Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(2): 3204–3209.
DOI: 10.3934/math.2023164
Received: 28 August 2022
Revised: 01 November 2022
Accepted: 08 November 2022
Published: 16 November 2022

Research article

Z_3-connectivity of graphs with independence number at most 3

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Abstract: It was conjectured by Jaeger et al. that all 5-edge-connected graphs are Z_3-connected. In this paper, we confirm this conjecture for all 5-edge-connected graphs with independence number at most 3.

Keywords: nowhere-zero flows; Z_3-connectivity; independence number
Mathematics Subject Classification: 05C21

1. Introduction

Graphs considered in this paper are finite, loopless and mutilple edges are allowed. Terminologies and notations not defined here can be found in [1].

Let Γ be a graph. For vertex subsets $U, W \subseteq V(\Gamma)$, denote by $e_\Gamma(U, W)$ the number of edges with one end in U and the other in W. For convenience, we write $e_\Gamma(U)$ and $e_\Gamma(x)$ for $e_\Gamma(U, V(\Gamma) \setminus U)$ and $e_\Gamma(\{x\})$, respectively. For a graph Γ, let $\alpha(\Gamma)$ denote the *independence number* of Γ.

Let D be an orientation of Γ. Let $d = xy$ be an edge in Γ directed from x to y. Then we call x and y the tail and head of d, respectively. For a vertex $x \in V(\Gamma)$, denote $E^T(x) = \{e|x \text{ is tail of } e\}$ and $E^H(x) = \{e|x \text{ is head of } e\}$.

Let Z_k be the cyclic group with order k and $Z_k^* = Z_k \setminus \{0\}$. Denote $M(\Gamma, Z_k) = \{g|g : E(\Gamma) \to Z_k\}$ and $M^*(\Gamma, Z_k) = \{g|g : E(\Gamma) \to Z_k^*\}$. Given a mapping $g \in M(\Gamma, Z_k)$, for each vertex $x \in V(\Gamma)$, define

$$\partial g(x) = \sum_{e \in E^T(x)} g(e) - \sum_{e \in E^H(x)} g(e).$$

The value $\partial g(x)$ is said to be the *outflow at x of g*.

Suppose that Γ is a graph and β is a mapping from $V(\Gamma)$ to Z_k. If $\sum_{x \in V(\Gamma)} \beta(x) = 0$, then β is said to be a *zero-sum mapping*. Set $O(\Gamma, Z_k) = \{\beta|\beta \text{ is zero-sum}\}$. Given a mapping β of $O(\Gamma, Z_k)$, a mapping $g \in M^*(\Gamma, Z_k)$ is called *nowhere-zero* (Z_k, β)*-flow* if $\partial g = \beta$ under some orientation of Γ. When $\beta = 0$, it is called a *nowhere-zero Z_k-flow*. If there is a nowhere-zero (Z_k, β)-flow in Γ for each $\beta \in O(\Gamma, Z_k)$, then Γ is called *Z_k-connected*.

Given a graph Γ, let $e = xy$ be an edge of Γ. Define *contraction*: remove edge e and identify x and y to be one vertex. Suppose that K is a subgraph of Γ. Use Γ/K to denote the resulting graph contracting all edges of K. A graph Γ is called *Z_k-reduced* if Γ does not contain nontrivial Z_k-connected subgraph. A Z_k-reduced graph Γ^* is called a *Z_k-reduction* of Γ if we can get Γ^* by contracting each nontrivial Z_k-connected subgraph in Γ. Obviously the Z_k-reduction of a Z_k-reduced graph is itself.

Tutte [10, 11] introduced integer flow problem and Jaeger et al. [4] generalized this concept and proposed group connectivity. Jaeger et al. [4] also gave the following conjecture, which is still widely open.

Conjecture 1.1. *A graph is Z_3-connected if it is 5-edge-connected.*

This conjecture has aroused the interest of scholars and many families of Z_3-connected graphs have been discovered. Luo et al. [8] proved that a bridgeless graph Γ admits a nowhere-zero 3-flow if $\alpha(\Gamma) \leq 2$ and Γ is not reduced to K_4 or not one of the 5 specified graphs. Yang et al. [12] studied the Z_3-connectivity of 3-edge-connected graphs with independence number at most 2.

Recently, Li et al. [6] extended the result of Luo et al. [8] and researched the existence of nowhere-zero 3-flows in the graphs whose independence number is at most 4. Since the 4-edge-connected graph Γ with $|V(\Gamma)| = 12$ and $\alpha(\Gamma) = 3$ constructed by Jaeger et al. [4] is not Z_3-connected, we investigate the Z_3-connectivity of graphs with edge-connectivity 5 and independence number at most 3. We prove the following theorem.

Theorem 1.2. *Let Γ be a 5-edge-connected graph with independence number at most 3. Then Γ is Z_3-connected.*

2. Preliminaries

In this section, we will introduce some lemmas and theorems that will be needed in the proof of our main theorem.

Lemma 2.1. [2] *Let k and n be positive integers. Then we have the following:*
(1) *if $n \geq 5$, then K_n and K_n^- are Z_3-connected.*
(2) *C_n is Z_k-connected if and only if $k > n$.*
(3) *W_{2k} is Z_3-connected and W_{2k+1} is not Z_3-connected.*

Lemma 2.2. [5] *Suppose that Γ have a subgraph K and x is a vertex in $V(\Gamma) \setminus V(K)$ with $e_\Gamma(x, V(K)) \geq 2$. If K is Z_3-connected, then the subgraph induced by $V(K) \cup \{x\}$ is Z_3-connected.*

Lemma 2.3. [2, 3] *Suppose that K is a subgraph of Γ. Then Γ is Z_3-connected if both Γ and Γ/K are Z_3-connected.*

Lemma 2.4. [12] *Suppose that Γ is a 2-connected simple graph. If $\delta(\Gamma) \geq 4$ and $\alpha(\Gamma) \leq 2$, then Γ is Z_3-connected.*

Let $v, v_1, v_2 \in V(\Gamma)$ and $vv_1, vv_2 \in E(\Gamma)$. Removing edges vv_1, vv_2 and adding new edge v_1v_2 in Γ, the resulting graph is denoted by $\Gamma_{[vv_1,vv_2]}$. Obviously $\Gamma_{[vv_1,vv_2]} = \Gamma \cup \{v_1v_2\} - \{vv_1, vv_2\}$.

Lemma 2.5. [2] *Suppose that* Γ *is a graph and* $v \in V(\Gamma)$ *with* $d_\Gamma(v) \geq 4$. *Then* Γ *is* Z_3*-connected if* $\Gamma_{[vv_1,vv_2]}$ *is* Z_3*-connected, where* v_1, v_2 *are two neighbors of* v.

Lemma 2.6. [6] *Suppose that* K *is a* Z_3*-reduced graph with* $\alpha(K) \leq 3$. *Then the order of* K *is at most 14. Furthermore,* K *is 5-edge-connected and contains a* K_4 *if* $|V(K)| = 14$.

Lemma 2.7. [6] *Suppose that* Γ *is a* Z_3*-reduction of a connected graph. If* $|V(\Gamma)| \leq 15$ *and* $\delta(\Gamma) > 4$, *then* Γ *is essentially 8-edge-connected and 5-edge-connected.*

Lemma 2.8. [6] *Suppose that* W_{2k+1} *is a proper subgraph of the graph* Γ *and* U, W *are two subsets of* $V(W_{2k+1})$ *with* $U \cup W = V(W_{2k+1})$. *Denote by* $\Gamma_{[U,W]}$ *the graph obtained from* Γ *by contracting* U *and* W *into* u *and* w*, respectively, and then deleting the loops and replacing the edges between* u *and* w *by one edge* uw. *Then* Γ *is* Z_3*-connected if* $\Gamma_{[U,W]}$ *is* Z_3*-connected.*

A simple graph Γ is said to satisfy the *Ore-condition* if for every pair of nonadjacent vertices x and y in Γ, $d_\Gamma(x) + d_\Gamma(y) \geq |V(\Gamma)|$.

Theorem 2.9. [9] *Suppose that* Γ *is a simple graph satisfying Ore-condition. If* $|V(\Gamma)| > 6$, *then* Γ *is* Z_3*-connected.*

Theorem 2.10. [7] *A graph is* Z_3*-connected if it is 6-edge-connected.*

3. Proof of Theorem 1.2

The proof of Theorem 1.2 will be given in this section.

Proof of Theorem 1.2. Suppose that Γ is a 5-edged-connected graph with $\alpha(\Gamma) \leq 3$. Suppose that K is Z_3-reduction of Γ. If $K = K_1$, then we have done. Thus in the following, we assume that $K \neq K_1$. Hence K is not Z_3-connected and is a 5-edged-connected graph with $\alpha(K) \leq 3$.

Claim 1. K is simple. Thus $\delta(K) \geq 5$ and $\alpha(K) = 3$.

Proof of Claim 1. By the definition of the reduction, it is clear that K is simple and $\delta(K) \geq 5$. If $\alpha(K) \leq 2$, then by Lemma 2.4 we get that K is Z_3-connected. That contradicts our assumption. Therefore $\alpha(K) = 3$.

Claim 2. $11 \leq |V(K)| \leq 14$ and K contains a triangle as subgraph.

Proof of Claim 2. If $|V(K)| \leq 10$, then K satisfies conditions of Theorem 2.9 since $\delta(K) \geq 5$. Then K is Z_3-connnected, a contradiction. Thus $|V(K)| \geq 11$. By Lemmas 2.6 and 2.7, we get that K is a 5-edge-connected graph with $|V(K)| \leq 14$. Hence $11 \leq |V(K)| \leq 14$.

Take an arbitrary vertex, say w, of the graph K and denote $N_K(w) = \{w_1, w_2, \ldots, w_t\}$. where $t = d_K(w) \geq 5$. Since $\alpha(K) = 3$, $K[N(w)]$ has at least two edges. Thus K contains a triangle.

Claim 3. $|V(K)| \geq 12$.

Proof of Claim 3. Suppose to the contrary that $|V(K)| \leq 11$. Then by Claim 2, $|V(K)| = 11$.

We first assume that K contains a K_4 with vertex set $\{u_1, u_2, w_1, w_2\}$. Let $U = \{u_1, u_2\}$ and $W = \{w_1, w_2\}$. Clearly $d_{K_{[U,W]}}(v_1) + d_{K_{[U,W]}}(v_2) \geq |V(K_{[U,W]})|$, where v_1, v_2 are arbitrary nonadjacent vertices in $K_{[U,W]}$. By Theorem 2.9, $K_{[U,W]}$ is Z_3-connected. Thus, it follows that K is Z_3-connected from Lemma 2.8. It contradicts our assumption.

Next we assume that K contains no K_4. Let $S = uvwu$ be a triangle in the graph K with $d_K(u) \geq 6$. Since $e_K(S) \geq 10$, there are two vertices z_1, z_2 which are adjacent to two of u, v, w in K. Suppose that $z_1w, z_1v \in E(K)$. It follows that the graph $K_{[z_1w, z_1v]}$ has a 2-cycle wvw. The resulting graph by repeatedly contracting 2-cycles in $K_{[z_1w, z_1v]}$ is a K_1, which is Z_3-connected. Then from Lemmas 2.3 and 2.5 K is Z_3-connected. Then we get a contradiction. Therefore $|V(K)| \geq 12$.

Claim 4. If $|V(K)| \in \{12, 13\}$, then K doesn't contain a K_4 or K_4^-.

Proof of Claim 4. Suppose to the contrary that K contains a K_4 or K_4^-. We first assume that H is a K_4 of K. Let U and W be a partition of $V(H)$ with $|U| = |W|$. Thus $|V(K_{[U,W]})| \leq 11$ and $\delta(K_{[U,W]}) \geq 5$. By Claims 2 and 3, $K_{[U,W]}$ is not a reduced graph. Thus $K_{[U,W]}$ must contain a nontrivial Z_3-connected graph. Then the resulting graph obtained by contracting this subgraph and repeatedly contracting 2-cycles generated in the processing is a K_1, which is Z_3-connected. By Lemmas 2.3 and 2.8, K is Z_3-connected. This contradiction proves that K doesn't contain a K_4.

Now we assume that J is a K_4^- of K. Let $V(J) = \{v_1, v_2, u_1, u_2\}$ and $E(J) = \{v_1v_2, v_1u_1, v_1u_2, v_2u_1, v_2u_2\}$. We claim that $|N(v_1) \cap N(v_2)| \leq 3$. Since K doesn't contain a K_4, every two vertices of $N(v_1) \cap N(v_2)$ are nonadjacent. If v_1 and v_2 have at least 4 common neighbors, then $N(v_1) \cap N(v_2)$ is an independent set with at least 4 vertices. It contradicts $\alpha(K) \leq 3$. Thus, $|N(v_1) \cap N(v_2)| = 2$, or 3. We consider these two cases in the following.

Case 1. $N(v_1) \cap N(v_2) = \{u_1, u_2, u_3\}$.

Let us consider the graph $K_{[v_1u_3, v_2u_3]}$. Note that $K_{[v_1u_3, v_2u_3]}$ contains the 2-cycle $v_1v_2v_1$. Since a 2-cycle is Z_3-connected from Lemma 2.1, $K_{[v_1u_3, v_2u_3]}$ contains a maximal Z_3-connected graph, say W, that contains $v_1v_2v_1$. Let $K^* = K_{[v_1u_3, v_2u_3]}/W$ and W be contracted to the new vertex v^*. We can get that $V(J) \subseteq V(W)$, $e_{K^*}(u_3, V(J)) = 0$ and $d_{K^*}(u_3) \geq 3$. Thus, we have $|V(K^*)| \leq 10$.

When $|V(K^*)| = 10$, then we have $|V(K)| = 13$, $d_{K^*}(v^*) \geq 8$ and $V(W) = V(J)$. Since $e_{K^*}(u_3, V(J)) = 0$, we have $d_{K^*}(v^*) = 8$. Set $N(v^*) = \{z_1, z_2, \ldots, z_8\}$. Then $V(K^*) = N[v^*] \cup \{u_3\}$. If $d_{K^*}(u_3) \geq 5$, then K^* satisfies Ore-condition. Therefore from Theorem 2.9, K^* is Z_3-connected. It contradicts our assumption. Thus $d_{K^*}(u_3) \leq 4$. Suppose that none of $M = \{z_1, z_2, z_3, z_4\}$ is adjacent to u_3. By our assumption, there are $i, j \in \{1, 2, 3, 4\}$ such that $z_iz_j \notin E(K^*)$. This implies that z_i and z_j have at least two common neighbors in $N(v^*)$. Therefore we obtain a W_4 with center at v^*. It contradicts that W is maximal.

When $|V(K^*)| = 9$, we have $|V(K)| = 12$ or 13. In the former case, $d_{K^*}(v^*) \geq 8$ and $V(W) = V(\Gamma)$. Since $e_{H^*}(u_3, V(J)) = 0$, there exists a vertex which has two neighbors in $V(W)$. That contradicts that K^* is simple. The proof of the case when $|V(K)| = 13$ is similar to that of the case when $|V(K)| = 12$.

When $|V(K^*)| \leq 8$, the graph K^* satisfies Ore-condition. Therefore K^* is Z_3-connected by Theorem 2.9. It follows that K is a Z_3-connected graph by Lemma 2.5. It contradicts our assumption.

Case 2. $N(v_1) \cap N(v_2) = \{u_1, u_2\}$.

If v_i and u_j have a common neighbor $z \notin \{v_1, v_2\}$ for $i, j \in \{1, 2\}$, then we can prove K is Z_3-connected by a similar proof of the above case. This implies that v_i and u_j have only one common neighbor. Set $Y = N(v_1) \cup N(v_2) \setminus V(J) = \{z_1, z_2, \ldots, z_t\}$. Since $\delta(K) \geq 5$, we have $t \geq 4$. Since K contains no K_4, there are two nonadjacent vertices, say z_i, z_j. It follows that $\{z_i, z_j, u_1, u_2\}$ is an independent set. That contradicts $\alpha(K) \leq 3$.

By Case 1 and Case 2, we get that if $|V(K)| \in \{12, 13\}$, then K doesn't contain a K_4 or K_4^-.

Claim 5. $|V(K)| = 14$.

Proof of Claim 5. Since $12 \leq |V(K)| \leq 14$, by Claim 4, we only need to show that if K does not contains a K_4^- with $|V(K)| \in \{12, 13\}$, then K is Z_3-connected. Since the proofs of $|V(K)| = 12$ and $|V(K)| = 13$ are similar, we only prove the case when $|V(K)| = 13$.

Let $Q = v_1v_2v_3v_1$ be a triangle of K with $d(v_1) \geq 6$. It follows that the degree of v_2, v_3 is 5 and the degree of v_1 is 6 by our assumption and Theorem 2.10. Now we suppose that the intersection of $N(v_1)$ and $N(v_2)$ is $\{v_3\}$. Then $N(v_1)\backslash V(Q) = \{v_{11}, v_{12}, v_{13}, v_{14}\}$ and $N(v_i)\backslash V(Q) = \{v_{i1}, v_{i2}, v_{i3}\}$, where $i = 2, 3$. Since K contains no K_4^- and $\alpha(K) = 3$, the graph induced by $N(v_i)$ contains only isolated edges. Suppose that $v_{11}v_{12}, v_{13}v_{14}, v_{21}v_{22}, v_{31}v_{32} \in E(K)$. Similarly, we get $e(v_{ij}, N(v_1)) \leq 2$, where $i = 2, 3$; $j = 1, 2, 3$. It follows that $e(v_{23}, N(v_3)) \geq 2$, $e(v_{33}, N(v_2)) \geq 2$, $e(v_{2j}, N(v_3)) \geq 1$, and $e(v_{3j}, N(v_2)) \geq 1$, where $j = 1, 2$. Since K does not contain a K_4^-, we may assume $v_{23}v_{33}, v_{23}v_{32}, v_{33}v_{22} \in E(K)$. If $v_{21}v_{31} \in E(K)$, then $v_{21}v_{31}v_{32}v_{23}v_{33}v_{22}v_{21}$ is a 6-cycle. Otherwise, $v_{21}v_{32}, v_{22}v_{31} \in E(K)$. Therefore, we get a 4-cycle $v_{21}v_{32}v_{31}v_{22}v_{21}$. Contracting the 2-cycle $v_2v_3v_2$ in $K_{[v_1v_2, v_1v_3]}$, the resulting graph is denoted by K^*. Suppose that the cycle is contracted into new vertex v^*. Note that $v_{ij} \in N(v^*)$, where $i = 2, 3, j = 1, 2, 3$. Thus, K^* contains a 6-wheel or 4-wheel, which is Z_3-connected. Contracting the wheel and repeatedly contracting 2-cycles in K^*, the resulting graph must be a K_1. It follows that K is Z_3-connected from Lemmas 2.3 and 2.5. We also get a contradiction. This completes the proof of Claim 5.

The final step. By Claim 5, $|V(K)| = 14$. Thus K contains a K_4 by Lemma 2.6. Let $V(K_4) = \{u_1, u_2, w_1, w_2\}$ and let $U = \{u_1, u_2\}$ and $W = \{w_1, w_2\}$. In this case, we consider the graph $K_{[U,W]}$. Note that the order of $K_{[U,W]}$ is 12. If $K_{[U,W]}$ contains a Z_3-connected subgraph, then the resulting graph obtained by contacting it and repeatedly contracting 2-cycles is a K_1. We know that $K_{[U,W]}$ is a Z_3-connected subgraph by Lemmas 2.3 and 2.5. Otherwise, $K_{[U,W]}$ is a reduced graph. Thus, $K_{[U,W]}$ is also Z_3-connected by Claims 4 and 5. By Lemma 2.8, K is Z_3-connected, a contradiction. This completes our proof.

4. Conclusions

Jaeger et al. [4] constructed a 4-edge-connected graph Γ with $\alpha(\Gamma) = 3$, which is not Z_3-connected. They further conjectured that each 5-edge-connected graph is Z_3-connected. If this conjecure is correct, then so is Tutte's 3-Flow Conjecture. The article confirm this conjecture for all 5-edge-connected graphs with independence number at most 3.

Acknowledgments

The second author is supported by the Natural Science Foundation of China (Grant No. 11701496) and Nanhu Scholars Program for Young Scholars of XYNU. The third author is supported by the Natural Science Foundation of China (Grant No.12271438) and Basic Research Foundation of Henan Educational Committee (No. 20ZX004).

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. B. A. Bondy, U. S. R. Murty, *Graph Theory*, London: Springer, 2008.
2. J. Chen, E. M. Eschen, H. J. Lai, Group connectivity of certain graphs, *Ars Combinatoria*, **89** (2008), 141–158.
3. M. DeVos, R. Xu, G. Yu, Nowhere-zero Z_3-flows through Z_3-connectivity, *Discrete Math.*, **306** (2006), 26–30. https://doi.org/10.1016/j.disc.2005.10.019
4. F. Jaeger, N. Linial, C. Payan, M. Tarsi, Group connectivity of graphs-a nonhomogeneous analogue of nowhere-zero flow properties, *J. Comb. Theory B*, **56** (1992), 165–182. https://doi.org/10.1016/0095-8956(92)90016-Q
5. H. J. Lai, Group connectivity of 3-edge-connected chordal graphs, *Graph. Combinator.*, **16** (2000), 165–176. https://doi.org/10.1007/PL00021177
6. J. Li, R. Luo, Y. Wang, Nowhere-zero 3-flow of graphs with small independence number, *Discrete Math.*, **341** (2018), 42–50. https://doi.org/10.1016/j.disc.2017.06.022
7. L. M. Lovász, C. Thomassen, Y. Wu, C. Q. Zhang, Nowhere-zero 3-flows and modulo k-orientations, *J. Comb. Theory B*, **103** (2013), 587–598. https://doi.org/10.1016/j.jctb.2013.06.003
8. R. Luo, Z. Miao, R. Xu, Nowhere-zero 3-flows of graphs with independence number two, *Graph. Combinator.*, **29** (2013), 1899–1907. https://doi.org/10.1007/s00373-012-1238-z
9. R. Luo, R. Xu, J. Yin, G. Yu, Ore-condition and Z_3-connectivity, *Eur. J. Comb.*, **29** (2008), 1587–1595. https://doi.org/10.1016/j.ejc.2007.11.014
10. W. T. Tutte, A contribution to the theory of chromatic polynomials, *Can. J. Math.*, **6** (1954), 80–91. https://doi.org/10.4153/CJM-1954-010-9
11. W. T. Tutte, On the algebraic theory of graph colorings, *J. Comb. Theory*, **1** (1966), 15–50. https://doi.org/10.1016/S0021-9800(66)80004-2
12. F. Yang, X. Li, L. Li, Z_3-connectivity with independent number 2, *Graph. Combinator.*, **32** (2016), 419–429. https://doi.org/10.1007/s00373-015-1556-z

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