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Research article

Analysis of soliton phenomena in (2+1)-dimensional Nizhnik-Novikov-Veselov model via a modified analytical technique

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Abstract: The present research applies an improved version of the modified Extended Direct Algebraic Method (mEDAM) called r+mEDAM to examine soliton phenomena in a notable mathematical model, namely the (2+1)-dimensional Nizhnik-Novikov-Veselov Model (NNVM), which possesses potential applications in exponentially localized structure interactions. The generalized hyperbolic and trigonometric functions are used to disclose a variety of soliton solutions, including kinks, anti-kink, bell-shaped and periodic soliton. Some 3D graphs are plotted for visual representations of these solutions which highlight their adaptability. The results provide a basis for practical usage and expansions to related mathematical models or physical systems. They also expand our understanding of the NNVM's dynamics, providing insights into its behavior and prospective applications.

Keywords: partial differential equations; Nizhnik-Novikov-Veselov model; modified extended direct algebraic method; variable transformation; soliton; incompressible fluid; shallow water wave
Mathematics Subject Classification: 33B15, 34A34, 35A20, 44A10, 35A22

1. Introduction

The use of Differential Equations (DEs) is widespread in many fields, including engineering and mathematical physics [1–5]. In the investigation of nonlinear science, the explicit solutions of Nonlinear Partial Differential Equations (NPDEs) hold prominent position. The inverse scattering transform [6], the Bäcklund transformation method [7], the Darboux transformation [8], exp-function and Jacobi elliptic function method [9–11], (G'/G)-expansion method [12, 13], Riccati mapping method [14], the unified method (UM) and its generalized form (GUM) [15,16] and the Hirota bilinear method [17–20], sech-function method [21], mEDAM [22] and many others [23–32] are a few efficient methods that have been developed to solve NPDEs. Among these techniques, mEDAM stands out as a very straightforward and practical method for finding precise soliton solutions for NPDEs [33–35].

via Due to its occurrence in a variety of contexts, including shallow-water waves, nonlinear optics, plasma and Bose-Einstein condensates, soliton solutions have recently attracted a great deal of interest [36–41]. Helal, for instance, provides a thorough discussion of the soliton solutions for various well-known NPDEs, such as the KdV, mKdV, Sine-Gordon, and nonlinear Schrödinger equations, in [42]. By utilizing Wronskian determinants, Freeman's work [43] displays soliton solutions for three important nonlinear evolution equations: Kadomtsev-Petviashvili, nonlinear Schrödinger and Davey-Stewartson. This work reveals difficulties in the Davey-Stewartson case with phase variables linked to soliton count, necessitating different Bäcklund transformations. This illustrates the complexity of these equations' soliton solutions. Burger's, Zakharov-Kuznetsov (ZK) and Korteweg-de Vries (KdV) equations may be precisely solved using the exponential function approach, according to Shumaila et al. [44], which relies on a sequence of exponential functions. Wang et al. in [45] successfully deduced the three-component coupled Hirota hierarchy by employing the dressing method. Tian et al. [46] proposed an effective and direct approach to study the symmetry-preserving discretization for a class of generalized higher order equations, and proposed an open problem about symmetries and the multipliers of conservation law. In addition, Li and Tian [47] solved the Cauchy problem of the general n-component nonlinear Schrödinger equations, and gave the N-soliton solutions. Besides, a conjecture about the law of nonlinear wave propagation was proposed. Moreover, with respect to deriving the solutions of Wadati-Konno-Ichikawa equation and complex short pulse equation, Li et al. have done some interesting work by employing the steepest descent method [48]. They solved the long-time asymptotic behavior of the solutions of these equations, and proved the soliton resolution conjecture and the asymptotic stability of solutions of these equations.

In this paper, we will explore and analyze the soliton solutions for (2+1)-dimensional NNVM using the r+mEDAM. This model is mathematically presented as [49]:

$$\begin{aligned} &u_t + bu_{yyy} + au_{xxx} + eu_y + du_x - 3b(uw)_y - 3a(uv)_x = 0, \\ &v_y - u_x = 0, \quad w_x - u_y = 0, \end{aligned} \tag{1.1}$$

where $u = u(x, y, t)$, $v = v(x, y, t)$ & $w = w(x, y, t)$ and are the components of the (dimensionless) velocity [50], a, b, d & e are constant coefficients. NNVM is the only known isotropic Lax extension of the Korteweg-de Vries (KdV) equation [51]. The NNVM is a theoretical physics and nonlinear dynamics mathematical model. It is largely used to describe specific types of integrable systems, most notably soliton theory and integrable hierarchies. Nizhnik, Novikov and Veselov proposed the model in the 1980s [52], and it has been intensively investigated in the field of mathematical physics. It is useful in understanding the behavior of many physical systems and has applications in domains such as fluid dynamics and plasma physics. For phenomena like shallow-water waves, lengthy internal waves and acoustic waves that occur in the world of incompressible fluids, the NNVM is important. An acoustic wave is a mechanical disturbance in an incompressible fluid that propagates as a pressure fluctuation without generating substantial density changes in the fluid [53–55]. Many academics have used various methods to investigate the NNVM. Utilizing different analytical methods, Osman et al.

tackled this equation [56–58]. The inner parameter-dependent symmetry constraint of the KP equation is another source from which NNVM can be obtained [59].

In view of this literature, our main goal of this work is to present another straightforward method r+mEDAM for creating different soliton solutions for NNVM, such as kink wave solutions, anti-kink wave solutions, periodic wave solutions and bell-shaped wave solutions. The suggested r+mEDAM for generating soliton solutions in the context of the NNVM is very resilient. This is due to its simple and efficient analytical method, which eliminates the need for linearization or sophisticated numerical procedures. The method's direct computations assure dependability and precision, while its ability to produce different families of soliton solutions gives significant insights into the NNVM model's underlying features. The r+mEDAM is a strong instrument for examining the complicated dynamics of acoustic waves in the world of incompressible fluid, with potential applications spanning numerous areas of shallow-water waves, lengthy internal waves and related sciences due to its simplicity and large solution space [60–62].

The rest of this paper is structured as follows: In Section 2, we describe the r+mEDAM's operating procedure. In Section 3, we use the r+mEDAM to find soliton solutions for the (2+1)-dimensional NNVM problem. Section 4 focuses on the presentation of graphs and the discussion of these graphical representations. The last section presents an overview and a summary of the results of our investigation.

2. The working strategy of r+mEDAM

In this part, we present the r+mEDAM method. Let us consider the PDE in the format described below [63, 64]:

$$E(u, v, w, u_t, v_x, w_y, u_x v_t, w_x u_y \ldots) = 0, \tag{2.1}$$

where $u = u(x, y, t)$, $v = v(x, y, t)$ & $w = w(x, y, t)$.

The following approaches are used to solve Eq (2.1):

(1) A variable transformation is first carried out as follows:

$$\begin{aligned} u(x, y, t) &= U(\sigma), \\ v(x, y, t) &= V(\sigma), \\ w(x, y, t) &= W(\sigma), \\ \sigma &= \mu(-ct + x + y). \end{aligned} \tag{2.2}$$

This transformation converts Eq (2.1) into the following NODE:

$$F(U, U', V, V', W, W', UV', W'U', VU', \ldots) = 0, \tag{2.3}$$

where primes represents derivatives of U, V & W with respect to σ in Eq (2.2). For the purpose of figuring out the constant(s) of integration, (2.3) may occasionally be integrated once or more.

(2) Then, we propose that Eq (2.3) contains the following solution (in our case):

$$U(\sigma) = V(\sigma) = W(\sigma) = \sum_{i=0}^{N} s_i (r + A(\sigma))^i. \tag{2.4}$$

The parameters s_i (where $i = 0, \ldots, N$) are to be driven, and $A(\sigma)$ satisfies the subsequent NODE:

$$A'(\sigma) = \ln(\lambda)(j + kA(\sigma) + l(A(\sigma))^2), \tag{2.5}$$

where $\lambda \neq 0, 1$ & j, k, l are constants.
(3) By seeking a homogeneous balance between the primary nonlinear component and the highest order derivative in Eq (2.3), we may get the positive integer N presented in Eq (2.4).
(4) We first substitute Eq (2.4) into Eq (2.3), or the equation resulting from integrating (2.3), in order to build a polynomial expression in $(A(\sigma))$, and then we arrange all of the terms of $(A(\sigma))$ in the same order. The coefficients of this derivation polynomial are then equal to zero in order to determine a system of nonlinear algebraic equations in s_i (where $i = 0, \ldots, N$) & other associated parameters.
(5) This system of algebraic equations is resolved using the MAPLE software.
(6) The analytical solutions for Eq (2.1), which are acquired from Eqs (2.4) and (2.3), are then produced by accounting for the unknown parameters and substituting the $A(\sigma)$ solution into Eq (2.4). Using the generic solution of Eq (2.2), we may get the families of soliton solutions that are depicted below:
Family 1: For $\Omega < 0$ & $l \neq 0$,

$$A_1(\sigma) = -\frac{k}{2l} + \frac{\sqrt{-\Omega}\tan_\lambda\left(\frac{1}{2}\sqrt{-\Omega}\sigma\right)}{2l},$$

$$A_2(\sigma) = -\frac{k}{2l} - \frac{\sqrt{-\Omega}\cot_\lambda\left(\frac{1}{2}\sqrt{-\Omega}\sigma\right)}{2l},$$

$$A_3(\sigma) = -\frac{k}{2l} + \frac{\sqrt{-\Omega}\left(\tan_\lambda\left(\sqrt{-\Omega}\sigma\right) \pm \left(\sqrt{pq}\sec_\lambda\left(\sqrt{-\Omega}\sigma\right)\right)\right)}{2l},$$

$$A_4(\sigma) = -\frac{k}{2l} - \frac{\sqrt{-\Omega}\left(\cot_\lambda\left(\sqrt{-\Omega}\sigma\right) \pm \left(\sqrt{pq}\csc_\lambda\left(\sqrt{-\Omega}\sigma\right)\right)\right)}{2l},$$

&

$$A_5(\sigma) = -\frac{k}{2l} + \frac{\sqrt{-\Omega}\left(\tan_\lambda\left(\frac{1}{4}\sqrt{-\Omega}\sigma\right) - \cot_\lambda\left(\frac{1}{4}\sqrt{-\Omega}\sigma\right)\right)}{4c}.$$

Family 2: For $\Omega > 0$ & $l \neq 0$,

$$A_6(\sigma) = -\frac{k}{2l} - \frac{\sqrt{\Omega}\tanh_\lambda\left(\frac{1}{2}\sqrt{\Omega}\sigma\right)}{2l},$$

$$A_7(\sigma) = -\frac{k}{2l} - \frac{\sqrt{\Omega}\coth_\lambda\left(\frac{1}{2}\sqrt{\Omega}\sigma\right)}{2l},$$

$$A_8(\sigma) = -\frac{k}{2l} - \frac{\sqrt{\Omega}\left(\tanh_\lambda\left(\sqrt{\Omega}\sigma\right) \pm \left(\sqrt{pq}sech_\lambda\left(\sqrt{\Omega}\sigma\right)\right)\right)}{2l},$$

$$A_9(\sigma) = -\frac{k}{2l} - \frac{\sqrt{\Omega}\left(\coth_\lambda\left(\sqrt{\Omega}\sigma\right) \pm \left(\sqrt{pq}csch_\lambda\left(\sqrt{\Omega}\sigma\right)\right)\right)}{2l},$$

&

$$A_{10}(\sigma) = -\frac{k}{2l} - \frac{\sqrt{\Omega}\left(\tanh_\lambda\left(\frac{1}{4}\sqrt{\Omega}\sigma\right) - \coth_\lambda\left(\frac{1}{4}\sqrt{\Omega}\sigma\right)\right)}{4l}.$$

Family 3: For $jl > 0$ & $k = 0$,

$$A_{11}(\sigma) = \sqrt{\frac{j}{l}}\tan_\lambda\left(\sqrt{jl}\sigma\right),$$

$$A_{12}(\sigma) = -\sqrt{\frac{j}{l}}\cot_\lambda\left(\sqrt{jl}\sigma\right),$$

$$A_{13}(\sigma) = \sqrt{\frac{j}{l}}\left(\tan_\lambda\left(2\sqrt{jl}\sigma\right) \pm \left(\sqrt{pq}\sec_\lambda\left(2\sqrt{jl}\sigma\right)\right)\right),$$

$$A_{14}(\sigma) = -\sqrt{\frac{j}{l}}\left(\cot_\lambda\left(2\sqrt{jl}\sigma\right) \pm \left(\sqrt{pq}\csc_\lambda\left(2\sqrt{jl}\sigma\right)\right)\right),$$

&

$$A_{15}(\sigma) = \frac{1}{2}\sqrt{\frac{j}{l}}\left(\tan_\lambda\left(\frac{1}{2}\sqrt{jl}\sigma\right) - \cot_\lambda\left(\frac{1}{2}\sqrt{jl}\sigma\right)\right).$$

Family 4: For $jl < 0$ & $k = 0$,

$$A_{16}(\sigma) = -\sqrt{-\frac{j}{l}}\tanh_\lambda\left(\sqrt{-jl}\sigma\right),$$

$$A_{17}(\sigma) = -\sqrt{-\frac{j}{l}}\coth_\lambda\left(\sqrt{-jl}\sigma\right),$$

$$A_{18}(\sigma) = -\sqrt{-\frac{j}{l}}\left(\tanh_\lambda\left(2\sqrt{-jl}\sigma\right) \pm \left(i\sqrt{pq}sech_\lambda\left(2\sqrt{-jl}\sigma\right)\right)\right),$$

$$A_{19}(\sigma) = -\sqrt{-\frac{j}{l}}\left(\coth_\lambda\left(2\sqrt{-jl}\sigma\right) \pm \left(\sqrt{pq}csch_\lambda\left(2\sqrt{-jl}\sigma\right)\right)\right),$$

&

$$A_{20}(\sigma) = -\frac{1}{2}\sqrt{-\frac{j}{l}}\left(\tanh_\lambda\left(\frac{1}{2}\sqrt{-jl}\sigma\right) + \coth_\lambda\left(\frac{1}{2}\sqrt{-jl}\sigma\right)\right).$$

Family 5: For $l = j$ & $k = 0$,

$$A_{21}(\sigma) = \tan_\lambda(j\sigma),$$

$$A_{22}(\sigma) = -\cot_\lambda(j\sigma),$$

$$A_{23}(\sigma) = \tan_\lambda(2j\sigma) \pm \left(\sqrt{pq}\sec_\lambda(2j\sigma)\right),$$

$$A_{24}(\sigma) = -\cot_\lambda(2j\sigma) \pm \left(\sqrt{pq}\csc_\lambda(2j\sigma)\right),$$

&

$$A_{25}(\sigma) = \frac{1}{2}\tan_\lambda\left(\frac{1}{2}j\sigma\right) - \frac{1}{2}\cot_\lambda\left(\frac{1}{2}j\sigma\right).$$

Family 6: For $l = -j$ & $k = 0$,

$$A_{26}(\sigma) = -\tanh_\lambda(j\sigma),$$

$$A_{27}(\sigma) = -\coth_\lambda(j\sigma),$$

$$A_{28}(\sigma) = -\tanh_\lambda(2j\sigma) \pm \left(i\sqrt{pq}sech_\lambda(2j\sigma)\right),$$

$$A_{29}(\sigma) = -\coth_{\lambda}(2\,j\sigma) \pm \left(\sqrt{pq}csch_{\lambda}(2\,j\sigma)\right),$$

&

$$A_{30}(\sigma) = -\frac{1}{2}\tanh_{\lambda}\left(\frac{1}{2}j\sigma\right) - \frac{1}{2}\coth_{\lambda}\left(\frac{1}{2}j\sigma\right).$$

Family 7: For $J = 0$,

$$A_{31}(\sigma) = -2\,\frac{j\,(k\sigma\ln\lambda + 2)}{k^2\sigma\ln\lambda}.$$

Family 8: For $k = h$, $j = sh(s \neq 0)$ & $l = 0$,

$$A_{32}(\sigma) = \lambda^{\sigma\,\sigma} - s.$$

Family 9: For $k = l = 0$,

$$A_{33}(\sigma) = j\sigma\ln(\lambda).$$

Family 10: For $k = j = 0$,

$$A_{34}(\sigma) = -\frac{1}{l\sigma\ln(\lambda)}.$$

Family 11: For $j = 0$, $k \neq 0$ & $l \neq 0$,

$$A_{35}(\sigma) = -\frac{pk}{l\,(\cosh_{\lambda}(k\sigma) - \sinh_{\lambda}(k\sigma) + p)},$$

&

$$A_{36}(\sigma) = -\frac{k\,(\cosh_{\lambda}(k\sigma) + \sinh_{\lambda}(k\sigma))}{l\,(\cosh_{\lambda}(k\sigma) + \sinh_{\lambda}(k\sigma) + q)}.$$

Family 12: For $k = h$, $l = sh(s \neq 0)$ & $j = 0$,

$$A_{37}(\sigma) = \frac{p\lambda^{\sigma h}}{p - sq\lambda^{\sigma h}},$$

$q, p > 0$ & are known as deformation parameters when $\Omega = k^2 - 4jl$ is satisfied. The following is a description of the hyperbolic & trigonometric functions that are generalized:

$$\sin_{\lambda}(\sigma) = \frac{pA^{i\sigma} - qA^{-i\sigma}}{2i}, \quad \cos_{\lambda}(\sigma) = \frac{pA^{i\sigma} + qA^{-i\sigma}}{2},$$
$$\sec_{\lambda}(\sigma) = \frac{1}{\cos_{\lambda}(\sigma)}, \quad \csc_{\lambda}(\sigma) = \frac{1}{\sin_{\lambda}(\sigma)},$$
$$\tan_{\lambda}(\sigma) = \frac{\sin_{\lambda}(\sigma)}{\cos_{\lambda}(\sigma)}, \quad \cot_{\lambda}(\sigma) = \frac{\cos_{\lambda}(\sigma)}{\sin_{\lambda}(\sigma)}.$$

Similarly,

$$\sinh_{\lambda}(\sigma) = \frac{pA^{\sigma} - qA^{-\sigma}}{2}, \quad \cosh_{\lambda}(\sigma) = \frac{pA^{\sigma} + qA^{-\sigma}}{2},$$
$$sech_{\lambda}(\sigma) = \frac{1}{\cosh_{\lambda}(\sigma)}, \quad csch_{\lambda}(\sigma) = \frac{1}{\sinh_{\lambda}(\sigma)},$$
$$\tanh_{\lambda}(\sigma) = \frac{\sinh_{\lambda}(\sigma)}{\cosh_{\lambda}(\sigma)}, \quad \coth_{\lambda}(\sigma) = \frac{\cosh_{\lambda}(\sigma)}{\sinh_{\lambda}(\sigma)}.$$

3. Application of r+mEDAM

We are going to tackle NNVM with r+mEDAM in this part. We begin by transforming the variables as described in (2.3) in order to obtain the soliton solution for (1.1). The following system of NODEs is created by transforming (1.1):

$$\begin{aligned}&(-c+e+d)U'+\mu^2(b+a)U'''-3[a(UV)'+b(UW)']=0,\\&\mu U'=\mu V',\\&\mu U'=\mu W'.\end{aligned}\tag{3.1}$$

By integrating all equations in (3.1) w.r.t σ and keeping the constant of integration zero yields:

$$\begin{aligned}&(-c+e+d)U+\mu^2(b+a)U''-3[a(UV)+b(UW)]=0,\\&U=V,\\&U=W.\end{aligned}\tag{3.2}$$

Putting second & third parts of (3.2) in first part gives following single NODE:

$$(-c+e+d)U+\mu^2(b+a)U''-3(a+b)U^2=0.\tag{3.3}$$

Establishing homogenous balance between highest order derivative U'' and nonlinear term U^2 gives $N+2=2N$ which implies $N=2$. By substituting $N=2$ in (), we get the following series form solution for (3.3):

$$U(\sigma)=\sum_{i=0}^{2}s_i(r+A(\sigma))^i=s_0+s_1(r+A(\sigma))^1+s_2(r+A(\sigma))^2.\tag{3.4}$$

By putting (3.4) in (3.3) and collecting all terms with same powers of $A(\sigma)$, we get an expression in $A(\sigma)$. Equating all coefficients to zero gives a system of nonlinear algebraic equations. Employing Maple and solving the system gives the following two cases of solutions:

Case 1.

$$\begin{aligned}&s_0=s_0,\, s_1=\frac{s_0\,(-2\,rl+k)}{j-kr+lr^2},\, s_2=\frac{s_0 l}{j-kr+lr^2},\mu=\frac{1}{\sqrt{2}\ln(\lambda)}\sqrt{\frac{s_0}{l\,(j-kr+lr^2)}}\\&c=-\frac{-2\,lje+4\,s_0ajl+4\,s_0bjl-2\,ljd+2\,erkl+2\,drkl-2\,el^2r^2-2\,dl^2r^2-s_0bk^2-s_0ak^2}{2l\,(j-kr+lr^2)}.\end{aligned}\tag{3.5}$$

Case 2.

$$\begin{aligned}&s_0=s_0,\, s_1=6\,\frac{s_0 l\,(-2\,rl+k)}{k^2+2\,lj-6\,rlk+6\,l^2r^2},\, s_2=6\,\frac{s_0 l^2}{k^2+2\,lj-6\,rlk+6\,l^2r^2},\\&c=\frac{k^2e-3\,s_0k^2(a+b)+k^2d+2\,lje+12\,s_0ajl+12\,s_0bjl+2\,ljd-6\,erkl-6\,drkl+6\,l^2r^2(e+d)}{k^2+2\,lj-6\,rlk+6\,l^2r^2},\\&\mu=\frac{\sqrt{3}}{\ln(\lambda)}\sqrt{\frac{s_0}{k^2+2\,lj-6\,rlk+6\,l^2r^2}}.\end{aligned}\tag{3.6}$$

By taking Case 1 into account and using (2.3), (3.4) & the corresponding solution of (3.3), we get the following families of soliton solutions for (1.1):

Family 1.1. For $\Omega < 0$ and $l \neq 0$,

$$u_{1,1}(t,x,y) = -\frac{1}{4}\frac{s_0\left(-4\,lj + k^2 + \Omega\left(\tan_\lambda\left(\frac{1}{2}\sqrt{-\Omega}\sigma\right)\right)^2\right)}{l\,(j - rk + lr^2)}, \tag{3.7}$$

$$u_{1,2}(t,x,y) = -\frac{1}{4}\frac{s_0\left(-4\,lj + k^2 + \Omega\left(\cot_\lambda\left(\frac{1}{2}\sqrt{-\Omega}\sigma\right)\right)^2\right)}{l\,(j - rk + lr^2)}, \tag{3.8}$$

$$u_{1,3}(t,x,y) = \frac{1}{4}\frac{s_0\left(-\Omega - 2\,\Omega\,\sin_\lambda\left(\sqrt{-\Omega}\sigma\right)\sqrt{pq} - \Omega\,pq\right)}{l\left(\cos_\lambda\left(\sqrt{-\Omega}\sigma\right)\right)^2(j - rk + lr^2)}, \tag{3.9}$$

$$u_{1,4}(t,x,y) = \frac{1}{4}\frac{s_0\left(\Omega + 2\,\Omega\,\cos_\lambda\left(\sqrt{-\Omega}\sigma\right)\sqrt{pq} + \Omega\,pq\right)}{l\left(-1 + \left(\cos_\lambda\left(\sqrt{-\Omega}\sigma\right)\right)^2\right)(j - rk + lr^2)}, \tag{3.10}$$

and

$$u_{1,5}(t,x,y) = \frac{1}{16}\frac{s_0\left(\Omega - 4\,\Omega\left(\cos_\lambda\left(\frac{1}{4}\sqrt{-\Omega}\sigma\right)\right)^2 + 4\,\Omega\left(\cos_\lambda\left(\frac{1}{4}\sqrt{-\Omega}\sigma\right)\right)^4\right)}{\left(\cos_\lambda\left(1/4\sqrt{-\Omega}\sigma\right)\right)^2\left(-1 + \left(\cos_\lambda\left(\frac{1}{4}\sqrt{-\Omega}\sigma\right)\right)^2\right)l\,(j - rk + lr^2)}. \tag{3.11}$$

Family 1.2. For $\Omega > 0$ and $l \neq 0$,

$$u_{1,6}(t,x,y) = \frac{1}{4}\frac{s_0\left(4\,lj - k^2 + \Omega\left(\tanh_\lambda\left(\frac{1}{2}\sqrt{\Omega}\sigma\right)\right)^2\right)}{l\,(j - rk + lr^2)}, \tag{3.12}$$

$$u_{1,7}(t,x,y) = \frac{1}{4}\frac{s_0\left(4\,lj - k^2 + \Omega\left(\coth_\lambda\left(\frac{1}{2}\sqrt{\Omega}\sigma\right)\right)^2\right)}{l\,(j - rk + lr^2)}, \tag{3.13}$$

$$u_{1,8}(t,x,y) = \frac{1}{4}\frac{s_0\left(-\Omega + 2\,\Omega\,\sinh_\lambda\left(\sqrt{\Omega}\sigma\right)\sqrt{pq} + \Omega\,pq\right)}{l\left(\cosh_\lambda\left(\sqrt{\Omega}\sigma\right)\right)^2(j - rk + lr^2)}, \tag{3.14}$$

$$u_{1,9}(t,x,y) = \frac{1}{4}\frac{s_0\left(\Omega + 2\,\Omega\,\cosh_\lambda\left(\sqrt{\Omega}\sigma\right)\sqrt{pq} + \Omega\,pq\right)}{l\left(\left(\cosh_\lambda\left(\sqrt{\Omega}\sigma\right)\right)^2 - 1\right)(j - rk + lr^2)}, \tag{3.15}$$

and

$$u_{1,10}(t,x,y) = \frac{1}{16}\frac{s_0\,(\Omega)}{\left(\cosh_\lambda\left(\frac{1}{4}\sqrt{\Omega}\sigma\right)\right)^2\left(\left(\cosh_\lambda\left(\frac{1}{4}\sqrt{\Omega}\sigma\right)\right)^2 - 1\right)l\,(j - rk + lr^2)}. \tag{3.16}$$

Family 1.3. For $jl > 0$ and $k = 0$,

$$u_{1,11}(t,x,y) = \frac{s_0 j\left(1+\left(\tan_\lambda\left(\sqrt{lj}\sigma\right)\right)^2\right)}{j+lr^2}, \tag{3.17}$$

$$u_{1,12}(t,x,y) = \frac{s_0 j\left(1+\left(\cot_\lambda\left(\sqrt{lj}\sigma\right)\right)^2\right)}{j+lr^2}, \tag{3.18}$$

$$u_{1,13}(t,x,y) = \frac{s_0 j\left(1+2\ \sin_\lambda\left(2\ \sqrt{lj}\sigma\right)\sqrt{pq}+pq\right)}{\left(\cos_\lambda\left(2\ \sqrt{lj}\sigma\right)\right)^2(j+lr^2)}, \tag{3.19}$$

$$u_{1,14}(t,x,y) = -\frac{s_0 j\left(1+2\ \cos_\lambda\left(2\ \sqrt{lj}\sigma\right)\sqrt{pq}+pq\right)}{\left(-1+\left(\cos_\lambda\left(2\ \sqrt{lj}\sigma\right)\right)^2\right)(j+lr^2)}, \tag{3.20}$$

and

$$u_{1,15}(t,x,y) = -\frac{1}{4}\frac{js_0}{\left(\cos_\lambda\left(\frac{1}{2}\sqrt{lj}\sigma\right)\right)^2\left(-1+\left(\cos_\lambda\left(\frac{1}{2}\sqrt{lj}\sigma\right)\right)^2\right)(j+lr^2)}. \tag{3.21}$$

Family 1.4. For $jl < 0$ and $k = 0$,

$$u_{1,16}(t,x,y) = -\frac{s_0 j\left(-1+\left(\tanh_\lambda\left(\sqrt{-lj}\sigma\right)\right)^2\right)}{j+lr^2}, \tag{3.22}$$

$$u_{1,17}(t,x,y) = -\frac{s_0 j\left(-1+\left(\coth_\lambda\left(\sqrt{-lj}\sigma\right)\right)^2\right)}{j+lr^2}, \tag{3.23}$$

$$u_{1,18}(t,x,y) = -\frac{s_0 j\left(-1+2\ \sinh_\lambda\left(2\ \sqrt{-lj}\sigma\right)i\sqrt{pq}+i^2 pq\right)}{\left(\cosh_\lambda\left(2\ \sqrt{-lj}\sigma\right)\right)^2(j+lr^2)}, \tag{3.24}$$

$$u_{1,19}(t,x,y) = -\frac{s_0 j\left(1+2\ \cosh_\lambda\left(2\ \sqrt{-lj}\sigma\right)\sqrt{pq}+pq\right)}{\left(\left(\cosh_\lambda\left(2\ \sqrt{-lj}\sigma\right)\right)^2-1\right)(j+lr^2)}, \tag{3.25}$$

and

$$u_{1,20}(t,x,y) = -\frac{1}{4}\frac{js_0}{\left(\cosh_\lambda\left(\frac{1}{2}\sqrt{-lj}\sigma\right)\right)^2\left(\left(\cosh_\lambda\left(\frac{1}{2}\sqrt{-lj}\sigma\right)\right)^2-1\right)(j+lr^2)}. \tag{3.26}$$

Family 1.5. For $l = j$ and $k = 0$,

$$u_{1,21}(t,x,y) = \frac{s_0}{(1+r^2)\left(\cos_\lambda(j\sigma)\right)^2}, \tag{3.27}$$

$$u_{1,22}(t,x,y) = -\frac{s_0}{(1+r^2)\left(-1+(\cos_\lambda(j\sigma))^2\right)}, \tag{3.28}$$

$$u_{1,23}(t,x,y) = \frac{s_0\left(1+2\,\sin_\lambda(2\,j\sigma)\,\sqrt{pq}+pq\right)}{(\cos_\lambda(2\,j\sigma))^2\,(1+r^2)}, \tag{3.29}$$

$$u_{1,24}(t,x,y) = -\frac{s_0\left(1+2\,\cos_\lambda(2\,j\sigma)\,\sqrt{pq}+pq\right)}{\left(-1+(\cos_\lambda(2\,j\sigma))^2\right)(1+r^2)}, \tag{3.30}$$

and

$$u_{1,25}(t,x,y) = -\frac{1}{4}\frac{s_0}{\left(\cos_\lambda\left(\frac{1}{2}j\sigma\right)\right)^2\left(-1+\left(\cos_\lambda\left(\frac{1}{2}j\sigma\right)\right)^2\right)(1+r^2)}. \tag{3.31}$$

Family 1.6. For $l = -j$ and $k = 0$,

$$u_{1,26}(t,x,y) = -\frac{s_0}{(-1+r^2)\,(\cosh_\lambda(j\sigma))^2}, \tag{3.32}$$

$$u_{1,27}(t,x,y) = \frac{s_0}{(-1+r^2)\left((\cosh_\lambda(j\sigma))^2-1\right)}, \tag{3.33}$$

$$u_{1,28}(t,x,y) = \frac{s_0\left(-1+2\,\sinh_\lambda(2\,j\sigma)\,i\,\sqrt{pq}+i^2pq\right)}{(\cosh_\lambda(2\,j\sigma))^2\,(-1+r^2)}, \tag{3.34}$$

$$u_{1,29}(t,x,y) = \frac{s_0\left(1+2\,\cosh_\lambda(2\,j\sigma)\,\sqrt{pq}+pq\right)}{\left((\cosh_\lambda(2\,j\sigma))^2-1\right)(-1+r^2)}, \tag{3.35}$$

and

$$u_{1,30}(t,x,y) = \frac{1}{4}\frac{s_0}{\left(\cosh_\lambda\left(\frac{1}{2}j\sigma\right)\right)^2\left(\left(\cosh_\lambda\left(\frac{1}{2}j\sigma\right)\right)^2-1\right)(-1+r^2)}. \tag{3.36}$$

Family 1.7. For $\Omega = 0$,

$$u_{1,31}(t,x,y) = \frac{s_0\left(k^4\sigma^2\,(\ln(\lambda))^2\,j-2\,k^3\sigma\,\ln(\lambda)\,j\,(k\sigma\,\ln(\lambda)+2)+4\,l\,(j\,(k\sigma\,\ln(\lambda)+2))^2\right)}{k^4\sigma^2\,(\ln(\lambda))^2\,(j-rk+lr^2)}. \tag{3.37}$$

Family 1.8. For $k = h$, $a = sh(s \neq 0)$ and $l = 0$,

$$u_{1,32}(t,x,y) = s_0 + \frac{s_0\left(r+\lambda^{h\sigma}\right)}{s-r}. \tag{3.38}$$

Family 1.9. For $j = k = 0$,

$$u_{1,33}(t,x,y) = \frac{s_0}{l^2\sigma^2\,(\ln(\lambda))^2\,r^2}. \tag{3.39}$$

Family 1.10. For $j = 0$, $k \neq 0$ and $l \neq 0$,

$$u_{1,34}(t,x,y) = \frac{s_0 p k^2 \left(\cosh_\lambda(k\sigma) - \sinh_\lambda(k\sigma)\right)}{l\left(\cosh_\lambda(k\sigma) - \sinh_\lambda(k\sigma) + p\right)^2 r(k - rl)}, \tag{3.40}$$

and

$$u_{1,35}(t,x,y) = \frac{s_0 k^2 q \left(\cosh_\lambda(k\sigma) + \sinh_\lambda(k\sigma)\right)}{l\left(\cosh_\lambda(k\sigma) + \sinh_\lambda(k\sigma) + q\right)^2 r(k - rl)}. \tag{3.41}$$

Family 1.11. For $k = h$, $l = sh(s \neq 0)$ and $j = 0$,

$$u_{1,36}(t,x,y) = \frac{p s_0 \left(p\lambda^{h\sigma} - \lambda^{2h\sigma} r + \lambda^{2h\sigma} sp\right)}{\left(-p + r\lambda^{h\sigma}\right)^2 r(-1 + sr)}. \tag{3.42}$$

where $\sigma = \frac{s_0}{\sqrt{2}\ln(\lambda)}(-(\frac{-2\,lje+4\,s_0ajl+4\,s_0bjl-2\,ljd+2\,erkl+2\,drkl-2\,el^2r^2-2\,dl^2r^2-s_0bk^2-s_0ak^2}{2\sqrt{(l(j-kr+lr^2))^2}})t + x + y)$.

By taking Case 2 into account and using (2.2), (3.4) & the corresponding solution of (3), we get the following families of soliton solutions for (1.1):

Family 2.1. For $\Omega < 0$ and $l \neq 0$,

$$u_{2,1}(t,x,y) = -\frac{1}{2}\frac{s_0\left(k^2 - 4\,jl + 3\,\Omega\left(\tan_\lambda\left(\frac{1}{2}\sqrt{-\Omega}\sigma\right)\right)^2\right)}{k^2 + 2\,jl - 6\,rlk + 6\,l^2r^2}, \tag{3.43}$$

$$u_{2,2}(t,x,y) = -\frac{1}{2}\frac{s_0\left(k^2 - 4\,jl + 3\,\Omega\left(\cot_\lambda\left(\frac{1}{2}\sqrt{-\Omega}\sigma\right)\right)^2\right)}{k^2 + 2\,jl - 6\,rlk + 6\,l^2r^2}, \tag{3.44}$$

$$u_{2,3}(t,x,y) = -\frac{1}{2}\frac{s_0\left(3\,\Omega - 2\,\Omega\left(\cos_\lambda\left(\sqrt{-\Omega}\sigma\right)\right)^2 + 6\,\Omega\,\sin_\lambda\left(\sqrt{-\Omega}\sigma\right)\sqrt{pq} + 3\,\Omega\,pq\right)}{\left(\cos_\lambda\left(\sqrt{-\Omega}\sigma\right)\right)^2(k^2 + 2\,jl - 6\,rlk + 6\,l^2r^2)}, \tag{3.45}$$

$$u_{2,4}(t,x,y) = \frac{1}{2}\frac{s_0\left(\Omega + 2\,\Omega\left(\cos_\lambda\left(\sqrt{-\Omega}\sigma\right)\right)^2 + 6\,\Omega\,\cos_\lambda\left(\sqrt{-\Omega}\sigma\right)\sqrt{pq} + 3\,\Omega\,pq\right)}{\left(-1 + \left(\cos_\lambda\left(\sqrt{-\Omega}\sigma\right)\right)^2\right)(k^2 + 2\,jl - 6\,rlk + 6\,l^2r^2)}, \tag{3.46}$$

and

$$u_{2,5}(t,x,y) = \frac{1}{8}\frac{s_0\left(3\,\Omega - 12\,\Omega\left(\cos_\lambda\left(\frac{1}{4}\sqrt{-\Omega}\sigma\right)\right)^2 + 12\,\Omega\left(\cos_\lambda\left(\frac{1}{4}\sqrt{-\Omega}\sigma\right)\right)^4\right)}{\left(\cos_\lambda\left(1/4\sqrt{-\Omega}\sigma\right)\right)^2\left(-1 + \left(\cos_\lambda\left(\frac{1}{4}\sqrt{-\Omega}\sigma\right)\right)^2\right)(k^2 + 2\,jl - 6\,rlk + 6\,l^2r^2)}. \tag{3.47}$$

Family 2.2. For $\Omega > 0$ and $l \neq 0$,

$$u_{2,6}(t,x,y) = \frac{1}{2}\frac{s_0\left(-k^2 + 4\,jl + 3\,\Omega\left(\tanh_\lambda\left(\frac{1}{2}\sqrt{\Omega}\sigma\right)\right)^2\right)}{k^2 + 2\,jl - 6\,rlk + 6\,l^2r^2}, \tag{3.48}$$

$$u_{2,7}(t,x,y)=\frac{1}{2}\frac{s_0\left(-k^2+4\,jl+3\,\Omega\left(\coth_\lambda\left(\frac{1}{2}\sqrt{\Omega}\sigma\right)\right)^2\right)}{k^2+2\,jl-6\,rlk+6\,l^2r^2},\tag{3.49}$$

$$u_{2,8}(t,x,y)=\frac{1}{2}\frac{s_0\left(2\,\Omega\left(\cosh_\lambda\left(\sqrt{\Omega}\sigma\right)\right)^2-3\,\Omega+6\,\Omega\,\sinh_\lambda\left(\sqrt{\Omega}\sigma\right)\sqrt{pq}+3\,\Omega\,pq\right)}{\left(\cosh_\lambda\left(\sqrt{\Omega}\sigma\right)\right)^2(k^2+2\,jl-6\,rlk+6\,l^2r^2)},\tag{3.50}$$

$$u_{2,9}(t,x,y)=\frac{1}{2}\frac{s_0\left(\Omega+2\,\Omega\left(\cosh_\lambda\left(\sqrt{\Omega}\sigma\right)\right)^2+6\,\Omega\,\cosh_\lambda\left(\sqrt{\Omega}\sigma\right)\sqrt{pq}+3\,\Omega\,pq\right)}{\left(\left(\cosh_\lambda\left(\sqrt{\Omega}\sigma\right)\right)^2-1\right)(k^2+2\,jl-6\,rlk+6\,l^2r^2)},\tag{3.51}$$

and

$$u_{2,10}(t,x,y)=\frac{1}{8}\frac{s_0\,(3\,\Omega)}{\left(\cosh_\lambda\left(\frac{1}{4}\sqrt{\Omega}\sigma\right)\right)^2\left(\left(\cosh_\lambda\left(\frac{1}{4}\sqrt{\Omega}\sigma\right)\right)^2-1\right)(k^2+2\,jl-6\,rlk+6\,l^2r^2)}.\tag{3.52}$$

Family 2.3. For $jl>0$ and $k=0$,

$$u_{2,11}(t,x,y)=\frac{s_0j\left(1+3\left(\tan_\lambda\left(\sqrt{jl}\sigma\right)\right)^2\right)}{j+3\,lr^2},\tag{3.53}$$

$$u_{2,12}(t,x,y)=\frac{s_0j\left(1+3\left(\cot_\lambda\left(\sqrt{jl}\sigma\right)\right)^2\right)}{j+3\,lr^2},\tag{3.54}$$

$$u_{2,13}(t,x,y)=\frac{s_0j\left(-2\left(\cos_\lambda\left(2\sqrt{jl}\sigma\right)\right)^2+3+6\,\sin_\lambda\left(2\sqrt{jl}\sigma\right)\sqrt{pq}+3\,pq\right)}{\left(\cos_\lambda\left(2\sqrt{jl}\sigma\right)\right)^2(j+3\,lr^2)},\tag{3.55}$$

$$u_{2,14}(t,x,y)=-\frac{s_0j\left(1+2\left(\cos_\lambda\left(2\sqrt{jl}\sigma\right)\right)^2+6\,\cos_\lambda\left(2\sqrt{jl}\sigma\right)\sqrt{pq}+3\,pq\right)}{\left(-1+\left(\cos_\lambda\left(2\sqrt{jl}\sigma\right)\right)^2\right)(j+3\,lr^2)},\tag{3.56}$$

and

$$u_{2,15}(t,x,y)=-\frac{1}{4}\frac{s_0j\left(-8\left(\cos_\lambda\left(\frac{1}{2}\sqrt{jl}\sigma\right)\right)^2+8\left(\cos_\lambda\left(\frac{1}{2}\sqrt{jl}\sigma\right)\right)^4+3\right)}{\left(\cos_\lambda\left(\frac{1}{2}\sqrt{jl}\sigma\right)\right)^2\left(-1+\left(\cos_\lambda\left(1/2\sqrt{jl}\sigma\right)\right)^2\right)(j+3\,lr^2)}.\tag{3.57}$$

Family 2.4. For $jl<0$ and $k=0$,

$$u_{2,16}(t,x,y)=-\frac{s_0j\left(-1+3\left(\tanh_\lambda\left(\sqrt{-jl}\sigma\right)\right)^2\right)}{j+3\,lr^2},\tag{3.58}$$

$$u_{2,17}(t,x,y)=-\frac{s_0j\left(-1+3\left(\coth_\lambda\left(\sqrt{-jl}\sigma\right)\right)^2\right)}{j+3\,lr^2},\tag{3.59}$$

$$u_{2,18}(t,x,y) = -\frac{s_0 j\left(2\left(\cosh_\lambda\left(2\sqrt{-jl}\sigma\right)\right)^2 - 3 + 6\sinh_\lambda\left(2\sqrt{-jl}\sigma\right) i\sqrt{pq} + 3\,i^2 pq\right)}{\left(\cosh_\lambda\left(2\sqrt{-jl}\sigma\right)\right)^2 (j + 3\,lr^2)}, \tag{3.60}$$

$$u_{2,19}(t,x,y) = -\frac{s_0 j\left(2\left(\cosh_\lambda\left(2\sqrt{-jl}\sigma\right)\right)^2 + 1 + 6\cosh_\lambda\left(2\sqrt{-jl}\sigma\right)\sqrt{pq} + 3\,pq\right)}{\left(\left(\cosh_\lambda\left(2\sqrt{-jl}\sigma\right)\right)^2 - 1\right)(j + 3\,lr^2)}, \tag{3.61}$$

and

$$u_{2,20}(t,x,y) = -\frac{1}{4}\frac{s_0 j\left(8\left(\cosh_\lambda\left(\frac{1}{2}\sqrt{-jl}\sigma\right)\right)^4 - 8\left(\cosh_\lambda\left(\frac{1}{2}\sqrt{-jl}\sigma\right)\right)^2 + 3\right)}{\left(\cosh_\lambda\left(\frac{1}{2}\sqrt{-jl}\sigma\right)\right)^2\left(\left(\cosh_\lambda\left(1/2\sqrt{-jl}\sigma\right)\right)^2 - 1\right)(j + 3\,lr^2)}. \tag{3.62}$$

Family 2.5. For $l = j$ and $k = 0$,

$$u_{2,21}(t,x,y) = -\frac{s_0\left(2\,(\cos_\lambda(j\sigma))^2 - 3\right)}{(1 + 3\,r^2)(\cos_\lambda(j\sigma))^2}, \tag{3.63}$$

$$u_{2,22}(t,x,y) = -\frac{s_0\left(1 + 2\,(\cos_\lambda(j\sigma))^2\right)}{(1 + 3\,r^2)\left(-1 + (\cos_\lambda(j\sigma))^2\right)}, \tag{3.64}$$

$$u_{2,23}(t,x,y) = \frac{s_0\left(-2\,(\cos_\lambda(2\,j\sigma))^2 + 3 + 6\sin_\lambda(2\,j\sigma)\sqrt{pq} + 3\,pq\right)}{(\cos_\lambda(2\,j\sigma))^2(1 + 3\,r^2)}, \tag{3.65}$$

$$u_{2,24}(t,x,y) = -\frac{s_0\left(1 + 2\,(\cos_\lambda(2\,j\sigma))^2 + 6\cos_\lambda(2\,j\sigma)\sqrt{pq} + 3\,pq\right)}{\left(-1 + (\cos_\lambda(2\,j\sigma))^2\right)(1 + 3\,r^2)}, \tag{3.66}$$

and

$$u_{2,25}(t,x,y) = -\frac{1}{4}\frac{s_0\left(-8\left(\cos_\lambda\left(\frac{1}{2}j\sigma\right)\right)^2 + 8\left(\cos_\lambda\left(\frac{1}{2}j\sigma\right)\right)^4 + 3\right)}{\left(\cos_\lambda\left(\frac{1}{2}j\sigma\right)\right)^2\left(-1 + \left(\cos_\lambda\left(\frac{1}{2}j\sigma\right)\right)^2\right)(1 + 3\,r^2)}. \tag{3.67}$$

Family 2.6. For $k = -j$ and $k = 0$,

$$u_{2,26}(t,x,y) = \frac{s_0\left(2\,(\cosh_\lambda(j\sigma))^2 - 3\right)}{(-1 + 3\,r^2)(\cosh_\lambda(j\sigma))^2}, \tag{3.68}$$

$$u_{2,27}(t,x,y) = \frac{s_0\left(2\,(\cosh_\lambda(j\sigma))^2 + 1\right)}{(-1 + 3\,r^2)\left((\cosh_\lambda(j\sigma))^2 - 1\right)}, \tag{3.69}$$

$$u_{2,28}(t,x,y) = \frac{s_0\left(2\,(\cosh_\lambda(2\,j\sigma))^2 - 3 + 6\sinh_\lambda(2\,j\sigma)\,i\sqrt{pq} + 3\,i^2 pq\right)}{(\cosh_\lambda(2\,j\sigma))^2(-1 + 3\,r^2)}, \tag{3.70}$$

$$u_{2,29}(t,x,y) = \frac{s_0\left(2\,(\cosh_\lambda(2\,j\sigma))^2 + 1 + 6\,\cosh_\lambda(2\,j\sigma)\,\sqrt{pq} + 3\,pq\right)}{\left((\cosh_\lambda(2\,j\sigma))^2 - 1\right)(-1+3\,r^2)}, \tag{3.71}$$

and

$$u_{2,30}(t,x,y) = \frac{1}{4}\frac{s_0\left(8\,\left(\cosh_\lambda\left(\frac{1}{2}j\sigma\right)\right)^4 - 8\,\left(\cosh_\lambda\left(\frac{1}{2}j\sigma\right)\right)^2 + 3\right)}{\left(\cosh_\lambda\left(\frac{1}{2}j\sigma\right)\right)^2\left(\left(\cosh_\lambda\left(\frac{1}{2}j\sigma\right)\right)^2 - 1\right)(-1+3\,r^2)}. \tag{3.72}$$

Family 2.7. For $\Omega = 0$,

$$u_{2,31}(t,x,y) = \frac{s_0\left((\sigma\ln(\lambda))^2(k^6 + 2\,k^4jl) - 12\,lk^3\sigma\,\ln(\lambda)\,j\Phi + 24\,l^2\,(j\Phi)^2\right)}{k^4\sigma^2(\ln(\lambda))^2(k^2 + 2\,jl - 6\,rlk + 6\,l^2r^2)}, \tag{3.73}$$

where $\Phi = (k\sigma\,\ln(\lambda) + 2)$.
Family 2.8. For $k = j = 0$,

$$u_{2,32}(t,x,y) = \frac{s_0}{l^2\sigma^2(\ln(\lambda))^2\,r^2}. \tag{3.74}$$

Family 2.9. For $j = 0$, $k \neq 0$ and $l \neq 0$,

$$u_{2,33}(t,x,y) = \frac{s_0k^2\left(-\Psi - 4\,\cosh_\lambda(k\sigma)\,p + 4\,\sinh_\lambda(k\sigma)\,p + 2\,(\cosh_\lambda(k\sigma))^2 - 1 + p^2\right)}{(\cosh_\lambda(k\sigma) - \sinh_\lambda(k\sigma) + p)^2(k^2 - 6\,rlk + 6\,l^2r^2)}, \tag{3.75}$$

and

$$u_{2,34}(t,x,y) = \frac{s_0k^2\left(\Psi + 2\,(\cosh_\lambda(k\sigma))^2 - 1 - 4\,\sinh_\lambda(k\sigma)\,q + q^2 - 4\,\cosh_\lambda(k\sigma)\,q\right)}{(\cosh_\lambda(k\sigma) + \sinh_\lambda(k\sigma) + q)^2(k^2 - 6\,rlk + 6\,l^2r^2)}, \tag{3.76}$$

where $\Psi = 2\,\cosh_\lambda(k\sigma)\sinh_\lambda(k\sigma)$.
Family 2.10. For $k = h$, $l = sh(s \neq 0)$ and $j = 0$,

$$u_{2,35}(t,x,y) = \frac{s_0\left(-2\,pr\lambda^{h\sigma} + 6\,sp^2\lambda^{h\sigma} + 6\,s^2p^2\lambda^{2\,h\sigma} - 6\,sp\lambda^{2\,h\sigma}r + r^2\lambda^{2\,h\sigma} + p^2\right)}{(-p + r\lambda^{h\sigma})^2(1 - 6\,rs + 6\,s^2r^2)}, \tag{3.77}$$

where $\sigma = \frac{\sqrt{3}s_0}{\ln(\lambda)}(-(\frac{k^2e - 3\,s_0ak^2 - 3\,s_0bk^2 + k^2d + 2\,lje + 12\,s_0ajl + 12\,s_0bjl + 2\,ljd - 6\,erkl - 6\,drkl + 6\,el^2r^2 + 6\,dl^2r^2}{\sqrt{(k^2 + 2\,lj - 6\,rlk + 6\,l^2r^2)^3}})t + x + y)$.

4. Discussion and graphs

This section includes graphical depictions of the several wave forms seen in the system under study. Using the improved version of EDAM known as r+mEDAM, we extract and depict these wave structures in 3D formats. Different parameter choices provide clear and educational visualizations. Furthermore, it is notable that the findings of this work are novel, and to the best of our knowledge, there has never been any prior documentation in the literature of the use of these mathematical methods for the (2+1)-dimensional NNVM.

The finding of many families of soliton solutions, including periodic, hyperbolic, rational, generalised trigonometric and hyperbolic families, is what makes our current work unique. Furthermore, our analytical methodology has the special benefit of allowing us to deduce solutions acquired by other analytical methods, such as the tan-method, (G'/G)-expansion method and subequation method, to name a few. Our results, for example, may be used to derive the three families of solitary solutions produced by the (G'/G)-expansion approach. Similarly, by substituting generalized trigonometric and hyperbolic functions for conventional trigonometric and hyperbolic functions, we may derive all solution families found using the tan-method. This adaptability in acquiring different answers improves our comprehension of the problem from numerous angles and broadens the usefulness of our discoveries in various settings.

This has the potential to significantly improve our comprehension of how physical phenomena behave. These soliton solutions are anticipated to have a significant influence, altering how we describe and understand the fundamental physics of time evolution processes. Some results are illustrated visually in Figures 1–5, which show different solitary wave profiles. Wave phenomena such as kink waves, anti-kink waves, bell-shaped soliton and periodic waves are crucial in studying the NNVM and related mathematical frameworks. A kink wave is a form of wave disturbance or structure which involves an abrupt shift in the wave's shape or direction. Kink waves are frequently found in nonlinear wave systems where characteristics such as dispersion or nonlinearity impact the wave's behavior. The opposites of kinks, anti-kink waves, aid in the understanding of field interactions in NNVM model. A bell-shaped soliton is a solitary wave or localized pulse with a distinctive bell-shaped form. It is a self-sustaining wave that keeps its form as it moves across a medium. Periodic waves, which repeat their patterns on a regular basis, help to understand periodic behaviors and system stability in NNVM, such as periodic soliton solutions. These wave types allow us to gain a better grasp of the dynamics and interactions inside these mathematical models, hence improving our knowledge of the corresponding physical systems.

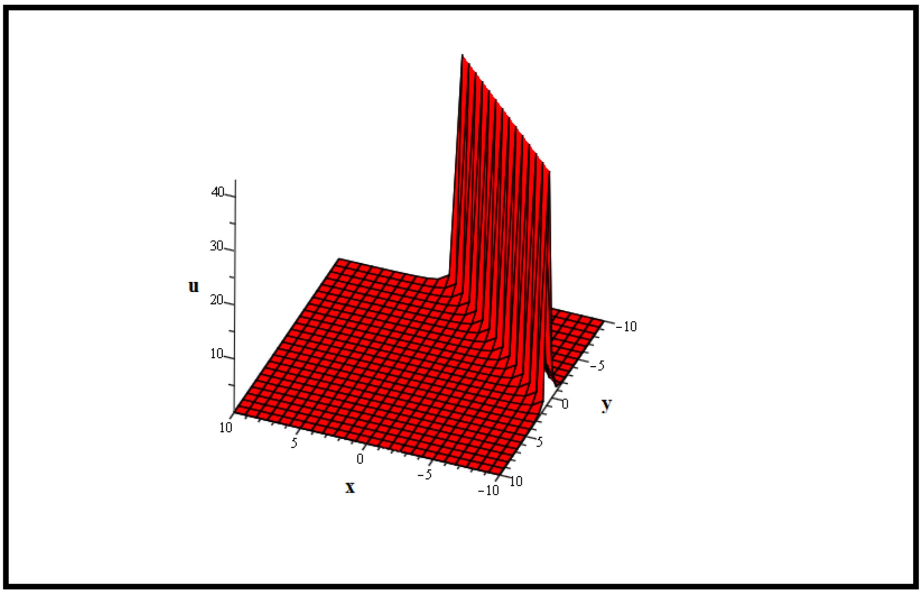

Figure 1. The 3D graph of $u_{1,11}$ in (3.17) is depicted for $j = 1, k = 0, l = 3, \lambda = \mathrm{e}, s_0 = 5, r = 10, a = 3, b = 5, d = 10, e = 1, t = 20$.

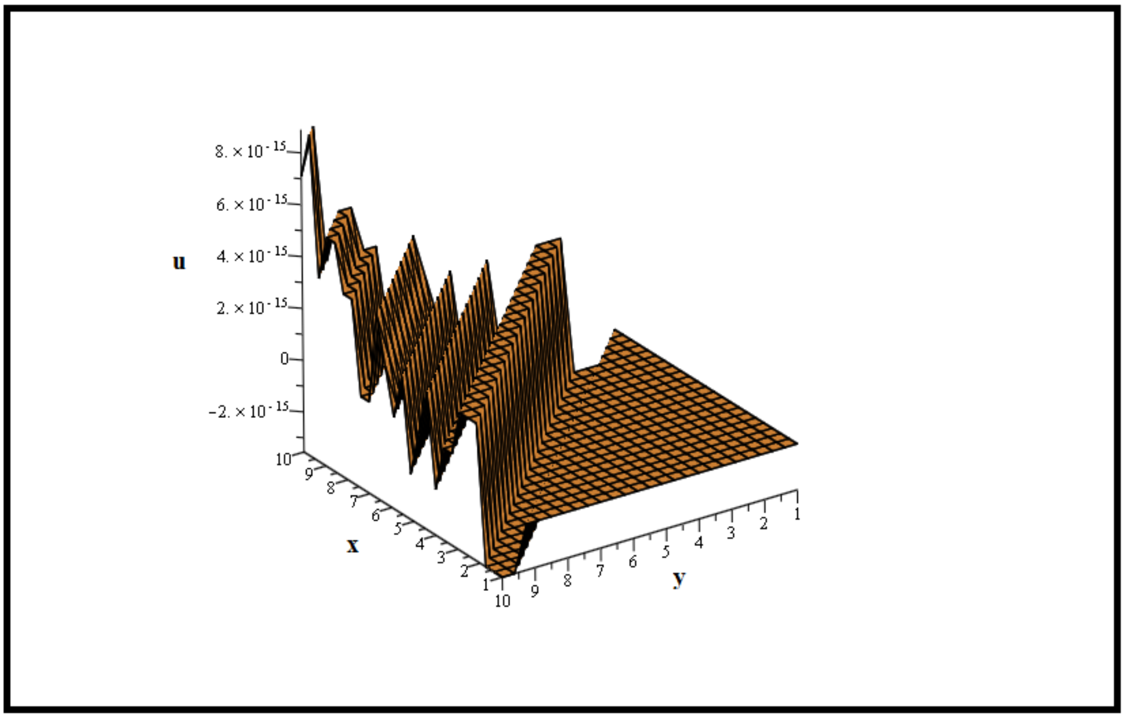

Figure 2. The 3D graph of $u_{1,27}$ in (3.33) is designed for $j = 3, k = 0, l = -3, \lambda = \mathrm{e}, s_0 = 10, r = 10, a = 20, b = 15, d = 1, e = 10, t = 5$.

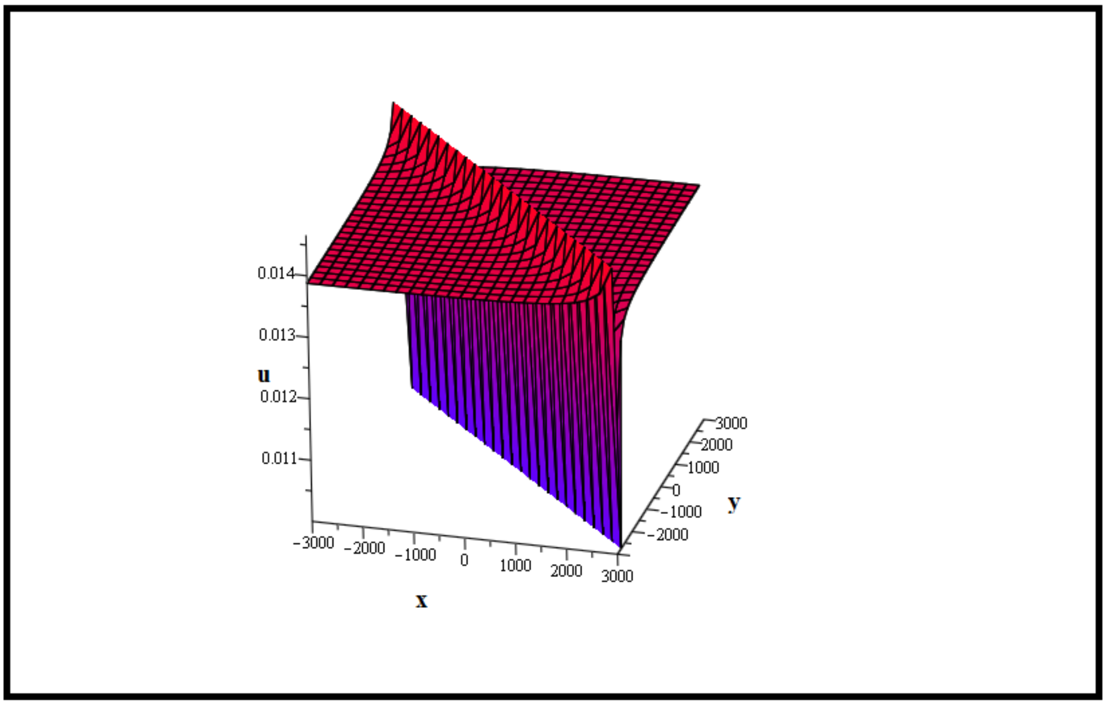

Figure 3. The 3D of $u_{1,31}$ in (3.37) is depicted for $j = 1, k = 2, l = 1, \lambda = 3, s_0 = 5, r = 20, a = 3, b = 7, d = 5, e = 2, t = 30$.

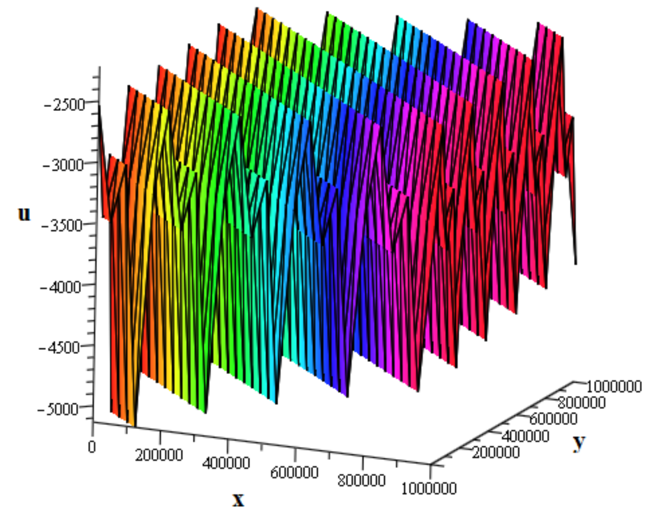

Figure 4. The 3D graph of $u_{2,5}$ in (3.47) is depicted for $j = 1, k = 1, l = 1, \lambda = \mathrm{e}, s_0 = 11, r = 9, a = 1, b = 2, d = 3, e = 4, t = 0$.

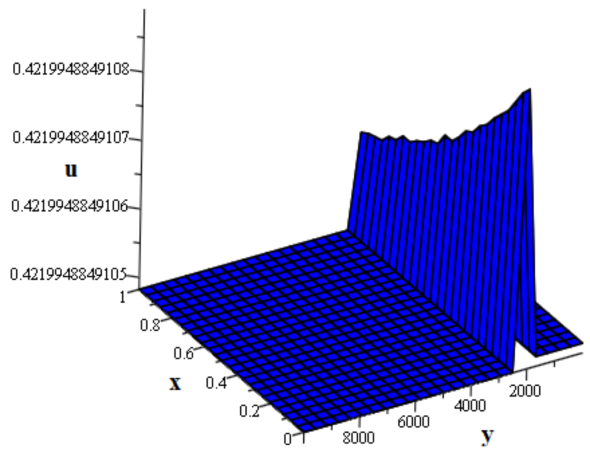

Figure 5. The 3D graphs of $u_{2,17}$ in (3.57) is depicted for $j = 11, k = 0, l = -21, \lambda = \mathrm{e}, s_0 = 30, r = 5, a = 25, b = 12, d = 34, e = 54, t = 50$.

5. Conclusions

Our research on the (2+1)-dimensional NNVM revealed important results. We have successfully discovered a wide spectrum of soliton solutions, including kinks, anti-kink, bell-shaped soliton and periodic wave solutions, using the r+mEDAM approach, all of which are critical to understanding the NNVM's behavior. The visual representations in the form of 3D graphs graphically highlight the approach's adaptability and usefulness. Furthermore, these discoveries add to our understanding of the dynamics of acoustic waves in incompressible fluids, with ramifications ranging from shallow-water waves to long interior waves and beyond. We emphasize the NNVM's critical position in theoretical physics and nonlinear dynamics, emphasizing its practical application and the promise for future advances in associated mathematical models and physical systems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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