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Research article

Derived equivalence, recollements under H-Galois extensions

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Abstract: In this paper, assume that H is a Hopf algebra and A/B is an H-Galois extension. Firstly, by introducing the concept of an H-stable tilting complex $T_\bullet$ over B, we show that $T_\bullet \otimes_B A$ is a tilting complex over A and a derived equivalence between two H-module algebras can be extended to smash product algebras under some conditions. Then we observe that $0 \to \mathrm{End}_{\mathcal{D}^b(B)}(T_\bullet) \to \mathrm{End}_{\mathcal{D}^b(A)}(T_\bullet \otimes_B A)$ is an H-Galois Frobenius extension if A/B is an H-Galois Frobenius extension. Finally, for any perfect recollement of derived categories of H-module algebras, we apply the above results to construct a perfect recollement of derived categories of their smash product algebras and generalize it to n-recollements.

Keywords: tilting complex; H-Galois extension; H-Frobenius extension; recollement
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1. Introduction

Tilting complexes were introduced by Rickard [1] and play an important role in representation theory, since then various results on extensions of algebras and extensions of tilting complexes (modules) have been given in representation theory. For more detailed information, please see [1–6]. However, in this paper, we consider the extension of tilting complexes in some new aspects.

It is well known that $A_1 \otimes A_3$ and $A_2 \otimes A_3$ are derived equivalent if A_1 and A_2 are derived equivalent, where A_1, A_2 and A_3 are three finite dimensional k-algebras (see [7]). Now if A and C are left H-module algebras, then there exist two smash product algebras $A\#H$ and $C\#H$. Since $A \otimes_k H$ can be thought as trivial smash product, we naturally ask the following questions:

(1) If A and C are derived equivalent, does it follow that $A\#H$ and $C\#H$ are also derived equivalent?

(2) Conversely, if $A\#H$ and $C\#H$ are derived equivalent, does it follow that A and C are also derived equivalent?

In general, we know that $A\#H$ is an H-Galois extension of A. Hence we can also ask whether the relation of derived equivalence is invariant under H-Galois extension. Let A and C be right faithful flat

H-Galois extensions of A^{coH^*} and C^{coH^*}, respectively.

(1) If A and C are derived equivalent, does it follow that A^{coH^*} and C^{coH^*} are also derived equivalent?

(2) Conversely, if A^{coH^*} and C^{coH^*} are derived equivalent, does it follow that A and C are also derived equivalent?

In fact, the above questions (1) and (2) have been considered in the context of strongly group graded algebras [8], and the questions (1) and (2) in the case of Morita equivalence have been considered in [9]. The aim of this paper is to answer the above question (2) (Corollary 3.1).

The recollements of triangulated categories were introduced by Beilinson-Bernstein-Deligne [10] and play an important role in algebraic geometry [10], representation theory [11]. Han constructed perfect recollements of derived categories of tensor product algebras from a perfect recollement of derived categories of algebras in [12]. In this paper we generalize this construction to smash product algebras. Furthermore, This result can be generalized to the n-recollement which is a natural generalization of the recollement.

This paper is organized as follows. In Section 2, we recall some basic definitions and results. In Section 3, according to Theorem 2.1, we show that a derived equivalence between two H-module algebras can be extended to their smash product algebras under some conditions. In Section 4, we prove that $0 \to \mathrm{End}_{\mathcal{D}^b(B)}(T_\bullet) \to \mathrm{End}_{\mathcal{D}^b(A)}(T_\bullet \otimes_B A)$ is an H-Galois Frobenius extension if A/B is an H-Galois Frobenius extension. In Section 5, for any perfect recollement of derived categories of H-module algebras, we construct a perfect recollement of derived categories of their smash product algebras. Moreover, we prove the similar result for n-recollements.

2. Preliminaries

In this section we introduce some basic notations, definitions and results needed in this paper.

Throughout this paper, k is a fixed field and all algebras are finite dimensional k-algebras. Given an algebra A, we denote by Mod$A$ the category of right A-modules, by mod$A$ the full subcategory of finite dimensional right A-modules and by Proj-A (resp. $\mathcal{P}(A)$) the category of (resp. finitely generated) projective right A-modules.

Let $\mathcal{A}$ be an abelian category. A complex $X_\bullet$ over $\mathcal{A}$ is a sequence of morphisms d_i between objects X_i in $\mathcal{A}$, that is,

$$\cdots \longrightarrow X_{i+1} \xrightarrow{d_i} X_i \xrightarrow{d_{i-1}} X_{i-1} \longrightarrow \cdots$$

such that $d_{i-1}d_i = 0$ for all $i \in \mathbb{Z}$. The category of all complexes over $\mathcal{A}$ with the usual complex maps of degree zero is denoted by $C(\mathcal{A})$. The homotopy category and the derived category of complexes over $\mathcal{A}$ is denoted by $\mathcal{K}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$, respectively. The full subcategory of $C(\mathcal{A})$ consisting of bounded (resp. bounded above) complexes over $\mathcal{A}$ is denoted by $C^b(\mathcal{A})$ (resp. $C^-(\mathcal{A})$). Similarly, we have full subcategories $\mathcal{D}^b(\mathcal{A})$, $\mathcal{D}^-(\mathcal{A})$ in $\mathcal{D}(\mathcal{A})$ and $\mathcal{K}^b(\mathcal{A})$, $\mathcal{K}^-(\mathcal{A})$ in $\mathcal{K}(\mathcal{A})$. In particular, if $\mathcal{A} =$ mod$A$ with A a finite dimensional k-algebra, then we briefly write $C(A)$, $C^b(A)$ for $C(\mathcal{A})$, $C^b(\mathcal{A})$, and so on.

Let H be a Hopf algebra over k and H^* be the dual Hopf algebra of H. We use the Sweedler notation for the comultiplication on H: $\triangle(h) = \sum h_1 \otimes h_2$. A k-algebra A is called a **left H-module algebra** if A is a left H-module such that $h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b)$ and $h \cdot 1_A = \epsilon(h) 1_A$ for all $a, b \in A$ and $h \in H$. Given any left H-module M, the submodule of **H-invariants** is the set $M^H = \{m \in M | h \cdot m =$

$\epsilon(h)m$ *for all* $h \in H\}$. If A is an H-module algebra, then A^H is a subalgebra of A. The dual notion of left H-module algebra is the right H^*-comodule algebra induced naturally from the left H-module.

Definition 2.1. *[13, Definitions 7.2.1 and 8.1.1] Let A be a k-algebra, B be a subalgebra of A, and H be a Hopf k-algebra.*

(1) $B \subset A$ is called a right H-extension if A is a right H-comodule algebra with structure map ρ satisfying $B = A^{coH}$, where A^{coH} is defined as the subcomodule $\{a \in A \mid \rho(a) = a \otimes 1\}$;

(2) A right H-extension $B \subset A$ is called *H-cleft* if there exists a right H-comodule map $\gamma : H \to A$ which is (convolution) invertible;

(3) A right H-extension $B \subset A$ is called *right H-Galois* if the map $\beta : A \otimes_B A \to A \otimes H$ given by $\beta(a \otimes b) = (a \otimes 1)\rho(b)$ is bijective.

The next proposition is taken from [13, Theorem 8.2.4].

Proposition 2.1. *[13] Let $A \subset B$ be an H-extension. Then the following are equivalent:*
(1) $A \subset B$ is H-cleft;
(2) $A \subset B$ is H-Galois and has the normal basis property.

If A is an H-module algebra, then A may be considered as an H^*-comodule algebra. Let ${}_A\mathcal{M}^{H^*}$ be the category of relative Hopf modules (see [13, Definition 8.5.1]). When H is a finite dimensional Hopf algebra over k, we may identify ${}_A\mathcal{M}^{H^*}$ with the category Mod($A\#H^{**}$)=Mod($A\#H$).

In this paper we are interested in giving some properties of derived categories under smash product. For the background on derived categories, we refer to [14]. The following Rickard's Morita theorem on derived categories is useful for our discussion in the sequel.

Theorem 2.1. *[7] For two finite dimensional k-algebras A and B, the following are equivalent:*

(a) $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ are equivalent as triangulated categories;

(b) $\mathcal{K}^b(\mathcal{P}(A))$ and $\mathcal{K}^b(\mathcal{P}(B))$ are equivalent as triangulated categories;

(c) $B \cong \mathrm{End}_{\mathcal{D}^b(A)}(T_\bullet)$, where $T_\bullet$ is a complex in $\mathcal{K}^b(\mathcal{P}(A))$ satisfying

(1) $T_\bullet$ is self-orthogonal in $\mathcal{K}^b(\mathcal{P}(A))$: $\mathrm{Hom}_{\mathcal{K}^b(\mathcal{P}(A))}(T_\bullet, T_\bullet[i]) = 0$ for all $i \neq 0$,

(2) add $(T_\bullet)$ generates $\mathcal{K}^b(\mathcal{P}(A))$ as a triangulated category.

Remark 2.1. *Rickard [5] also showed that (b) can be replaced by the following condition:*
For each non-zero object $X_\bullet$ of $\mathcal{K}^-(Proj\text{-}A)$, there is some i such that $\mathrm{Hom}_{\mathcal{K}(A)}(T_\bullet, X_\bullet[i]) \neq 0$.

If two finite dimensional k-algebras A and B satisfy one of the equivalent conditions in Theorem 2.1, then A and B are said to be derived equivalent. A complex $T_\bullet$ in $\mathcal{K}^b(\mathcal{P}(A))$ satisfying the conditions (1) and (2) in Theorem 2.1 is called a tilting complex over A. Given a derived equivalence F between A and B, there is a unique (up to isomorphism) tilting complex $T_\bullet$ over A such that $F(T_\bullet) = B$. This complex $T_\bullet$ is called a tilting complex associated to F.

Here we recall some homological results which we need in the sequel.

Lemma 2.1. *Let A and B be two finite dimensional k-algebras.*

(1) In the situation $(P_{A}, {}_B U_{A}, {}_B Z)$, if P_A is finitely generated and projective, then

$$P \otimes_A \mathrm{Hom}_B(U, Z) \cong \mathrm{Hom}_B(\mathrm{Hom}_A(P, U), Z).$$

(2) In the situation $(P_A, X_{B,A} Y_B)$, if P_A is finitely generated projective, or if X_B is finitely generated projective, then

$$P \otimes_A \mathrm{Hom}_B(X_{B,A} Y_B) \cong \mathrm{Hom}_B(X_B, P \otimes_A Y_B).$$

Dually, in the situation $({}_AP,{}_B X,{}_B Y_A)$, if ${}_AP$ is finitely generated projective, or if ${}_BX$ is finitely generated projective, then

$$\mathrm{Hom}_B({}_BX,{}_B Y_A) \otimes_A P \cong \mathrm{Hom}_B({}_BX,{}_B Y \otimes_A P).$$

3. Derived equivalence for smash product algebra

In this section, assume that $B \subset A$ is a right H-Galois extension. Now we consider the following question: how to construct a derived equivalence between two smash product algebras from a derived equivalence between two left H-module algebras. An efficient way is to construct tilting complex. In fact, a similar problem has been considered by many authors ([1–6, 15, 16]). But nobody considered to relate this problem with Hopf algebra. Now we give some results from this viewpoint.

Firstly, following [17], we recall the definition of H-stable module.

Definition 3.1. [17] Let A/B be an H-Galois extension, and M a right B-module. M is called H-stable or stable if there is a right B-linear and right H-colinear isomorphism $M \otimes_B A \cong M \otimes H$, where the module and comodule structure on $M \otimes H$ are defined by $(m \otimes h) \cdot b = m \cdot b \otimes h$ and $id \otimes \Delta$, respectively. If $H = kG$ is a group algebra, then H-stable modules are called G-invariant in [18].

Example 3.1. (1) Note that B is H-stable if and only if $B \subset A$ satisfies the normal basis property, see [13]. Thus following Proposition 2.1, we see that if $B \subset A$ is a right H-cleft extension, then B is H-stable. This means that each projective B-module is H-stable.

(2) Let G be a group and $G' \trianglelefteq G$ be a normal subgroup. Consider the $k[G/G']$-Galois extension $B = k[G'] \subset A = k[G]$, $\Delta_A(g) = g \otimes \overline{g}$ for all $g \in G$, where $\overline{g} \in G/G'$. In this case, any B-module M is stable if and only if for all $g \in G$, M is B-isomorphic to the twisted B-module ${}_gM$.

(3) Any left A-module N is H-stable over B (by restriction).

Inspired by Definition 3.1, we introduce the following definition.

Definition 3.2. Let A/B be an H-Galois extension, and $X_\bullet$ a right B-complex. $X_\bullet$ is called H-stable or stable if there is a right B-linear and right H-colinear isomorphism $X_\bullet \otimes_B A \cong X_\bullet \otimes H$.

Now a tilting complex $T_\bullet$ over B is called an H-stable tilting complex if there is a right B-linear and right H-colinear isomorphism $X_\bullet \otimes_B A \cong X_\bullet \otimes H$.

Clearly, if $B \subset A$ is a right H-cleft extension, then each tilting B-complex is an H-stable tilting complex.

Note that, if an $A\#H$-complex $T_\bullet$ is a tilting complex over A (by restriction), then $T_\bullet$ is an H-stable tilting complex over A. In this case, we say $T_\bullet$ to be an H-tilting complex over A.

Lemma 3.1. *Let A/B be an H-Galois extension. If $T_\bullet$ is a H-stable tilting complex, then $T_\bullet \otimes_B A$ is a tilting complex over A.*

Proof. Set

$$T_\bullet = \cdots \longrightarrow T_{i+1} \xrightarrow{d_{i+1}} T_i \xrightarrow{d_i} T_{i-1} \longrightarrow \cdots$$

and $T_\bullet^{'} = T_\bullet \otimes_B A$. By the definition of tilting complex, T_i is a finitely generated projective B-module for each $i \in \mathbb{Z}$. Hence $T_\bullet^{'}$ is in $\mathcal{K}^b(\mathcal{P}(A))$.

Since $\mathrm{Hom}_A(T_\bullet \otimes_B A, T_\bullet \otimes_B A) = \mathrm{Hom}_B(T_\bullet, T_\bullet \otimes_B A)$, we have the following isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}^b(A)}(T_\bullet^{'}, T_\bullet^{'}[i]) &= \mathrm{Hom}_{\mathcal{D}^b(A)}(T_\bullet \otimes_B A, T_\bullet \otimes_B A[i]) \\
&\cong H^i\mathrm{RHom}_A(T_\bullet \otimes_B A, T_\bullet \otimes_B A) \\
&\cong H^i\mathrm{RHom}_B(T_\bullet, T_\bullet \otimes_B A) \\
&\cong H^i\mathrm{RHom}_B(T_\bullet, T_\bullet \otimes H) \\
&\cong H^i\mathrm{RHom}_B(T_\bullet, T_\bullet) \otimes H \\
&\cong \mathrm{Hom}_{\mathcal{D}^b(B)}(T_\bullet, T_\bullet[i]) \otimes H \\
&\cong 0,
\end{aligned}$$

for any $i \neq 0$, where RHom_A means the derived functor of Hom_A in $\mathcal{D}(A)$ and the third isomorphism holds since $T_\bullet$ is a H-stable tilting complex over B.

Let X be an object of $K^-(\text{Proj-A})$ such that $\mathrm{Hom}_{K(A)}(T_\bullet \otimes_B A, X[i]) = 0$ for all i. Then we have the following isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}^b(B)}(T_\bullet, X \otimes_A A_B[i]) &= H^i\mathrm{RHom_B(T_\bullet, X \otimes_A A_B)} \\
&\cong H^i\mathrm{RHom}_A(T_\bullet \otimes_B A, X) \\
&\cong \mathrm{Hom}_{\mathcal{D}^b(A)}(T_\bullet \otimes_B A, X[i]) \\
&\cong 0
\end{aligned}$$

for all i. Since $T_\bullet$ is a tilting complex over B, we have $X \otimes_A A_B \cong 0$ in $\mathcal{D}^b(B)$, that is, $H^i(X_B) = H^i(X_A) = 0$ for all i. Thus $X \cong 0$ in $\mathcal{D}^b(A)$. This means that $T_\bullet^{'} = T_\bullet \otimes_B A$ is a tilting complex over A.

Remark 3.1. *In [4], Miyachi showed that (in our terminology), if $B \to A$ is a ring homomorphism and $T_\bullet$ is a tilting complex over B with $\mathrm{Hom}_{\mathcal{D}^b(B)}(T_\bullet, T_\bullet \otimes_B A_B[i]) = 0$ for all $i \neq 0$, then $T_\bullet \otimes_B A$ is a tilting complex over A. Here we can view H-stable tilting complex as a concrete example that satisfies the assumption given by Miyachi above.*

Let A/B be an H-Galois extension with the canonical map $\beta : A \otimes_B A \longrightarrow A \otimes H$. For any $h \in H$, we write $\beta^{-1}(1 \otimes h) = \sum X_i^h \otimes Y_i^h$. Following [19, Lemma 2.1] and [17, Remark 3.4], we recall the properties of the elements X_i^h and Y_i^h.

Lemma 3.2. *[17] Following the notations above, let $a \in A, b \in B$ and $h, l \in H$. Then the following statements hold:*

(1) $\sum bX_i^h \otimes Y_i^h = \sum X_i^h \otimes Y_i^h b$;
(2) $\sum a_0 X_i^{a_1} \otimes Y_i^{a_1} = 1 \otimes a$;
(3) $\sum X_i^h Y_i^h = \varepsilon(h) 1_A$;
(4) $\sum X_i^h \otimes Y_{i,0}^h \otimes Y_{i,1}^h = \sum X_i^{h_1} \otimes Y_i^{h_1} \otimes h_2$;
(5) $\sum X_{i,0}^h \otimes Y_i^h \otimes X_{i,1}^h = \sum X_i^{h_2} \otimes Y_i^{h_2} \otimes S(h_1)$;

(6) $\sum X_i^{hl} \otimes Y_i^{hl} = \sum X_i^l X_j^h \otimes Y_j^h Y_i^l$;
(7) $\sum X_i^{h_1} \otimes Y_i^{h_1} X_j^{h_2} \otimes Y_j^{h_2} = \sum X_i^h \otimes 1 \otimes Y_i^h$;
(8) $\sum X_{i,0}^h Y_{i,0}^h \otimes X_{i,1}^h \otimes Y_{i,1}^h = \sum 1 \otimes S(h_1) \otimes h_2$.

For any right A-modules X and Y, there is a natural left H-module action on $\mathrm{Hom}_B(X, Y) : (h \cdot f)(x) = \sum f(xX_i^{S(h)})Y_i^{S(h)}$, where $h \in H$, $x \in X$ and $f \in \mathrm{Hom}_B(X, Y)$. Moreover, if $X_\bullet$ and $Y_\bullet$ are complexes of right A-modules, then $\mathrm{Hom}_B(X_\bullet, Y_\bullet)$ is also a complex of left H-module.

It is known that for any right A-modules M, there exists an isomorphism of right H-comodules and right B-modules:

$$M \otimes_B A \to M \otimes H, \ \ m \otimes a \mapsto \sum ma_0 \otimes a_1, \tag{3.1}$$

whose inverse morphism is given by

$$M \otimes H \to M \otimes_B A, \ \ m \otimes h \mapsto \sum mX_i^h \otimes Y_i^h. \tag{3.2}$$

Similarly, these isomorphisms can be generalized to complexes.

Now, suppose that $X_\bullet$ is an A-complex. If $X_\bullet$ is a tilting complex over B (by restriction), then $X_\bullet$ is an H-stable tilting complex over B. So we have the following result.

Theorem 3.1. *Let A/B be an H-Galois extension and $T_\bullet$ an A-complex. If $T_\bullet$ is tilting complex over B (by restriction) with $C = \mathrm{Hom}_{\mathcal{D}^b(B)}(T_\bullet, T_\bullet)$, then $T_\bullet \otimes_B A$ is a tilting complex over A and* $\mathrm{End}_{\mathcal{D}^b(\mathrm{A})}(\mathrm{T}_\bullet \otimes_\mathrm{B} \mathrm{A}) \cong \mathrm{C\#H}$ *as algebras. Therefore we have the derived equivalence between A and $C\#H$.*

Proof. By Lemma 3.1, $T_\bullet \otimes_B A$ is a tilting complex over A. Thus it remains only to show that $\mathrm{End}_{\mathcal{D}^b(\mathrm{A})}(\mathrm{T}_\bullet \otimes_\mathrm{B} \mathrm{A}) \cong \mathrm{C\#H}$ as algebras. Consider the following isomorphism:

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{D}^b(\mathrm{B})}(\mathrm{T}_\bullet, \mathrm{T}_\bullet) \otimes \mathrm{H} & \cong & \mathrm{Hom}_{\mathcal{D}^b(\mathrm{B})}(\mathrm{T}_\bullet, \mathrm{T}_\bullet \otimes \mathrm{H}) \\ & \cong & \mathrm{Hom}_{\mathcal{D}^b(\mathrm{B})}(\mathrm{T}_\bullet, \mathrm{T}_\bullet \otimes_\mathrm{B} \mathrm{A}) \\ & \cong & \mathrm{Hom}_{\mathcal{D}^b(\mathrm{A})}(\mathrm{T}_\bullet \otimes_\mathrm{B} \mathrm{A}, \mathrm{T}_\bullet \otimes_\mathrm{B} \mathrm{A}). \end{array} \tag{3.3}$$

Thus $\mathrm{End}_{\mathcal{D}^b(\mathrm{A})}(\mathrm{T}_\bullet \otimes_\mathrm{B} \mathrm{A}) \cong \mathrm{C\#H}$ as vector spaces.

From the above isomorphism, we also obtain an isomorphism of complexes

$$\varphi : \mathrm{Hom}_{\mathcal{D}^b(B)}(T_\bullet, T_\bullet) \otimes H \longrightarrow \mathrm{Hom}_{\mathcal{D}^b(A)}(T_\bullet \otimes_B A, T_\bullet \otimes_B A),$$

such that

$$\varphi(f \otimes h)(t \otimes a) = \sum f(tX_i^h) \otimes Y_i^h a,$$

where $f \in \mathrm{Hom}_{\mathcal{D}^b(B)}(T_\bullet, T_\bullet), h \in H, a \in A$ and $t \in T_\bullet$.

Note that $\beta : A \otimes_B A \to A \otimes H$ is bijective, which induces an isomorphism $(\beta \otimes id_H)(id_A \otimes \beta)$: $A \otimes_B A \otimes_B A \to A \otimes H \otimes H$. One easily show that

$$(\beta \otimes id_H)(id_A \otimes \beta)(\sum X_j^{h_2} X_k^{S(h_1)} \otimes Y_k^{S(h_1)} \otimes Y_j^{h_2}) = (\beta \otimes id_H)(id_A \otimes \beta)(\sum 1 \otimes X_i^h \otimes Y_i^h).$$

Thus $\sum X_j^{h_2} X_k^{S(h_1)} \otimes Y_k^{S(h_1)} \otimes Y_j^{h_2} = \sum 1 \otimes X_i^h \otimes Y_i^h$.

According to Lemma 3.2, we have the following equalities.

$$\varphi((f\#h)(g\#l))(t \otimes a) \quad = \quad \varphi(f(h_1 \cdot g)\#h_2 l)(t \otimes a)$$

$$
\begin{aligned}
&= \sum f(h_1 \cdot g)(tX_i^{h_2 l}) \otimes Y_i^{h_2 l} a \\
&= \sum f(g(tX_i^{h_2 l} X_j^{S(h_1)}) Y_j^{S(h_1)}) \otimes Y_i^{h_2 l} a \\
&= \sum f(g(tX_i^{l} X_j^{h_2} X_k^{S(h_1)}) Y_k^{S(h_1)}) \otimes Y_j^{h_2} Y_i^{l} a \\
&= \sum f(g(tX_i^{l}) X_j^{h}) \otimes Y_j^{h} Y_i^{l} a \\
&= \varphi(f\#h)\varphi(g\#l))(t \otimes a),
\end{aligned}
$$

where $f, g \in \mathrm{Hom}_{\mathcal{D}^b(B)}(T_\bullet, T_\bullet)$, $h, l \in H$, $a \in A$ and $t \in T_\bullet$. Thus φ is an isomorphism of algebras.

By Theorem 2.1, we get the desired result.

Let A_1, A_2 and A_3 be finite dimensional k-algebras. Following [7] we know that $\mathcal{D}^b(A_1 \otimes A_3) \simeq \mathcal{D}^b(A_2 \otimes A_3)$ if $\mathcal{D}^b(A_1) \simeq \mathcal{D}^b(A_2)$. Inspired by this result, it is natural to consider the following question: Are $A\#H$ and $C\#H$ derived equivalent when two H-module algebras A and C are derived equivalent as k-algebras? The following corollary is the answer to this question.

Corollary 3.1. *Let H be a finite dimensional Hopf algebra, and A a left H-module algebra. If there exists an H-tilting complex $T_\bullet$ over A with $C = \mathrm{Hom}_{\mathcal{D}^b(A)}(T_\bullet, T_\bullet)$, then $\mathcal{D}^b(A\#H)$ and $\mathcal{D}^b(C\#H)$ are equivalent.*

Proof. Set $T'_\bullet = T_\bullet \otimes_A (A\#H)$. By Lemma 3.1, $T_\bullet \otimes_A (A\#H)$ is a tilting complex over $A\#H$. Thus $\mathcal{D}^b(A\#H)$ and $\mathcal{D}^b(C\#H)$ are derived equivalent.

Let A be a right H-comodule algebra, and $\mathcal{M}_A^H$ the category of right relative Hopf modules. Then we have a pair of adjoint functors $(F = - \otimes_{A^{coH}} A, G = (\ \ )^{coH})$ between the categories $\mathrm{Mod}A^{coH}$ and $\mathcal{M}_A^H$. If the extension $A^{coH} \subset A$ is right faithful flat H-Galois, then $\mathrm{Mod}A^{coH}$ and $\mathcal{M}_A^H$ are equivalent as abelian categories.

Corollary 3.2. *Let H be a finite dimensional Hopf algebra, and A a left H-module algebra. Assume that there exists an H-tilting complex $T_\bullet$ over A with $C = \mathrm{Hom}_{\mathcal{D}^b(A)}(T_\bullet, T_\bullet)$. If A and C are right faithful flat H^*-Galois extensions of A^H and C^H respectively, then $\mathcal{D}^b(A^H)$ and $\mathcal{D}^b(C^H)$ are equivalent.*

Proof. From Corollary 3.1 we have that $\mathcal{D}^b(A\#H) \simeq \mathcal{D}^b(C\#H)$.

Since H is a finite dimensional Hopf algebra over a field k, it is not difficult to see that $\mathrm{Mod}(A\#H) \simeq \mathcal{M}_A^{H^*}$ and $\mathrm{Mod}(C\#H) \simeq \mathcal{M}_C^{H^*}$. Therefore we have $\mathcal{D}^b(\mathcal{M}_A^{H^*}) \simeq \mathcal{D}^b(\mathcal{M}_C^{H^*})$.

According to [13], by the assumption that A and C are right faithful flat H^*-Galois extensions of A^H and C^H respectively, we have the following equivalences:

$$\mathcal{M}_A^{H^*} \simeq \mathrm{Mod}A^H \quad \text{and} \quad \mathcal{M}_C^{H^*} \simeq \mathrm{Mod}C^H.$$

Thus $\mathcal{D}^b(A^H) \simeq \mathcal{D}^b(C^H)$.

Following [4] we recall the definition of cotilting complex. Denote by $D = \mathrm{Hom}_k(-, k)$ the standard duality from $\mathcal{D}^b(\mathrm{modA})$ to $\mathcal{D}^b(\text{A-mod})$. A complex $T_\bullet$ is called a cotilting complex if the following statements hold:

(1) $T_\bullet \in K^b(\mathcal{I}_{\mathcal{A}})$, where $\mathcal{I}_{\mathcal{A}}$ is the category of finitely generated injective right A-modules;

(2) $\mathrm{Hom}_{\mathcal{D}^b(A)}(T_\bullet, T_\bullet[i]) = 0$ for all $i \neq 0$;

(3) $D(A) \in \mathcal{I}(\text{add } \mathcal{T}_\bullet)$, where $\mathcal{I}(\text{add } \mathcal{T}_\bullet)$ is the triangulated subcategory of $K^b(\mathcal{I}_{\mathcal{A}})$ generated by objects in add $T_\bullet$.

Recall that if $X_\bullet$ belongs to $K^b(\mathcal{P}(\mathcal{A}))$, then there exists an Auslander-Reiten translation $\tau_A(X_\bullet)$ which is isomorphic to $\nu(X_\bullet)[-1]$, where $\nu_A = - \otimes_A^L D(A)$, and then there exists an Auslander-Reiten triangle $\tau_A X_\bullet \to Y_\bullet \to X_\bullet \to \tau_A X_\bullet[1]$ in $\mathcal{D}^b(A)$, see [20]. Then $\tau_A(T_\bullet)$ is a cotilting complex over A if $T_\bullet$ is a tilting complex over A.

Proposition 3.1. *Let A/B be an H-Galois extension. If $T_\bullet$ is an H-stable tilting complex over B, then* $\mathrm{RHom}_B({}_AA_B, \tau_B(T_\bullet))$ *is a cotilting complex over A.*

Proof. Consider the following isomorphisms:

$$\begin{aligned} \tau_A(T_\bullet \otimes_B^L A) &\cong (T_\bullet \otimes_B A) \otimes_A D(A)[-1] \\ &\cong T_\bullet \otimes_B D(A)[-1] \\ &\cong D\mathrm{Hom}_B(T_\bullet, A_B)[-1] && \text{by Proposition 2.1(1)} \\ &\cong D({}_AA \otimes_B \mathrm{Hom}_B(T_\bullet, B))[-1] && \text{by Proposition 2.1(2)} \\ &\cong \mathrm{Hom}_A({}_AA_B, D\mathrm{Hom}_B(T_\bullet, B))[-1] \\ &\cong \mathrm{RHom}_A({}_AA_B, \tau_B T_\bullet). \end{aligned}$$

By Lemma 3.1, we get that $\mathrm{RHom}_B({}_AA_B, \tau_B T_\bullet)$ is a cotilting complex over A.

4. H-Frobenius extension

In this section, suppose that $B \subset A$ is a right H-Galois extension. The endomorphism ring extension of an A-module M has been studied by Oystaeyen and Zhang in [19]. It was proved that there is an isomorphism of algebras $\mathrm{End}_A(M \otimes_B A) = \mathrm{End}_B(M)\#H$. The necessary and sufficient conditions for an endomorphism ring extension to be a H-Galois extension were also induced in [19]. In this section we generalize this idea to derived endomorphism ring of H-stable tilting complex. Furthermore, we show that $0 \to \mathrm{End}_{\mathcal{D}^b(B)}(T_\bullet) \to \mathrm{End}_{\mathcal{D}^b(A)}(T_\bullet \otimes_B A)$ is an H-Frobenius extension if A/B is an H-Frobenius extension.

Recall that A/B is H-Frobenius if A is a finitely generated projective right B-module and $A \cong \mathrm{Hom}_B(A, B)$ as B-A-bimodules, see [21]. If A/B is H-Galois as well as H-Frobenius, we say the extension A/B to be H-Galois Frobenius. Let $T_\bullet$ be a tilting complex over A and $X_\bullet \in \mathcal{K}^b(\mathcal{P}(\mathcal{A}))$. Recall that $X_\bullet$ is isomorphic to a direct summand of a finite direct sum of $T_\bullet$ if $\mathrm{Hom}_{\mathcal{D}^b(A)}(T_\bullet, X_\bullet[i]) = \mathrm{Hom}_{\mathcal{D}^b(A)}(X_\bullet[i],\ T_\bullet) = 0$ for all $i \neq 0$, see [4, Lemma 3.2].

Theorem 4.1. *Let A/B be an H-Galois extension such that $0 \to B \to A \to C \to 0$ is an exact sequence of B-bimodule. Assume that an A-complex $T_\bullet$ is a tilting complex over B (by restriction). Let $B' = \mathrm{End}_{\mathcal{D}^b(B)}(T_\bullet)$, $A' = \mathrm{End}_{\mathcal{D}^b(A)}(T_\bullet \otimes_B A)$ and $C' = \mathrm{Hom}_{\mathcal{D}^b(B)}(T_\bullet, T_\bullet \otimes_B C)$. Then A' is an H-Galois extension of the algebra B' such that $0 \to B' \to A' \to C' \to 0$ is an exact sequence of B'-bimodule. Moreover, if A/B is H-Galois Frobenius, then A'/B' is also H-Galois Frobenius.*

Proof. Since $T_\bullet$ is a complex of projective B-modules, from the assumption we have the following short exact sequence of B-complex

$$0 \to T_\bullet \to T_\bullet \otimes_B A_B \to T_\bullet \otimes_B C_B \to 0.$$

Clearly, we also have the following exact sequence

$$0 \to \mathrm{Hom}_B(T_\bullet, T_\bullet) \to \mathrm{Hom}_B(T_\bullet, T_\bullet \otimes_B A_B) \to \mathrm{Hom}_B(T_\bullet, T_\bullet \otimes_B C_B) \to 0.$$

Since $\mathrm{Hom}_{\mathcal{D}^b(B)}(T_\bullet, T_\bullet[i]) = \mathrm{Hom}_{\mathcal{D}^b(B)}(T_\bullet, T_\bullet \otimes_B A[i]) = 0$, we obtain that $\mathrm{Hom}_{\mathcal{D}^b(B)}(T_\bullet, T_\bullet \otimes_B C_B[i]) = 0$ for all $i \neq 0$. This implies that

$$0 \to \mathrm{Hom}_{\mathcal{D}^b(B)}(T_\bullet, T_\bullet) \to \mathrm{Hom}_{\mathcal{D}^b(B)}(T_\bullet, T_\bullet \otimes_B A_B) \to \mathrm{Hom}_{\mathcal{D}^b(B)}(T_\bullet, T_\bullet \otimes_B C_B) \to 0$$

is exact, which is also a sequence of B-bimodules.

Now, it remains to show that A' is a Frobenius extension of B' if A/B is an H-Frobenius extension. Clearly $T_\bullet \otimes_B A_B$ is a complex of projective B-module since $T_\bullet$ is a tilting complex over B and A/B is an H-Frobenius extension. And we obtain that ${}_BA_A \cong \mathrm{Hom}_B({}_AA_B, {}_BB_B)$ as B-A-bimodules. Consider following isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}^b(B)}(T_\bullet \otimes_B A_B, T_\bullet[i]) &\cong H^i\mathrm{RHom}_B(T_\bullet \otimes_B A_B, T_\bullet) \\
&\cong H^i\mathrm{RHom}_B(T_\bullet, \mathrm{Hom}_B({}_BA_B, T_\bullet)) \\
&\cong H^i\mathrm{RHom}_B(T_\bullet, T_\bullet \otimes_B A_B) \\
&\cong \mathrm{Hom}_{\mathcal{D}^b(B)}(T_\bullet, T_\bullet \otimes_B A_B[i]) \\
&\cong 0 \qquad \text{for all} \ \ i \neq 0.
\end{aligned}$$

Similarly, we can prove $\mathrm{Hom}_{\mathcal{D}^b(B)}(T_\bullet[i], T_\bullet \otimes_B A_B) = 0$, thus $T_\bullet \otimes_B A_B$ is isomorphic to a direct summand of a finite direct sum of $T_\bullet$ according to [4, Lemma 3.2], as mentioned above. Since ${}_{A'}A'_{B'}$ is isomorphic to ${}_{A'}\mathrm{Hom}_{\mathcal{D}^b(B)}(T_\bullet, T_\bullet \otimes_B A_B)_{B'}$ as A'-B'-bimodules, $A'_{B'}$ is a finitely generated projective B'-module. Now we consider the following isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{B'}({}_{A'}A'_{B'}, {}_{B'}B'_{B'}) &\cong \mathrm{Hom}_{B'}({}_{A'}\mathrm{Hom}_{\mathcal{D}^b(B)}(T_\bullet, T_\bullet \otimes_B A_B)_{B'}, {}_{B'}\mathrm{Hom}_{\mathcal{D}^b(A)}(T_\bullet, T_\bullet)_{B'}) \\
&\cong \mathrm{Hom}_{\mathcal{D}^b(B)}(T_\bullet \otimes_B A_B, T_\bullet) \\
&\cong \mathrm{Hom}_{\mathcal{D}^b(A)}(T_\bullet \otimes_B A, T_\bullet \otimes_B A) \\
&\cong {}_{B'}A'_{A'}.
\end{aligned}$$

This completes the proof.

5. Recollement and H-Galois extension

In this section, for any perfect recollement of derived categories of H-module algebras, we give a way to construct a perfect recollement of derived categories of their smash product algebras. Firstly, following [10] we recall the definition of recollements of triangulated categories.

Definition 5.1. (Beilinson-Bernstein-Deligne [10]) Let $\mathcal{T}_1$, $\mathcal{T}$ and $\mathcal{T}_2$ be triangulated categories. A recollement of $\mathcal{T}$ relative to $\mathcal{T}_1$ and $\mathcal{T}_2$ is given by

$$\begin{tikzcd}
\mathcal{T}_1 \arrow[r, "i_* = i_!"] & \mathcal{T} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^! = j^*"] & \mathcal{T}_2 \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd}$$

and denoted by 9-tuple $(\mathcal{T}_1, \mathcal{T}, \mathcal{T}_2, i^*, i_* = i_!, i^!, j_!, j^! = j^*, j_*)$ such that

(R1) $(i^*, i_*), (i_!, i^!), (j_!, j^!)$ and (j^*, j_*) are adjoint pairs of triangulated functors;

(R2) i_*, $j_!$ and j_* are full embeddings;

(R3) $j^! i_* = 0$ (and thus also $i^! j_* = 0$ and $i^* j_! = 0$);

(R4) for each $X \in \mathcal{T}$, there are triangles

$$j_! j^! X \to X \to i_* i^* X \to$$

$$i_! i^! X \to X \to j_* j^* X \to .$$

Let X be an object in $\mathcal{D}(A)$. Define $X^{\perp} = \{Y \in \mathcal{D}(A) | \mathrm{Hom}_{\mathcal{D}(A)}(X, Y[n]) = 0, \forall n \in \mathbb{Z}\}$, and TriaX to be the smallest full triangulated subcategory of $\mathcal{D}(A)$ which contains X and is closed under small coproducts. We say X to be exceptional if $\mathrm{Hom}_{\mathcal{D}(A)}(X, X[n]) = 0$ for all $n \neq 0$. We say X to be compact if the functor $\mathrm{Hom}_{\mathcal{D}(A)}(X, -)$ preserves small coproduct, or equivalently, X to be perfect, if X is isomorphic to an object in $\mathcal{K}^b(\mathcal{P}(A))$.

A recollement $(\mathcal{D}(A_1), \mathcal{D}(A), \mathcal{D}(A_2), i^*, i_* = i_!, i^!, j_!, j^! = j^*, j_*)$ is said to be perfect if $i_*(A_1)$ is perfect, where A_1, A and A_2 are three k-algebras.

Definition 5.2. *[22] Let $\mathcal{T}_1$, $\mathcal{T}$ and $\mathcal{T}_2$ be triangulated categories, and n a positive integer. An n-***recollement** *of $\mathcal{T}$ relative to $\mathcal{T}_1$ and $\mathcal{T}_2$ is given by $n+2$ layers of triangle functors*

$$\begin{tikzcd}
\mathcal{T}_1 \arrow[r, leftarrow, shift left=4.5] \arrow[r, shift left=1.5] \arrow[r, leftarrow, shift right=1.5] \arrow[r, shift right=4.5] & \mathcal{T} \arrow[r, leftarrow, shift left=4.5] \arrow[r, shift left=1.5] \arrow[r, leftarrow, shift right=1.5] \arrow[r, shift right=4.5] & \mathcal{T}_2 \\
& \vdots \quad\quad \vdots &
\end{tikzcd}$$

such that every consecutive three layers form a recollement.

According to [22], if we have a perfect recollement $(\mathcal{D}(A_1), \mathcal{D}(A), \mathcal{D}(A_2), i_1, i_2, i_3, j_1, j_2, j_3)$, then it can be extended one step downwards by choosing the right adjoint functors of i_3, j_3, which is a 2-recollement; on the other hand, if we have a n-recollement, the first three layers form a perfect recollement. Thus a perfect recollement is equivalent to a 2-recollement.

There is an important criterion for the derived category of an algebra to admit a perfect recollement in [12].

Theorem 5.1. *[12] Let A_1, A and A_2 be algebras. Then $\mathcal{D}(A)$ admits a perfect recollement relative to $\mathcal{D}(A_1)$ and $\mathcal{D}(A_2)$ if and only if there are objects X_i, $i = 1, 2$, in $\mathcal{D}(A)$ such that*

(1) $\mathrm{End}_{\mathcal{D}(A)}(X_i) = A_i$ *as algebras,* $\forall\ i = 1, 2$*;*

(2) X_i is exceptional and perfect, $\forall\ i = 1, 2$*;*

(3) $X_1 \in X_2^{\perp}$;

(4) $X_1^{\perp} \cap X_2^{\perp} = \{0\}$.

Following [12] we see that $\mathcal{D}(C \otimes A)$ admits a perfect recollement relative to $\mathcal{D}(C \otimes A_1)$ and $\mathcal{D}(C \otimes A_2)$ if $\mathcal{D}(A)$ admits a perfect recollement relative to $\mathcal{D}(A_1)$ and $\mathcal{D}(A_2)$, where C is an algebra. Now we generalize this idea to smash product algebras.

Theorem 5.2. *Let A/B be an H-Galois extension, B_1 and B_2 two k-algebras. Suppose that there exist A-complexes X_i, $i = 1,2$, in $\mathcal{D}(A)$ such that X_1 and X_2 as B-complex induce the following perfect recollement*

$$\begin{tikzcd} \mathcal{D}(B_1) \arrow[r] & \mathcal{D}(B) \arrow[l, shift right=3ex] \arrow[l, shift left=3ex] \arrow[r] & \mathcal{D}(B_2). \arrow[l, shift right=3ex] \arrow[l, shift left=3ex] \end{tikzcd}$$

Then $\mathcal{D}(A)$ admits a perfect recollement relative to $\mathcal{D}(B_1\#H)$ and $\mathcal{D}(B_2\#H)$.

Proof. Set $X_1' = X_1 \otimes_B A$ and $X_2' = X_2 \otimes_B A$. Clearly, by Theorem 5.1, we shall show that X_i', $i = 1,2$ satisfy the conditions in Theorem 5.1. Since A/B is an H-Galois extension and $X_i (i = 1,2)$ is perfect, X_i' is perfect for $i = 1,2$.

Consider the following isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}(A)}(X_i', X_i'[n]) &= \mathrm{Hom}_{\mathcal{D}(A)}(X_i \otimes_B A, X_i \otimes_B A[n]) \\
&\cong H^n\mathrm{RHom}_A(X_i \otimes_B A, X_i \otimes_B A) \\
&\cong H^n\mathrm{RHom}_B(X_i, X_i \otimes_B A) \\
&\cong H^n\mathrm{RHom}_B(X_i, X_i \otimes H) \\
&\cong H^n\mathrm{RHom}_B(X_i, X_i) \otimes H \\
&\cong \mathrm{Hom}_{\mathcal{D}(B)}(X_i, X_i[n]) \otimes H \\
&\cong \begin{cases} 0, & n \neq 0 \\ \mathrm{Hom}_{\mathcal{D}(B)}(X_i, X_i) \otimes H, & n = 0 \end{cases}
\end{aligned}$$

Thus X_i' is exceptional for $i = 1,2$. As the same as the proof in Theorem 3.1, we can show that $\mathrm{End}_{\mathcal{D}(A)}(X_i \otimes_B A) \cong B_i\#H$ as algebras for $i = 1,2$.

Since $X_1 \in X_2^{\perp}$ and X_2 is perfect, we obtain the following isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}(A)}(X_2', X_1'[n]) &= \mathrm{Hom}_{\mathcal{D}(A)}(X_2 \otimes_B A, X_1 \otimes_B A[n]) \\
&\cong H^n\mathrm{RHom}_A(X_2 \otimes_B A, X_1 \otimes_B A) \\
&\cong H^n\mathrm{RHom}_B(X_2, X_1 \otimes_B A) \\
&\cong H^n\mathrm{RHom}_B(X_2, X_1 \otimes H) \\
&\cong H^n\mathrm{RHom}_B(X_2, X_1) \otimes H \\
&\cong \mathrm{Hom}_{\mathcal{D}(B)}(X_2, X_1[n]) \otimes H \\
&\cong 0, \quad \text{for all } \; n.
\end{aligned}$$

Hence $X_1' \in X_2'^{\perp}$. Now it remains to show that $X_1'^{\perp} \cap X_2'^{\perp} = \{0\}$. Let $X \in X_1'^{\perp} \cap X_2'^{\perp}$. Then we have the following isomorphisms:

$$\begin{aligned}
0 &= \mathrm{Hom}_{\mathcal{D}(A)}(X_i \otimes_B A, X[n]) \\
&\cong H^n\mathrm{RHom}_A(X_i \otimes_B A, X) \\
&\cong H^n\mathrm{RHom}_B(X_i, X) \\
&\cong \mathrm{Hom}_{\mathcal{D}(B)}(X_i, X[n]), \quad \text{for all } \; n \in \mathbb{Z} \; \text{ and } i = 1,2.
\end{aligned} \tag{5.1}$$

This implies that $X \cong 0$ in $\mathcal{D}(A)$. This completes the proof.

Suppose A-complex $T_\bullet$ is a tilting complex over B, then $T_\bullet$ and 0 are two exceptional and perfect objects satisfy the conditions in Theorem 5.1. Setting $B_1 = \mathrm{End}_{\mathcal{D}^b(B)}(T_\bullet), B_2 = 0$, according to Theorem 5.1, $\mathcal{D}^b(A)$ admits a perfect recollement relative to $\mathcal{D}^b(B_1\#H)$ and $\mathcal{D}^b(B_2\#H)$. And in this case,$\mathcal{D}^b(A)$ is equivalent to $\mathcal{D}^b(B_1\#H)$. Thus Theorem 5.2 is a generalization of Theorem 3.1.

Assume that B is a left H-module algebra. Since $B\#H$ is an H-Galois extension of B, we have the following corollary.

Corollary 5.1. *Let H be a finite dimensional Hopf algebra, B be a left H-module algebra and B_1, B_2 be two k-algebras. Suppose that there exist $B\#H$-complexes X_i, $i = 1, 2$, in $\mathcal{D}(A)$ such that X_1 and X_2 as B-complex induce the following perfect recollement*

$$\begin{tikzcd} \mathcal{D}(B_1) \arrow[r] & \mathcal{D}(B) \arrow[l, shift right=2ex] \arrow[l, shift left=2ex] \arrow[r] & \mathcal{D}(B_2). \arrow[l, shift right=2ex] \arrow[l, shift left=2ex] \end{tikzcd}$$

Then $\mathcal{D}(B\#H)$ admits a perfect recollement relative to $\mathcal{D}(B_1\#H)$ and $\mathcal{D}(B_2\#H)$.

Recall that an algebra A is said to be **smooth** if it has a finite Hochschild dimension, i.e., $\mathrm{pd}_{A^e}A < \infty$, see [23], or equivalently A is isomorphic to an object in $\mathcal{K}^b(\text{Proj-}A^e)$.

According to [12, Theorem 3], if $\mathcal{D}(B)$ admit a recollement relative to $\mathcal{D}(B_1)$ and $\mathcal{D}(B_2)$, Then B is smooth if and only if so are B_1 and B_2. Thus we have the following corollary.

Corollary 5.2. *Let A/B be an H-Galois extension, B_1 and B_2 two k-algebras. Suppose that there exist A-complexes X_i, $i = 1, 2$, in $\mathcal{D}(A)$ such that X_1 and X_2 as B-complex induce the following perfect recollement*

$$\begin{tikzcd} \mathcal{D}(B_1) \arrow[r] & \mathcal{D}(B) \arrow[l, shift right=2ex] \arrow[l, shift left=2ex] \arrow[r] & \mathcal{D}(B_2). \arrow[l, shift right=2ex] \arrow[l, shift left=2ex] \end{tikzcd}$$

Then A is smooth if and only if so are $B_1\#H$ and $B_2\#H$.

Definition 5.3. *[22] Let B, B_1 and B_2 be algebras. An n-recollement*

$$(\mathcal{D}(B_1), \mathcal{D}(B), \mathcal{D}(B_2), i_1, i_2, ..., i_{n+2}, j_1, j_2, ..., j_{n+2})$$

is said to be standard via defined by $Y \in \mathcal{D}(B^{\mathrm{op}} \otimes B_1)$ and $Y_2 \in \mathcal{D}(B_2^{\mathrm{op}} \otimes B)$ if $i_1 \cong - \otimes^L_B Y$, $j_1 \cong - \otimes^L_{B_2} Y_2$.

According to [22, Proposition 1, Remark 2], if $\mathcal{D}(B)$ admits an n-recollement relative to $\mathcal{D}(B_1)$ and $\mathcal{D}(B_2)$. then $\mathcal{D}(B)$ admits a standard n-recollement relative to $\mathcal{D}(B_1)$ and $\mathcal{D}(B_2)$, defined by $Y \in \mathcal{D}(B^{\mathrm{op}} \otimes B_1)$ and $Y_2 \in \mathcal{D}(B_2^{\mathrm{op}} \otimes B)$ as follows:

$$\begin{array}{lll}
i_1 \cong - \otimes^L_B Y, & j_1 \cong - \otimes^L_{B_2} Y_2, & \\
i_2 \cong - \otimes^L_{B_1} Y^{*B_1}, & j_2 \cong - \otimes^L_B Y_2^{*B}, & \\
i_3 \cong - \otimes^L_B Y^{*B_1*B}, & j_3 \cong - \otimes^L_{B_2} Y_2^{*B*B_2}, & \\
\vdots & \vdots & \\
i_{n+1} \cong - \otimes^L_{B_1} Y^{*B_1(*B*B_1)^{\frac{n-1}{2}}}, & j_{n+1} \cong - \otimes^L_B Y_2^{*B(*B_2*B)^{\frac{n-1}{2}}}, & \text{if } n \text{ is odd,} \\
i_{n+1} \cong - \otimes^L_B Y^{(*B_1*B)^{\frac{n}{2}}}, & j_{n+1} \cong - \otimes^L_{B_2} Y_2^{(*B*B_2)^{\frac{n}{2}}}, & \text{if } n \text{ is even,} \\
i_{n+2} \cong \mathrm{RHom}_B(Y^{*B_1(*B*B_1)^{\frac{n-1}{2}}}, -), & j_{n+2} \cong \mathrm{RHom}_{B_2}(Y_2^{*B(*B_2*B)^{\frac{n-1}{2}}}, -), & \text{if } n \text{ is odd,} \\
i_{n+2} \cong \mathrm{RHom}_{B_1}(Y^{(*B_1*B)^{\frac{n}{2}}}, -), & j_{n+2} \cong \mathrm{RHom}_B(Y_2^{(*B*B_2)^{\frac{n}{2}}}, -), & \text{if } n \text{ is even,}
\end{array}$$

where $X^{*B} := \mathrm{RHom_B(X, B)}$.

Finally, we generalize Theorem 5.2 to n-recollements.

Theorem 5.3. *Let A/B be an H-Galois extension, B_1 and B_2 two k-algebras. Suppose $\mathcal{D}(B)$ admits an n-recollement relative to $\mathcal{D}(B_1)$ and $\mathcal{D}(B_2)$ such that B-complexes X_i, $i = 1, 2$ induced by first four layers are in $\mathcal{D}(A)$. Then $\mathcal{D}(A)$ admits an n-recollement relative to $\mathcal{D}(B_1\#H)$ and $\mathcal{D}(B_2\#H)$.*

Proof. Since $\mathcal{D}(B)$ admits an n-recollement relative to $\mathcal{D}(B_1)$ and $\mathcal{D}(B_2)$, $\mathcal{D}(B)$ admits a standard n-recollement as above. Setting $Y_1 = Y^{*B_1} \in \mathcal{D}(B_1^{\mathrm{op}} \otimes B)$, $(Y_1)_B \cong X_1, (Y_2)_B \cong X_2$ in $\mathcal{D}(B)$ and $Y_i, i = 1, 2$ are also A-complexes according to [14].

Furthermore, we may assume $Y_1, Y_1^{*B}, ..., Y_1^{(*B*B_1)^s}$ as well as $Y_2, Y_2^{*B}, ..., Y_2^{(*B*B_2)^t}$ are exceptional and perfect for $s = [\frac{n}{2} - 1], t = [\frac{n+1}{2} - 1]$, and $Y_1^{(*B*B_1)^i} \in (Y_2^{(*B*B_2)^i})^\perp$, $(Y_1^{(*B*B_1)^i})^\perp \cap (Y_2^{(*B*B_2)^i})^\perp = \{0\}$ as well as $Y_2^{(*B*B_2)^j} \in (Y_1^{(*B*B_1)^{j-1}})^\perp$, $(Y_2^{(*B*B_2)^j})^\perp \cap (Y_1^{(*B*B_1)^{j-1}})^\perp = \{0\}$ for $0 \le i \le s$ and $1 \le j \le t$.

Now we set $Y_1' = Y_1 \otimes_B A$, $Y_2' = Y_2 \otimes_B A$. According to [14, 22], if we can prove via replacing Y_i by Y_i', B_i by $B_i\#H$, $i = 1, 2$ and B by A, the assumptions mentioned in the previous paragraph still hold, then we complete the proof.

Since $Y_i' \cong X_1 \otimes_{B_i} A$ in $\mathcal{D}(A)$, $i = 1, 2$, the statements hold for Y_i have been proved in Theorem 5.2. Thus we only consider the rest complexes.

Firstly, we prove $Y_1'^{*A}$ to be exceptional and perfect. In fact, similar to (3.3) in the proof of Theorem 3.1, we can show that

$$Y_1'^{*A} = \mathrm{RHom_A}(Y_1 \otimes_B A, B \otimes_B A) \cong \mathrm{RHom_B}(Y_1, B) \otimes H. \tag{5.2}$$

Besides,

$$\mathrm{RHom_B}(Y_1, B) \otimes H \cong \mathrm{RHom_B}(Y_1, B) \otimes_{B_1} B_1\#H = Y_1^{*B} \otimes_{B_1} B_1\#H \tag{5.3}$$

in $\mathcal{D}(B_1\#H)$, where the first isomorphism is given by (3.2) above Theorem 3.1. Since $B_1\#H/B_1$ is an H-Galois extension and $Y_1^{*B} \otimes_{B_1} B_1$ is perfect as B_1-complex, $Y_1^{*B} \otimes_{B_1} B_1\#H$ is also perfect as $B_1\#H$-complex. As the same as the proof in Theorem 5.2, we can prove

$$\mathrm{Hom}_{\mathcal{D}(B_1\#H)}(Y_1'^{*A}, Y_1'^{*A}[n]) \cong \mathrm{Hom}_{\mathcal{D}(B_1\#H)}(Y_1^{*B} \otimes_{B_1} B_1\#H, Y_1^{*B} \otimes_{B_1} B_1\#H[n]) = 0 \ \text{ if } n \neq 0,$$

since Y_1^{*B} is exceptional.

Similarly, we can prove inductively $Y_1'^{*A*B_1\#H}, ..., Y_1'^{(*A*B_1\#H)^s}$ as well as $Y_2'^{*A}, ..., Y_2'^{(*A*B_2\#H)^t}$ are exceptional and perfect for $s = [\frac{n}{2} - 1], t = [\frac{n+1}{2} - 1]$, and

$$Y_2'^{(*A*B_2\#H)^i} \cong Y_1^{(*B*B_1)^i} \otimes_B A, \ \text{ for } \ i = 1, ..., s;$$

$$Y_2'^{*A} \cong Y_2^{*B} \otimes_{B_2} B_2\#H, \quad Y_2'^{(*A*B_2\#H)^i} \cong Y_2^{(*B*B_2)^i} \otimes_B A, \ \text{ for } \ i = 1, ..., t,$$

whose proof is similar to (5.2) and (5.3). Finally, we prove the rest statements. As the same as the proof of Theorem 5.2, we have

$$\mathrm{Hom}_{\mathcal{D}(A)}(Y_1', Y_2'^{*A*B_2\#H}[n]) \cong \mathrm{Hom}_{\mathcal{D}(A)}(Y_1 \otimes_B A, Y_2^{*B*B_2} \otimes_B A[n]) = 0, \ \ n \in \mathbb{Z},$$

since $Y_2^{*B*B_2} \in Y_1^\perp$, which means $Y_2'^{*A*B_2\#H} \in Y_1'^\perp$. And we have

$$\mathrm{Hom}_{\mathcal{D}(A)}(Y_1', X[n]) \cong \mathrm{Hom}_{\mathcal{D}(B)}(Y_1, X[n]),$$

$$\mathrm{Hom}_{\mathcal{D}(A)}(Y_2'^{*A*B_2\#H}, X[n]) \cong \mathrm{Hom}_{\mathcal{D}(B)}(Y_2^{*B*B_2}, X[n]),$$

which can be proved as the same as (5.1) in the proof of Theorem 5.2 for any complex X. Thus $Y_1'^{\perp} \cap (Y_2'^{*A*B_2\#H})^{\perp} = Y_1^{\perp} \cap (Y_2'^{*A*B_2\#H})^{\perp} = 0$.

Similarly, we can prove

$$Y_1'^{(*A*B_1\#H)^i} \in (Y_2'^{(*A*B_2\#H)^i})^{\perp}, \quad \text{and} \quad (Y_1'^{(*A*B_1\#H)^i})^{\perp} \cap (Y_2'^{(*A*B_2\#H)^i})^{\perp} = \{0\}$$

as well as

$$Y_2'^{(*A*B_2\#H)^j} \in (Y_1'^{(*A*B_1\#H)^{j-1}})^{\perp}, \quad \text{and} \quad (Y_2'^{(*A*B_2\#H)^j})^{\perp} \cap (Y_1'^{(*A*B_1\#H)^{j-1}})^{\perp} = \{0\}$$

for $1 \le i \le s$ and $2 \le j \le t$. This completes the proof.

6. Conclusions

In this paper, we mainly find some homological invariants under H-Galois extensions. In the first part, we prove derived equivalences are invariant under the H-Galois extensions with proper conditions.

As a generalization of the derived equivalence, the recollement is an important concept in category theory. Hence in the second part, as a development of the previous result, we further prove the recollements is invariant under the H-Galois extensions with proper conditions. In our proof, we firstly prove the case for 2-recollements, i.e., perfect recollements, then inductively prove the result for general cases.

Considering the importance of H-Frobenius extensions, we study derived equivalences under H-Galois Frobenius extensions. It is found that if A/B is H-Galois Frobenius, then $\mathrm{End}_{\mathcal{D}^{\mathrm{b}}(\mathrm{A})}(\mathrm{T}_{\bullet} \otimes_{\mathrm{B}} \mathrm{A})$ is an H-Galois Frobenius extension of $\mathrm{End}_{\mathcal{D}^b(B)}(T_{\bullet})$, following the fact that $\mathrm{End}_{\mathcal{D}^b(B)}(T_{\bullet})$ is derive equivalent to B.

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Conflict of interest

The authors declare that there is no conflict of interest.

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