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Research article

Rational contractions on complex-valued extended b-metric spaces and an application

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Abstract: The aim of this article is to obtain common fixed point results on complex-valued extended b-metric spaces for rational contractions involving control functions of two variables. Our theorems generalize some famous results in the literature. We supply an example to show the originality of our main result. As an application, we develop common fixed point results for rational contractions involving control functions of one variable in the class of complex-valued extended b-metric spaces.

Keywords: complex-valued extended b-metric space; common fixed point; control functions; rational expressions
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1. Introduction

Fixed point theory is one of the celebrated and conventional theories in mathematics which has comprehensive applications in different fields. In this theory, the first and pioneer result is the Banach contraction principle in which the underlying space is the complete metric space. The fundamental establishment of a metric space was naturally accomplished by M. Frechet in 1906. Motivated from the influence of this genuine idea to fixed point theory, various authors have undertaken several extensions of this idea in recent years.

In 2011, Azam et al. [1] gave the notion of a complex-valued metric space (CVMS) and proved common fixed points of two single-valued mappings. Since the concept to introduce complex-valued metric spaces is designed to define rational expressions that cannot be defined in cone metric spaces, and therefore several results of fixed point theory cannot be proved to cone metric spaces, so complex-valued metric spaces form a special class of cone metric spaces. Actually, the definition of a cone metric space banks on the underlying Banach space which is not a division Ring. However, we can study generalizations of many results of fixed point theory involving divisions in complex-valued metric spaces. Moreover, this idea is also used to define complex-valued Banach spaces [2] which offer a lot of scopes for further investigations.

Subsequently, Rouzkard et al. [3] added a rational expression in the Azam's contraction and generalized the main work of Azam et al. [1]. Later on, Sintunavarat et al. [4] replaced the constants involved in the contraction with control functions of one variable and generalized the papers of Azam et al. [1] and Rouzkard et al. [3]. Sitthikul et al. [5] used control functions of two variables in the contraction and established common fixed point theorems in the context of a CVMS. In 2014, Mukheimer [6] extended this concept to complex-valued b-metric spaces (CVbMSs). In recent times, Naimatullah et al. [7] gave the concept of a complex-valued extended b-metric space (CVEbMS) as an extension of a CVbMS and proved some fixed point results for generalized contractions. For more details in this direction, we refer the readers to [8–15].

In this article, we utilize the notion of a complex-valued extended b-metric space and obtain common fixed point results for rational contractions involving control functions of two variables. We also provide a non-trivial example to show the originality of our main results. As an application, we investigate the solution of a Urysohn integral equation.

2. Preliminaries

The notion of a complex-valued metric space (CVMS) is given as follows:

Definition 2.1. *(See [1]) Let* $\upsilon_1, \upsilon_2 \in \mathbb{C}$. *A partial order* $\precsim$ *on* $\mathbb{C}$ *is defined as follows:*

$$\upsilon_1 \precsim \upsilon_2 \ \Leftrightarrow \ Re(\upsilon_1) \leqslant Re(\upsilon_2),\ Im(\upsilon_1) \leqslant Im(\upsilon_2).$$

It follows that

$$\upsilon_1 \precsim \upsilon_2,$$

if one of these assertions is satisfied:

$$\begin{aligned}
\text{(a) } \mathrm{Re}(\upsilon_1) &= \mathrm{Re}(\upsilon_2),\ \mathrm{Im}(\upsilon_1) < \mathrm{Im}(\upsilon_2),\\
\text{(b) } \mathrm{Re}(\upsilon_1) &< \mathrm{Re}(\upsilon_2),\ \mathrm{Im}(\upsilon_1) = \mathrm{Im}(\upsilon_2),\\
\text{(c) } \mathrm{Re}(\upsilon_1) &< \mathrm{Re}(\upsilon_2),\ \mathrm{Im}(\upsilon_1) < \mathrm{Im}(\upsilon_2),\\
\text{(d) } \mathrm{Re}(\upsilon_1) &= \mathrm{Re}(\upsilon_2),\ \mathrm{Im}(\upsilon_1) = \mathrm{Im}(\upsilon_2).
\end{aligned}$$

Definition 2.2. *(See [1]) Let* $\mathfrak{F} \neq \emptyset$ *and* $\varsigma : \mathfrak{F} \times \mathfrak{F} \to \mathbb{C}$ *be a mapping satisfying*

(i) $0 \precsim \varsigma(\upsilon, \hbar)$, *and* $\varsigma(\upsilon, \hbar) = 0 \Leftrightarrow \upsilon = \hbar$;
(ii) $\varsigma(\upsilon, \hbar) = \varsigma(\hbar, \upsilon)$;
(iii) $\varsigma(\upsilon, \hbar) \precsim \varsigma(\upsilon, \varrho) + \varsigma(\varrho, \hbar)$,

for all $\upsilon, \hbar, \varrho \in \mathfrak{F}$, *then* $(\mathfrak{F}, \varsigma)$ *is said to be a complex-valued metric space.*

Example 2.1. (See [1]) Let $\mathfrak{F} = [0, 1]$ and $\upsilon, \hbar \in \mathfrak{F}$. Define $\varsigma : \mathfrak{F} \times \mathfrak{F} \to \mathbb{C}$ by

$$\varsigma(\upsilon, \hbar) = \begin{cases} 0, & \text{if } \upsilon = \hbar, \\ \frac{i}{2}, & \text{if } \upsilon \neq \hbar. \end{cases}$$

Then $(\mathfrak{F}, \varsigma)$ is a complex-valued metric space.

Mukheimer [6] presented the concept of a complex-valued b-metric space (CVbMS).

Definition 2.3. *(See [6]) Let* $\mathfrak{F} \neq \emptyset$, $\pi \geq 1$ *and* $\varsigma : \mathfrak{F} \times \mathfrak{F} \to \mathbb{C}$ *be a mapping satisfying*

(i) $0 \precsim \varsigma(\upsilon, \hbar)$ *and* $\varsigma(\upsilon, \hbar) = 0 \Leftrightarrow \upsilon = \hbar$;
(ii) $\varsigma(\upsilon, \hbar) = \varsigma(\hbar, \upsilon)$;
(iii) $\varsigma(\upsilon, \hbar) \precsim \pi \left[\varsigma(\upsilon, \varrho) + \varsigma(\varrho, \hbar)\right]$,

for all $\upsilon, \hbar, \varrho \in \mathfrak{F}$, *then* $(\mathfrak{F}, \varsigma)$ *is said to be a complex-valued b- metric space (CVbMS).*

Example 2.2. *(See [6]) Let* $\mathfrak{F} = [0, 1]$. *Define* $\varsigma : \mathfrak{F} \times \mathfrak{F} \to \mathbb{C}$ *by*

$$\varsigma(\upsilon, \hbar) = |\upsilon - \hbar|^2 + i|\upsilon - \hbar|^2$$

for all $\upsilon, \hbar \in \mathfrak{F}$. *Then* $(\mathfrak{F}, \varsigma)$ *is a complex-valued b-metric space with* $\pi = 2$.

In 2019, Naimatullah et al. [7] defined the concept of a complex valued extended b-metric space as follows:

Definition 2.4. *(See [7]) Let* $\mathfrak{F} \neq \emptyset$, $\varphi : \mathfrak{F} \times \mathfrak{F} \to [1, \infty)$ *and* $\varsigma : \mathfrak{F} \times \mathfrak{F} \to \mathbb{C}$ *be a mapping satisfying*

(i) $0 \precsim \varsigma(\upsilon, \hbar)$, *for all* $\upsilon, \hbar \in \mathfrak{F}$ *and* $\varsigma(\upsilon, \hbar) = 0$ *if and only if* $\upsilon = \hbar$;
(ii) $\varsigma(\upsilon, \hbar) = \varsigma(\hbar, \upsilon)$ *for all* $\upsilon, \hbar \in \mathfrak{F}$;
(iii) $\varsigma(\upsilon, \hbar) \precsim \varphi(\upsilon, \hbar)\left[\varsigma(\upsilon, \varrho) + \varsigma(\varrho, \hbar)\right]$, *for all* $\upsilon, \hbar, \varrho \in \mathfrak{F}$.

Then $(\mathfrak{F}, \varsigma)$ *is said to be a complex-valued extended b-metric space (CVEbMS).*

Example 2.3. *(See [7]) Given* $\mathfrak{F} \neq \emptyset$. *Let* $\varphi : \mathfrak{F} \times \mathfrak{F} \to [1, \infty)$ *be defined by*

$$\varphi(\upsilon, \hbar) = \frac{1 + \upsilon + \hbar}{\upsilon + \hbar}$$

and $\varsigma : \mathfrak{F} \times \mathfrak{F} \to \mathbb{C}$ *by*
(i) $\varsigma(\upsilon, \hbar) = \frac{i}{\upsilon\hbar}$ *for all* $0 < \upsilon, \hbar \leq 1$;
(ii) $\varsigma(\upsilon, \hbar) = 0 \Leftrightarrow \upsilon = \hbar$ *for all* $0 \leq \upsilon, \hbar \leq 1$;
(iii) $\varsigma(\upsilon, 0) = \varsigma(0, \upsilon) = \frac{i}{\upsilon}$ *for all* $0 < \upsilon \leq 1$.
Then $(\mathfrak{F}, \varsigma)$ *is a CVEbMS.*

Lemma 2.1. *(See [7]) Let $(\mathfrak{F}, \varsigma)$ be a CVEbMS and let $\{\upsilon_\iota\} \subseteq \mathfrak{F}$. Then $\{\upsilon_\iota\}$ converges to υ if and only if $|\varsigma(\upsilon_\iota, \upsilon)| \to 0$ when $\iota \to \infty$.*

Lemma 2.2. *(See [7]) Let $(\mathfrak{F}, \varsigma)$ be a CVEbMS and let $\{\upsilon_\iota\} \subseteq \mathfrak{F}$. Then $\{\upsilon_\iota\}$ is a Cauchy sequence if and only if $|\varsigma(\upsilon_\iota, \upsilon_{\iota+m})| \to 0$ when $\iota \to \infty$, for each $m \in \mathbb{N}$.*

In this paper, we obtain common fixed point results on a complex-valued extended b-metric space (CVEbMS) for rational contractions involving control functions of two variables. Applying our results, we derive the main theorems of Azam et al. [1], Rouzkard et al. [3], Sintunavarat et al. [4] and Sitthikul et al. [5] for single-valued mappings in CVEbMSs.

3. Main results

We state and prove the following proposition which is required in the proof of our main result.

Proposition 3.1. *Let $\upsilon_0 \in \mathfrak{F}$. Define the sequence $\{\upsilon_\iota\}$ by*

$$\upsilon_{2\iota+1} = \mathfrak{I}_1 \upsilon_{2\iota} \text{ and } \upsilon_{2\iota+2} = \mathfrak{I}_2 \upsilon_{2\iota+1}$$

for all $\iota = 0, 1, \cdots$. Assume that there exists $\rho : \mathfrak{F} \times \mathfrak{F} \to [0, 1)$ satisfying

$$\rho(\mathfrak{I}_2\mathfrak{I}_1\upsilon, \hbar) \leq \rho(\upsilon, \hbar) \text{ and } \rho(\upsilon, \mathfrak{I}_1\mathfrak{I}_2\hbar) \leq \rho(\upsilon, \hbar)$$

for all $\upsilon, \hbar \in \mathfrak{F}$. Then

$$\rho(\upsilon_{2\iota}, \hbar) \leq \rho(\upsilon_0, \hbar) \text{ and } \rho(\upsilon, \upsilon_{2\iota+1}) \leq \rho(\upsilon, \upsilon_1)$$

for all $\upsilon, \hbar \in \mathfrak{F}$ and $\iota = 0, 1, \cdots$.

Lemma 3.1. *Let $\rho, \kappa : \mathfrak{F} \times \mathfrak{F} \to [0, 1)$ and $\upsilon, \hbar \in \mathfrak{F}$. If $\mathfrak{I}_1, \mathfrak{I}_2 : \mathfrak{F} \to \mathfrak{F}$ satisfies*

$$\varsigma(\mathfrak{I}_1\upsilon, \mathfrak{I}_2\mathfrak{I}_1\upsilon) \leq \rho(\upsilon, \mathfrak{I}_1\upsilon)\varsigma(\upsilon, \mathfrak{I}_1\upsilon) + \kappa(\upsilon, \mathfrak{I}_1\upsilon)\frac{\varsigma(\upsilon, \mathfrak{I}_1\upsilon)\varsigma(\mathfrak{I}_1\upsilon, \mathfrak{I}_2\mathfrak{I}_1\upsilon)}{1 + \varsigma(\upsilon, \mathfrak{I}_1\upsilon)},$$

$$\varsigma(\mathfrak{I}_1\mathfrak{I}_2\hbar, \mathfrak{I}_2\hbar) \leq \rho(\mathfrak{I}_2\hbar, \hbar)\varsigma(\mathfrak{I}_2\hbar, \hbar) + \kappa(\mathfrak{I}_2\hbar, \hbar)\frac{\varsigma(\mathfrak{I}_2\hbar, \mathfrak{I}_1\mathfrak{I}_2\hbar)\varsigma(\hbar, \mathfrak{I}_2\hbar)}{1 + \varsigma(\mathfrak{I}_2\hbar, \hbar)},$$

then

$$\left|\varsigma(\mathfrak{I}_1\upsilon, \mathfrak{I}_2\mathfrak{I}_1\upsilon)\right| \leq \rho(\upsilon, \mathfrak{I}_1\upsilon)\left|\varsigma(\upsilon, \mathfrak{I}_1\upsilon)\right| + \kappa(\upsilon, \mathfrak{I}_1\upsilon)\left|\varsigma(\mathfrak{I}_1\upsilon, \mathfrak{I}_2\mathfrak{I}_1\upsilon)\right|,$$

$$\left|\varsigma(\mathfrak{I}_1\mathfrak{I}_2\hbar, \mathfrak{I}_2\hbar)\right| \leq \rho(\mathfrak{I}_2\hbar, \hbar)\left|\varsigma(\mathfrak{I}_2\hbar, \hbar)\right| + \kappa(\mathfrak{I}_2\hbar, \hbar)\left|\varsigma(\mathfrak{I}_2\hbar, \mathfrak{I}_1\mathfrak{I}_2\hbar)\right|.$$

Proof. We can write

$$\begin{aligned}
\left|\varsigma(\mathfrak{I}_1\upsilon, \mathfrak{I}_2\mathfrak{I}_1\upsilon)\right| &\leq \left|\rho(\upsilon, \mathfrak{I}_1\upsilon)\varsigma(\upsilon, \mathfrak{I}_1\upsilon) + \kappa(\upsilon, \mathfrak{I}_1\upsilon)\frac{\varsigma(\upsilon, \mathfrak{I}_1\upsilon)\varsigma(\mathfrak{I}_1\upsilon, \mathfrak{I}_2\mathfrak{I}_1\upsilon)}{1 + \varsigma(\upsilon, \mathfrak{I}_1\upsilon)}\right| \\
&\leq \rho(\upsilon, \mathfrak{I}_1\upsilon)\left|\varsigma(\upsilon, \mathfrak{I}_1\upsilon)\right| + \kappa(\upsilon, \mathfrak{I}_1\upsilon)\left|\frac{\varsigma(\upsilon, \mathfrak{I}_1\upsilon)}{1 + \varsigma(\upsilon, \mathfrak{I}_1\upsilon)}\right|\left|\varsigma(\mathfrak{I}_1\upsilon, \mathfrak{I}_2\mathfrak{I}_1\upsilon)\right| \\
&\leq \rho(\upsilon, \mathfrak{I}_1\upsilon)\left|\varsigma(\upsilon, \mathfrak{I}_1\upsilon)\right| + \kappa(\upsilon, \mathfrak{I}_1\upsilon)\left|\varsigma(\mathfrak{I}_1\upsilon, \mathfrak{I}_2\mathfrak{I}_1\upsilon)\right|.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}\left|\varsigma(\Im_1\Im_2\hbar,\Im_2\hbar)\right| &\leq \left|\rho(\Im_2\hbar,\hbar)\varsigma(\Im_2\hbar,\hbar)+\kappa(\Im_2\hbar,\hbar)\frac{\varsigma(\Im_2\hbar,\Im_1\Im_2\hbar)\varsigma(\hbar,\Im_2\hbar)}{1+\varsigma(\Im_2\hbar,\hbar)}\right| \\ &\leq \rho(\Im_2\hbar,\hbar)\left|\varsigma(\Im_2\hbar,\hbar)\right|+\kappa(\Im_2\hbar,\hbar)\left|\frac{\varsigma(\hbar,\Im_2\hbar)}{1+\varsigma(\Im_2\hbar,\hbar)}\right|\left|\varsigma(\Im_2\hbar,\Im_1\Im_2\hbar)\right| \\ &\leq \rho(\Im_2\hbar,\hbar)\left|\varsigma(\Im_2\hbar,\hbar)\right|+\kappa(\Im_2\hbar,\hbar)\left|\varsigma(\Im_2\hbar,\Im_1\Im_2\hbar)\right|.\end{aligned}$$

□

Theorem 3.1. *Let $(\mathfrak{F},\varsigma)$ be a complete CVEbMS, $\varphi:\mathfrak{F}\times\mathfrak{F}\to[1,\infty)$ and let $\Im_1,\Im_2$: $\mathfrak{F}\to\mathfrak{F}$. If there exist mappings $\rho,\kappa,\mu:\mathfrak{F}\times\mathfrak{F}\to[0,1)$ such that for all $\upsilon,\hbar\in\mathfrak{F}$,*

(a) $\rho(\Im_2\Im_1\upsilon,\hbar)\leq\rho(\upsilon,\hbar)$ *and* $\rho(\upsilon,\Im_1\Im_2\hbar)\leq\rho(\upsilon,\hbar)$,
$\kappa(\Im_2\Im_1\upsilon,\hbar)\leq\kappa(\upsilon,\hbar)$ *and* $\kappa(\upsilon,\Im_1\Im_2\hbar)\leq\kappa(\upsilon,\hbar)$,
$\mu(\Im_2\Im_1\upsilon,\hbar)\leq\mu(\upsilon,\hbar)$ *and* $\mu(\upsilon,\Im_1\Im_2\hbar)\leq\mu(\upsilon,\hbar)$,

(b) $\rho(\upsilon,\hbar)+\kappa(\upsilon,\hbar)+\mu(\upsilon,\hbar)<1$,

(c)

$$\varsigma(\Im_1\upsilon,\Im_2\hbar)\leq\rho(\upsilon,\hbar)\varsigma(\upsilon,\hbar)+\kappa(\upsilon,\hbar)\frac{\varsigma(\upsilon,\Im_1\upsilon)\varsigma(\hbar,\Im_2\hbar)}{1+\varsigma(\upsilon,\hbar)}+\mu(\upsilon,\hbar)\frac{\varsigma(\hbar,\Im_1\upsilon)\varsigma(\upsilon,\Im_2\hbar)}{1+\varsigma(\upsilon,\hbar)}, \tag{3.1}$$

(d) For each $\upsilon_0\in\mathfrak{F}$, $\lambda=\frac{\rho(\upsilon_0,\upsilon_1)}{1-\kappa(\upsilon_0,\upsilon_1)}<1$ and $\lim_{\iota,m\to\infty}\varphi(\upsilon_\iota,\upsilon_m)\lambda<1$ hold, whenever the sequence $\{\upsilon_\iota\}$ is defined by

$$\upsilon_{2\iota+1}=\Im_1\upsilon_{2\iota} \text{ and } \upsilon_{2\iota+2}=\Im_2\upsilon_{2\iota+1}, \tag{3.2}$$

then $\Im_1$ and $\Im_2$ have a unique common fixed point.

Proof. Let $\upsilon,\hbar\in\mathfrak{F}$. From (3.1), we have

$$\begin{aligned}\varsigma(\Im_1\upsilon,\Im_2\Im_1\upsilon) &\leq \rho(\upsilon,\Im_1\upsilon)\varsigma(\upsilon,\Im_1\upsilon)+\kappa(\upsilon,\Im_1\upsilon)\frac{\varsigma(\upsilon,\Im_1\upsilon)\varsigma(\Im_1\upsilon,\Im_2\Im_1\upsilon)}{1+\varsigma(\upsilon,\Im_1\upsilon)} \\ &= \rho(\upsilon,\Im_1\upsilon)\varsigma(\upsilon,\Im_1\upsilon)+\kappa(\upsilon,\Im_1\upsilon)\frac{\varsigma(\upsilon,\Im_1\upsilon)\varsigma(\Im_1\upsilon,\Im_2\Im_1\upsilon)}{1+\varsigma(\upsilon,\Im_1\upsilon)}.\end{aligned}$$

By Lemma 3.1, we get

$$\left|\varsigma(\Im_1\upsilon,\Im_2\Im_1\upsilon)\right|\leq\rho(\upsilon,\Im_1\upsilon)\left|\varsigma(\upsilon,\Im_1\upsilon)\right|+\kappa(\upsilon,\Im_1\upsilon)\left|\varsigma(\Im_1\upsilon,\Im_2\Im_1\upsilon)\right|. \tag{3.3}$$

Similarly, we have

$$\begin{aligned}\varsigma(\Im_1\Im_2\hbar,\Im_2\hbar) &\leq \rho(\Im_2\hbar,\hbar)\varsigma(\Im_2\hbar,\hbar)+\kappa(\Im_2\hbar,\hbar)\frac{\varsigma(\Im_2\hbar,\Im_1\Im_2\hbar)\varsigma(\hbar,\Im_2\hbar)}{1+\varsigma(\Im_2\hbar,\hbar)} \\ &\quad +\mu(\upsilon,\hbar)\frac{\varsigma(\hbar,\Im_1\Im_2\hbar)\varsigma(\Im_2\hbar,\Im_2\hbar)}{1+\varsigma(\Im_2\hbar,\hbar)} \\ &= \rho(\Im_2\hbar,\hbar)\varsigma(\Im_2\hbar,\hbar)+\kappa(\Im_2\hbar,\hbar)\frac{\varsigma(\Im_2\hbar,\Im_1\Im_2\hbar)\varsigma(\hbar,\Im_2\hbar)}{1+\varsigma(\Im_2\hbar,\hbar)}.\end{aligned}$$

By Lemma 3.1, we get

$$\left|\varsigma\left(\mathfrak{I}_1\mathfrak{I}_2\hbar,\mathfrak{I}_2\hbar\right)\right| \leq \rho\left(\mathfrak{I}_2\hbar,\hbar\right)\left|\varsigma\left(\mathfrak{I}_2\hbar,\hbar\right)\right| + \kappa\left(\mathfrak{I}_2\hbar,\hbar\right)\left|\varsigma\left(\mathfrak{I}_2\hbar,\mathfrak{I}_1\mathfrak{I}_2\hbar\right)\right|. \tag{3.4}$$

Let υ_0 be an arbitrary point in $\mathfrak{F}$ and the sequence $\{\upsilon_\iota\}$ be defined by (3.2). From Proposition 3.1, (3.3) and (3.4) and for all $\iota = 0, 1, 2, \cdots$,

$$\begin{aligned}
\left|\varsigma\left(\upsilon_{2\iota+1},\upsilon_{2\iota}\right)\right| &= \left|\varsigma\left(\mathfrak{I}_1\mathfrak{I}_2\upsilon_{2\iota-1},\mathfrak{I}_2\upsilon_{2\iota-1}\right)\right| \leq \rho\left(\mathfrak{I}_2\upsilon_{2\iota-1},\upsilon_{2\iota-1}\right)\left|\varsigma\left(\mathfrak{I}_2\upsilon_{2\iota-1},\upsilon_{2\iota-1}\right)\right| \\
&\quad +\kappa\left(\mathfrak{I}_2\upsilon_{2\iota-1},\upsilon_{2\iota-1}\right)\left|\varsigma\left(\mathfrak{I}_2\upsilon_{2\iota-1},\mathfrak{I}_1\mathfrak{I}_2\upsilon_{2\iota-1}\right)\right| \\
&= \rho\left(\upsilon_{2\iota},\upsilon_{2\iota-1}\right)\left|\varsigma\left(\upsilon_{2\iota},\upsilon_{2\iota-1}\right)\right| + \kappa\left(\upsilon_{2\iota},\upsilon_{2\iota-1}\right)\left|\varsigma\left(\upsilon_{2\iota},\upsilon_{2\iota+1}\right)\right| \\
&\leq \rho\left(\upsilon_0,\upsilon_{2\iota-1}\right)\left|\varsigma\left(\upsilon_{2\iota},\upsilon_{2\iota-1}\right)\right| + \kappa\left(\upsilon_0,\upsilon_{2\iota-1}\right)\left|\varsigma\left(\upsilon_{2\iota},\upsilon_{2\iota+1}\right)\right| \\
&\leq \rho\left(\upsilon_0,\upsilon_1\right)\left|\varsigma\left(\upsilon_{2\iota},\upsilon_{2\iota-1}\right)\right| + \kappa\left(\upsilon_0,\upsilon_1\right)\left|\varsigma\left(\upsilon_{2\iota},\upsilon_{2\iota+1}\right)\right|,
\end{aligned}$$

which implies that

$$\left|\varsigma\left(\upsilon_{2\iota+1},\upsilon_{2\iota}\right)\right| \leq \frac{\rho\left(\upsilon_0,\upsilon_1\right)}{1-\kappa\left(\upsilon_0,\upsilon_1\right)}\left|\varsigma\left(\upsilon_{2\iota},\upsilon_{2\iota-1}\right)\right|. \tag{3.5}$$

Similarly, we have

$$\begin{aligned}
\left|\varsigma\left(\upsilon_{2\iota+2},\upsilon_{2\iota+1}\right)\right| &= \left|\varsigma\left(\mathfrak{I}_2\mathfrak{I}_1\upsilon_{2\iota},\mathfrak{I}_1\upsilon_{2\iota}\right)\right| \leq \rho\left(\upsilon_{2\iota},\mathfrak{I}_1\upsilon_{2\iota}\right)\left|\varsigma\left(\upsilon_{2\iota},\mathfrak{I}_1\upsilon_{2\iota}\right)\right| \\
&\quad +\kappa\left(\upsilon_{2\iota},\mathfrak{I}_1\upsilon_{2\iota}\right)\left|\varsigma\left(\mathfrak{I}_1\upsilon_{2\iota},\mathfrak{I}_2\mathfrak{I}_1\upsilon_{2\iota}\right)\right| \\
&= \rho\left(\upsilon_{2\iota},\upsilon_{2\iota+1}\right)\left|\varsigma\left(\upsilon_{2\iota},\upsilon_{2\iota+1}\right)\right| + \kappa\left(\upsilon_{2\iota},\upsilon_{2\iota+1}\right)\left|\varsigma\left(\upsilon_{2\iota+1},\upsilon_{2\iota+2}\right)\right| \\
&\leq \rho\left(\upsilon_0,\upsilon_{2\iota+1}\right)\left|\varsigma\left(\upsilon_{2\iota},\upsilon_{2\iota+1}\right)\right| + \kappa\left(\upsilon_0,\upsilon_{2\iota+1}\right)\left|\varsigma\left(\upsilon_{2\iota+1},\upsilon_{2\iota+2}\right)\right| \\
&\leq \rho\left(\upsilon_0,\upsilon_1\right)\left|\varsigma\left(\upsilon_{2\iota},\upsilon_{2\iota+1}\right)\right| + \kappa\left(\upsilon_0,\upsilon_1\right)\left|\varsigma\left(\upsilon_{2\iota+1},\upsilon_{2\iota+2}\right)\right|.
\end{aligned}$$

It implies that

$$\begin{aligned}
\left|\varsigma\left(\upsilon_{2\iota+2},\upsilon_{2\iota+1}\right)\right| &\leq \frac{\rho\left(\upsilon_0,\upsilon_1\right)}{1-\kappa\left(\upsilon_0,\upsilon_1\right)}\left|\varsigma\left(\upsilon_{2\iota},\upsilon_{2\iota+1}\right)\right| \\
&= \frac{\rho\left(\upsilon_0,\upsilon_1\right)}{1-\kappa\left(\upsilon_0,\upsilon_1\right)}\left|\varsigma\left(\upsilon_{2\iota+1},\upsilon_{2\iota}\right)\right|.
\end{aligned} \tag{3.6}$$

Let $\lambda = \frac{\rho(\upsilon_0,\upsilon_1)}{1-\kappa(\upsilon_0,\upsilon_1)} < 1$. Then from (3.5) and (3.6), we have

$$\left|\varsigma\left(\upsilon_{\iota+1},\upsilon_\iota\right)\right| \leq \lambda\left|\varsigma\left(\upsilon_\iota,\upsilon_{\iota-1}\right)\right|$$

for all $\iota \in \mathbb{N}$. By induction, we build a sequence $\{\upsilon_\iota\}$ in $\mathfrak{F}$ so that

$$\begin{aligned}
\left|\varsigma\left(\upsilon_{\iota+1},\upsilon_\iota\right)\right| &\leq \lambda\left|\varsigma\left(\upsilon_\iota,\upsilon_{\iota-1}\right)\right|, \\
\left|\varsigma\left(\upsilon_{\iota+1},\upsilon_\iota\right)\right| &\leq \lambda^2\left|\varsigma\left(\upsilon_{\iota-1},\upsilon_{\iota-2}\right)\right|, \\
&\cdot \\
&\cdot \\
&\cdot \\
\left|\varsigma\left(\upsilon_{\iota+1},\upsilon_\iota\right)\right| &\leq \lambda^\iota\left|\varsigma\left(\upsilon_1,\upsilon_0\right)\right| = \lambda^\iota\left|\varsigma\left(\upsilon_0,\upsilon_1\right)\right|,
\end{aligned}$$

$\forall\ \iota \in \mathbb{N}$. Now, for $m > \iota$, we have

$$\begin{aligned}
|\varsigma(\upsilon_\iota, \upsilon_m)| &\leq \varphi(\upsilon_\iota, \upsilon_m)\lambda^\iota |\varsigma(\upsilon_0, \upsilon_1)| + \varphi(\upsilon_\iota, \upsilon_m)\varphi(\upsilon_{\iota+1}, \upsilon_m)\lambda^{\iota+1}|\varsigma(\upsilon_0, \upsilon_1)| \\
&\quad + \cdots + \varphi(\upsilon_\iota, \upsilon_m)\varphi(\upsilon_{\iota+1}, \upsilon_m)\cdots\varphi(\upsilon_{m-2}, \upsilon_m)\varphi(\upsilon_{m-1}, \upsilon_m)\lambda^{m-1}|\varsigma(\upsilon_0, \upsilon_1)| \\
&\leq |\varsigma(\upsilon_0, \upsilon_1)| \begin{bmatrix} \varphi(\upsilon_\iota, \upsilon_m)\lambda^\iota \\ +\varphi(\upsilon_\iota, \upsilon_m)\varphi(\upsilon_{\iota+1}, \upsilon_m)\lambda^{\iota+1} + \cdots + \\ \varphi(\upsilon_\iota, \upsilon_m)\varphi(\upsilon_{\iota+1}, \upsilon_m)\cdots\varphi(\upsilon_{m-2}, \upsilon_m)\varphi(\upsilon_{m-1}, \upsilon_m)\lambda^{m-1} \end{bmatrix}.
\end{aligned}$$

Since $\lim_{\iota,m\to\infty} \varphi(\upsilon_\iota, \upsilon_m)\lambda < 1$, the series $\sum_{\iota=1}^{\infty} \lambda^\iota \prod_{i=1}^{p} \varphi(\upsilon_i, \upsilon_m)$ converges by ratio test for each $m \in \mathbb{N}$. Let

$$S = \sum_{\iota=1}^{\infty} \lambda^\iota \prod_{i=1}^{p} \varphi(\upsilon_i, \upsilon_m), \ S_\iota = \sum_{j=1}^{\iota} \lambda^j \prod_{i=1}^{p} \varphi(\upsilon_i, \upsilon_m).$$

For $m > \iota$, one writes

$$|\varsigma(\upsilon_\iota, \upsilon_m)| \leq |\varsigma(\upsilon_0, \upsilon_1)| \left[S_{m-1} - S_\iota\right].$$

When $\iota \to \infty$, we have

$$|\varsigma(\upsilon_\iota, \upsilon_m)| \to 0.$$

Using Lemma 2.2, one asserts that the sequence $\{\upsilon_\iota\}$ is Cauchy. Since $\mathfrak{F}$ is complete, there is υ^* so that $\upsilon_\iota \to \upsilon^* \in \mathfrak{F}$ as $\iota \to \infty$.

Now, we show that υ^* is a fixed point of $\Im_1$. From (3.1), we have

$$\begin{aligned}
\varsigma(\upsilon^*, \Im_1\upsilon^*) &\leq \varphi(\upsilon^*, \Im_1\upsilon^*)\left(\varsigma(\upsilon^*, \Im_2\upsilon_{2\iota+1}) + \varsigma(\Im_2\upsilon_{2\iota+1}, \Im_1\upsilon^*)\right) \\
&= \varphi(\upsilon^*, \Im_1\upsilon^*)\left(\varsigma(\upsilon^*, \Im_2\upsilon_{2\iota+1}) + \varsigma(\Im_1\upsilon^*, \Im_2\upsilon_{2\iota+1})\right) \\
&\leq \varphi(\upsilon^*, \Im_1\upsilon^*) \begin{pmatrix} \varsigma(\upsilon^*, \upsilon_{2\iota+2}) + \rho(\upsilon^*, \upsilon_{2\iota+1})\varsigma(\upsilon^*, \upsilon_{2\iota+1}) \\ +\kappa(\upsilon^*, \upsilon_{2\iota+1})\frac{\varsigma(\upsilon^*, \Im_1\upsilon^*)\varsigma(\upsilon_{2\iota+1}, \Im_2\upsilon_{2\iota+1})}{1+\varsigma(\upsilon^*, \upsilon_{2\iota+1})} \\ +\mu(\upsilon^*, \upsilon_{2\iota+1})\frac{\varsigma(\upsilon_{2\iota+1}, \Im_1\upsilon^*)\varsigma(\upsilon^*, \Im_2\upsilon_{2\iota+1})}{1+\varsigma(\upsilon^*, \upsilon_{2\iota+1})} \end{pmatrix} \\
&\leq \varphi(\upsilon^*, \Im_1\upsilon^*) \begin{pmatrix} \varsigma(\upsilon^*, \upsilon_{2\iota+2}) + \rho(\upsilon^*, \upsilon_{2\iota+1})\varsigma(\upsilon^*, \upsilon_{2\iota+1}) \\ +\kappa(\upsilon^*, \upsilon_{2\iota+1})\frac{\varsigma(\upsilon^*, \Im_1\upsilon^*)\varsigma(\upsilon_{2\iota+1}, \upsilon_{2\iota+2})}{1+\varsigma(\upsilon^*, \upsilon_{2\iota+1})} \\ +\mu(\upsilon^*, \upsilon_{2\iota+1})\frac{\varsigma(\upsilon_{2\iota+1}, \Im_1\upsilon^*)\varsigma(\upsilon^*, \upsilon_{2\iota+2})}{1+\varsigma(\upsilon^*, \upsilon_{2\iota+1})} \end{pmatrix}.
\end{aligned}$$

This implies that

$$\left|\varsigma(\upsilon^*, \Im_1\upsilon^*)\right| \leq \varphi(\upsilon^*, \Im_1\upsilon^*) \begin{pmatrix} |\varsigma(\upsilon^*, \upsilon_{2\iota+2})| + \rho(\upsilon^*, \upsilon_{2\iota+1})|\varsigma(\upsilon^*, \upsilon_{2\iota+1})| \\ +\kappa(\upsilon^*, \upsilon_{2\iota+1})\frac{\left|\varsigma(\upsilon^*, \Im_1\upsilon^*)\right||\varsigma(\upsilon_{2\iota+1}, \upsilon_{2\iota+2})|}{|1+\varsigma(\upsilon^*, \upsilon_{2\iota+1})|} \\ +\mu(\upsilon^*, \upsilon_{2\iota+1})\frac{\left|\varsigma(\upsilon_{2\iota+1}, \Im_1\upsilon^*)\right||\varsigma(\upsilon^*, \upsilon_{2\iota+2})|}{|1+\varsigma(\upsilon^*, \upsilon_{2\iota+1})|} \end{pmatrix}.$$

Letting $\iota \to \infty$, we have $\left|\varsigma(\upsilon^*, \Im_1\upsilon^*)\right| = 0$. Thus $\upsilon^* = \Im_1\upsilon^*$. Now, we prove that υ^* is a fixed point of $\Im_2$. By (3.1), we have

$$
\begin{aligned}
\varsigma\left(\upsilon^*, \Im_2\upsilon^*\right) &\leq \varphi\left(\upsilon^*, \Im_2\upsilon^*\right)\left(\varsigma\left(\upsilon^*, \Im_1\upsilon_{2\iota}\right) + \varsigma\left(\Im_1\upsilon_{2\iota}, \Im_2\upsilon^*\right)\right) \\
&\leq \varphi\left(\upsilon^*, \Im_2\upsilon^*\right)\begin{pmatrix} \varsigma\left(\upsilon^*, \Im_1\upsilon_{2\iota}\right) + \rho\left(\upsilon_{2\iota}, \upsilon^*\right)\varsigma\left(\upsilon_{2\iota}, \upsilon^*\right) \\ +\kappa\left(\upsilon_{2\iota}, \upsilon^*\right)\frac{\varsigma\left(\upsilon_{2\iota}, \Im_1\upsilon_{2\iota}\right)\varsigma\left(\upsilon^*, \Im_2\upsilon^*\right)}{1+\varsigma\left(\upsilon_{2\iota}, \upsilon^*\right)} \\ +\mu\left(\upsilon_{2\iota}, \upsilon^*\right)\frac{\varsigma\left(\upsilon^*, \Im_1\upsilon_{2\iota}\right)\varsigma\left(\upsilon_{2\iota}, \Im_2\upsilon^*\right)}{1+\varsigma\left(\upsilon_{2\iota}, \upsilon^*\right)} \end{pmatrix} \\
&\leq \varphi\left(\upsilon^*, \Im_2\upsilon^*\right)\begin{pmatrix} \varsigma\left(\upsilon^*, \upsilon_{2\iota+1}\right) + \rho\left(\upsilon_{2\iota}, \upsilon^*\right)\varsigma\left(\upsilon_{2\iota}, \upsilon^*\right) \\ +\kappa\left(\upsilon_{2\iota}, \upsilon^*\right)\frac{\varsigma\left(\upsilon_{2\iota}, \upsilon_{2\iota+1}\right)\varsigma\left(\upsilon^*, \Im_2\upsilon^*\right)}{1+\varsigma\left(\upsilon_{2\iota}, \upsilon^*\right)} \\ +\mu\left(\upsilon_{2\iota}, \upsilon^*\right)\frac{\varsigma\left(\upsilon^*, \upsilon_{2\iota+1}\right)\varsigma\left(\upsilon_{2\iota}, \Im_2\upsilon^*\right)}{1+\varsigma\left(\upsilon_{2\iota}, \upsilon^*\right)} \end{pmatrix}.
\end{aligned}
$$

This implies that

$$
\left|\varsigma\left(\upsilon^*, \Im_2\upsilon^*\right)\right| \leq \varphi\left(\upsilon^*, \Im_2\upsilon^*\right)\begin{pmatrix} \left|\varsigma\left(\upsilon^*, \upsilon_{2\iota+1}\right)\right| + \rho\left(\upsilon_{2\iota}, \upsilon^*\right)\left|\varsigma\left(\upsilon_{2\iota}, \upsilon^*\right)\right| \\ +\kappa\left(\upsilon_{2\iota}, \upsilon^*\right)\frac{\left|\varsigma\left(\upsilon_{2\iota}, \upsilon_{2\iota+1}\right)\right|\left|\varsigma\left(\upsilon^*, \Im_2\upsilon^*\right)\right|}{\left|1+\varsigma\left(\upsilon_{2\iota}, \upsilon^*\right)\right|} \\ +\mu\left(\upsilon_{2\iota}, \upsilon^*\right)\frac{\left|\varsigma\left(\upsilon^*, \upsilon_{2\iota+1}\right)\right|\left|\varsigma\left(\upsilon_{2\iota}, \Im_2\upsilon^*\right)\right|}{\left|1+\varsigma\left(\upsilon_{2\iota}, \upsilon^*\right)\right|} \end{pmatrix}.
$$

Letting $\iota \to \infty$, we have $\left|\varsigma\left(\upsilon^*, \Im_2\upsilon^*\right)\right| = 0$. Thus, $\upsilon^* = \Im_2\upsilon^*$. Now, we prove that υ^* is unique. We assume that there exists another common fixed of $\upsilon^/$ of $\Im_1$ and $\Im_2$, i.e.,

$$
\upsilon^/ = \Im_1\upsilon^/ = \Im_2\upsilon^/,
$$

but $\upsilon^* \neq \upsilon^/$. Now, from (3.1), we have

$$
\begin{aligned}
\varsigma\left(\upsilon^*, \upsilon^/\right) &= \varsigma\left(\Im_1\upsilon^*, \Im_2\upsilon^/\right) \\
&\leq \rho\left(\upsilon^*, \upsilon^/\right)\varsigma\left(\upsilon^*, \upsilon^/\right) + \kappa\left(\upsilon^*, \upsilon^/\right)\frac{\varsigma\left(\upsilon^*, \Im\upsilon^*\right)\varsigma\left(\upsilon^/, \Im_2\upsilon^/\right)}{1+\varsigma\left(\upsilon^*, \upsilon^/\right)} \\
&\quad +\mu\left(\upsilon^*, \upsilon^/\right)\frac{\varsigma\left(\upsilon^/, \Im_1\upsilon^*\right)\varsigma\left(\upsilon^*, \Im_2\upsilon^/\right)}{1+\varsigma\left(\upsilon^*, \upsilon^/\right)} \\
&= \rho\left(\upsilon^*, \upsilon^/\right)\varsigma\left(\upsilon^*, \upsilon^/\right) + \kappa\left(\upsilon^*, \upsilon^/\right)\frac{\varsigma\left(\upsilon^*, \upsilon^*\right)\varsigma\left(\upsilon^/, \upsilon^/\right)}{1+\varsigma\left(\upsilon^*, \upsilon^/\right)} \\
&\quad +\mu\left(\upsilon^*, \upsilon^/\right)\frac{\varsigma\left(\upsilon^/, \upsilon^*\right)\varsigma\left(\upsilon^*, \upsilon^/\right)}{1+\varsigma\left(\upsilon^*, \upsilon^/\right)}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\left|\varsigma\left(\upsilon^*, \upsilon^/\right)\right| &\leq \rho\left(\upsilon^*, \upsilon^/\right)\left|\varsigma\left(\upsilon^*, \upsilon^/\right)\right| + \mu\left(\upsilon^*, \upsilon^/\right)\left|\varsigma\left(\upsilon^*, \upsilon^/\right)\right|\left|\frac{\varsigma\left(\upsilon^*, \upsilon^/\right)}{1+\varsigma\left(\upsilon^*, \upsilon^/\right)}\right| \\
&\leq \rho\left(\upsilon^*, \upsilon^/\right)\left|\varsigma\left(\upsilon^*, \upsilon^/\right)\right| + \mu\left(\upsilon^*, \upsilon^/\right)\left|\varsigma\left(\upsilon^*, \upsilon^/\right)\right| \\
&= \left(\rho\left(\upsilon^*, \upsilon^/\right) + \mu\left(\upsilon^*, \upsilon^/\right)\right)\left|\varsigma\left(\upsilon^*, \upsilon^/\right)\right|.
\end{aligned}
$$

As $\rho\left(\upsilon^*,\upsilon^/\right)+\mu\left(\upsilon^*,\upsilon^/\right)<1$, we have

$$\left|\varsigma\left(\upsilon^*,\upsilon^/\right)\right|=0.$$

Thus, $\upsilon^*=\upsilon^/$. □

By setting $\mu=0$ in Theorem 3.1, we state the following result.

Corollary 3.1. *Let $(\mathfrak{F},\varsigma)$ be a complete CVEbMS, $\varphi:\mathfrak{F}\times\mathfrak{F}\to[1,\infty)$ and let $\mathfrak{I}_1,\mathfrak{I}_2:\mathfrak{F}\to\mathfrak{F}$. If there exist mappings $\rho,\kappa:\mathfrak{F}\times\mathfrak{F}\to[0,1)$ such that for all $\upsilon,\hbar\in\mathfrak{F}$,*

(a) $\rho(\mathfrak{I}_2\mathfrak{I}_1\upsilon,\hbar)\leq\rho(\upsilon,\hbar)$ and $\rho(\upsilon,\mathfrak{I}_1\mathfrak{I}_2\hbar)\leq\rho(\upsilon,\hbar)$,
$\kappa(\mathfrak{I}_2\mathfrak{I}_1\upsilon,\hbar)\leq\kappa(\upsilon,\hbar)$ and $\kappa(\upsilon,\mathfrak{I}_1\mathfrak{I}_2\hbar)\leq\kappa(\upsilon,\hbar)$,

(b) $\rho(\upsilon,\hbar)+\kappa(\upsilon,\hbar)<1$,

(c)

$$\varsigma(\mathfrak{I}_1\upsilon,\mathfrak{I}_2\hbar)\leq\rho(\upsilon,\hbar)\varsigma(\upsilon,\hbar)+\kappa(\upsilon,\hbar)\frac{\varsigma(\upsilon,\mathfrak{I}_1\upsilon)\varsigma(\hbar,\mathfrak{I}_2\hbar)}{1+\varsigma(\upsilon,\hbar)},$$

(d) For each $\upsilon_0\in\mathfrak{F}$, $\lambda=\frac{\rho(\upsilon_0,\upsilon_1)}{1-\kappa(\upsilon_0,\upsilon_1)}<1$ and $\lim_{\iota,m\to\infty}\varphi(\upsilon_\iota,\upsilon_m)\lambda<1$ hold, whenever the sequence $\{\upsilon_\iota\}$ is defined by

$$\upsilon_{2\iota+1}=\mathfrak{I}_1\upsilon_{2\iota}\text{ and }\upsilon_{2\iota+2}=\mathfrak{I}_2\upsilon_{2\iota+1},$$

then $\mathfrak{I}_1$ and $\mathfrak{I}_2$ have a unique common fixed point.

By taking $\kappa=0$ in Theorem 3.1, we get the following corollary:

Corollary 3.2. *Let $(\mathfrak{F},\varsigma)$ be a complete CVEbMS, $\varphi:\mathfrak{F}\times\mathfrak{F}\to[1,\infty)$ and let $\mathfrak{I}_1,\mathfrak{I}_2:\mathfrak{F}\to\mathfrak{F}$. If there exist mappings $\rho,\mu:\mathfrak{F}\times\mathfrak{F}\to[0,1)$ such that for all $\upsilon,\hbar\in\mathfrak{F}$,*

(a) $\rho(\mathfrak{I}_2\mathfrak{I}_1\upsilon,\hbar)\leq\rho(\upsilon,\hbar)$ and $\rho(\upsilon,\mathfrak{I}_1\mathfrak{I}_2\hbar)\leq\rho(\upsilon,\hbar)$,
$\mu(\mathfrak{I}_2\mathfrak{I}_1\upsilon,\hbar)\leq\mu(\upsilon,\hbar)$ and $\mu(\upsilon,\mathfrak{I}_1\mathfrak{I}_2\hbar)\leq\mu(\upsilon,\hbar)$,

(b) $\rho(\upsilon,\hbar)+\mu(\upsilon,\hbar)<1$,

(c)

$$\varsigma(\mathfrak{I}_1\upsilon,\mathfrak{I}_2\hbar)\leq\rho(\upsilon,\hbar)\varsigma(\upsilon,\hbar)+\mu(\upsilon,\hbar)\frac{\varsigma(\hbar,\mathfrak{I}_1\upsilon)\varsigma(\upsilon,\mathfrak{I}_2\hbar)}{1+\varsigma(\upsilon,\hbar)},$$

(d) For each $\upsilon_0\in\mathfrak{F}$, $\lambda=\rho(\upsilon_0,\upsilon_1)<1$ and $\lim_{\iota,m\to\infty}\varphi(\upsilon_\iota,\upsilon_m)\lambda<1$ hold, whenever the sequence $\{\upsilon_\iota\}$ is defined by

$$\upsilon_{2\iota+1}=\mathfrak{I}_1\upsilon_{2\iota}\text{ and }\upsilon_{2\iota+2}=\mathfrak{I}_2\upsilon_{2\iota+1},$$

then $\mathfrak{I}_1$ and $\mathfrak{I}_2$ have a unique common fixed point.

By setting $\kappa=\mu=0$ in Theorem 3.1, we have the following corollary.

Corollary 3.3. *Let $(\mathfrak{F},\varsigma)$ be a complete CVEbMS, $\varphi:\mathfrak{F}\times\mathfrak{F}\to[1,\infty)$ and let $\mathfrak{I}_1,\mathfrak{I}_2:\mathfrak{F}\to\mathfrak{F}$. If there exists the mapping $\rho:\mathfrak{F}\times\mathfrak{F}\to[,g)$ such that for all $\upsilon,\hbar\in\mathfrak{F}$,*

(a) $\rho(\mathfrak{I}_2\mathfrak{I}_1\upsilon,\hbar)\leq\rho(\upsilon,\hbar)$ and $\rho(\upsilon,\mathfrak{I}_1\mathfrak{I}_2\hbar)\leq\rho(\upsilon,\hbar)$,

(b) $\varsigma(\mathfrak{I}_1\upsilon,\mathfrak{I}_2\hbar)\leq\rho(\upsilon,\hbar)\varsigma(\upsilon,\hbar)$,

(c) For each $\upsilon_0\in\mathfrak{F}$, $\lambda=\rho(\upsilon_0,\upsilon_1)<1$ and $\lim_{\iota,m\to\infty}\varphi(\upsilon_\iota,\upsilon_m)\lambda<1$ hold, whenever the sequence $\{\upsilon_\iota\}$ is defined by

$$\upsilon_{2\iota+1}=\mathfrak{I}_1\upsilon_{2\iota}\text{ and }\upsilon_{2\iota+2}=\mathfrak{I}_2\upsilon_{2\iota+1},$$

then $\mathfrak{I}_1$ and $\mathfrak{I}_2$ have a unique common fixed point.

By setting $\Im_1 = \Im_2 = \Im$ in Theorem 3.1, we have the following corollary.

Corollary 3.4. *Let* $(\mathfrak{F}, \varsigma)$ *be a complete CVEbMS,* $\varphi : \mathfrak{F} \times \mathfrak{F} \to [1, \infty)$ *and let* $\Im : \mathfrak{F} \to \mathfrak{F}$. *If there exist mappings* $\rho, \kappa, \mu : \mathfrak{F} \times \mathfrak{F} \to [0, 1)$ *such that for all* $\upsilon, \hbar \in \mathfrak{F}$,

(a) $\rho(\Im\upsilon, \hbar) \leq \rho(\upsilon, \hbar)$ *and* $\rho(\upsilon, \Im\hbar) \leq \rho(\upsilon, \hbar)$,
$\kappa(\Im\upsilon, \hbar) \leq \kappa(\upsilon, \hbar)$ *and* $\kappa(\upsilon, \Im\hbar) \leq \kappa(\upsilon, \hbar)$,
$\mu(\Im\upsilon, \hbar) \leq \mu(\upsilon, \hbar)$ *and* $\mu(\upsilon, \Im\hbar) \leq \mu(\upsilon, \hbar)$,

(b) $\rho(\upsilon, \hbar) + \kappa(\upsilon, \hbar) + \mu(\upsilon, \hbar) < 1$,

(c)

$$\varsigma(\Im\upsilon, \Im\hbar) \leq \rho(\upsilon, \hbar)\varsigma(\upsilon, \hbar) + \kappa(\upsilon, \hbar)\frac{\varsigma(\upsilon, \Im\upsilon)\varsigma(\hbar, \Im\hbar)}{1 + \varsigma(\upsilon, \hbar)} + \mu(\upsilon, \hbar)\frac{\varsigma(\hbar, \Im\upsilon)\varsigma(\upsilon, \Im\hbar)}{1 + \varsigma(\upsilon, \hbar)},$$

(d) For each $\upsilon_0 \in \mathfrak{F}$, $\lambda = \frac{\rho(\upsilon_0, \upsilon_1)}{1 - \kappa(\upsilon_0, \upsilon_1)} < 1$ *and* $\lim_{\iota, m \to \infty} \varphi(\upsilon_\iota, \upsilon_m)\lambda < 1$ *hold, whenever the sequence* $\{\upsilon_\iota\}$ *is defined by*

$$\upsilon_{\iota+1} = \Im\upsilon_\iota,$$

then $\Im$ *has a unique fixed point.*

Example 3.1. Let $\mathfrak{F} = [0, 1)$ and $\varphi : \mathfrak{F} \times \mathfrak{F} \to [1, \infty)$ be a function defined by

$$\varphi(\upsilon, \hbar) = \frac{5 + \upsilon + \hbar}{1 + \upsilon + \hbar}$$

and $\varsigma : \mathfrak{F} \times \mathfrak{F} \to \mathbb{C}$ by

$$\varsigma(\upsilon, \hbar) = |\upsilon - \hbar|^2 + i\,|\upsilon - \hbar|^2$$

for all $\upsilon, \hbar \in \mathfrak{F}$. Then $(\mathfrak{F}, \varsigma)$ is a complete CVEbMS. Define a self mapping $\Im : \mathfrak{F} \to \mathfrak{F}$ by

$$\Im\upsilon = \frac{1}{3}(2 - \upsilon).$$

Consider

$$\rho, \kappa, \mu : \mathfrak{F} \times \mathfrak{F} \to [, g)$$

by

$$\rho(\upsilon, \hbar) = \frac{1}{1 + \upsilon + \hbar}$$

and

$$\kappa(\upsilon, \hbar) = \mu(\upsilon, \hbar) = 0.$$

Then, it is very simple to prove that all the hypothesis of Corollary 3.4 are hold and $\frac{1}{2}$ is a fixed point of the mapping $\Im$.

Corollary 3.5. *Let* $(\mathfrak{F}, \varsigma)$ *be a complete CVEbMS,* $\varphi : \mathfrak{F} \times \mathfrak{F} \to [1, \infty)$ *and let* $\Im : \mathfrak{F} \to \mathfrak{F}$. *If there exist mappings* $\rho, \kappa, \mu : \mathfrak{F} \times \mathfrak{F} \to [0, 1)$ *such that for all* $\upsilon, \hbar \in \mathfrak{F}$,

(a) $\rho(\Im\upsilon, \hbar) \leq \rho(\upsilon, \hbar)$ *and* $\rho(\upsilon, \Im\hbar) \leq \rho(\upsilon, \hbar)$,
$\kappa(\Im\upsilon, \hbar) \leq \kappa(\upsilon, \hbar)$ *and* $\kappa(\upsilon, \Im\hbar) \leq \kappa(\upsilon, \hbar)$,
$\mu(\Im\upsilon, \hbar) \leq \mu(\upsilon, \hbar)$ *and* $\mu(\upsilon, \Im\hbar) \leq \mu(\upsilon, \hbar)$,

(b) $\rho(\upsilon,\hbar)+\kappa(\upsilon,\hbar)+\mu(\upsilon,\hbar)<1$,

(c)

$$\varsigma(\Im^n\upsilon,\Im^n\hbar)\leq\rho(\upsilon,\hbar)\varsigma(\upsilon,\hbar)+\kappa(\upsilon,\hbar)\frac{\varsigma(\upsilon,\Im^n\upsilon)\varsigma(\hbar,\Im^n\hbar)}{1+\varsigma(\upsilon,\hbar)}+\mu(\upsilon,\hbar)\frac{\varsigma(\hbar,\Im^n\upsilon)\varsigma(\upsilon,\Im^n\hbar)}{1+\varsigma(\upsilon,\hbar)}, \tag{3.7}$$

(d) For each $\upsilon_0\in\mathfrak{F}$, $\lambda=\frac{\rho(\upsilon_0,\upsilon_1)}{1-\kappa(\upsilon_0,\upsilon_1)}<1$ *and* $\lim_{\iota,m\to\infty}\varphi(\upsilon_\iota,\upsilon_m)\lambda<1$ *hold, whenever the sequence* $\{\upsilon_\iota\}$ *is defined by*

$$\upsilon_{\iota+1}=\Im\upsilon_\iota,$$

then there exists a unique point $\upsilon^*\in\mathfrak{F}$ *such that* $\Im\upsilon^*=\upsilon^*$.

Proof. From Corollary 3.4, we have $\upsilon\in\mathfrak{F}$ such that $\Im^n\upsilon=\upsilon$. Now, from

$$\begin{aligned}\varsigma(\Im\upsilon,\upsilon) &= \varsigma(\Im\Im^n\upsilon,\Im^n\upsilon)=\varsigma(\Im^n\Im\upsilon,\Im^n\upsilon)\\ &\leq \rho(\Im\upsilon,\upsilon)\varsigma(\Im\upsilon,\upsilon)+\kappa(\Im\upsilon,\upsilon)\frac{\varsigma(\Im\upsilon,\Im^n\Im\upsilon)\varsigma(\upsilon,\Im^n\upsilon)}{1+\varsigma(\Im\upsilon,\upsilon)}\\ &\quad+\mu(\Im\upsilon,\upsilon)\frac{\varsigma(\upsilon,\Im^n\Im\upsilon)\varsigma(\Im\upsilon,\Im^n\upsilon)}{1+\varsigma(\Im\upsilon,\upsilon)}\\ &\leq \rho(\Im\upsilon,\upsilon)\varsigma(\Im\upsilon,\upsilon)+\kappa(\Im\upsilon,\upsilon)\frac{\varsigma(\Im\upsilon,\Im\upsilon)\varsigma(\upsilon,\upsilon)}{1+\varsigma(\Im\upsilon,\upsilon)}+\mu(\Im\upsilon,\upsilon)\frac{\varsigma(\upsilon,\Im\upsilon)\varsigma(\Im\upsilon,\upsilon)}{1+\varsigma(\Im\upsilon,\upsilon)}\\ &= \rho(\Im\upsilon,\upsilon)\varsigma(\Im\upsilon,\upsilon)+\mu(\Im\upsilon,\upsilon)\frac{\varsigma(\upsilon,\Im\upsilon)\varsigma(\Im\upsilon,\upsilon)}{1+\varsigma(\Im\upsilon,\upsilon)},\end{aligned}$$

which implies that

$$\begin{aligned}\left|\varsigma(\Im\upsilon,\upsilon)\right| &\leq \rho(\Im\upsilon,\upsilon)\left|\varsigma(\Im\upsilon,\upsilon)\right|+\mu(\Im\upsilon,\upsilon)\left|\varsigma(\upsilon,\Im\upsilon)\right|\left|\frac{\varsigma(\Im\upsilon,\upsilon)}{1+\varsigma(\Im\upsilon,\upsilon)}\right|\\ &\leq \rho(\Im\upsilon,\upsilon)\left|\varsigma(\Im\upsilon,\upsilon)\right|+\mu(\Im\upsilon,\upsilon)\left|\varsigma(\upsilon,\Im\upsilon)\right|\\ &= (\rho(\Im\upsilon,\upsilon)+\mu(\Im\upsilon,\upsilon))\left|\varsigma(\upsilon,\Im\upsilon)\right|.\end{aligned}$$

It is possible only whenever $\left|\varsigma(\Im\upsilon,\upsilon)\right|=0$. Thus, $\Im\upsilon=\upsilon$. □

Corollary 3.6. *Let* $(\mathfrak{F},\varsigma)$ *be a complete CVEbMS,* $\varphi:\mathfrak{F}\times\mathfrak{F}\to[1,\infty)$ *and let* $\Im_1,\Im_2:\mathfrak{F}\to\mathfrak{F}$. *If there exist mappings* $\rho,\kappa,\mu:\mathfrak{F}\to[0,1)$ *such that for all* $\upsilon,\hbar\in\mathfrak{F}$,

(a) $\rho(\Im_2\Im_1\upsilon)\leq\rho(\upsilon)$, $\kappa(\Im_2\Im_1\upsilon)\leq\kappa(\upsilon)$, $\mu(\Im_2\Im_1\upsilon)\leq\mu(\upsilon)$,

(b) $\rho(\upsilon)+\kappa(\upsilon)+\mu(\upsilon)<1$,

(c)

$$\varsigma(\Im_1\upsilon,\Im_2\hbar)\leq\rho(\upsilon)\varsigma(\upsilon,\hbar)+\kappa(\upsilon)\frac{\varsigma(\upsilon,\Im_1\upsilon)\varsigma(\hbar,\Im_2\hbar)}{1+\varsigma(\upsilon,\hbar)}+\mu(\upsilon)\frac{\varsigma(\hbar,\Im_1\upsilon)\varsigma(\upsilon,\Im_2\hbar)}{1+\varsigma(\upsilon,\hbar)},$$

(d) For each $\upsilon_0\in\mathfrak{F}$, $\lambda=\frac{\rho(\upsilon_0)}{1-\kappa(\upsilon_0)}<1$ *and* $\lim_{\iota,m\to\infty}\varphi(\upsilon_\iota,\upsilon_m)\lambda<1$ *hold, whenever the sequence* $\{\upsilon_\iota\}$ *is defined by*

$$\upsilon_{2\iota+1}=\Im_1\upsilon_{2\iota}\text{ and }\upsilon_{2\iota+2}=\Im_2\upsilon_{2\iota+1},$$

then $\Im_1$ *and* $\Im_2$ *have a unique common fixed point.*

Proof. Define $\rho, \kappa, \mu : \mathfrak{F} \times \mathfrak{F} \rightarrow [0, 1)$ by

$$\rho(\upsilon, \hbar) = \rho(\upsilon), \quad \kappa(\upsilon, \hbar) = \kappa(\upsilon) \quad \text{and} \quad \mu(\upsilon, \hbar) = \mu(\upsilon)$$

for all $\upsilon, \hbar \in \mathfrak{F}$. Then for all $\upsilon, \hbar \in \mathfrak{F}$, we have

(a) $\rho(\Im_2\Im_1\upsilon, \hbar) = \rho(\Im_2\Im_1\upsilon) \leq \rho(\upsilon) = \rho(\upsilon, \hbar)$ and $\rho(\upsilon, \Im_1\Im_2\hbar) = \rho(\upsilon) = \rho(\upsilon, \hbar)$,
$\kappa(\Im_2\Im_1\upsilon, \hbar) = \kappa(\Im_2\Im_1\upsilon) \leq \kappa(\upsilon) = \kappa(\upsilon, \hbar)$ and $\kappa(\upsilon, \Im_1\Im_2\hbar) = \kappa(\upsilon) = \kappa(\upsilon, \hbar)$,
$\mu(\Im_2\Im_1\upsilon, \hbar) = \mu(\Im_2\Im_1\upsilon) \leq \mu(\upsilon) = \mu(\upsilon, \hbar)$ and $\mu(\upsilon, \Im_1\Im_2\hbar) = \mu(\upsilon) = \mu(\upsilon, \hbar)$,

(b) $\rho(\upsilon, \hbar) + \kappa(\upsilon, \hbar) + \mu(\upsilon, \hbar) = \rho(\upsilon) + \kappa(\upsilon) + \mu(\upsilon) < 1$,

(c)

$$\begin{aligned} \varsigma(\Im_1\upsilon, \Im_2\hbar) &\leq \rho(\upsilon)\varsigma(\upsilon, \hbar) + \kappa(\upsilon)\frac{\varsigma(\upsilon, \Im_1\upsilon)\varsigma(\hbar, \Im_2\hbar)}{1 + \varsigma(\upsilon, \hbar)} + \mu(\upsilon)\frac{\varsigma(\hbar, \Im_1\upsilon)\varsigma(\upsilon, \Im_2\hbar)}{1 + \varsigma(\upsilon, \hbar)} \\ &= \rho(\upsilon, \hbar)\varsigma(\upsilon, \hbar) + \kappa(\upsilon, \hbar)\frac{\varsigma(\upsilon, \Im_1\upsilon)\varsigma(\hbar, \Im_2\hbar)}{1 + \varsigma(\upsilon, \hbar)} + \mu(\upsilon, \hbar)\frac{\varsigma(\hbar, \Im_1\upsilon)\varsigma(\upsilon, \Im_2\hbar)}{1 + \varsigma(\upsilon, \hbar)}, \end{aligned}$$

(d) $\lambda = \frac{\rho(\upsilon_0, \upsilon_1)}{1 - \kappa(\upsilon_0, \upsilon_1)} = \frac{\rho(\upsilon_0)}{1 - \kappa(\upsilon_0)} < 1.$ □

By Theorem 3.1, $\Im_1$ and $\Im_2$ have a unique common fixed point.

If we take $\rho(\cdot) = \rho$, $\kappa(\cdot) = \kappa$ and $\mu(\cdot) = \mu$ in Corollary 3.6, we have the following corollaries.

Corollary 3.7. *Let $(\mathfrak{F}, \varsigma)$ be a complete CVEbMS, $\varphi : \mathfrak{F} \times \mathfrak{F} \rightarrow [1, \infty)$ and let $\Im_1, \Im_2 : \mathfrak{F} \rightarrow \mathfrak{F}$. If there exist nonnegative real numbers ρ, κ and μ with $\rho + \kappa + \mu < 1$ such that*

$$\varsigma(\Im_1\upsilon, \Im_2\hbar) \leq \rho\varsigma(\upsilon, \hbar) + \kappa\frac{\varsigma(\upsilon, \Im_1\upsilon)\varsigma(\hbar, \Im_2\hbar)}{1 + \varsigma(\upsilon, \hbar)} + \mu\frac{\varsigma(\hbar, \Im_1\upsilon)\varsigma(\upsilon, \Im_2\hbar)}{1 + \varsigma(\upsilon, \hbar)}$$

for all $\upsilon, \hbar \in \mathfrak{F}$, and for each $\upsilon_0 \in \mathfrak{F}$, $\lambda = \frac{\rho}{1-\kappa} < 1$ and $\lim_{\iota, m\rightarrow\infty} \varphi(\upsilon_\iota, \upsilon_m)\lambda < 1$ hold, whenever the sequence $\{\upsilon_\iota\}$ is defined by

$$\upsilon_{2\iota+1} = \Im_1\upsilon_{2\iota} \text{ and } \upsilon_{2\iota+2} = \Im_2\upsilon_{2\iota+1},$$

then $\Im_1$ and $\Im_2$ have a unique common fixed point.

Corollary 3.8. *Let $(\mathfrak{F}, \varsigma)$ be a complete CVEbMS, $\varphi : \mathfrak{F} \times \mathfrak{F} \rightarrow [1, \infty)$ and let $\Im_1, \Im_2 : \mathfrak{F} \rightarrow \mathfrak{F}$. If there are nonnegative reals ρ and κ with $\rho + \kappa < 1$ such that*

$$\varsigma(\Im_1\upsilon, \Im_2\hbar) \leq \rho\varsigma(\upsilon, \hbar) + \kappa\frac{\varsigma(\upsilon, \Im_1\upsilon)\varsigma(\hbar, \Im_2\hbar)}{1 + \varsigma(\upsilon, \hbar)}$$

for all $\upsilon, \hbar \in \mathfrak{F}$, and for each $\upsilon_0 \in \mathfrak{F}$, $\lambda = \frac{\rho}{1-\kappa} < 1$ and $\lim_{\iota, m\rightarrow\infty} \varphi(\upsilon_\iota, \upsilon_m)\lambda < 1$ hold, whenever the sequence $\{\upsilon_\iota\}$ is defined by

$$\upsilon_{2\iota+1} = \Im_1\upsilon_{2\iota} \text{ and } \upsilon_{2\iota+2} = \Im_2\upsilon_{2\iota+1},$$

then $\Im_1$ and $\Im_2$ have a unique common fixed point.

4. Application

The main result of this section is as follows:

Theorem 4.1. *Let $\mathfrak{F} = C([a,b],\mathbb{R}^n)$, $a > 0$ and $d : \mathfrak{F} \times \mathfrak{F} \to \mathbb{C}$ be given as*

$$d(\upsilon,\hbar) = \max_{t\in[a,b]} \|\upsilon(t) - \hbar(t)\|^2 \sqrt{1+a^2}e^{i\tan^{-1}a}$$

and $\varphi : \mathfrak{F} \times \mathfrak{F} \to [1,\infty)$ be defined by $\varphi(\upsilon,\hbar) = 2$. Then $(\mathfrak{F}, d)$ is complete CVEbMS. Considering the set of Urysohn integral equations

$$\upsilon(t) = \int_a^b K_1(t,s,\upsilon(s))ds + \varpi(t), \tag{4.1}$$

and

$$\upsilon(t) = \int_a^b K_2(t,s,\upsilon(s))ds + \phi(t), \tag{4.2}$$

for $\iota \in [a,b] \subset \mathbb{R}$, $\upsilon, \varpi, \phi \in \mathfrak{F}$.

Assume that $K_1, K_2 : [a,b]\times[a,b]\times\mathbb{R}^n \to \mathbb{R}^n$ are such that $F_\upsilon, G_\upsilon \in \mathfrak{F}$, $\forall\ \upsilon \in \mathfrak{F}$, where

$$F_\upsilon(t) = \int_a^b K_1(t,s,\upsilon(s))ds, \quad G_\upsilon(t) = \int_a^b K_2(t,s,\upsilon(s))ds,$$

$\forall\ t \in [a,b]$.

If there exist ρ,$\kappa \in [0,1)$ with $\rho+\kappa < 1$ such that for every $\upsilon,\hbar \in \mathfrak{F}$,

$$\|F_\upsilon(t) - G_\hbar(t) + \varpi(t) - \phi(t)\|^2 \sqrt{1+a^2}e^{i\tan^{-1}a} \precsim \rho A(\upsilon,\hbar)(t) + \kappa B(\upsilon,\hbar)(t),$$

where

$$\begin{aligned} A(\upsilon,\hbar)(t) &= \|\upsilon(t) - \hbar(t)\|^2 \sqrt{1+a^2}e^{i\tan^{-1}a}, \\ B(\upsilon,\hbar)(t) &= \frac{\|F_\upsilon(t) + \varpi(t) - \upsilon(t)\|^2 \|G_\hbar(t) + \phi(t) - \hbar(t)\|^2}{1 + \max\limits_{t\in[a,b]} A(\upsilon,\hbar)(t)} \sqrt{1+a^2}e^{i\tan^{-1}a}, \end{aligned}$$

then (4.1) and (4.2) have a unique common solution.

Proof. Define $\mathfrak{I}_1, \mathfrak{I}_2 : \mathfrak{F} \to \mathfrak{F}$ by

$$\mathfrak{I}_1\upsilon = F_\upsilon + \varpi, \quad \mathfrak{I}_2\upsilon = G_\upsilon + \phi.$$

Then,

$$d(\mathfrak{I}_1\upsilon, \mathfrak{I}_2\hbar) = \max_{t\in[a,b]} \|F_\upsilon(t) - G_\hbar(t) + \varpi(t) - \phi(t)\|^2 \sqrt{1+a^2}e^{i\tan^{-1}a},$$

$$d(\upsilon,\hbar) = \max_{t\in[a,b]} A(\upsilon,\hbar)(t),$$

$$\frac{d(\upsilon,\mathfrak{I}_1\upsilon)d(\hbar,\mathfrak{I}_2\hbar)}{1+d(\upsilon,\hbar)} = \max_{t\in[a,b]} B(\upsilon,\hbar)(t).$$

It is easily seen that

$$d(\mathfrak{I}_1\upsilon, \mathfrak{I}_2\hbar) \precsim \rho d(\upsilon,\hbar) + \kappa\frac{d(\upsilon,\mathfrak{I}_1\upsilon)d(\hbar,\mathfrak{I}_2\hbar)}{1+d(\upsilon,\hbar)}$$

for every $\upsilon,\hbar \in \mathfrak{F}$. By Corollary 3.8, the Urysohn integral equations (4.1) and (4.2) have a unique common solution. □

5. Conclusions

In this article, we have utilized the notion of a complex-valued extended b-metric space (CVEbMS) and obtained common fixed point results for rational contractions involving control functions of two variables. We have derived common fixed points and fixed points of single-valued mappings for contractions involving control functions of one variable and constants. We hope that the obtained results in this article will form new relations for those who are employing in a CVEbMS.

The future works in this way will target studying the common fixed points of single-valued and multivalued mappings in the setting of a CVEbMS. Differential and integral equations can be solved as applications of these results.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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