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Research article

Jensen and Hermite-Hadamard type inclusions for harmonical h-Godunova-Levin functions

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Abstract: The role of integral inequalities can be seen in both applied and theoretical mathematics fields. According to the definition of convexity, it is possible to relate both concepts of convexity and integral inequality. Furthermore, convexity plays a key role in the topic of inclusions as a result of its definitional behavior. The importance and superior applications of convex functions are well known, particularly in the areas of integration, variational inequality, and optimization. In this paper, various types of inequalities are introduced using inclusion relations. The inclusion relation enables us firstly to derive some Hermite-Hadamard inequalities (H.H-inequalities) and then to present Jensen inequality for harmonical h-Godunova-Levin interval-valued functions (GL-IVFS) via Riemann integral operator. Moreover, the findings presented in this study have been verified with the use of useful examples that are not trivial.

Keywords: Jensen inequality; Hermite-Hadamard inequality; Godunova-Levin function; harmonic convexity; interval valued functions
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1. Introduction

Since we know that interval analysis has a veritably broad history, Moore [1] first developed the interval and interval-valued functions in his work in the 1950s. This research field has attracted the attention of the mathematical community since it was established. There are many applications of interval analysis in global optimization algorithms and constraint solving algorithms. In contrast, calculation error has long been a problematic issue in numerical analysis. The accumulation of calculation errors can make the calculation result meaningless, so interval analysis has attracted much attention as a tool to solve uncertainty problems [2, 3]. For the last five decades, It has been used in a variety of fields, including neural network output optimization [4], computer graphic [5], interval differential equation [6], aeroelasticity [7] and so on. The fusion of integral inequalities with interval-valued functions (IVFS) has resulted in many insightful findings in recent decades. Since the invention of interval analysis, researchers studying inequalities have been interested in seeing if the inequalities found in the below results can be substituted with inclusions. Among these are Beckenbach type inequalities and Minkowski types for IVFS developed by Roman-Flores [8]. Moreover, Costa [9] established Opial-type inclusion for IVFS.

Classical Hermite-Hadamard inequality (H-H) is as follows:

$$\frac{f(r)+f(s)}{2} \geq \frac{1}{s-r}\int_r^s f(\alpha)d\alpha \geq f\left(\frac{r+s}{2}\right). \tag{1.1}$$

Due to its geometrical interpretation, the H-H inequality is considered one of the classics of elementary mathematics. In addition to being generalized and refined, the function is now extended to cover various classes of convexity. Many inequalities have been revealed by the convexity of functions over time in mathematics and other scientific fields, including economics, probability theory, and optimal control theory, as well as in economics. In probability theory, a convex function applied to the expected value of a random variable is always bound by the expected value of its convex function. Further, Jensen's inequality is a probabilistic inequality, and its beauty lies in the fact that several well-known inequalities can be deduced from it, including the arithmetic-geometric mean inequality and Holder's inequality based on the expected values for convex and concave transforms of random variables. For different extensions and conceptions of these inequalities [10–19]. Initially, Işcan present the concept of harmonical convexity in 2014 and created various H-H inequalities for this form of convexity [20]. In the case of harmonical convex functions, some refinements of such inequalities have been investigated [21–26].

Noor et al. [27] established harmonical h-convex functions and developed a revised form of H-H inequalities in 2015. In addition to interval analysis, Dafang et al. extended H-H and Jensen type inequalities to h-convex and harmonic h-convex in the context of IVFS [28, 29]. We refer interested readers to some new research on harmonical h-convexity [30–35]. Based on the notion of the h-GL function, Kilicman et al. developed the following inequality [36]. As a step forward, Afzal et al. developed these inequalities in 2022 for the generalized class of h-Godunova-Levin functions and (h_1, h_2)-Godunova-Levin functions in the context of interval-valued functions using inclusion relation [37, 38]. The beauty of this class of convexity is that inequality terms are straightforward to deduce and generalize. Moreover, Baloch et al. developed the Jensen-type inequality for harmonic h-convex functions [39].

Inspired by [29, 37–39], we present harmonical h-Godunova-Levin functions as a new class of convexity based on inclusion relation for IVFS. As part of our analysis, we first derived new variants of the H-H inequality, and then we used this new class to represent the Jensen inequality. Additionally, we provide several examples to illustrate how our key findings can be applied.

Finally, the rest of the paper is organized as follows. In Section 2, preliminary information is provided. The key conclusions are described in Section 3. Section 4 contains the conclusion.

2. Preliminaries

For the notions which are used in this paper and are not defined here, we refer [28]. Let's say I represent a set of real numbers in the form of a pack of all intervals of $\mathcal{R}$, $[s] \in I$ is defined as

$$[s] = [\underline{s}, \overline{s}] = \{x \in \mathbb{R} | \underline{s} \leq x \leq \overline{s}\}, \ \underline{s}, \overline{s} \in \mathbb{R},$$

where real interval $[s]$ is compact subset of $\mathcal{R}$. There is a degeneration of the interval $[s]$ when $\underline{s} = \overline{s}$. In this case, we are denoting the bundle of all intervals in $\mathcal{R}$ by $\mathcal{R}_{\mathcal{I}}$ and use $\mathcal{R}_{\mathcal{I}}{}^{+}$ for the collection of all positive intervals. The inclusion "$\subseteq$" is established as

$$[s] \subseteq [r] \Longleftrightarrow [\underline{s}, \overline{s}] \subseteq [\underline{r}, \overline{r}] \Longleftrightarrow \underline{r} \leq \underline{s}, \overline{s} \leq \overline{r}.$$

For any arbitrary $\kappa \in \mathbb{R}$ and $[s]$, the $\kappa[s]$ is defined as

$$\kappa.[\underline{s}, \overline{s}] = \begin{cases} [\kappa\underline{s}, \kappa\overline{s}], & \text{if } \kappa > 0; \\ \{0\}, & \text{if } \kappa = 0; \\ [\kappa\overline{s}, \kappa\underline{s}], & \text{if } \kappa < 0. \end{cases}$$

For $[s] = [\underline{s}, \overline{s}]$, and $[r] = [\underline{r}, \overline{r}]$, defining arithmetic operators as

$$\begin{aligned} [s] + [r] &= \left[\underline{s} + \underline{r}, \overline{s} + \overline{r}\right], \\ [s] - [r] &= \left[\underline{s} - \underline{r}, \overline{s} - \overline{r}\right], \\ [s] \cdot [r] &= \left[\min\left\{\underline{sr}, \underline{s}\overline{r}, \overline{s}\underline{r}, \overline{sr}\right\}, \max\left\{\underline{sr}, \underline{s}\overline{r}, \overline{sr}, \overline{s}\underline{r}\right\}\right], \\ [s]/[r] &= \left[\min\left\{\underline{s}/\underline{r}, \underline{s}/\overline{r}, \overline{s}/\underline{r}, \overline{s}/\overline{r}\right\}, \max\left\{\underline{s}/\underline{r}, \underline{s}/\overline{r}, \overline{s}/\overline{r}, \overline{s}/\overline{r}\right\}\right], \end{aligned}$$

where

$$0 \notin [\underline{s}, \overline{s}].$$

In intervals, Hausdorff distance is calculated as follows:

$$d\left([\underline{s}, \overline{s}], [\underline{r}, \overline{r}]\right) = \max\left\{|\underline{s} - \underline{r}|, |\overline{s} - \overline{r}|\right\}.$$

As far as we know, the entire metric space (R_I, d) is completed moreover, $\mathcal{IR}$ denote the Riemann integrable.

Definition 2.1. *[40] Let $f : [s, r] \to \mathcal{R}_I$ be defined as $f(q) = [\underline{f}(q), \overline{f}(q)]$ for any $q \in [s, r]$ and $\underline{f}, \overline{f}$ are $\mathcal{IR}$ over interval $[s, r]$. Consequently, we say that our function f is $\mathcal{IR}$ over $[s, r]$ and defined as*

$$\int_s^r f(q)dq = \left[\int_s^r \underline{f}(q)dq, \int_s^r \overline{f}(q)dq\right].$$

Definition 2.2. *[41] A set $S \subset \mathcal{R}^n - \{0\}$ is called harmonical convex, if*

$$\frac{sr}{\kappa s + (1-\kappa)r} \in S,$$

for all $s, r \in S$ and $\kappa \in [0, 1]$.

Definition 2.3. *[42] A function $f : S \to \mathcal{R}^+$ is called GL-function, if*

$$f(\kappa s + (1-\kappa)r) \le \frac{f(s)}{\kappa} + \frac{f(r)}{(1-\kappa)},$$

for all $s, r \in S$ and $\kappa \in (0, 1)$.

Definition 2.4. *[20] A function $f : S \to \mathcal{R}$ is called harmonically convex, if*

$$f\left(\frac{sr}{\kappa s + (1-\kappa)r}\right) \le \kappa f(s) + (1-\kappa) f(r),$$

for all $s, r \in S$ and $\kappa \in [0, 1]$.

Definition 2.5. *[43] Consider $h : [0, 1] \subseteq S \to \mathcal{R}$ with $h \neq 0$ be a nonnegative function. We say $f : S \to \mathcal{R}$ is called harmonical h-convex, if*

$$f\left(\frac{sr}{\kappa s + (1-\kappa)r}\right) \le h(\kappa) f(s) + h(1-\kappa) f(r),$$

for all $s, r \in S$ and $\kappa \in [0, 1]$.

Definition 2.6. *[44] Consider $h : (0, 1) \subseteq S \to \mathcal{R}$ be a nonnegative function. We say $f : S \to \mathcal{R}$ is called h-GL function, if*

$$f(\kappa s + (1-\kappa)r) \le \frac{f(s)}{h(\kappa)} + \frac{f(r)}{h(1-\kappa)},$$

for all $s, r \in S$ and $\kappa \in (0, 1)$.

Definition 2.7. *[45] Consider $h : (0, 1) \subseteq S \to \mathcal{R}$ be a nonnegative function. We say $f : S \to \mathcal{R}$ is called harmonical h-GL function, if*

$$f\left(\frac{sr}{\kappa s + (1-\kappa)r}\right) \le \frac{f(s)}{h(\kappa)} + \frac{f(r)}{h(1-\kappa)},$$

for all $s, r \in S$ and $\kappa \in (0, 1)$.

Remark 2.1. *(1) If $h(\kappa) = \frac{1}{\kappa}$, then Definition 2.7 provides a harmonical convex function [20];*
(2) If $h(\kappa) = 1$, then Definition 2.7 provides a harmonical p-convex function [43];
(3) If $h(\kappa) = \kappa^s$, then Definition 2.7 provides a harmonical s-GL function [43].

3. Main results

In this section firstly we define a novel class of convexity called harmonic h-GL IVFS.

Definition 3.1. *Consider $h : (0,1) \subseteq S \to \mathcal{R}$ such that $h \neq 0$. A function $f : S \to \mathcal{R}_I^+$ is called harmonical h-GL IVF, if*

$$\frac{f(s)}{h(\kappa)} + \frac{f(r)}{h(1-\kappa)} \subseteq f\left(\frac{sr}{\kappa s + (1-\kappa)r}\right), \tag{3.1}$$

for all $s, r \in S$ and $\kappa \in (0,1)$. If the inclusion is change from $\subseteq$ to $\supseteq$ in Definition 3.1, then f is called harmonical h-GL concave IVF. Harmonical h-GL convex and concave IVFS are represented by $SGHX\left(\left(\frac{1}{h}\right), S, \mathcal{R}_I{}^+\right)$ and $SGHV\left(\left(\frac{1}{h}\right), S, \mathcal{R}_I{}^+\right)$, respectively.

Proposition 3.1. *Consider $f : [s,r] \to \mathcal{R}_I{}^+$ be harmonical h-GL convex IVF defined as $f(\kappa) = [\underline{f}(\kappa), \overline{f}(\kappa)]$. Then, if $f \in SGHX\left(\left(\frac{1}{h}\right), S, \mathcal{R}_I{}^+\right)$ iff $\underline{f} \in SGHX\left(\left(\frac{1}{h}\right), [s,r], \mathcal{R}^+\right)$ and if $\overline{f} \in SGHV\left(\left(\frac{1}{h}\right), [s,r], \mathcal{R}^+\right)$.*

Proof. Suppose f be be harmonical h-GL convex IVF and consider $x, y \in [s,r], \kappa \in (0,1)$, we have

$$\frac{f(x)}{h(\kappa)} + \frac{f(y)}{h(1-\kappa)} \subseteq f\left(\frac{xy}{\kappa x + (1-\kappa)y}\right), \tag{3.2}$$

that is,

$$\left[\frac{\underline{f}(x)}{h(\kappa)} + \frac{\underline{f}(y)}{h(1-\kappa)}, \frac{\overline{f}(x)}{h(\kappa)} + \frac{\overline{f}(x)}{h(1-\kappa)}\right] \subseteq \left[\underline{f}\left(\frac{xy}{\kappa x + (1-\kappa)y}\right), \overline{f}\left(\frac{xy}{\kappa x + (1-\kappa)y}\right)\right].$$

Consequently, we have

$$\frac{\underline{f}(x)}{h(\kappa)} + \frac{\underline{f}(y)}{h(1-\kappa)} \geq \underline{f}\left(\frac{xy}{\kappa x + (1-\kappa)y}\right)$$

and

$$\frac{\overline{f}(x)}{h(\kappa)} + \frac{\overline{f}(y)}{h(1-\kappa)} \leq \overline{f}\left(\frac{xy}{\kappa x + (1-\kappa)y}\right).$$

It shows that $\underline{f} \in SGHX\left(\left(\frac{1}{h}\right), [s,r], \mathcal{R}^+\right)$ and $\overline{f} \in SGHV\left(\left(\frac{1}{h}\right), [s,r], \mathcal{R}^+\right)$. Conversely, suppose that if $\underline{f} \in SGHX\left(\left(\frac{1}{h}\right), [s,r], R_I{}^+\right)$ and $\overline{f} \in SGHV\left(\left(\frac{1}{h}\right), [s,r], R_I{}^+\right)$. According to the above definition and set inclusion, we can say that $f \in SGHX\left(\left(\frac{1}{h}\right), [s,r], R_I{}^+\right)$. This completes the proof. □

Proposition 3.2. *Suppose $f : [s,r] \to \mathcal{R}_I{}^+$ be harmonical h-GL concave IVF defined as $f(\kappa) = [\underline{f}(\kappa), \overline{f}(\kappa)]$. Then if $f \in SGHV\left(\left(\frac{1}{h}\right), [s,r], \mathcal{R}_I{}^+\right)$ iff $\underline{f} \in SGHV\left(\left(\frac{1}{h}\right), [s,r], \mathcal{R}^+\right)$ and if $\overline{f} \in SGHX\left(\left(\frac{1}{h}\right), [s,r], \mathcal{R}^+\right)$.*

Proof. The proof similar to Proposition 3.1. □

3.1. Hermite-Hadamard inequalities

Theorem 3.1. *Consider $h : (0,1) \to \mathcal{R}^+$ such that $h \neq 0$. Let $f : [s,r] \to \mathcal{R}_I{}^+$. If $f \in SGHX\left(\left(\frac{1}{h}\right), [s,r], \mathcal{R}_I{}^+\right)$ and $f \in \mathcal{IR}_{[s,r]}$, we have*

$$\frac{\left[h\left(\frac{1}{2}\right)\right]}{2} f\left(\frac{2sr}{s+r}\right) \supseteq \frac{sr}{r-s}\int_s^r \frac{f(\kappa)}{\kappa^2}d\kappa \supseteq [f(s) + f(r)]\int_0^1 \frac{dx}{h(x)}. \tag{3.3}$$

Proof. We begin by assuming that $f \in SGHX\left(\left(\frac{1}{h}\right), [s,r], \mathcal{R}_I{}^+\right)$, then

$$\frac{f(a_1)}{h\left(\frac{1}{2}\right)} + \frac{f(b_1)}{h\left(\frac{1}{2}\right)} \subseteq f\left(\frac{2a_1b_1}{a_1+b_1}\right),$$

where

$$a_1 = \frac{sr}{xs+(1-x)r},$$

and

$$b_1 = \frac{sr}{(1-x)s+xr}.$$

Then

$$\frac{1}{h\left(\frac{1}{2}\right)}\left[f\left(\frac{sr}{xs+(1-x)r}\right) + f\left(\frac{sr}{(1-x)s+xr}\right)\right] \subseteq f\left(\frac{2sr}{s+r}\right). \tag{3.4}$$

Multiplying both sides by $h\left(\frac{1}{2}\right)$, we have

$$\left[f\left(\frac{sr}{xs+(1-x)r}\right) + f\left(\frac{sr}{(1-x)s+xr}\right)\right] \subseteq h\left(\frac{1}{2}\right)f\left(\frac{2sr}{s+r}\right). \tag{3.5}$$

The above inequality is integrated over (0, 1), we have

$$\int_0^1 \left[f\left(\frac{sr}{xs+(1-x)r}\right) + f\left(\frac{sr}{(1-x)s+xr}\right)\right]dx \subseteq h\left(\frac{1}{2}\right)\int_0^1 f\left(\frac{2sr}{s+r}\right)dx.$$

So

$$\int_0^1 \underline{f}\left(\frac{sr}{xs+(1-x)r}\right)dx + \int_0^1 \underline{f}\left(\frac{sr}{(1-x)s+xr}\right)dx \geq h\left(\frac{1}{2}\right)\int_0^1 \underline{f}\left(\frac{2sr}{s+r}\right)dx,$$

and

$$\int_0^1 \overline{f}\left(\frac{sr}{xs+(1-x)r}\right)dx + \int_0^1 \overline{f}\left(\frac{sr}{(1-x)s+xr}\right)dx \leq h\left(\frac{1}{2}\right)\int_0^1 \overline{f}\left(\frac{2sr}{s+r}\right)dx.$$

It follows that

$$\frac{2sr}{r-s}\int_s^r \frac{\underline{f}(\kappa)}{\kappa^2}d\kappa \geq h\left(\frac{1}{2}\right)\int_0^1 \underline{f}\left(\frac{2sr}{s+r}\right)dx = h\left(\frac{1}{2}\right)\underline{f}\left(\frac{2sr}{s+r}\right).$$

Similarly,

$$\frac{2sr}{r-s}\int_s^r \frac{\overline{f}(\kappa)}{\kappa^2}d\kappa \leq h\left(\frac{1}{2}\right)\int_0^1 \overline{f}\left(\frac{2sr}{s+r}\right)dx = h\left(\frac{1}{2}\right)\overline{f}\left(\frac{2sr}{s+r}\right).$$

This implies that

$$\left[h\left(\frac{1}{2}\right)\right]\left[\underline{f}\left(\frac{2sr}{s+r}\right), \overline{f}\left(\frac{2sr}{s+r}\right)\right] \supseteq \frac{2sr}{r-s}\int_s^r \frac{f(\kappa)}{\kappa^2}d\kappa.$$

Divide both sides by $\frac{1}{2}$ first inclusion of (3.3) is proved,

$$\frac{\left[h\left(\frac{1}{2}\right)\right]}{2}\left[\underline{f}\left(\frac{2sr}{s+r}\right),\overline{f}\left(\frac{2sr}{s+r}\right)\right] \supseteq \frac{sr}{r-s}\int_s^r \frac{f(\kappa)}{\kappa^2}d\kappa. \tag{3.6}$$

According to our hypothesis,

$$\frac{f(s)}{h(1-x)}+\frac{f(r)}{h(x)} \subseteq f\left(\frac{sr}{(1-x)s+xr}\right),$$

and

$$\frac{f(s)}{h(x)}+\frac{f(r)}{h(1-x)} \subseteq f\left(\frac{sr}{xs+(1-x)r}\right).$$

Adding above two inclusions and integrate over $(0,1)$, we have

$$\begin{aligned}&[f(s)+f(r)]\int_0^1 \frac{1}{h(x)}dx+[f(s)+f(r)]\int_0^1 \frac{1}{h(1-x)}dx\\ \subseteq&\int_0^1\left[f\left(\frac{sr}{xs+(1-x)r}\right)+f\left(\frac{sr}{(1-x)s+xr}\right)\right]dx.\end{aligned}$$

Since at $x=\frac{1}{2}$ both integrals

$$\int_0^1 \frac{1}{h(x)}dx=\int_0^1 \frac{1}{h(1-x)}dx$$

are equal, which implies that

$$2\left[f(s)+f(r)\right]\int_0^1 \frac{1}{h(x)}dx \subseteq \frac{2sr}{r-s}\int_s^r \frac{f(\kappa)}{\kappa^2}d\kappa.$$

Dividing by 2, we obtain the desired result,

$$[f(s)+f(r)]\int_0^1 \frac{1}{h(x)}dx \subseteq \frac{sr}{r-s}\int_s^r \frac{f(\kappa)}{\kappa^2}d\kappa. \tag{3.7}$$

By combining (3.6) and (3.7), we obtain the desired result

$$\frac{\left[h\left(\frac{1}{2}\right)\right]}{2}f\left(\frac{2sr}{s+r}\right) \supseteq \frac{sr}{r-s}\int_s^r \frac{f(\kappa)d\kappa}{\kappa^2} \supseteq \ [f(s)+f(r)]\int_0^1 \frac{dx}{h(x)}.$$

This completes the proof. □

Remark 3.1. *It is shown that Theorem 3.1 can be reduced to harmonical p-IVFS, if $h(x)=1$, i.e.,*

$$\frac{1}{2}f\left(\frac{2sr}{s+r}\right) \supseteq \frac{sr}{r-s}\int_s^r \frac{f(\kappa)d\kappa}{\kappa^2} \supseteq \ [f(s)+f(r)].$$

If $h(x)=\dfrac{1}{x}$, then Theorem 3.1 reduces to harmonical convex IVFS:

$$f\left(\frac{2sr}{s+r}\right) \supseteq \frac{sr}{r-s}\int_s^r \frac{f(\kappa)d\kappa}{\kappa^2} \supseteq \ \frac{[f(s)+f(r)]}{2}.$$

If $h(x)=\dfrac{1}{x^{s}}$, *then Theorem 3.1 reduces to harmonical s-IVFS:*

$$2^{s-1}f\left(\frac{2sr}{s+r}\right)\supseteq\frac{sr}{r-s}\int_{s}^{r}\frac{f(\kappa)d\kappa}{\kappa^{2}}\supseteq\frac{[f(s)+f(r)]}{s+1}.$$

Example 3.1. *Let us define* $h(x)=\dfrac{1}{x}$ *for* $x\in(0,1),[r,s]=\left[\dfrac{1}{2},1\right]$ *and* $f:[r,s]\to\mathcal{R}_I{}^{+}$ *be defined as* $f(\kappa)=[2\kappa^{2},4-e^{\kappa}]$. *Then*

$$\begin{aligned}&\frac{\left[h(\frac{1}{2})\right]}{2}f\left(\frac{2sr}{s+r}\right)=f\left(\frac{2}{3}\right)=\left[\frac{8}{9},4-e^{\frac{2}{3}}\right],\\&\frac{sr}{r-s}\int_{s}^{r}\frac{f(\kappa)}{\kappa^{2}}d\kappa=\left[\int_{\frac{1}{2}}^{1}2d\kappa,\int_{\frac{1}{2}}^{1}\frac{(4-e^{\kappa})}{\kappa^{2}}d\kappa\right]=[1,1.979941375566026]\,,\end{aligned}$$

and

$$[f(s)+f(r)]\int_{0}^{1}\frac{dx}{h(x)}=[f(s)+f(r)]\int_{0}^{1}xdx=\left[\frac{5}{4},4-\frac{\sqrt{e}}{2}-\frac{e}{2}\right].$$

Thus, we obtain

$$\left[\frac{8}{9},4-e^{\frac{2}{3}}\right]\supseteq[1,1.979941375566026]\supseteq\left[\frac{5}{4},4-\frac{\sqrt{e}}{2}-\frac{e}{2}\right],$$

which demonstrates the result described in Theorem 3.1.

Theorem 3.2. *Consider* $h:(0,1)\to\mathcal{R}^{+}$ *such that* $h\neq 0$. *Let* $f:[s,r]\to\mathcal{R}_I{}^{+}$. *If* $f\in SGHX\left(\left(\frac{1}{h}\right),[s,r],\mathcal{R}_I{}^{+}\right)$ *and* $f\in\mathcal{IR}_{[s,r]}$, *we have*

$$\frac{\left[h\left(\frac{1}{2}\right)\right]^{2}}{4}f\left(\frac{2sr}{s+r}\right)\supseteq\triangle_1\supseteq\frac{sr}{r-s}\int_{s}^{r}\frac{f(\kappa)}{\kappa^{2}}d\kappa\supseteq\triangle_2\supseteq\left\{[f(s)+f(r)]\left[\frac{1}{2}+\frac{1}{h\left(\frac{1}{2}\right)}\right]\right\}\int_{0}^{1}\frac{dx}{h(x)},$$

where

$$\triangle_1=\frac{[h(\frac{1}{2})]}{4}\left[f\left(\frac{4sr}{s+3r}\right)+f\left(\frac{4sr}{r+3s}\right)\right],$$

and

$$\triangle_2=\left[f\left(\frac{2sr}{s+r}\right)+\left(\frac{f(s)+f(r)}{2}\right)\right]\int_{0}^{1}\frac{dx}{h(x)}.$$

Proof. Consider $f\in SGHX\left(\left(\frac{1}{h}\right),[s,r],\mathcal{R}_I{}^{+}\right)$ and $f\in IR_{[s,r]}$, for $[s,\frac{2sr}{s+r}]$, we have

$$\frac{f\left(\frac{s\frac{2sr}{s+r}}{xs+(1-x)\frac{2sr}{s+r}}\right)}{h\left(\frac{1}{2}\right)}+\frac{f\left(\frac{s\frac{2sr}{s+r}}{(1-x)s+x\frac{2sr}{s+r}}\right)}{h\left(\frac{1}{2}\right)}\subseteq f\left(\frac{4sr}{s+3r}\right),$$

we get

$$\frac{1}{h\left(\frac{1}{2}\right)}\left[f\left(\frac{s\frac{2sr}{s+r}}{xs+(1-x)\frac{2sr}{s+r}}\right)+f\left(\frac{s\frac{2sr}{s+r}}{(1-x)s+x\frac{2sr}{s+r}}\right)\right]\subseteq f\left(\frac{4sr}{s+3r}\right).$$

On integration over $(0,1)$, we have

$$\frac{1}{h(\frac{1}{2})}\int_0^1\left[\underline{f}\left(\frac{s\frac{2sr}{s+r}}{xs+(1-x)\frac{2sr}{s+r}}\right)dx+\underline{f}\left(\frac{s\frac{2sr}{s+r}}{(1-x)s+x\frac{2sr}{s+r}}\right)dx\right.$$
$$\left.\overline{f}\left(\frac{s\frac{2sr}{s+r}}{xs+(1-x)\frac{2sr}{s+r}}\right)dx+\overline{f}\left(\frac{s\frac{2sr}{s+r}}{(1-x)s+x\frac{2sr}{s+r}}\right)dx\right]\subseteq f\left(\frac{4sr}{s+3r}\right).$$

Then, above inequality become as

$$\begin{aligned}
&\frac{1}{h\left(\frac{1}{2}\right)}\left[\frac{2sr}{r-s}\int_s^{\frac{2sr}{s+r}}\frac{\underline{f}(\kappa)}{\kappa^2}d\kappa+\frac{2sr}{r-s}\int_s^{\frac{2sr}{s+r}}\frac{\underline{f}(\kappa)}{\kappa^2}d\kappa,\ \frac{2sr}{r-s}\int_s^{\frac{2sr}{s+r}}\frac{\overline{f}(\kappa)}{\kappa^2}d\kappa+\frac{2sr}{r-s}\int_s^{\frac{2sr}{s+r}}\frac{\overline{f}(\kappa)}{\kappa^2}d\kappa\right]\\
&\subseteq f\left(\frac{4sr}{s+3r}\right)\\
&=\frac{1}{h\left(\frac{1}{2}\right)}\left[\frac{4sr}{r-s}\int_s^{\frac{2sr}{s+r}}\frac{\underline{f}(\kappa)}{\kappa^2}d\kappa,\frac{4sr}{r-s}\int_s^{\frac{2sr}{s+r}}\frac{\overline{f}(\kappa)}{\kappa^2}d\kappa\right]\subseteq f\left(\frac{4sr}{s+3r}\right)\\
&=\frac{4}{h\left(\frac{1}{2}\right)}\left[\frac{sr}{r-s}\int_s^{\frac{2sr}{s+r}}\frac{\underline{f}(\kappa)}{\kappa^2}d\kappa,\frac{sr}{r-s}\int_s^{\frac{2sr}{s+r}}\frac{\overline{f}(\kappa)}{\kappa^2}d\kappa\right]\subseteq f\left(\frac{4sr}{s+3r}\right)\\
&=\frac{4}{h\left(\frac{1}{2}\right)}\left[\frac{sr}{r-s}\int_s^{\frac{2sr}{s+r}}\frac{f(\kappa)}{\kappa^2}d\kappa\right]\subseteq f\left(\frac{4sr}{s+3r}\right)\\
&=\frac{sr}{r-s}\int_s^{\frac{2sr}{s+r}}\frac{f(\kappa)}{\kappa^2}d\kappa\subseteq\frac{\left[h\left(\frac{1}{2}\right)\right]}{4}f\left(\frac{4sr}{s+3r}\right).
\end{aligned}\tag{3.8}$$

Similarly for interval $\left[\frac{2sr}{s+r},r\right]$, we have

$$\frac{sr}{r-s}\int_{\frac{2sr}{s+r}}^{r}\frac{f(\kappa)}{\kappa^2}d\kappa\subseteq\frac{\left[h\left(\frac{1}{2}\right)\right]}{4}f\left(\frac{4sr}{r+3s}\right).\tag{3.9}$$

Adding above inclusions (3.8) and (3.9), we have

$$\begin{aligned}
\triangle_1&=\frac{\left[h\left(\frac{1}{2}\right)\right]}{4}\left[f\left(\frac{4sr}{3s+r}\right)+f\left(\frac{4sr}{s+3r}\right)\right]\supseteq\left[\frac{sr}{r-s}\int_s^{r}\frac{f(\kappa)}{\kappa^2}d\kappa\right]\\
&=\frac{1}{2}\left[\frac{2sr}{r-s}\int_s^{\frac{2sr}{s+r}}\frac{f(\kappa)}{\kappa^2}d\kappa+\frac{2sr}{r-s}\int_{\frac{2sr}{s+r}}^{r}\frac{f(\kappa)}{\kappa^2}d\kappa\right]\\
&\supseteq\frac{1}{2}\left[\left[f(s)+f\left(\frac{2sr}{s+r}\right)\right]\int_0^1\frac{dx}{h(x)}\right]+\frac{1}{2}\left[\left[f(r)+f\left(\frac{2sr}{s+r}\right)\right]\int_0^1\frac{dx}{h(x)}\right]\\
&=\frac{1}{2}\left[\left\{f(s)+f(r)+2f\left(\frac{2sr}{s+r}\right)\right\}\int_0^1\frac{dx}{h(x)}\right]\\
&=\left[\frac{f(s)+f(r)}{2}+f\left(\frac{2sr}{s+r}\right)\right]\int_0^1\frac{dx}{h(x)}=\triangle_2.
\end{aligned}$$

Now,

$$
\begin{aligned}
\frac{\left[h\left(\frac{1}{2}\right)\right]^2}{4} f\left(\frac{2sr}{s+r}\right) &= \frac{\left[h\left(\frac{1}{2}\right)\right]^2}{4} f\left(\frac{2\frac{4sr}{s+3r}\frac{4sr}{r+3s}}{\frac{4sr}{s+3r}+\frac{4sr}{r+3s}}\right) \supseteq \frac{\left[h\left(\frac{1}{2}\right)\right]^2}{4}\left[\frac{f\left(\frac{4sr}{s+3r}\right)}{h(\frac{1}{2})}+\frac{f\left(\frac{4sr}{r+3s}\right)}{h\left(\frac{1}{2}\right)}\right] \\
&= \frac{\left[h\left(\frac{1}{2}\right)\right]^2}{4h(\frac{1}{2})}\left[f\left(\frac{4sr}{s+3r}\right)+f\left(\frac{4sr}{r+3s}\right)\right] \\
&= \frac{\left[h\left(\frac{1}{2}\right)\right]}{4}\left[f\left(\frac{4sr}{s+3r}\right)+f\left(\frac{4sr}{s+3r}\right)\right] = \triangle_1 \\
&\supseteq \frac{\left[h\left(\frac{1}{2}\right)\right]}{4}\left\{\frac{1}{h\left(\frac{1}{2}\right)}\left[f(s)+f\left(\frac{2sr}{s+r}\right)\right]+\frac{1}{h(\frac{1}{2})}\left[f(r)+f\left(\frac{2sr}{s+r}\right)\right]\right\} \\
&= \frac{\left[h\left(\frac{1}{2}\right)\right]}{4}\left\{\frac{1}{h\left(\frac{1}{2}\right)}\left[f(s)+f(r)+2f\left(\frac{2sr}{s+r}\right)\right]\right\} \\
&= \frac{1}{4}\left\{f(s)+f(r)+2f\left(\frac{2sr}{s+r}\right)\right\} = \frac{1}{2}\left[\frac{f(s)+f(r)}{2}+f\left(\frac{2sr}{s+r}\right)\right] \\
&\supseteq \left[\frac{f(s)+f(r)}{2}+f\left(\frac{2sr}{s+r}\right)\right]\int_0^1 \frac{dx}{h(x)} = \triangle_2 \\
&\supseteq \left[\frac{f(s)+f(r)}{2}+\frac{f(s)}{h\left(\frac{1}{2}\right)}+\frac{f(r)}{h\left(\frac{1}{2}\right)}\right]\int_0^1 \frac{dx}{h(x)} \\
&= \left[\frac{f(s)+f(r)}{2}+\frac{1}{h\left(\frac{1}{2}\right)}\left[f(s)+f(r)\right]\right]\int_0^1 \frac{dx}{h(x)} \\
&= \left\{\left[f(s)+f(r)\right]\left[\frac{1}{2}+\frac{1}{h\left(\frac{1}{2}\right)}\right]\right\}\int_0^1 \frac{dx}{h(x)}.
\end{aligned}
$$

This completes the proof. □

Example 3.2. *Recall to Example 3.1, we have*

$$
\frac{\left[h\left(\frac{1}{2}\right)\right]^2}{4} f\left(\frac{2sr}{s+r}\right) \supseteq \triangle_1 \supseteq \frac{sr}{r-s}\int_s^r \frac{f(\kappa)}{\kappa^2}d\kappa \supseteq \triangle_2 \supseteq \left\{\left[f(s)+f(r)\right]\left[\frac{1}{2}+\frac{1}{h(\frac{1}{2})}\right]\right\}\int_0^1 \frac{dx}{h(x)},
$$

where

$$
\frac{\left[h\left(\frac{1}{2}\right)\right]^2}{4} f\left(\frac{2sr}{s+r}\right) = f\left(\frac{2}{3}\right) = \left[\frac{8}{9}, 4-e^{\frac{2}{3}}\right],
$$

$$\triangle_1 = \frac{\left[h\left(\frac{1}{2}\right)\right]}{4}\left[f\left(\frac{4sr}{s+3r}\right)+f\left(\frac{4sr}{r+3s}\right)\right] = \frac{1}{2}\left[\left[\frac{32}{49}, 4-e^{\frac{4}{7}}\right]+\left[\frac{32}{25}, 4-e^{\frac{4}{5}}\right]\right] = \left[\frac{1184}{1225}, 4-\frac{e^{\frac{4}{7}}}{2}-\frac{e^{\frac{4}{5}}}{2}\right],$$

$$\triangle_2 = \left[f\left(\frac{2sr}{s+r}\right)+\left(\frac{f(s)+f(r)}{2}\right)\right]\int_0^1 \frac{dx}{h(x)} = \frac{1}{2}\left[f\left(\frac{2}{3}\right)+\left(\frac{f(\frac{1}{2})+f(1)}{2}\right)\right] = \left[\frac{77}{72}, -\frac{e}{4}-\frac{\sqrt{e}}{4}-\frac{e^{\frac{2}{3}}}{2}+4\right].$$

Thus, we obtain

$$\left[\frac{8}{9}, 4-e^{\frac{2}{3}}\right] \supseteq \left[\frac{1184}{1225}, 4-\frac{e^{\frac{4}{7}}}{2}-\frac{e^{\frac{4}{5}}}{2}\right] \supseteq [1, 1.979941375566026] \supseteq \left[\frac{77}{72}, -\frac{e}{4}-\frac{\sqrt{e}}{4}-\frac{e^{\frac{2}{3}}}{2}+4\right],$$

which demonstrates the result described in Theorem 3.2.

Theorem 3.3. *Consider $h_1, h_2 : (0,1) \to \mathbb{R}^+$ such that $h_1, h_2 \neq 0$. Let $f : [s,r] \to \mathbb{R}_I{}^+$. If $f \in SGHX\left(\left(\frac{1}{h_1}\right), [s,r], \mathbb{R}_I{}^+\right)$, $g \in SGHX\left(\left(\frac{1}{h_2}\right), [s,r], \mathbb{R}_I{}^+\right)$ and $f, g \in \mathbb{IR}_{[s,r]}$, we have*

$$\frac{sr}{r-s}\int_s^r \frac{f(\kappa)g(\kappa)}{\kappa^2}d\kappa \supseteq M(s,r)\int_0^1 \frac{1}{h_1(x)h_2(x)}dx + N(s,r)\int_0^1 \frac{1}{h_1(x)h_2(1-x)}dx,$$

where

$$M(s,r) = f(s)g(s) + f(r)g(r)$$

and

$$N(s,r) = f(s)g(r) + f(r)g(s).$$

Proof. We assume that $f \in SGHX\left(\left(\frac{1}{h_1}\right), [s,r], \mathbb{R}_I{}^+\right)$, $g \in SGHX\left(\left(\frac{1}{h_2}\right), [s,r], \mathcal{R}_I{}^+\right)$, then

$$\frac{f(s)}{h_1(x)} + \frac{f(r)}{h_1(1-x)} \subseteq f\left(\frac{sr}{xs+(1-x)r}\right)$$

and

$$\frac{g(s)}{h_2(x)} + \frac{g(r)}{h_2(1-x)} \subseteq g\left(\frac{sr}{xs+(1-x)r}\right).$$

Then

$$f\left(\frac{sr}{xs+(1-x)r}\right)g\left(\frac{sr}{xs+(1-x)r}\right)$$
$$\supseteq \frac{f(s)g(s)}{h_1(x)h_2(x)} + \frac{f(s)g(r)}{h_1(x)h_2(1-x)} + \frac{f(r)g(s)}{h_1(1-x)h_2(x)} + \frac{f(r)g(r)}{h_1(1-x)h_2(1-x)}.$$

On integration over $(0,1)$, we have

$$
\begin{aligned}
&\int_0^1 f\left(\frac{sr}{xs+(1-x)r}\right) g\left(\frac{sr}{xs+(1-x)r}\right) dx\\
=&\left[\int_0^1 \underline{f}\left(\frac{sr}{xs+(1-x)r}\right) \underline{g}\left(\frac{sr}{xs+(1-x)r}\right) dx, \int_0^1 \overline{f}\left(\frac{sr}{xs+(1-x)r}\right) \overline{g}\left(\frac{sr}{xs+(1-x)r}\right) dx\right]\\
=&\left[\frac{sr}{r-s}\int_s^r \frac{\underline{f}(\kappa)\underline{g}(\kappa)}{\kappa^2} d\kappa, \frac{sr}{r-s}\int_s^r \frac{\overline{f}(\kappa)\overline{g}(\kappa)}{\kappa^2} d\kappa\right]\\
=&\frac{sr}{r-s}\int_s^r \frac{f(\kappa)g(\kappa)}{\kappa^2} d\kappa\\
\supseteq&\int_0^1 \frac{[f(s)g(s)+f(r)g(r)]}{h_1(x)h_2(x)} dx + \int_0^1 \frac{[f(s)g(r)+f(r)g(s)]}{h_1(x)h_2(1-x)} dx.
\end{aligned}
$$

It follows that

$$
\frac{sr}{r-s}\int_s^r \frac{f(\kappa)g(\kappa)}{\kappa^2} d\kappa \supseteq M(s,r)\int_0^1 \frac{1}{h_1(x)h_2(x)} dx + N(s,r)\int_0^1 \frac{dx}{h_1(x)h_2(1-x)}.
$$

The proof is completed. □

Example 3.3. *Suppose that* $h_1(x) = \frac{1}{x}$, $h_2(x) = 2$ *for* $x \in (0,1)$, $[s,r] = \left[\frac{1}{2}, 1\right]$ *and*

$$
f(\kappa) = \left[\kappa^2, 6 - e^{\kappa}\right], \ g(\kappa) = \left[\kappa, 5 - \kappa^2\right].
$$

Then,

$$
\frac{sr}{r-s}\int_s^r f(\kappa)g(\kappa) d\kappa = \left[\int_{\frac{1}{2}}^1 \kappa d\kappa, \int_{\frac{1}{2}}^1 \frac{(6-e^{\kappa})(5-\kappa^2)}{\kappa^2} d\kappa\right] = \left[\frac{3}{8}, \frac{-16e + 15\sqrt{e} + 53}{4}\right],
$$

$$
M(s,r)\int_0^1 \frac{1}{h_1(x)h_2(x)} dx = \frac{M\left(\frac{1}{2},1\right)}{2}\int_0^1 x dx = \left[\frac{9}{32}, \frac{19}{4} - \frac{\sqrt{e}}{2} - \frac{11e}{16}\right]
$$

and

$$
N(s,r)\int_0^1 \frac{1}{h_1(x)h_2(1-x)} dx = \frac{N\left(\frac{1}{2},1\right)}{2}\int_0^1 x dx = \left[\frac{3}{16}, \frac{19}{4} - \frac{11\sqrt{e}}{16} - \frac{e}{2}\right].
$$

It follows that

$$
\begin{aligned}
\left[\frac{3}{8}, \frac{-16e + 15\sqrt{e} + 53}{4}\right] &\supseteq \left[\frac{9}{32}, \frac{19}{4} - \frac{\sqrt{e}}{2} - \frac{11e}{16}\right] + \left[\frac{3}{16}, \frac{19}{4} - \frac{11\sqrt{e}}{16} - \frac{e}{2}\right]\\
&= \left[\frac{15}{32}, \frac{19}{2} + \frac{-19e - 19\sqrt{e}}{16}\right],
\end{aligned}
$$

which demonstrates the result described in Theorem 3.3.

Theorem 3.4. *Consider $h_1, h_2 : (0,1) \to \mathbb{R}^+$ such that $h_1, h_2 \neq 0$. Let $f : [s,r] \to \mathbb{R}_I^+$. If $f \in SGHX\left(\left(\frac{1}{h_1}\right), [s,r], \mathcal{R}_I^+\right)$, $g \in SGHX\left(\left(\frac{1}{h_2}\right), [s,r], \mathcal{R}_I^+\right)$ and $f, g \in \mathbb{IR}_{[s,r]}$, we have*

$$\frac{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}{2} f\left(\frac{2sr}{s+r}\right) g\left(\frac{2sr}{s+r}\right)$$
$$\supseteq \frac{sr}{r-s}\int_s^r \frac{f(\kappa)g(\kappa)}{\kappa^2}d\kappa + M(s,r)\int_0^1 \frac{1}{h_1(x)h_2(x)}dx + N(s,r)\int_0^1 \frac{1}{h_1(x)h_2(1-x)}dx.$$

Proof. According to our hypothesis, we have

$$f\left(\frac{2sr}{s+r}\right) \supseteq \frac{f\left(\frac{sr}{xs+(1-x)r}\right)}{h_1\left(\frac{1}{2}\right)} + \frac{f\left(\frac{sr}{xs+(1-x)r}\right)}{h_1\left(\frac{1}{2}\right)}$$

and

$$g\left(\frac{2sr}{s+r}\right) \supseteq \frac{g\left(\frac{sr}{xs+(1-x)r}\right)}{h_2\left(\frac{1}{2}\right)} + \frac{g\left(\frac{sr}{xs+(1-x)r}\right)}{h_2\left(\frac{1}{2}\right)}.$$

Then

$$\begin{aligned}
& f\left(\frac{2sr}{s+r}\right) g\left(\frac{2sr}{s+r}\right) \\
\supseteq & \frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\left[f\left(\frac{sr}{xs+(1-x)r}\right)g\left(\frac{sr}{xs+(1-x)r}\right) + f\left(\frac{sr}{xr+(1-x)s}\right)g\left(\frac{sr}{xr+(1-x)s}\right)\right] \\
+ & \frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\left[f\left(\frac{sr}{xs+(1-x)r}\right)g\left(\frac{sr}{xr+(1-x)s}\right) + f\left(\frac{sr}{xr+(1-x)s}\right)g\left(\frac{sr}{xs+(1-x)r}\right)\right] \\
\supseteq & \frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\left[f\left(\frac{sr}{xs+(1-x)r}\right)g\left(\frac{sr}{xs+(1-x)r}\right) + f\left(\frac{sr}{xr+(1-x)s}\right)g\left(\frac{sr}{xr+(1-x)s}\right)\right] \\
+ & \frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\left[\left(\frac{f(s)}{h_1(x)} + \frac{f(r)}{h_1(1-x)}\right)\left(\frac{g(s)}{h_2(1-x)} + \frac{g(r)}{h_2(x)}\right) + \left(\frac{f(s)}{h_1(1-x)} + \frac{f(r)}{h_1(x)}\right)\left(\frac{g(s)}{h_2(x)} + \frac{g(r)}{h_2(1-x)}\right)\right] \\
= & \frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\left[f\left(\frac{sr}{xs+(1-x)r}\right)g\left(\frac{sr}{xs+(1-x)r}\right) + f\left(\frac{sr}{xr+(1-x)s}\right)g\left(\frac{sr}{xr+(1-x)s}\right)\right] \\
+ & \frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\left[\left(\frac{1}{h_1(x)h_2(1-x)} + \frac{1}{h_1(1-x)h_2(x)}\right)M(s,r) + \left(\frac{1}{h_1(x)h_2(x)} + \frac{1}{h_1(1-x)h_2(1-x)}\right)N(s,r)\right].
\end{aligned}$$

On integration over $(0,1)$, we have

$$\int_0^1 f\left(\frac{2sr}{s+r}\right)g\left(\frac{2sr}{s+r}\right)dx = \left[\int_0^1 \underline{f}\left(\frac{2sr}{s+r}\right)g\left(\frac{2sr}{s+r}\right)dx, \int_0^1 \overline{f}\left(\frac{2sr}{s+r}\right)g\left(\frac{2sr}{s+r}\right)dx\right]$$
$$= f\left(\frac{2sr}{s+r}\right)g\left(\frac{2sr}{s+r}\right)$$
$$\supseteq \frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\left[\frac{2sr}{r-s}\int_s^r \frac{f(\kappa)g(\kappa)}{\kappa^2}d\kappa\right]$$
$$+ \frac{2}{h_1\frac{1}{2})h_2(\frac{1}{2})}\left[M(s,r)\int_0^1 \frac{1}{h_1(x)h_2(1-x)}dx + N(s,r)\int_0^1 \frac{1}{h_1(x)h_2(x)}dx\right].$$

Multiply both sides by $\frac{h_1(\frac{1}{2})h_2(\frac{1}{2})}{2}$, above equation we get

$$\frac{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}{2}f\left(\frac{2sr}{s+r}\right)g\left(\frac{2sr}{s+r}\right)$$
$$\supseteq\frac{sr}{r-s}\int_s^r \frac{f(\kappa)g(\kappa)}{\kappa^2}d\kappa + M(s,r)\int_0^1 \frac{1}{h_1(x)h_2(1-x)}dx + N(s,r)\int_0^1 \frac{1}{h_1(x)h_2(x)}dx.$$

Therefore, the proof is completed. □

Example 3.4. *Consider* $h_1(x) = \frac{1}{x},\ h_2(x) = 2$ *for* $x \in\ (0,1),\ [s,r] = [\frac{1}{2},1]$ *and*

$$f(\kappa) = [\kappa^2, 6-e^{\kappa}],\ g(\kappa) = [\kappa, 5-\kappa^2].$$

Then

$$\frac{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}{2}f\left(\frac{2sr}{s+r}\right)g\left(\frac{2sr}{s+r}\right) = 2f\left(\frac{2}{3}\right)g\left(\frac{2}{3}\right) = \left[\frac{16}{27}, \frac{492-82e^{\frac{2}{3}}}{9}\right],$$
$$\frac{sr}{r-s}\int_s^r f(\kappa)g(\kappa)d\kappa = \left[\int_{\frac{1}{2}}^1 \kappa d\kappa, \int_{\frac{1}{2}}^1 \frac{(6-e^{\kappa})(5-\kappa^2)}{\kappa^2}d\kappa\right] = \left[\frac{3}{8}, \frac{-16e+15\sqrt{e}+53}{4}\right],$$
$$M(s,r)\int_0^1 \frac{1}{h_1(x)h_2(x)}dx = \frac{M\left(\frac{1}{2},1\right)}{2}\int_0^1 xdx = \left[\frac{9}{32}, \frac{19}{4} - \frac{\sqrt{e}}{2} - \frac{11e}{16}\right],$$

and

$$N(s,r)\int_0^1 \frac{1}{h_1(x)h_2(1-x)}dx\frac{N\left(\frac{1}{2},1\right)}{2}\int_0^1 xdx = \left[\frac{3}{16}, \frac{19}{4} - \frac{11\sqrt{e}}{16} - \frac{e}{2}\right]. \tag{3.10}$$

It follows that

$$\left[\frac{16}{27}, \frac{492-82e^{\frac{2}{3}}}{9}\right] \supseteq \left[\frac{3}{8}, \frac{-16e+15\sqrt{e}+53}{4}\right] + \left[\frac{9}{32}, \frac{19}{4} - \frac{\sqrt{e}}{2} - \frac{11e}{16}\right] + \left[\frac{3}{16}, \frac{19}{4} - \frac{11\sqrt{e}}{16} - \frac{e}{2}\right]$$
$$= \left[\frac{27}{32}, \frac{19}{2} + \frac{-83e+41\sqrt{e}+212}{16}\right].$$

This verifies the above theorem.

3.2. Jensen type inequality

Theorem 3.5. *Let $t_1, t_2, t_3, \dots, t_l \in \mathbb{R}^+$ with $l \geq 2$. Let f be non-negative harmonical h-GL IVF or $f \in SGX\left((\frac{1}{h}), [s, r], \mathcal{R}_I{}^+\right)$ and h is non-negative super multiplicative function with $b_1, b_2, b_3, \dots, b_l \in I \subseteq R_I{}^+$. Then one has*

$$f\left(\frac{1}{\frac{1}{T_l}\sum_{i=1}^{l}\frac{t_i}{b_i}}\right) \supseteq \sum_{i=1}^{l}\left[\frac{f(b_i)}{h\left(\frac{t_i}{T_l}\right)}\right], \tag{3.11}$$

where $T_l = \sum_{i=1}^{l} t_i$.

Proof. For $l = 2$, the inequality (3.11) is trivially true. Now, we assume that it also works for $l - 1$. Consider

$$\begin{aligned}
f\left(\frac{1}{\frac{1}{T_l}\sum_{i=1}^{l}\frac{t_i}{b_i}}\right) &= f\left(\frac{1}{\frac{t_l}{T_l b_l} + \sum_{i=1}^{l-1}\frac{t_i}{T_l b_i}}\right) \\
&= f\left(\frac{1}{\frac{t_l}{T_l b_l} + \frac{T_{l-1}}{T_l}\sum_{i=1}^{l-1}\frac{t_i}{T_{l-1} b_i}}\right) \\
&\supseteq \frac{f(b_l)}{h\left(\frac{t_l}{T_l}\right)} + \frac{f\left(\frac{1}{\sum_{i=1}^{l-1}\frac{t_i}{T_{l-1}b_i}}\right)}{h\left(\frac{T_{l-1}}{T_l}\right)} \\
&\supseteq \frac{f(b_l)}{h\left(\frac{t_l}{T_l}\right)} + \sum_{i=1}^{l-1}\left[\frac{f(b_i)}{h\left(\frac{t_i}{T_{l-1}}\right)}\right]\frac{1}{h\left(\frac{T_{l-1}}{T_l}\right)} \\
&\supseteq \frac{f(b_l)}{h\left(\frac{t_l}{T_l}\right)} + \sum_{i=1}^{l-1}\left[\frac{f(b_i)}{h(\frac{t_i}{T_l})}\right] \\
&= \sum_{i=1}^{l}\left[\frac{f(b_i)}{h\left(\frac{t_i}{T_l}\right)}\right].
\end{aligned}$$

Therefore, the result is proven using mathematical induction. □

4. Conclusions

The purpose of this paper is to introduce the harmonical h-GL concept for IVFS. Our goal with the above concept was to study Jensen and H-H inequalities for IVFS. The inequalities recently developed by Kiliman [36] and Dafang et al. [29] are generalized in this study. Additionally, some useful examples are provided to support our main findings. This is an interesting topic that can be explored in the future to determine equivalent inequalities for different types of convexity. Using these concepts, convex optimization theory and fuzzy convex analysis take a new direction. Additionally, we will explore the generalizations of this concept by using various other types of integral operators in the future. Hopefully, this concept will be useful to other authors in various scientific fields.

Conflict of interest

The authors declare no conflicts of interest.

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