Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(12): 29517–29534.
DOI: 10.3934/math.20231511
Received: 01 September 2023
Revised: 10 October 2023
Accepted: 23 October 2023
Published: 31 October 2023

Research article

A fixed point iterative scheme based on Green's function for numerical solutions of singular BVPs

Junaid Ahmad[1], **Muhammad Arshad**[1] **and Reny George**[2,*]

[1] Department of Mathematics and Statistics, International Islamic University, H-10, Islamabad-44000, Pakistan

[2] Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam Bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia

* **Correspondence:** Email: renygeorge02@yahoo.com.

Abstract: We suggest a novel iterative scheme for solutions of singular boundary value problems (SBVPs) that is obtained by embedding Green's function into the Picard-Mann Hybrid (PMH) iterative scheme. This new scheme we call PMH-Green's iterative scheme and prove its convergence towards a sought solution of certain SBVPs. We impose possible mild conditions on the operator or on the parameters involved in our scheme to obtain our main outcome. After this, we prove that this new iterative scheme is weak w^2-stable. Eventually, using two different numerical examples of SBVPs, we show that our new approach suggests highly accurate numerical solutions as compared the corresponding Picard-Green's and Mann-Green's iterative schemes.

Keywords: solution; iteration scheme; boundary value problem; Green's function; Banach space
Mathematics Subject Classification: 47H09, 47H10

1. Introduction

In some sense, all the real world phenomena can be essentially modeled in the form of differential equations having certain boundary conditions [1]. This is one of the reasons that the study of differential equations is too important. One of the difficulties that arise naturally in studying differential equations is that their sought solutions are explicitly unknown (or it is very hard to solve them using available analytical approaches). In this case, the sought solution is thus reasonably possible to set the form of a fixed point problem of an operator (whose domain is possibly some distance space). However, when a fixed point of this operator exists, then one naturally thinks how its approximate value can be computed using an appropriate numerical scheme. In 1922, Banach [2] proved that if the domain of such an operator is complete normed space and the operator is contraction, then such operators admit a unique fixed point (which is the unique solution of the underlying problem) and the sequence of Picard [3] iterates essentially converge to this unique fixed point. This result has many useful applications in differential and integral equations because it gives the existence and approximation of a solution for these problems. Notice that if B denotes a Banach space with the norm $\|.\|$, then the operator $F : B \to B$ is called a contraction (sometimes called a Banach-contraction) if for all $v, w \in B$ it is possible to find a real constant $\mu \in [0, 1)$, such that

$$\|Fv - Fw\| \le \mu\|v - w\|. \tag{1.1}$$

We say that a point $s^* \in B$ is known as a fixed point of F when the equation $Fs^* = s^*$ holds and we write $fix(F)$ to denote a set of all fixed points. In this case, the Picard iteration [3] of F is defined as:

$$\begin{cases} w_0 \in B, \\ w_{m+1} = Fw_m, (m = 0, 1, 2, 3, ...). \end{cases} \tag{1.2}$$

We know that the selfmap F is known as a nonexapnsive mapping on B if the relation (1.1) holds for $\mu = 1$. Fixed point approximation under different iterative schemes is an active and important area of research on its own [4–7]. Browder's [8] (cf. also Gohde [9] and Kirk [10]) fixed point theorem suggests a fixed point (may not unique) for a certain nonexpansive operator in a Banach space setting. Moreover, there are some well-known numerical examples of nonexpansive operators, for which the Picard iteration is not convergent to its fixed point (see, e.g., [11] and others). To overcome the case of nonexpansive mappings, Mann [12] suggested a new iteration scheme which needs an initial value as well as a sequence of real numbers whose values are between 0 and 1.

The Mann iteration [12] recursively generates a sequence as:

$$\begin{cases} w_0 \in B, \\ w_{m+1} = (1 - \alpha_m)w_m + \alpha_m Fw_m, (m = 0, 1, 2, 3, ...), \end{cases} \tag{1.3}$$

where $\alpha_m \in [0, 1]$.

In the literature of iterative schemes, it is known that the speed of the both Picard [3] and Mann [12] iteration is slow. To achieve a better rate of convergence, Khan [13] combined the iterative schemes due to Picard and Mann and named the resultant iterative scheme as a Picard-Mann hybrid iterative scheme (PMH-iterative scheme).

Precisely, PMH-iterative scheme [13] recursively generates a sequence as:

$$\begin{cases} w_0 \in B, \\ v_m = (1 - \alpha_m)w_m + \alpha_m Fw_m, \\ w_{m+1} = Fv_m, (m = 0, 1, 2, 3, ...), \end{cases} \tag{1.4}$$

where $\alpha_m \in (0, 1)$.

Khan proved that the PMH-iteration scheme (1.4) essentially converges to a fixed point of a certain operator. Moreover, he proved analytically and numerically that this scheme suggests high accurate results corresponding to Mann and Picard iterations. Existence and approximation of solutions for BVPs is an important area on its own. Different techniques have been studied by authors for existence and approximation of solutions for various classes of BVPs [14]. On the other hand, Khuri and Sayfy [15–17] embedded Green's function into some well-known iterative schemes and proved that these new type of schemes suggest high accurate results corresponding to the other available methods of the literature. Motivated by Khuri and Sayfy, Assadi et al. [18] introduced Picard-Green's and Mann-Green's iterative schemes for a class of SBVPs and proved that both of these schemes are better than the many other previous iterative schemes studied for SBVPs. Thus, the challenging question is when is it possible to obtain a modified version of the scheme (1.4) based on Green's function for finding solution of SBVP? In this paper, we first obtain the requested version of this scheme for SBVPs and name it PMH-Green's iterative scheme and show that the PMH-Green produces very high accurate results compared to Mann-Green's and Picard-Green's iteration schemes. We also show that the PMH-Green's iterative scheme is weak w^2-stable in this case. The numerical computations given at the end of the paper supports the main outcome of the paper and suggests the numerical effectiveness of the proposed scheme.

2. Overview of the iterative scheme

To propose the desired PMH-Green's iterative scheme, first we define some elementary concepts and results that are necessary for the main work.

2.1. Construction of the Green's function

This subsection will establish the Green's function for a broad class of SBVPs. To succeed in this aim, suppose $t \in (a, b)$, we consider a linear second order equation which is mathematically written as following:

$$L(w) = w''(t) + p(t)w'(t) + q(t)w(t) = f(t), \tag{2.1}$$

and the associated boundary conditions (BCs) are the following:

$$\begin{cases} B_a[w] = \alpha_0 w(a) + \alpha_1 w'(a) = \xi, \\ B_b[w] = \beta_0 w(b) + \beta_1 w'(b) = \lambda. \end{cases} \tag{2.2}$$

It should be noted that the possible general solution is given as $w(t) = w_h(t) + w_p(t)$. Here, the function $w_h(t)$ is essentially the solution for the equation $L[w] = 0$ subjected to the BCs suggested in (2.2), and $w_p(t)$ is solution for the equation $L[w] = f(t)$ endowed with the homogeneous BCs given below:

$$B_a[w] = B_b[w] = 0. \tag{2.3}$$

Now, for finding $w_p(t)$, one needs a solution for

$$L[w] = \delta(t - s), \tag{2.4}$$

which is essentially subject to the BCs as given in (2.3) and in this case, such a solution is known as a Green's function, denoted normally by $G(t, s)$. Then

$$w_p = \int_a^b G(t, s) f(s) ds. \tag{2.5}$$

Let w_1, w_2 be two linearly independent solutions of $L[w] = 0$. Notice that the Green's function essentially obeys the homogeneous equation for each choice of $t \neq s$ and thus it will be a linear combination of w_1 and w_2:

$$G(t,s)=\begin{cases} c_1w_1(t)+c_2w_2(t) & \text{when } \ a<t<s, \\ d_1w_1(t)+d_2w_2(t) & \text{when } \ s<t<b, \end{cases}$$

the constants c_i and d_i, $(i=1,2)$ are determined using the following axioms:

(A_1) G satisfies the given homogeneous BCs, i.e.,

$$B_a[G(t,s)]=B_b[G(t,s)]=0. \tag{2.6}$$

(A_2) Continuity of G at $t=s$:

$$c_1w_1(s)+c_2w_2(s)=d_1w_1(s)+d_2w_2(s). \tag{2.7}$$

(A_3) Jump discontinuity of G' at $t=s$, i.e.,

$$d_1w_1'(s)+d_2w_2'(s)-c_1w_1'(s)-c_2w_2'(s)=1. \tag{2.8}$$

For nonlinear SBVPs

$$w''(t)+p(t)w'(t)+q(t)w(t)=f(t,w(t),w'(t)), \tag{2.9}$$

the particular solution satisfies

$$w_p=\int_a^b G(t,s)f(s,w_p(s),w_p'(s))ds, \tag{2.10}$$

and here G is the Green's function connected to (2.9).

2.2. PMH-Green's iterative scheme

Now we propose our desired PMH-Green's iterative scheme for approximate solutions of the following SBVPs of the form as follows:

$$L[w]=w''+\frac{q}{t}w'=f(t,w,w'), \tag{2.11}$$

and the associated BCs are as given in (2.2). Now consider the Green's function G corresponding to the linear term and we consider the following operator:

$$M[w_p]=\int_a^b G(t,s)L[w_p]ds. \tag{2.12}$$

Now from (2.10) and (2.12), we obtain

$$M[w_p]=\int_a^b G(t,s)[L(w_p)-f(s,w(s),w'(s)]ds+w_p. \tag{2.13}$$

Put $w_p=w$ in (2.13), one has

$$M(w)=w+\int_a^b G(t,s)[L(w)-f(s,w(s),w'(s)]ds. \tag{2.14}$$

Hence from (2.14), we obtain the modified form of PMH-iterative scheme given in (1.4) as follows:

$$\begin{cases} v_m = (1-\alpha_m)w_m + \alpha_m M[w_m], \\ w_{m+1} = M[v_m], \\ (m = 0, 1, 2, 3, ...), \end{cases} \tag{2.15}$$

which yields the following iterative procedure:

$$\begin{cases} v_m = (1-\alpha_m)w_m + \alpha_m [\int_a^b G(t,s)[L(w_m) - f(s, w_m(s), w'_m(s)]ds + w_m], \\ w_{m+1} = [\int_a^b G(t,s)[L(v_m) - f(s, v_m(s), v'_m(s)]ds + v_m], (m = 0, 1, 2, 3, ...). \end{cases} \tag{2.16}$$

It follows that

$$\begin{cases} v_m = w_m + \alpha_m [\int_a^b G(t,s)[L(w_m) - f(s, w_m(s), w'_m(s)]ds], \\ w_{m+1} = v_m + \int_a^b G(t,s)[L(v_m) - f(s, v_m(s), v'_m(s)]ds, (m = 0, 1, 2, 3, ...), \end{cases} \tag{2.17}$$

and here L denotes the linear term and the initial value to start the scheme, that is, w_0 must be chosen in a way that satisfies the Eq (2.11), $L[w] = 0$, and the given specified BCs.

3. Convergence result

We are now interested in establishing the main convergence result. For this, let $q \geq 2$ and consider a SBVP as provided below:

$$w''(t) + \frac{q}{t}w'(t) = f(t, w(t), w'(t)), \tag{3.1}$$

and the associated BCs are the following:

$$w'(0) = \alpha, \quad w(1) = \beta. \tag{3.2}$$

To construct the required Green's function associated with (3.1), we apply the axioms of Green's function which we gave in the last section. Hence, after solving the Eq (3.1), one has

$$G(t,s) = \begin{cases} A + Bt^{1-q}, & \text{when } 0 < t < s, \\ C + Dt^{1-q}, & \text{when } s < t < 1. \end{cases}$$

Accordingly, using homogenous BCs as given in (3.2), that is, $w'(0) = w(1) = 0$, one has the following

$$B = 0, \quad D + C = 0. \tag{3.3}$$

Applying the continuity axioms of Green's function, we obtain

$$A + Bs^{1-q} = C + Ds^{1-q}. \tag{3.4}$$

Using unit jump discontinuity connected to the first derivative of the Green's function, we have

$$D(1-q)s^{-q} - B(1-q)s^{-q} = 1. \tag{3.5}$$

After solving (3.3)–(3.5), we get the desired Green's function as follows:

$$G(t,s)=\begin{cases}\frac{s^q-s}{q-1}, & \text{when } 0<t<s,\\ \frac{s^q(1-t^{1-q})}{q-1}, & \text{when } s<t<1.\end{cases}$$

Now, embedding the above Green's function in the PMH-iterative scheme given in (2.17), we get the following PMH-iterative scheme:

$$\begin{cases} v_m = w_m + \alpha_m[\int_0^t \frac{s^q}{q-1}(t^{1-q}-1)[w''_m(s) + \frac{q}{s}w'_m(s) - f(s,w_m(s),w'_m(s)]ds, \\ + \int_t^1 \frac{1}{q-1}(s-s^q)[w''_m(s) + \frac{q}{s}w'_m(s) - f(s,w_m(s),w'_m(s)]ds], \\ w_{m+1} = v_m + \int_0^t \frac{s^q}{q-1}(t^{1-q}-1)[v''_m(s) + \frac{q}{s}v'_m(s) - f(s,v_m(s),v'_m(s)]ds, \\ + \int_t^1 \frac{1}{q-1}(s-s^q)[v''_m(s) + \frac{q}{s}v'_m(s) - f(s,v_m(s),v'_m(s)]ds, \\ (m=0,1,2,3,...). \end{cases} \tag{3.6}$$

It follows that

$$\begin{cases} v_m = w_m + \alpha_m[\int_0^1 G(t,s)[w''_m(s) + \frac{q}{s}w'_m(s) - f(s,w_m(s),w'_m(s)]ds], \\ w_{m+1} = v_m + \int_0^1 G(t,s)[v''_m(s) + \frac{q}{s}v'_m(s) - f(s,v_m(s),v'_m(s)]ds, \\ (m=0,1,2,3,...). \end{cases} \tag{3.7}$$

Set $F_G : C[0,1] \to C[0,1]$ by

$$F_G w(t) = w(t) + \int_0^1 G(t,s)[w''_m(s) + \frac{q}{s}w'_m(s) - f(s,w(s),w'(s)]ds. \tag{3.8}$$

Then (3.7) becomes

$$\begin{cases} v_m = (1-\alpha_m)w_m + \alpha_m F_G w_m, \\ w_{m+1} = F_G v_m, (m=0,1,2,3,...). \end{cases} \tag{3.9}$$

The main result of the paper is now ready to establish.

Theorem 3.1. *Consider a Banach space $B = C[0,1]$ with the supremum norm. Let $F_G : B \to B$ be the operator defined in (3.8) and $\{w_m\}$ be the sequence of PMH-Green's iterative scheme (3.9). Assume that the following conditions hold:*

(a) $\frac{1}{2(q-1)}M_c < 1$*, where* $M_c = \max_{[0,1]\times\mathbb{R}^2} |\frac{\partial f}{\partial w}|$.
(b) $\sum \alpha_m = \infty$ *or for some* α*,* $0 < \alpha \leq \alpha_m$.

Subsequently, $\{w_m\}$ *converges strongly to the unique solution of the problems (3.1) and (3.2).*

Proof. By using assumption (a), we show that F_G is a Banach-contraction, that is, $\|F_G v - F_G w\| \leq \mu\|v-w\|$ for all $v, w \in B$ and some fixed $\mu \in [0,1)$. To do this, direct integration gives us,

$$\begin{aligned} \int_0^t \frac{s^q}{1-q}(t^{1-q}-1)[w''(s) + \frac{q}{s}w'(s)]ds &= \frac{(t^{1-q}-1)}{1-q}\int_0^t [s^q w''(s) + qs^{q-1}w'(s)]ds \\ &= \frac{(t^{1-q}-1)}{1-q}\int_0^t [(s^q w'(s))']ds \\ &= \frac{t-t^q}{1-q}w'(t). \end{aligned} \tag{3.10}$$

Also, integration by parts twice, we get

$$\begin{aligned}\int_t^1 \frac{1}{q-1}(s^q-s)w''(s) &= \frac{1}{q-1}[(1-q)w(1)+(qt^{q-1}-1)w(t)+(t-t^q)w'(t)\\ &\quad +q(q-1)\int_t^1 s^{q-2}w(s)ds]. \end{aligned} \tag{3.11}$$

Now integrating once, we have

$$\int_t^1 \frac{1}{q-1}(s^q-s)\frac{q}{s}w(s)ds = \frac{q}{q-1}[(1-t^{q-1})w(t)-(q-1)\int_t^1 s^{q-2}w(s)ds]. \tag{3.12}$$

From (3.10)–(3.12), we have

$$F_G(w) = w(1) + \int_0^t \frac{s^q}{1-q}(t^{1-q}-1)f(s,w(s),w'(s))ds + \int_t^1 \frac{1}{q-1}(s^q-s)f(s,w(s),w')ds. \tag{3.13}$$

From (3.2), $w(1)=\beta$, therefore (3.13) takes the following form

$$F_G(w) = \beta + \int_0^1 G(t,s)f(s,w(s),w'(s),w'(s))ds. \tag{3.14}$$

Hence

$$\begin{aligned}|F_G(v)-F_G(w)| &= \left|\beta + \int_0^1 G(t,s)f(s,v,v')ds - \beta - \int_0^1 G(t,s)f(s,w,w')ds\right|\\ &= \left|\int_0^1 G(t,s)[f(s,v,v')-f(s,w,w')]\right|ds. \end{aligned} \tag{3.15}$$

Now a simple integration suggests

$$\int_0^1 G(t,s)ds = \frac{1}{2(q+1)}(t^2-1) = g(t). \tag{3.16}$$

It is easy to see that the $g(t)$ attains the maximum value in $[0,1]$ either at endpoints or on the critical points. Hence

$$|g(t)| \le \frac{1}{2(q+1)}. \tag{3.17}$$

By (3.15)–(3.17), one has

$$\begin{aligned}|F_G(v)-F_G(w)| &\le \frac{1}{2(q+1)}\max_{t\in[0,1]}|f(t,v,v')-f(t,w,w')|\\ &\le \frac{1}{2(q+1)}M_c\|v-w\|. \end{aligned} \tag{3.18}$$

But $\mu = \frac{1}{2(q+1)} M_c < 1$, it follows from (3.18), that F_G is a Banch contraction. Since B is complete and F_G is a Banach-contraction, thanks to the BCP [2], F_G admits essentially a unique fixed point in $B = C[0,1]$ and this point we shall denote by s^* and hence it follows that this s^* is a unique solution of the problems (3.1) and (3.2).

Moreover, from the assumption (b), we will prove that PMH-Green's iterative converges strongly to s^*. First, we assume the case when $\sum \alpha_m = \infty$. Now

$$\begin{aligned}
\|v_m - s^*\| &= \|(1-\alpha_m)w_m + \alpha_m F_G w_m - s^*\| \\
&= \|(1-\alpha_m)(w_m - s^*) + \alpha_m(F_G w_m - s^*)\| \\
&\leq (1-\alpha_m)\|w_m - s^*\| + \alpha_m\|F_G w_m - s^*\| \\
&\leq (1-\alpha_m)\|w_m - s^*\| + \alpha_m \mu\|w_m - s^*\| \\
&= [1 - \alpha_m(1-\mu)]\|w_m - s^*\|.
\end{aligned}$$

Hence, we obtain

$$\|v_m - s^*\| \leq [1 - \alpha_m(1-\mu)]\|w_m - s^*\|. \tag{3.19}$$

Finally, using (3.19), we compute $\|w_{m+1} - s^*\|$ as follows.

$$\begin{aligned}
\|w_{m+1} - s^*\| &= \|F_G v_m - s^*\| \\
&\leq \mu\|v_m - s^*\| \\
&\leq \mu[1 - \alpha_m(1-\mu)]\|v_m - s^*\|.
\end{aligned}$$

Hence, we obtain

$$\|v_m - s^*\| \leq \mu[1 - \alpha_m(1-\mu)]\|v_m - s^*\|. \tag{3.20}$$

Now, from (3.20), we step by step obtain the following

$$\begin{aligned}
\|w_{m+1} - s^*\| &\leq \mu[1 - \alpha_m(1-\mu)]\|w_m - s^*\| \\
&\leq \mu^2[1 - \alpha_m(1-\mu)][1 - \alpha_{m-1}(1-\mu)]\|w_{m-1} - s^*\| \\
&\leq \mu^3[1 - \alpha_m(1-\mu)][1 - \alpha_{m-1}(1-\mu)][1 - \alpha_{m-2}(1-\mu)]\|w_{m-2} - s^*\|.
\end{aligned}$$

It follows that

$$\|w_{m+1} - s^*\| \leq (\mu)^{m+1}\Pi_{i=0}^{m}[1 - \alpha_i(1-\mu)]\|w_0 - w^*\|. \tag{3.21}$$

Noting that $\lim_{m\to\infty}(\mu)^m = 0$ because $\mu \in [0,1)$. Also, it is well-known from the classical analysis that $1 - w \leq e^{-w}$ for all $w \in [0,1]$. Taking these facts into account with (3.21), we get

$$\|w_{m+1} - s^*\| \leq (\mu)^{m+1} e^{-(1-\mu)\sum_{i=0}^{m}\alpha_i}\|w_0 - s^*\|. \tag{3.22}$$

As supposed $\sum \alpha_m = \infty$ and μ lies in $[0,1)$, we have from (3.22) that

$$\lim_{m\to\infty}\|w_{m+1} - s^*\| = 0.$$

Accordingly, $\{w_m\}$ converges to a fixed point s^* of F_G which is the unique solution of the problems (3.1) and (3.2). The case when $0 < \alpha \leq \alpha_m$ is included already in the case (a) and hence omitted. □

4. Stability

In all branches of mathematics where iterative methods are used for finding approximate value of the sought solution, stability analysis is one of the desirable properties for such schemes (see [19–21] and others). Fixed point procedure may or may not stable when we implement them on a certain operator equation [22] (cf. also [23, 24] and others). Suppose a given iterative scheme of a certain operator is convergent to some of its fixed point. In this case, the iterative scheme is said to be stable if and only if the estimated error between two successive iterative strategy does not affect its so-called convergence. As many know, stability for fixed points iterations finds its initial roots in the paper due to Urabe [25]. Motivated by Urabe [25], Harder and Hicks [26] constructed mathematical definition for stability. Some basic concepts that we need in the work are recalled below.

Definition 4.1. *[26] Consider a mapping F of a Banach space B and suppose $\{w_m\} \subseteq B$ is a sequence generated from certain iterative scheme using the mapping F as follows:*

$$\begin{cases} w_0 \in B, \\ w_{m+1} = \gamma(F, w_m), \end{cases} \tag{4.1}$$

here, the element w_0 denotes starting point and γ is a function of F and w_m. Assume that the sequence of iterates $\{w_m\}$ converges to $s^* \in fix(F)$. In this case, $\{w_m\}$ is said to be stable if and only if

$$\lim_{m\to+\infty} \|s_{m+1} - \gamma(F, s_m)\| = 0 \Rightarrow \lim_{m\to+\infty} s_m = s^*,$$

where $\{s_m\}$ is any chosen sequence in the space B.

Definition 4.2. *[27] Suppose $\{s_m\}$ and $\{w_m\}$ are any two sequences in a Banach space. We say that these two sequences equivalent if and only if* $\lim_{m\to+\infty} \|s_m - w_m\| = 0$.

Opposed to the concept of arbitrary sequences, Timis [28], used the concept of equivalent sequences and obtained a new mathematical definition of the weak stability. This new type of stability is called the weak w^2-stability. The formal definition is given below.

Definition 4.3. *[28] Consider a Banach space B and F a selfmap on B. If $\{w_m\}$ is a sequence of iterates of F produced by the formula (4.1). Assume that $\{w_m\}$ is convergent to a point $s^* \in fix(F)$. Then $\{w_m\}$ is said to be weak w^2-stable if for every equivalent sequence $\{s_m\} \subseteq B$ of $\{w_m\}$, one has the following*

$$\lim_{m\to+\infty} \|s_{m+1} - f(F, s_m)\| = 0 \text{ implies } \lim_{m\to+\infty} s_m = s^*.$$

Using the above concepts, we now show that our PMH-Green's iterative scheme (3.9) is weak w^2-stable.

Theorem 4.4. *Let B, F_G and $\{w_m\}$ be as given in the Theorem 3.1. The $\{w_m\}$ is essentially weak w^2-stable with respect to F_G.*

Proof. To complete the proof, we consider any equivalent sequence $\{s_m\}$ of $\{w_m\}$, that is $\lim_{m\to\infty} \|s_m - w_m\| = 0$. Put

$$\epsilon_m = \|s_{m+1} - F_G r_m]\|,$$

where $r_m = (1 - \alpha_m)s_m + \alpha_m F_G s_m$.

Assumed that $\lim_{m\to+\infty} \epsilon_m = 0$. First we compute the estimate $\|r_m - v_m\|$. For this,

$$\begin{aligned}
\|r_m - v_m\| &= \|[(1-\alpha_m)s_m + \alpha_m F_G s_m] - [(1-\alpha_m)w_m + \alpha_m T_G w_m]\| \\
&= \|[(1-\alpha_m)(s_m - w_m) + \alpha_m(F_G s_m - F_{\mathcal{G}} w_m]\| \\
&\leq (1-\alpha_m)\|s_m - w_m\| + \alpha_m\|F_G s_m - F_G w_m\| \\
&\leq (1-\alpha_m)\|s_m - w_m\| + \alpha_m\mu\|s_m - w_m\| \\
&\leq [1-\alpha_m(1-\mu)]\|s_m - w_m\|.
\end{aligned}$$

Consequently, we find

$$\|r_m - v_m\| \leq [1-\alpha_m(1-\mu)]\|s_m - w_m\|. \tag{4.2}$$

Keeping (4.2) in mind, we can proceed as follows:

$$\begin{aligned}
\|s_{m+1} - s^*\| &\leq \|s_{m+1} - w_{m+1}\| + \|w_{m+1} - s^*\| \\
&\leq \|s_{m+1} - F_G r_m\| + \|F_G r_m - w_{m+1}\| + \|w_{m+1} - s^*\| \\
&= \epsilon_m + \|F_G r_m - w_{m+1}\| + \|w_{m+1} - s^*\| \\
&= \epsilon_m + \|F_G r_m - F_G v_m\| + \|w_{m+1} - s^*\| \\
&\leq \epsilon_m + \mu\|r_m - v_m\| + \|w_{m+1} - s^*\| \\
&\leq \epsilon_m + \mu[1-\alpha_m(1-\mu)]\|s_m - w_m\| + \|w_{m+1} - s^*\|.
\end{aligned}$$

Subsequently, we obtain

$$\|s_{m+1} - s^*\| \leq \epsilon_m + \mu[1-\alpha_m(1-\mu)]\|s_m - w_m\| + \|w_{m+1} - s^*\|. \tag{4.3}$$

By assumptions, $\lim_{m\to+\infty} \epsilon_m = 0$ and $\lim_{m\to\infty} \|s_m - w_m\| = 0$ because $\{s_m\}$ is an equivalent sequence for $\{w_m\}$. Also $\lim_{m\to+\infty} \|w_m - s^*\| = 0$ due to the convergence of $\{w_m\}$ towards s^*. Accordingly, from (4.3), $\lim_{m\to+\infty} \|s_m - s^*\| = 0$. This means that $\{w_m\}$ generated by PMH-Green's iterative scheme (3.9) is weak w^2-stable with respect to the mapping F_G. □

5. Numerical example and computations

In this section, we consider several numerical examples to show the high accurate numerical results produced by our proposed method.

Example 5.1. *First, we consider the following SBVP which represents the equilibrium of isothermal gas sphere [29]:*

$$w''(t) + \frac{2}{t}w'(t) = -w^5(t), \tag{5.1}$$

subjected with the BCs:

$$w'(0) = 0,\ w(1) = \sqrt{\frac{3}{4}}, \tag{5.2}$$

where $0 < t < 1$. *The exact solution of (5.1) and (5.2) is* $w(t) = \sqrt{\frac{3}{3+t^2}}$. *Take* $w_0(t) = \sqrt{\frac{3}{4}} = 0.866025$ *which satisfies the equation* $w'' = 0$ *and given BCs.*

Now using Example 5.1, the proposed scheme takes the following form:

$$\begin{cases} v_m = w_m - \alpha_m \int_0^t s^2(1-\frac{1}{t})[w_m''(s) + \frac{2}{s}w_m'(s) + w_m^5(s)]ds \\ -\alpha_m \int_t^1 s(s-1)[w_m''(s) + \frac{2}{s}w_m'(s) + w_m^5(s)]ds, \\ w_{m+1} = v_m - \int_0^t s^2(1-\frac{1}{t})[v_m''(s) + \frac{2}{s}v_m'(s) + v_m^5(s)]ds \\ - \int_t^1 s(s-1)[v_m''(s) + \frac{2}{s}v_m'(s) + v_m^5(s)]ds. \end{cases} \tag{5.3}$$

Now, for $\alpha_m = 0.99$, the values generated by Picard-Green's, Mann-Green's and PMH-Green's iterative schemes in Tables 1–3. Clrearly the PMH-Green's iterative scheme moving faster to the solution. While Green's function involved in scheme (5.3) is provided in Figure 1, the graphical comparison of the absolute errors in this case is given in Figure 2.

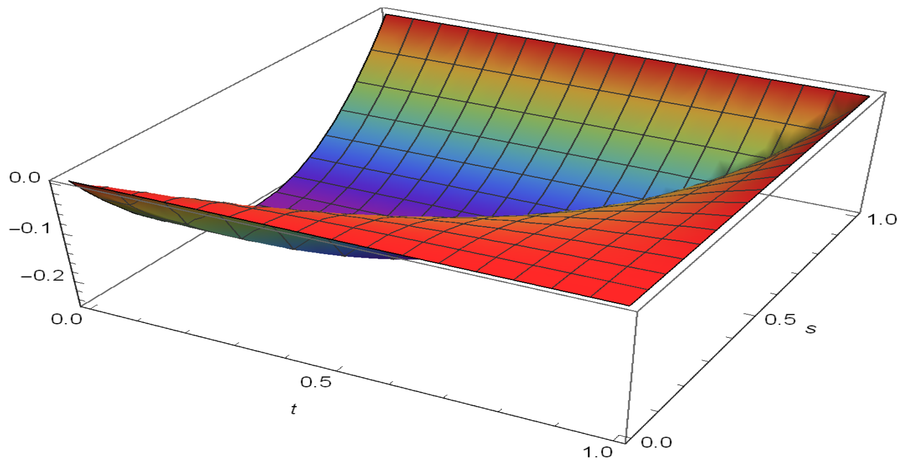

Figure 1. Plot of Green's function involved in the scheme (5.3).

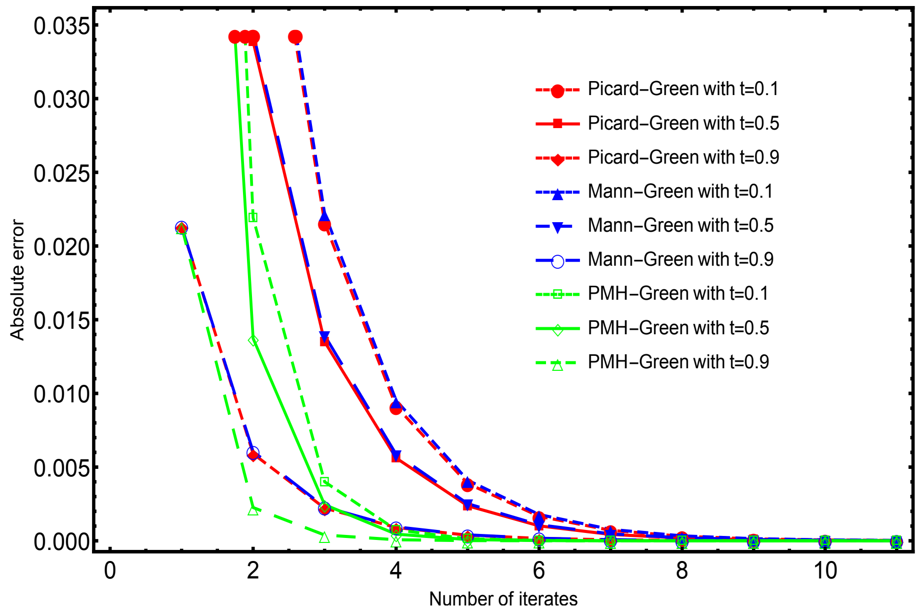

Figure 2. Comparison of various iterative schemes based on Green's function for Example 5.1.

Table 1. Convergence of iterates towards the numerical solution for $t = 0.1$.

m	**Picard – Green**	**Mann – Green**	**PMH – Green**
0	0.866025	0.866025	0.866025
1	0.946403	0.945599	0.976422
2	0.976769	0.976114	0.994334
3	0.989216	0.988807	0.997594
4	0.994451	0.994222	0.998199
5	0.996676	0.996554	0.998311
6	0.997626	0.997563	0.998332
7	0.998032	0.998001	0.998336
8	0.998206	0.998191	0.998337
9	0.998281	0.998273	0.998337
10	0.998313	0.998309	0.998337

Table 2. Convergence of iterates towards the numerical solution for $t = 0.5$.

m	**Picard – Green**	**Mann – Green**	**PMH – Green**
0	0.866025	0.866025	0.866025
1	0.926917	0.926308	0.866028
2	0.947268	0.946831	0.958322
3	0.955153	0.954895	0.960316
4	0.958392	0.958250	0.960684
5	0.959755	0.959681	0.960753
6	0.960335	0.960297	0.960765
7	0.960583	0.960564	0.960768
8	0.960689	0.960679	0.960768
9	0.960734	0.960730	0.960768
10	0.960754	0.960751	0.960768

Table 3. Convergence of iterates towards the numerical solution for $t = 0.9$.

m	**Picard – Green**	**Mann – Green**	**PMH – Green**
0	0.866025	0.866025	0.866025
1	0.881451	0.881297	0.885099
2	0.885140	0.885061	0.886964
3	0.886451	0.886409	0.887284
4	0.886976	0.886953	0.887343
5	0.887194	0.887182	0.887354
6	0.887287	0.887281	0.887356
7	0.887326	0.887323	0.887356
8	0.887343	0.887342	0.887356
9	0.887351	0.887350	0.887356
10	0.887354	0.887353	0.887356

We finish the section with following example.

Example 5.2. *Now we consider a SBVP whose exact solution is not known explicitly as follows:*

$$w''(t) + \frac{2}{t}w'(t) = -e^{-w(t)}, \tag{5.4}$$

subjected to the BCs:

$$w'(0) = 0, \;\; 2w(1) + w'(1) = 0, \tag{5.5}$$

where $0 < t < 1$.

The initial iterate $w_0(t) = 0$ *corresponding to* $w'' = 0$ *and given BCs.*

Now using Example 5.2, the proposed scheme takes the following form:

$$\begin{cases} v_m = w_m - \alpha_m \int_0^t s^2(\frac{1}{2} - \frac{1}{t})[w_m''(s) + \frac{2}{s}w_m'(s) + e^{-w_m(s)}]ds \\ -\alpha_m \int_t^1 s(\frac{s}{2} - 1)[w_m''(s) + \frac{2}{s}w_m'(s) + e^{-w_m(s)}]ds, \\ w_{m+1} = v_m - \int_0^t s^2(\frac{1}{2} - \frac{1}{t})[v_m''(s) + \frac{2}{s}v_m'(s) + e^{-v_m(s)}]ds \\ - \int_t^1 s(\frac{s}{2} - 1)[v_m''(s) + \frac{2}{s}v_m'(s) + e^{-v_m(s)}]ds. \end{cases} \tag{5.6}$$

Green's function involved in scheme (5.6) is provided in Figure 3. The absolute error in this case is given in Table 4. The graphical comparison of the absolute errors in this case is given in Figure 4. Again, we see that the PMH-Green's iterative approach is more accurate than the Picard-Green's and Mann-Green's iterative approaches for problems (5.4) and (5.5).

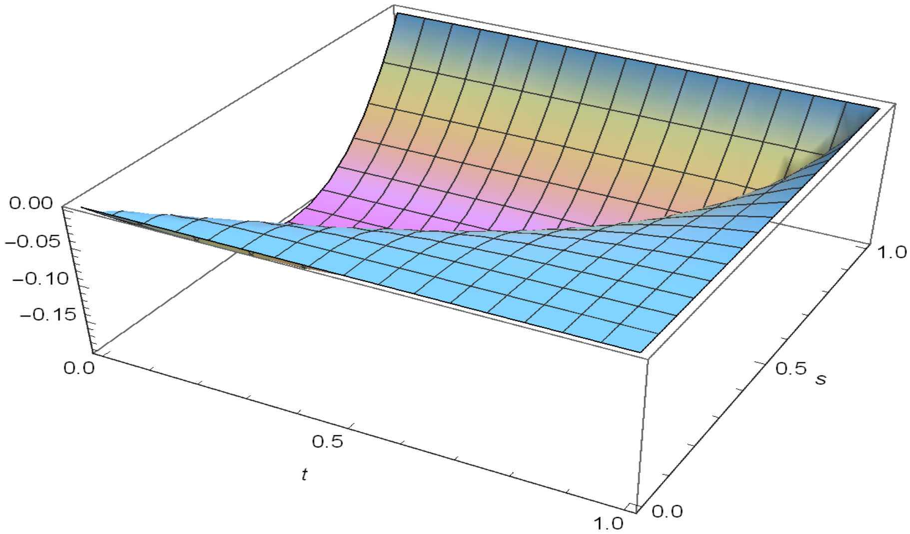

Figure 3. Plot of Green's function involved in the scheme (5.6).

Table 4. Absolute error between different iterations.

Values of t	**Picard − Green**	**Mann − Green**	**PMH − Green**
0.1	2.67164×10^{-8}	1.40478×10^{-8}	7.77156×10^{-16}
0.2	2.62110×10^{-8}	1.37820×10^{-8}	8.32667×10^{-16}
0.3	2.53779×10^{-8}	1.33440×10^{-8}	7.77156×10^{-16}
0.4	2.42312×10^{-8}	1.27410×10^{-8}	7.21645×10^{-16}
0.5	2.27905×10^{-8}	1.19835×10^{-8}	6.93889×10^{-16}
0.6	2.10813×10^{-8}	1.10848×10^{-8}	6.66134×10^{-16}
0.7	1.91348×10^{-8}	1.00613×10^{-8}	6.10623×10^{-16}
0.8	1.69877×10^{-8}	8.93231×10^{-9}	5.27356×10^{-16}
0.9	1.46826×10^{-8}	7.72027×10^{-9}	4.99600×10^{-16}

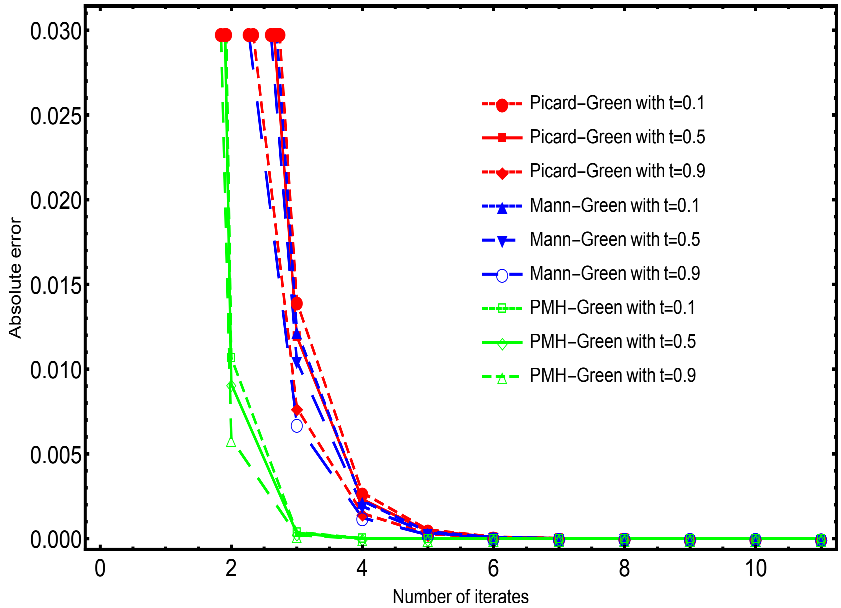

Figure 4. Comparison of various iterative schemes based on Green's function for Example 5.2.

6. Conclusions

We modified the PHM-iterative scheme by embedding a Green's function of a certain SBVP. The convergence to a sought solution to a given SBVP is proved under some possible mild conditions. We proved the proposed iterative scheme is weak w^2-stable. Some numerical experiments are performed and it has been shown the numerical accuracy of the PMH-Green's is more accurate and corresponds to the Picard-Green's and Mann-Green's iterative schemes studied by Assadi et al. [18]. Since the proposed scheme is stable and gives high accurate numerical solutions in the setting of SBVPs, we conclude that our results improve and extend many other results of the literature due to various authors. Our next aim is to use the proposed scheme for other BVPs that arise in nano-fluid and mathematical physics. Eventually, we point out the following:

(i) If we prove the mapping F_G a Kannan contraction, that is, for all v, w, there exists a constant $\mu \in [0, \frac{1}{2})$ such that $\|F_G v - F_G w\| \leq \mu[\|v - F_G v\| + \|w - F_G w\|]$, then F_G also admits a unique fixed point. Thus, our iterative scheme can be used to approximate the solutions. However, the Kannan mappings are sometimes not continuous [30]; therefore, this is not a good option in our case. Furthermore, the space $C[0, 1]$ contains only continuous functions.

(ii) Our iterative scheme uses only one sequence of scalars $\{\alpha_m\}$ and the condition we imposed on it is very simple in our convergence theorem. Thus, our iterative scheme needs few parameters to start. Moreover, it gives high accurate results for small values of this sequence.

(iii) The iterative scheme is proved weak w^2-stable. We know that every stable iterative scheme is weak w^2-stable but the converse is not generally true. Thus, our result of weak w^2-stability contains the case when one proves a stability result for our iterative scheme in the classical sense.

(iv) Since the Banach space used in this paper contains only continuous functions, it is a challenge for us to replace this Banach space by a Banach space that also contains some discontinuous functions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Authors contributions

Muhammad Arshad gave the idea as a supervisor. Junaid Ahmad wrote the initial draft. Reny George edited the final version and approved for submission.

Acknowledgments

The authors extend their appreciation to Prince Sattam bin Abdulaziz University, Saudi Arabia for funding this research work through the project number (PSAU/2023/01/9010).

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. K. Zhao, Existence and UH-stability of integral boundary problem for a class of nonlinear higher-order Hadamard fractional Langevin equation via Mittag-Leffler functions, *Filomat*, **37** (2023), 1053–1063. https://doi.org/10.2298/FIL2304053Z
2. S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrales, *Fund. Math.* **3** (1922), 133–181. https://doi.org/10.4064/fm-3-1-133-181
3. E. M. Picard, Memorie sur la theorie des equations aux derivees partielles et la methode des approximation ssuccessives, *J. Math. Pure Appl.*, **6** (1890), 145–210.
4. P. Cholamjiak, W. Cholamjiak, Y. J. Cho, S. Suantai, Weak and strong convergence to common fixed points of a countable family of multi-valued mappings in Banach spaces, *Thai J. Math.*, **9** (2011), 505–520.
5. R. Pandey, R. Pant, V. Rakocevie, R. Shukla, Approximating fixed points of a general class of nonexpansive mappings in Banach spaces with applications, *Results Math.* **74** (2018), 1–24. https://doi.org/10.1007/s00025-018-0930-6
6. I. Uddin, M. Imdad, Convergence of SP-iteration for generalized nonexpansive mapping in Hadamard spaces, *Hacet. J. Math. Stat.*, **47** (2018), 1595–1604.
7. H. Afsharia, H. Aydi, Some results about Krasnoselskii-Mann iteration process, *J. Nonlinear Sci. Appl.*, **9** (2016), 4852–4859. https://doi.org/10.22436/jnsa.009.06.120
8. F. E. Browder, Nonexpansive nonlinear operators in a Banach space, *P. Natl. Acad. Sci. USA*, **54** (1965), 1041–1044. https://doi.org/10.1073/pnas.54.4.1041
9. D. Gohde, Zum prinzip der kontraktiven abbildung, *Math. Nachr.*, **30** (1965), 251–258. https://doi.org/10.1002/mana.19650300312

10. W. A. Kirk, A fixed point theorem for mappings which do not increase distance, *Am. Math. Mon.*, **72** (1965), 1004–1006. https://doi.org/10.2307/2313345
11. V. Berinde, *Iterative approximation of fixed points*, 2 Eds., Lecture Notes in Mathematics, Berlin: Springer, 2007. https://doi.org/10.1109/SYNASC.2007.49
12. W. R. Mann, Mean value methods in iteration, *P. Am. Math. Soc.*, **4** (1953), 506–510. https://doi.org/10.1090/S0002-9939-1953-0054846-3
13. S. H. Khan, A Picard-Mann hybrid iterative process, *Fixed Point Theory A.*, **69** (2013), 1–10. https://doi.org/10.1186/1687-1812-2013-69
14. K. Zhao, Existence, stability and simulation of a class of nonlinear fractional Langevin equations involving nonsingular Mittag-Leffler kernel, *Fractal Fract.*, **6** (2022), 469. https://doi.org/10.3390/fractalfract6090469
15. S. A. Khuri, A. Sayfy, Variational iteration method: Green's functions and fixed point iterations perspective, *Appl. Math. Lett.*, **32** (2014), 24–34. https://doi.org/10.1016/j.aml.2014.01.006
16. S. A. Khuri, A. Sayfy, Generalizing the variational iteration method for BVPs: Proper setting of the correction functional, *Appl. Math. Lett.*, **68** (2017), 68–75. https://doi.org/10.1016/j.aml.2016.11.018
17. S. A. Khuri, A. Sayfy, An iterative method for boundary value problems, *Nonlinear Sci. Lett. A*, **8** (2017), 178–186.
18. R. Assadi, S. A. Khuri, A. Sayfy, Numerical solution of nonlinear second order singular BVPs based on Green's functions and fixed point Iterative schemes, *Int. J. Appl. Comput. Math.*, **4** (2018), 1–13. https://doi.org/10.1007/s40819-018-0569-8
19. K. Zhao, Stability of a nonlinear langevin system of ML-type fractional derivative affected by time-varying delays and differential feedback control, *Fractal Fract.*, **6** (2022), 725. https://doi.org/10.3390/fractalfract6120725
20. J. Ahmad, M. Arshad, A. Hussain, H. Al-Sulami, A Green's function based iterative approach for solutions of BVPs in symmetric spaces, *Symmetry*, **15** (2023), 1838.
21. K. Zhao, Stability of a nonlinear ML-nonsingular kernel fractional Langevin system with distributed lags and integral control, *Axioms*, **11** (2022), 350. https://doi.org/10.3390/axioms11070350
22. M. O. Osilike, Stability of the Mann and Ishikawa iteration procedures for ϕ-strong pseudocontractions and nonlinear equations of the ϕ-strongly accretive type, *J. Math. Anal. Appl.*, **227** (1998), 319–334. https://doi.org/10.1006/jmaa.1998.6075
23. A. Sahin, Some new results of M-iteration process in hyperbolic spaces, *Carpathian J. Math.*, **35** (2019), 221–232. https://doi.org/10.37193/CJM.2019.02.10
24. A. Sahin, Some results of the Picard-Krasnoselskii hybrid iterative process, *Filomat*, **33** (2019), 359–365. https://doi.org/10.2298/FIL1902359S
25. M. Urabe, Convergence of numerical iteration in solution of equations, *J. Sci. Hiroshima Univ. A*, **19** (1956), 479–489. https://doi.org/10.32917/hmj/1556071264
26. A. M. Harder, T. L. Hicks, Stability results for fixed point iteration procedures, *Math. Japonica*, **33** (1988), 693–706.

27. T. Cardinali, P. Rubbioni, A generalization of the Caristi fixed point theorem in metric spaces, *Fixed Point Theory*, **11** (2010), 3–10.

28. I. Timis, On the weak stability of Picard iteration for some contractive type mappings, *Ann. Univ. Craiova-Mat.*, **37** (2010), 106–114.

29. M. Chawla, R. Subramanian, H. Sathi, A fourth order method for a singular two-point boundary value problem, *BIT*, **28** (1988), 88–97. https://doi.org/10.1007/BF01934697

30. P. Debnath, N. Konwar, S. Radenovic, Metric fixed point theory: Applications in science, engineering and behavioural sciences, Singapore: Springer, 2023. https://doi.org/10.1007/978-981-16-4896-0