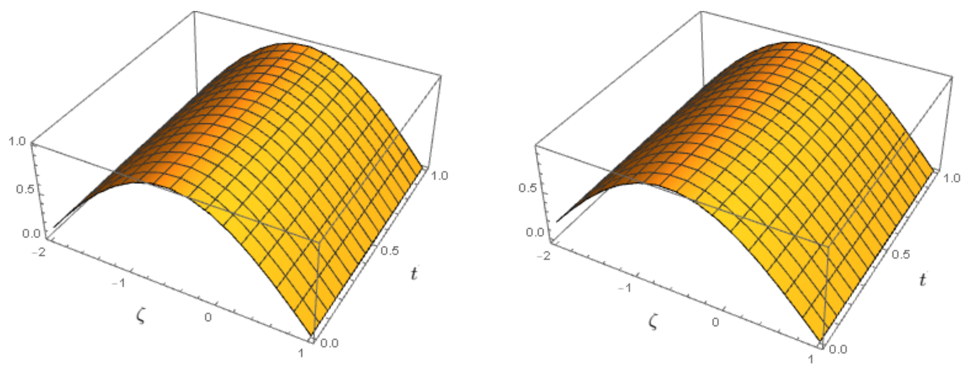

Figure 7. LRPSM solutions for $u(\zeta, t)$ at $\alpha = 2.5$ $\alpha = 2.8$, and $t = 0.3$.

5. Conclusions

In this article, the fractional partial differential equation has been solved analytically by employing the Laplace residual power series method in conjunction with the Caputo operator. To demonstrate the validity of the recommended method, we analyzed three distinct partial differential equation problems. The simulation results demonstrate that the outcomes of our method are in close accordance with the exact answer. The new method is highly straightforward, efficient, and suitable for getting numerical solutions to partial differential equations. The primary advantage of the proposed approach is the series form solution, which rapidly converges to the exact answer. We can therefore conclude that the suggested approach is quite methodical and efficient for a more thorough investigation of fractional-order mathematical models.

Conflict of interest

The authors declare no conflicts of interest.

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