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Research article

On the sixth power mean values of a generalized two-term exponential sums

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Abstract: This paper examines the evaluations of sixth power mean values of a generalized two-term exponential sums. In the case $p \equiv 3 \bmod 4$, we try to establish two precise formulas by applying the properties of character sums and the number of the solutions of relevant congruence equations modulo an odd prime p.

Keywords: the two-term exponential sums; sixth power mean values; elementary method; congruence equation; calculating formula
Mathematics Subject Classification: 11L03, 11L05

1. Introduction

Let p always denote an odd prime and let χ denote a Dirichlet character modulo p. For any integers $k > h \geq 1$, integer m and integer n, the generalized two-term exponential sums $S(m, n, k, h, \chi; p)$ are defined as follows:

$$S(m, n, k, h, \chi; p) = \sum_{a \bmod p} \chi(a) e\left(\frac{ma^k + na^h}{p}\right),$$

where $e(y) = e^{2\pi i y}$, $i^2 = -1$.

In the investigation of additive and analytic number theory, these sums are crucial. In reality, it is strongly related to a number of significant number theory issues, including the prime distribution and the Waring's problems. For example, the Waring-Goldbach problems is concerned with the representation of positive integers by the kth powers of primes, i.e.,

$$n = p_1^k + p_2^k + \cdots + p_s^k.$$

It is common to use exponential sums to study the number of solutions to the above equation. As a result, a large number of academics have researched the numerous classical results of $S(m, n, k, h, \chi; p)$, and have come to a number of insightful conclusions. For instance, Zhang and Zhang [1] shown that

$$\sum_{m \bmod p}{}' \left| \sum_{a \bmod p} e\left(\frac{ma^3+na}{p}\right) \right|^4 = \begin{cases} 2p^3-p^2, & \text{if } 3 \nmid p-1; \\ 2p^3-7p^2, & \text{if } 3 \mid p-1, \end{cases} \tag{1.1}$$

where n represents any integer with $(n,p)=1$.

Duan and Zhang [2] obtained the identities for $S(m,n,3,1,\chi;p)$ with $3 \nmid (p-1)$.

Recently, Zhang and Meng [3] also considered the sixth power mean of $S(m,n,3,1,\chi_0;p)$, and got that

$$\sum_{m \bmod p}{}' \left| \sum_{a \bmod p} e\left(\frac{ma^3+na}{p}\right) \right|^6 = \begin{cases} 5p^3\cdot(p-1), & \text{if } p\equiv 5(\bmod 6); \\ p^2\cdot\left(5p^2-23p-d^2\right), & \text{if } p\equiv 1(\bmod 6), \end{cases} \tag{1.2}$$

where $S(n,p)=1$, $4p=d^2+27\cdot b^2$, and d is solely determined by $d\equiv 1(\bmod 3)$ and $b>0$.

On the other hand, Chen and Wang [4] studied the fourth power mean of $S(m,1,4,1,\chi_0;p)$, and gave the exact calculation formulas for it.

Liu and Zhang [5] proved the following conclusion: when $3 \nmid (p-1)$,

$$\sum_{\chi \bmod p} \sum_{m \bmod p} \left| \sum_{a \bmod p}{}' \chi(a) e\left(\frac{ma^3+a}{p}\right) \right|^6 = p(p-1)\left(6p^3-28p^2+39p+5\right). \tag{1.3}$$

Some papers related to exponential sums can also be found in references [6–12].

It is clear from the formulas (1.1)–(1.3) that all of these publications have the same content: $h=1$ in $S(m,n,k,h,\chi;p)$. We cannot come across a study that discusses the 4th power mean of the generalized two-term exponential sums $S(m,n,k,2,\chi;p)$ in the literature. As a result, the research is challenging and rarely yields optimum results when $k>h=2$.

In this paper, we explore the calculation of $2k$-th power mean

$$\sum_{\chi \bmod p} \sum_{m \bmod p} \left| \sum_{a \bmod p} \chi(a) e\left(\frac{ma^4+a^2}{p}\right) \right|^{2k}, \tag{1.4}$$

and provide a precisely calculated formula for (1.4) with $p\equiv 3(\bmod 4)$ and $k=2$ or 3 using elementary and analytical approaches as well as the number of solutions to related congruence equations. Thus, we shall demonstrate two results:

Theorem 1. *Any odd prime p, the identities will be given*

$$\frac{1}{p(p-1)} \sum_{\chi \bmod p} \sum_{m \bmod p} \left| \sum_{a \bmod p} \chi(a) e\left(\frac{ma^4+a^2}{p}\right) \right|^4$$
$$= \begin{cases} 4(p-1)(p-2), & \text{if } p\equiv 3(\bmod 4); \\ 4\left(p^2-4p+6+\sqrt{p}\right), & \text{if } p\equiv 5(\bmod 8); \\ 4\left(p^2-4p+6-3\sqrt{p}\right), & \text{if } p\equiv 1(\bmod 8). \end{cases}$$

Theorem 2. *Let p be a prime with $p \equiv 3(\bmod 4)$. Then, we have the identity*

$$\frac{1}{p(p-1)} \sum_{\chi \bmod p} \sum_{m \bmod p} \left| \sum_{a \bmod p} \chi(a) e\left(\frac{ma^4 + a^2}{p}\right) \right|^6 = 23p^3 - 126p^2 + 179p + 8.$$

Some notes: In Theorem 2, we only discussed the case $p \equiv 3(\bmod 4)$. If $p \equiv 1(\bmod 4)$, then we could not get a satisfactory result. The reason is that we lack precise knowledge about the values or nontrivial upper bound estimation of

$$\sum_{a \bmod p}{}' \sum_{\substack{b \bmod p \\ abc \equiv de(\bmod p)}}{}' \sum_{c \bmod p}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' \left(\frac{a^2 + b^2 + c^2 - d^2 - e^2 - 1}{p}\right),$$

$a^4 + b^4 + c^4 \equiv d^4 + e^4 + 1(\bmod p)$.

Unsolved is the questions of whether (1.4) with $p \equiv 1(\bmod 4)$ and $k = 3$ can be calculated precisely.

Another interesting issue is if there is a precise method for calculating (1.4) with $p \equiv 3(\bmod 4)$ and $k \geq 4$.

2. Several lemmas

To establish our results, we require six fundamental lemmas. It is worth noting that these lemmas necessitate vast stores of knowledge of elementary or analytic number theory, which will be obviously seen through [13–15],

Lemma 1. *For an odd prime p, we have*

$$\sharp\{(a,b,c,d,e) \in \mathbb{Z}_p^* : a + b + c = d + e + 1, abc = de\} = p^3 - 3p^2 + 5p - 5.$$

Proof. Using the properties of the reduced residue system modulo p we have

$$\begin{aligned} & \sharp\{(a,b,c,d,e) \in \mathbb{Z}_p^* : a + b + c = d + e + 1, abc = de\} \\ = \ & \sharp\{(a,b,c,d,e) \in \mathbb{Z}_p^* : d(a-1) + e(b-1) + c - 1 = 0, abc = 1\}. \end{aligned} \tag{2.1}$$

To computing the values of (2.1), let us distinguish the following several cases:

If $a = b = c = 1$, then the congruence equations $da - d + eb - e + c - 1 \equiv 0(\bmod p)$ and $a \cdot b \cdot c \equiv 1(\bmod p)$ have $(p-1)^2$ solutions;

If $a = 1$, $b \neq 1$ and $c = \overline{b}$, then the congruence equations $da - d + eb - e + c - 1 \equiv 0(\bmod p)$ and $a \cdot b \cdot c \equiv 1(\bmod p)$ have $(p-1) \cdot (p-2)$ solutions;

Similarly, if $b = 1$, $a \neq 1$ and $c = \overline{a}$, then the congruence equations $da - d + eb - e + c - 1 \equiv 0(\bmod p)$ and $abc \equiv 1(\bmod p)$ also have $(p-1) \cdot (p-2)$ solutions;

If $c = 1$, $a \neq 1$ and $b = \overline{a}$, then the congruence equations $da - d + eb - e + c - 1 \equiv 0(\bmod p)$ and $a \cdot b \cdot c \equiv 1(\bmod p)$ also have $(p-1) \cdot (p-2)$ solutions;

If $a \neq 1$, $b \neq 1$, $c \neq 1$ and $abc \equiv 1(\bmod p)$, then the congruence equations $da - d + eb - e + c - 1 \equiv 0(\bmod p)$ and $abc \equiv 1(\bmod p)$ are equivalent to $d + e + 1 \equiv 0(\bmod p)$ and $a \cdot b \cdot c \equiv 1(\bmod p)$, and they have

$$(p-2)\left[(p-1)^2 - 3(p-2) - 1\right] = (p-2)^2(p-3)$$

solutions.

Note that if $a = 1$ and $b = 1$, then from $a \cdot b \cdot c \equiv 1(\bmod p)$ we can deduce $c = 1$. Now by applying (2.1) and synthesizing these results, we can get

$$\begin{aligned} & \sharp\{(a,b,c,d,e) \in \mathbb{Z}_p^* : a+b+c = d+e+1, abc = de\} \\ = \ & (p-2)^2(p-3) + (p-1)^2 + 3(p-1)(p-2) \\ = \ & p^3 - 3p^2 + 5p - 5. \end{aligned}$$

This provides proof of Lemma 1. □

Lemma 2. *For an odd prime p, then*

$$\sum_{a \bmod p}{}' \sum_{b \bmod p}{}' \sum_{\substack{c \bmod p \\ a+b+c \equiv d+e+1(\bmod p) \\ abc \equiv de(\bmod p)}}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' \left(\frac{a}{p}\right) = p^2 - 2p - 1.$$

Proof. Based on the reduced residue system, we have

$$\sum_{a \bmod p}{}' \sum_{b \bmod p}{}' \sum_{\substack{c \bmod p \\ a+b+c \equiv d+e+1(\bmod p) \\ abc \equiv de(\bmod p)}}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' \left(\frac{a}{p}\right) = \sum_{a \bmod p}{}' \sum_{b \bmod p}{}' \sum_{\substack{c \bmod p \\ a-1+d(b-1)+e(c-1) \equiv 0(\bmod p) \\ abc \equiv 1(\bmod p)}}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' \left(\frac{a}{p}\right). \tag{2.2}$$

If $a = b = c = 1$, then the congruence equations $a - 1 + db - d + ec - e \equiv 0(\bmod p)$ and $a \cdot b \cdot c \equiv 1(\bmod p)$ have $(p-1)^2$ solutions and $\left(\frac{1}{p}\right) = 1$;

If $a = 1$, $b \neq 1$ and $c = \overline{b}$, then the congruence equations $a - 1 + db - d + ec - e \equiv 0(\bmod p)$ and $a \cdot b \cdot c \equiv 1(\bmod p)$ have $(p-1) \cdot (p-2)$ solutions;

Similarly, if $b = 1$, $a \neq 1$ and $c = \overline{a}$, so we get

$$\sum_{a=2}^{p-1} \sum_{\substack{d \bmod p \\ a-1+e(\overline{a}-1) \equiv 0(\bmod p)}}{}' \sum_{e \bmod p}{}' \left(\frac{a}{p}\right) = (p-1) \sum_{a=2}^{p-1} \left(\frac{a}{p}\right) = -p + 1. \tag{2.3}$$

If $c = 1$, $a \neq 1$ and $b = \overline{a}$, then we also have

$$\sum_{a=2}^{p-1} \sum_{\substack{d \bmod p \\ a-1+d(\overline{a}-1) \equiv 0(\bmod p)}}{}' \sum_{e \bmod p}{}' \left(\frac{a}{p}\right) = (p-1) \sum_{a=2}^{p-1} \left(\frac{a}{p}\right) = -p + 1. \tag{2.4}$$

If $a \neq 1$, $b \neq 1$, $c \neq 1$ and $abc \equiv 1(\bmod p)$, then we can deduce that

$$\sum_{a=2}^{p-1} \sum_{b=2}^{p-1} \sum_{c=2}^{p-1} \sum_{\substack{d \bmod p \\ a-1+d(b-1)+e(c-1) \equiv 0(\bmod p) \\ abc \equiv 1(\bmod p)}}{}' \sum_{e \bmod p}{}' \left(\frac{a}{p}\right) = \sum_{a=2}^{p-1} \sum_{b=2}^{p-1} \sum_{c=2}^{p-1} \sum_{\substack{d \bmod p \\ 1+d+e \equiv 0(\bmod p) \\ abc \equiv 1(\bmod p)}}{}' \sum_{e \bmod p}{}' \left(\frac{a}{p}\right)$$

$$
\begin{aligned}
&= (p-2)\left[(p-1)\sum_{a \bmod p}{}' \left(\frac{a}{p}\right) - (p-2) - 2\sum_{a=2}^{p-1}\left(\frac{a}{p}\right) - 1\right] \\
&= -p^2 + 5p - 6. \qquad (2.5)
\end{aligned}
$$

Combining (2.2)–(2.5), we will have the result

$$
\sum_{a \bmod p}{}' \sum_{b \bmod p}{}' \sum_{\substack{c \bmod p \\ a+b+c\equiv d+e+1(\bmod p) \\ abc\equiv de(\bmod p)}}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' \left(\frac{a}{p}\right) = p^2 - 2p - 1.
$$

The proof of Lemma 2 is provided. □

Lemma 3. *If p is an odd prime with $p \equiv 3(\bmod 4)$, we will have*

$$
\begin{aligned}
&\sum_{a \bmod p}{}' \sum_{b \bmod p}{}' \sum_{\substack{c \bmod p \\ a+b+c\equiv d+e+1(\bmod p) \\ abc\equiv de(\bmod p)}}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' \left(\frac{d}{p}\right) \\
= &\sum_{a \bmod p}{}' \sum_{b \bmod p}{}' \sum_{\substack{c \bmod p \\ a+b+c\equiv d+e+1(\bmod p) \\ abc\equiv de(\bmod p)}}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' \left(\frac{de}{p}\right) = -(p-1).
\end{aligned}
$$

Proof. Firstly, using important properties related to the reduced reside system, we have

$$
\sum_{a \bmod p}{}' \sum_{b \bmod p}{}' \sum_{\substack{c \bmod p \\ a+b+c\equiv d+e+1(\bmod p) \\ abc\equiv de(\bmod p)}}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' \left(\frac{d}{p}\right) = \sum_{a \bmod p}{}' \sum_{b \bmod p}{}' \sum_{\substack{c \bmod p \\ d(a-1)+e(b-1)+c-1\equiv 0(\bmod p) \\ abc\equiv 1(\bmod p)}}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' \left(\frac{d}{p}\right). \qquad (2.6)
$$

If $a = b = c = 1$, then from (2.6) we can get

$$
\sum_{d \bmod p}{}' \sum_{e \bmod p}{}' \left(\frac{d}{p}\right) = 0. \qquad (2.7)
$$

If $a = 1$, $b \neq 1$ and $c = \overline{b}$, then from (2.6),

$$
\sum_{b=2}^{p-1} \sum_{\substack{c=2 \\ e(b-1)+c-1\equiv 0(\bmod p) \\ bc\equiv 1(\bmod p)}}^{p-1} \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' \left(\frac{d}{p}\right) = \sum_{b=2}^{p-1} \sum_{\substack{c=2 \\ e(b-1)+c-1\equiv 0(\bmod p) \\ bc\equiv 1(\bmod p)}}^{p-1} \sum_{e \bmod p}{}' \sum_{d \bmod p}{}' \left(\frac{d}{p}\right) = 0. \qquad (2.8)
$$

Similarly, if $b = 1$, $a \neq 1$ and $c = \overline{a}$, then from (2.6) we infer that

$$
\sum_{\substack{a=2 \\ d(a-1)+\overline{a}-1\equiv 0(\bmod p)}}^{p-1} \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' \left(\frac{d}{p}\right) = (p-1)\sum_{a=2}^{p-1}\left(\frac{\overline{a}}{p}\right) = -p + 1. \qquad (2.9)
$$

If $c=1$, $a\neq 1$ and $b=\overline{a}$, then we also can get

$$\sum_{a=2}^{p-1}\mathop{\sum\nolimits'}_{d \bmod p}\mathop{\sum\nolimits'}_{\substack{e \bmod p\\ d(a-1)+e(\overline{a}-1)\equiv 0(\bmod p)}}\left(\frac{d}{p}\right)=\sum_{a=2}^{p-1}\mathop{\sum\nolimits'}_{e \bmod p}\left(\frac{\overline{a}}{p}\right)\left(\frac{e}{p}\right)=0. \tag{2.10}$$

If $a\neq 1$, $b\neq 1$, $c\neq 1$ and $abc\equiv 1(\bmod p)$, then note that $\left(\frac{a}{p}\right)=\left(\frac{\overline{a}}{p}\right)$. It follows from (2.6) that

$$\begin{aligned}
&\sum_{a=2}^{p-1}\sum_{b=2}^{p-1}\sum_{c=2}^{p-1}\mathop{\sum\nolimits'}_{d \bmod p}\mathop{\sum\nolimits'}_{\substack{e \bmod p\\ d(a-1)+e(b-1)+(c-1)\equiv 0(\bmod p)\\ abc\equiv 1(\bmod p)}}\left(\frac{d}{p}\right)=\sum_{a=2}^{p-1}\sum_{b=2}^{p-1}\sum_{c=2}^{p-1}\mathop{\sum\nolimits'}_{d \bmod p}\mathop{\sum\nolimits'}_{\substack{e \bmod p\\ d+e+c-1\equiv 0(\bmod p)\\ abc\equiv 1(\bmod p)}}\left(\frac{(a-1)d}{p}\right)\\
=\;&\sum_{a=2}^{p-1}\sum_{b=2}^{p-1}\sum_{c=2}^{p-1}\mathop{\sum\nolimits'}_{\substack{e \bmod p\\ abc\equiv 1(\bmod p)}}\left(\frac{(a-1)(1-c-e)}{p}\right)=\sum_{a=1}^{p-1}\sum_{b=2}^{p-1}\mathop{\sum\nolimits'}_{\substack{c \bmod p\\ abc\equiv 1(\bmod p)}}\left(\frac{(a-1)(c-1)}{p}\right)\\
=\;&\mathop{\sum\nolimits'}_{a \bmod p}\mathop{\sum\nolimits'}_{c \bmod p}\left(\frac{(a-1)(c-1)}{p}\right)-\mathop{\sum\nolimits'}_{a \bmod p}\left(\frac{(a-1)(\overline{a}-1)}{p}\right)\\
=\;&-\mathop{\sum\nolimits'}_{c \bmod p}\left(\frac{c-1}{p}\right)-\sum_{a=2}^{p-1}\left(\frac{-\overline{a}}{p}\right)=0.
\end{aligned} \tag{2.11}$$

Combining (2.6)–(2.11) we can deduce that

$$\mathop{\sum\nolimits'}_{a \bmod p}\mathop{\sum\nolimits'}_{b \bmod p}\mathop{\sum\nolimits'}_{c \bmod p}\mathop{\sum\nolimits'}_{d \bmod p}\mathop{\sum\nolimits'}_{\substack{e \bmod p\\ a+b+c\equiv d+e+1(\bmod p)\\ abc\equiv de(\bmod p)}}\left(\frac{d}{p}\right)=-p+1. \tag{2.12}$$

Applying the reduced residue system modulo p and (2.12),

$$\begin{aligned}
&\mathop{\sum\nolimits'}_{a \bmod p}\mathop{\sum\nolimits'}_{b \bmod p}\mathop{\sum\nolimits'}_{c \bmod p}\mathop{\sum\nolimits'}_{d \bmod p}\mathop{\sum\nolimits'}_{\substack{e \bmod p\\ a+b+c\equiv d+e+1(\bmod p)\\ abc\equiv de(\bmod p)}}\left(\frac{de}{p}\right)\\
=\;&\mathop{\sum\nolimits'}_{a \bmod p}\mathop{\sum\nolimits'}_{b \bmod p}\mathop{\sum\nolimits'}_{c \bmod p}\mathop{\sum\nolimits'}_{d \bmod p}\mathop{\sum\nolimits'}_{\substack{e \bmod p\\ ad+bd+cd\equiv d+ed+1(\bmod p)\\ abcd^3\equiv d^2e(\bmod p)}}\left(\frac{d^2e}{p}\right)\\
=\;&\mathop{\sum\nolimits'}_{a \bmod p}\mathop{\sum\nolimits'}_{b \bmod p}\mathop{\sum\nolimits'}_{c \bmod p}\mathop{\sum\nolimits'}_{d \bmod p}\mathop{\sum\nolimits'}_{\substack{e \bmod p\\ a+b+c\equiv 1+e+\overline{d}(\bmod p)\\ abc\equiv \overline{d}e(\bmod p)}}\left(\frac{e}{p}\right)\\
=\;&\mathop{\sum\nolimits'}_{a \bmod p}\mathop{\sum\nolimits'}_{b \bmod p}\mathop{\sum\nolimits'}_{c \bmod p}\mathop{\sum\nolimits'}_{d \bmod p}\mathop{\sum\nolimits'}_{\substack{e \bmod p\\ a+b+c\equiv 1+e+d(\bmod p)\\ abc\equiv de(\bmod p)}}\left(\frac{e}{p}\right)=-p+1.
\end{aligned} \tag{2.13}$$

Now combing (2.12) and (2.13), we can easily prove Lemma 3. □

Lemma 4. *If $p \equiv 3(\bmod 4)$, then we will get*

$$\sum_{a \bmod p}{}' \sum_{b \bmod p}{}' \sum_{\substack{c \bmod p \\ a+b+c\equiv d+e+1(\bmod p) \\ abc\equiv de(\bmod p)}}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' \left(\frac{ab}{p}\right) = -p+1,$$

and

$$\sum_{a \bmod p}{}' \sum_{b \bmod p}{}' \sum_{\substack{c \bmod p \\ a+b+c\equiv d+e+1(\bmod p) \\ abc\equiv de(\bmod p)}}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' \left(\frac{ad}{p}\right) = p^2-2p-1.$$

Proof. Using important properties related to the reduced residue system and Lemma 3 we may immediately obtain

$$\begin{aligned}
&\sum_{a \bmod p}{}' \sum_{b \bmod p}{}' \sum_{\substack{c \bmod p \\ a+b+c\equiv d+e+1(\bmod p) \\ abc\equiv de(\bmod p)}}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' \left(\frac{ab}{p}\right) \\
=\ & \sum_{a \bmod p}{}' \sum_{b \bmod p}{}' \sum_{\substack{c \bmod p \\ ab+b+cb\equiv db+eb+1(\bmod p) \\ ab^3c\equiv b^2de(\bmod p)}}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' \left(\frac{ab^2}{p}\right) \\
=\ & \sum_{a \bmod p}{}' \sum_{b \bmod p}{}' \sum_{\substack{c \bmod p \\ a+1+c\equiv d+e+\overline{b}(\bmod p) \\ ac\equiv \overline{b}de(\bmod p)}}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' \left(\frac{a}{p}\right) \\
=\ & \sum_{a \bmod p}{}' \sum_{b \bmod p}{}' \sum_{\substack{c \bmod p \\ a+b+c\equiv d+e+1(\bmod p) \\ abc\equiv de(\bmod p)}}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' \left(\frac{d}{p}\right) = -p+1. \qquad (2.14)
\end{aligned}$$

Similarly, applying Lemma 2 we also have

$$\begin{aligned}
&\sum_{a \bmod p}{}' \sum_{b \bmod p}{}' \sum_{\substack{c \bmod p \\ a+b+c\equiv d+e+1(\bmod p) \\ abc\equiv de(\bmod p)}}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' \left(\frac{ad}{p}\right) \\
=\ & \sum_{a \bmod p}{}' \sum_{b \bmod p}{}' \sum_{\substack{c \bmod p \\ ad+bd+cd\equiv d+ed+1(\bmod p) \\ abcd^3\equiv d^2e(\bmod p)}}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' \left(\frac{ad^2}{p}\right) \\
=\ & \sum_{a \bmod p}{}' \sum_{b \bmod p}{}' \sum_{\substack{c \bmod p \\ a+b+c\equiv 1+e+\overline{d}(\bmod p) \\ abc\equiv \overline{d}e(\bmod p)}}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' \left(\frac{a}{p}\right)
\end{aligned}$$

$$= \sum_{a \bmod p}{}' \sum_{\substack{b \bmod p \\ a+b+c\equiv 1+e+d(\bmod p) \\ abc\equiv de(\bmod p)}}{}' \sum_{c \bmod p}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' \left(\frac{a}{p}\right)$$

$$= p^2 - 2p - 1. \tag{2.15}$$

Now Lemma 4 is proved. □

Lemma 5. *If p is an odd prime, then*

$$\sum_{a \bmod p}{}' \sum_{\substack{b \bmod p \\ a^4+b^4+c^4\equiv d^4+e^4+1(\bmod p) \\ a^2+b^2+c^2\equiv d^2+e^2+1(\bmod p) \\ abc\equiv de(\bmod p)}}{}' \sum_{c \bmod p}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' 1 = 8\left(3p^2 - 15p + 20\right).$$

Proof. Utilizing the properties of the congruence equation modulo p we infer that

$$\sum_{a \bmod p}{}' \sum_{\substack{b \bmod p \\ a^4+b^4+c^4\equiv d^4+e^4+1(\bmod p) \\ a^2+b^2+c^2\equiv d^2+e^2+1(\bmod p) \\ abc\equiv de(\bmod p)}}{}' \sum_{c \bmod p}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' 1 = \sum_{a \bmod p}{}' \sum_{\substack{b \bmod p \\ a^2+b^2+c^2\equiv d^2+e^2+1(\bmod p) \\ \left(a^2+b^2+c^2\right)^2-a^4-b^4-c^4\equiv\left(d^2+e^2+1\right)^2-d^4-e^4-1(\bmod p) \\ abc\equiv de(\bmod p)}}{}' \sum_{c \bmod p}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' 1$$

$$= \sum_{a \bmod p}{}' \sum_{\substack{b \bmod p \\ a^2+b^2+c^2\equiv d^2+e^2+1(\bmod p) \\ a^2b^2+a^2c^2+b^2c^2\equiv d^2+e^2+d^2e^2(\bmod p) \\ abc\equiv de(\bmod p)}}{}' \sum_{c \bmod p}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' 1 = \sum_{a \bmod p}{}' \sum_{\substack{b \bmod p \\ \left(a^2-1\right)\left(b^2-1\right)\left(c^2-1\right)\equiv 0(\bmod p) \\ a^2+b^2+c^2\equiv d^2+e^2+1(\bmod p) \\ abc\equiv de(\bmod p)}}{}' \sum_{c \bmod p}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' 1$$

$$= 3\sum_{a \bmod p}{}' \sum_{\substack{b \bmod p \\ a^2\equiv 1(\bmod p) \\ b^2+c^2\equiv d^2+e^2(\bmod p) \\ abc\equiv de(\bmod p)}}{}' \sum_{c \bmod p}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' 1 - 3\sum_{a \bmod p}{}' \sum_{\substack{b \bmod p \\ 1+c^2\equiv d^2+e^2 \bmod p \\ abc\equiv de \bmod p}}{}' \sum_{c \bmod p}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' 1$$

$$+ \sum_{a \bmod p}{}' \sum_{\substack{b \bmod p \\ a^2\equiv b^2\equiv c^2\equiv 1(\bmod p) \\ 2\equiv d^2+e^2(\bmod p) \\ abc\equiv de(\bmod p)}}{}' \sum_{c \bmod p}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' 1$$

$$= 3(p-1)\sum_{a \bmod p}{}' \sum_{\substack{b \bmod p \\ a^2\equiv 1(\bmod p) \\ b^2+c^2\equiv d^2+1(\bmod p) \\ abc\equiv d(\bmod p)}}{}' \sum_{c \bmod p}{}' \sum_{d \bmod p}{}' 1 - 3\sum_{a \bmod p}{}' \sum_{\substack{b \bmod p \\ a^2\equiv b^2\equiv 1(\bmod p) \\ \left(d^2-1\right)\left(e^2-1\right)\equiv 0(\bmod p) \\ abc\equiv de(\bmod p)}}{}' \sum_{c \bmod p}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' 1 + 16$$

$$= 3(p-1)\sum_{a \bmod p}{}' \sum_{\substack{b \bmod p \\ a^2\equiv 1(\bmod p) \\ \left(b^2-1\right)\left(c^2-1\right)\equiv 0(\bmod p) \\ abc\equiv d(\bmod p)}}{}' \sum_{c \bmod p}{}' \sum_{d \bmod p}{}' 1 - 48(p-2) + 16$$

$$= 24(p-1)(p-2) - 48(p-2) + 16 = 8\left(3p^2 - 15p + 20\right).$$

This completes the proof of Lemma 5. □

Lemma 6. *Assume that $p \equiv 3(\bmod 4)$, the identity will be given*

$$\sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p \\ a^4+b^4+c^4\equiv d^4+e^4+1(\bmod p) \\ abc\equiv de(\bmod p)}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p} 1 = p^3+6p^2-19p-8.$$

Proof. Since $p \equiv 3(\bmod 4)$, then we get

$$\begin{aligned}
&\sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p \\ a^4+b^4+c^4\equiv d^4+e^4+1(\bmod p) \\ abc\equiv de(\bmod p)}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p} 1 = \sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p \\ a^4+b^4+c^4\equiv d^4+e^4+1(\bmod p) \\ abc\equiv -de(\bmod p)}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p} 1 \\
= &\ \frac{1}{2}\sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p \\ a^4+b^4+c^4\equiv d^4+e^4+1(\bmod p) \\ a^4b^4c^4\equiv d^4e^4(\bmod p)}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p} 1 \\
= &\ \frac{1}{2}\sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p \\ a+b+c\equiv d+e+1(\bmod p) \\ abc\equiv de(\bmod p)}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p}\left(1+\left(\frac{a}{p}\right)\right)\left(1+\left(\frac{b}{p}\right)\right)\left(1+\left(\frac{c}{p}\right)\right) \\
&\times\left(1+\left(\frac{d}{p}\right)\right)\left(1+\left(\frac{e}{p}\right)\right) \\
= &\ \frac{1}{2}\sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p \\ a+b+c\equiv d+e+1(\bmod p) \\ abc\equiv de(\bmod p)}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p} 1 + \frac{3}{2}\sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p \\ a+b+c\equiv d+e+1(\bmod p) \\ abc\equiv de(\bmod p)}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p}\left(\frac{a}{p}\right) \\
&+\sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p \\ a+b+c\equiv d+e+1(\bmod p) \\ abc\equiv de(\bmod p)}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p}\left(\frac{d}{p}\right) + \frac{3}{2}\sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p \\ a+b+c\equiv d+e+1(\bmod p) \\ abc\equiv de(\bmod p)}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p}\left(\frac{ab}{p}\right) \\
&+\frac{1}{2}\sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p \\ a+b+c\equiv d+e+1(\bmod p) \\ abc\equiv de(\bmod p)}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p}\left(\frac{ed}{p}\right) + 3\sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p \\ a+b+c\equiv d+e+1(\bmod p) \\ abc\equiv de(\bmod p)}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p}\left(\frac{ad}{p}\right) \\
&+\frac{1}{2}\sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p \\ a+b+c\equiv d+e+1(\bmod p) \\ abc\equiv de(\bmod p)}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p}\left(\frac{abc}{p}\right) + 3\sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p \\ a+b+c\equiv d+e+1(\bmod p) \\ abc\equiv de(\bmod p)}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p}\left(\frac{abd}{p}\right) \\
&+\frac{3}{2}\sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p \\ a+b+c\equiv d+e+1(\bmod p) \\ abc\equiv de(\bmod p)}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p}\left(\frac{ade}{p}\right) + \sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p \\ a+b+c\equiv d+e+1(\bmod p) \\ abc\equiv de \bmod p}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p}\left(\frac{abcd}{p}\right)
\end{aligned}$$

$$+\frac{3}{2}\sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p\\ a+b+c\equiv d+e+1(\bmod p)\\ abc\equiv de(\bmod p)}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p}\left(\frac{abde}{p}\right)$$

$$+\frac{1}{2}\sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p\\ a+b+c\equiv d+e+1(\bmod p)\\ abc\equiv de(\bmod p)}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p}\left(\frac{abcde}{p}\right). \tag{2.16}$$

Note that the identities

$$\sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p\\ a+b+c\equiv d+e+1(\bmod p)\\ abc\equiv de(\bmod p)}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p}\left(\frac{abc}{p}\right)=\sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p\\ a+b+c\equiv d+e+1(\bmod p)\\ abc\equiv de(\bmod p)}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p}\left(\frac{de}{p}\right); \tag{2.17}$$

$$\sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p\\ a+b+c\equiv d+e+1(\bmod p)\\ abc\equiv de(\bmod p)}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p}\left(\frac{abd}{p}\right)=\sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p\\ a+b+c\equiv d+e+1(\bmod p)\\ abc\equiv de(\bmod p)}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p}\left(\frac{ec}{p}\right); \tag{2.18}$$

$$\sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p\\ a+b+c\equiv d+e+1(\bmod p)\\ abc\equiv de(\bmod p)}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p}\left(\frac{ade}{p}\right)=\sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p\\ a+b+c\equiv d+e+1(\bmod p)\\ abc\equiv de(\bmod p)}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p}\left(\frac{bc}{p}\right); \tag{2.19}$$

$$\sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p\\ a+b+c\equiv d+e+1(\bmod p)\\ abc\equiv de(\bmod p)}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p}\left(\frac{abcd}{p}\right)=\sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p\\ a+b+c\equiv d+e+1(\bmod p)\\ abc\equiv de(\bmod p)}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p}\left(\frac{e}{p}\right); \tag{2.20}$$

$$\sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p\\ a+b+c\equiv d+e+1(\bmod p)\\ abc\equiv de(\bmod p)}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p}\left(\frac{abde}{p}\right)=\sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p\\ a+b+c\equiv d+e+1(\bmod p)\\ abc\equiv de(\bmod p)}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p}\left(\frac{c}{p}\right); \tag{2.21}$$

$$\sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p\\ a+b+c\equiv d+e+1(\bmod p)\\ abc\equiv de(\bmod p)}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p}\left(\frac{abcde}{p}\right)=\sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p\\ a+b+c\equiv d+e+1(\bmod p)\\ abc\equiv de(\bmod p)}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p}1. \tag{2.22}$$

From Lemma 1 to Lemma 4, formulas (2.16)–(2.22) we have

$$\sideset{}{'}\sum_{a \bmod p}\sideset{}{'}\sum_{b \bmod p}\sideset{}{'}\sum_{\substack{c \bmod p\\ a^4+b^4+c^4\equiv d^4+e^4+1(\bmod p)\\ abc\equiv de(\bmod p)}}\sideset{}{'}\sum_{d \bmod p}\sideset{}{'}\sum_{e \bmod p}1=\frac{1}{2}\left(p^3-3p^2+5p-5\right)+\frac{3}{2}\left(p^2-2p-1\right)-(p-1)$$

$$-\frac{3}{2}(p-1)-\frac{1}{2}(p-1)+3\left(p^2-2p-1\right)-\frac{1}{2}(p-1)+3\left(p^2-2p-1\right)$$
$$-\frac{3}{2}(p-1)-(p-1)+\frac{3}{2}\left(p^2-2p-1\right)+\frac{1}{2}\left(p^3-3p^2+5p-5\right)$$
$$=p^3+6p^2-19p-8.$$

This proves Lemma 6. □

3. Proofs of the theorems

We utilize the lemmas presented in Section 2 to finalize the proof of theorems. Firstly, we prove Theorem 2. Use the identities

$$\sum_{a \bmod p} e\left(\frac{na}{p}\right)=\begin{cases} p, & \text{if } p \mid n; \\ 0, & \text{if } p \nmid n, \end{cases}$$

for $(n,p)=1$, we have

$$\sum_{a \bmod p} e\left(\frac{na^2}{p}\right)=1+\sum_{a \bmod p}{}' (1+\chi_2(a))\, e\left(\frac{na}{p}\right)=\left(\frac{n}{p}\right)\cdot\tau(\chi_2).$$

Then we calculate the equation.

$$\begin{aligned}
&\sum_{\chi \bmod p}\sum_{m \bmod p}\left|\sum_{a \bmod p}{}' \chi(a) e\left(\frac{ma^4+a^2}{p}\right)\right|^6 \\
=\;& p\cdot\sum_{\chi \bmod p}\mathop{\sum_{a \bmod p}{}'\sum_{b \bmod p}{}'\sum_{c \bmod p}{}'\sum_{d \bmod p}{}'\sum_{e \bmod p}{}'\sum_{f \bmod p}{}'}_{a^4+b^4+c^4\equiv d^4+e^4+f^4 (\bmod p)} \chi\left(abc\overline{def}\right) e\left(\frac{a^2+b^2+c^2-d^2-e^2-f^2}{p}\right) \\
=\;& p(p-1)\cdot\mathop{\sum_{a \bmod p}{}'\sum_{b \bmod p}{}'\sum_{c \bmod p}{}'\sum_{d \bmod p}{}'\sum_{e \bmod p}{}'\sum_{f \bmod p}{}'}_{\substack{a^4+b^4+c^4\equiv d^4+e^4+f^4 (\bmod p)\\ abc\equiv def (\bmod p)}} e\left(\frac{a^2+b^2+c^2-d^2-f^2-e^2}{p}\right) \\
=\;& p(p-1)\cdot\mathop{\sum_{a \bmod p}{}'\sum_{b \bmod p}{}'\sum_{c \bmod p}{}'\sum_{d \bmod p}{}'\sum_{e \bmod p}{}'\sum_{f \bmod p}{}'}_{\substack{a^4+b^4+c^4\equiv d^4+e^4+1 (\bmod p)\\ abc\equiv de (\bmod p)}} e\left(\frac{f^2\left(a^2+b^2+c^2-d^2-e^2-1\right)}{p}\right) \\
=\;& p^2(p-1)\cdot\mathop{\sum_{a \bmod p}{}'\sum_{b \bmod p}{}'\sum_{c \bmod p}{}'\sum_{d \bmod p}{}'\sum_{e \bmod p}{}'}_{\substack{a^4+b^4+c^4\equiv d^4+e^4+1 (\bmod p)\\ a^2+b^2+c^2\equiv d^2+e^2+1 (\bmod p)\\ abc\equiv de (\bmod p)}} 1 \\
&-p(p-1)\cdot\mathop{\sum_{a \bmod p}{}'\sum_{b \bmod p}{}'\sum_{c \bmod p}{}'\sum_{d \bmod p}{}'\sum_{e \bmod p}{}'}_{\substack{a^4+b^4+c^4\equiv d^4+e^4+1 (\bmod p)\\ abc\equiv de (\bmod p)}} 1
\end{aligned}$$

$$+p(p-1)\cdot\tau(\chi_2)\cdot\sum_{a \bmod p}{}' \sum_{\substack{b \bmod p\\ a^4+b^4+c^4\equiv d^4+e^4+1(\bmod p)\\ abc\equiv de(\bmod p)}}{}' \sum_{c \bmod p}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' \left(\frac{a^2+b^2+c^2-d^2-e^2-1}{p}\right). \tag{3.1}$$

Note that $p\equiv 3(\bmod 4)$ and the identity $\tau(\chi_2)=i\cdot\sqrt{p}$. It is clear that $\tau(\chi_2)$ is a purely imaginary number. But the left hand side of the formula (27) is a real number, it follows that

$$\sum_{a \bmod p}{}' \sum_{\substack{b \bmod p\\ a^4+b^4+c^4\equiv d^4+e^4+1(\bmod p)\\ abc\equiv de(\bmod p)}}{}' \sum_{c \bmod p}{}' \sum_{d \bmod p}{}' \sum_{e \bmod p}{}' \left(\frac{a^2+b^2+c^2-d^2-e^2-1}{p}\right)=0. \tag{3.2}$$

From (3.1), (3.2), Lemmas 5 and 6 we have the identity

$$\begin{aligned}
&\sum_{\chi \bmod p}\sum_{m \bmod p}\left|\sum_{a \bmod p}\chi(a)e\left(\frac{ma^4+a^2}{p}\right)\right|^6\\
=\ & 8p^2(p-1)\left(3p^2-15p+20\right)-p(p-1)\left(p^3+6p^2-19p-8\right)\\
=\ & p(p-1)\left(23p^3-126p^2+179p+8\right).
\end{aligned}$$

This completes the proof of Theorem 2.

Then we give the proof of Theorem 1.To prove Theorem 1, note that

$$\sum_{a \bmod p} e\left(\frac{na^2}{p}\right)=\left(\frac{n}{p}\right)\cdot\sum_{a \bmod p}\left(\frac{a}{p}\right)e\left(\frac{a}{p}\right)=\begin{cases}\left(\frac{n}{p}\right)\cdot i\cdot\sqrt{p}, & \text{if } p\equiv 3(\bmod 4);\\ \left(\frac{n}{p}\right)\cdot\sqrt{p}, & \text{if } p\equiv 1(\bmod 4),\end{cases}$$

where $(n,p)=1$, similar to the proof of Theorem 2, we have

$$\begin{aligned}
&\frac{1}{p(p-1)}\sum_{\chi \bmod p}\sum_{m \bmod p}\left|\sum_{a \bmod p}{}'\chi(a)e\left(\frac{ma^4+a^2}{p}\right)\right|^4\\
=\ &\sum_{a \bmod p}{}' \sum_{\substack{b \bmod p\\ ab\equiv c(\bmod p)\\ a^4+b^4\equiv c^4+1(\bmod p)}}{}' \sum_{c \bmod p}{}' \sum_{d \bmod p}{}' e\left(\frac{d^2\left(a^2+b^2-c^2-1\right)}{p}\right)\\
=\ &\sum_{a \bmod p}{}' \sum_{\substack{b \bmod p\\ \left(a^4-1\right)\left(b^4-1\right)\equiv 0(\bmod p)}}{}' \sum_{d \bmod p}{}' e\left(\frac{-d^2\left(a^2-1\right)\left(b^2-1\right)}{p}\right).
\end{aligned} \tag{3.3}$$

When $p\equiv 3 \bmod 4$, we know that -1 is a quadratic nonresidue modulo p, so we have

$$\sum_{a \bmod p}{}' \sum_{\substack{b \bmod p\\ \left(a^4-1\right)\left(b^4-1\right)\equiv 0(\bmod p)}}{}' \sum_{d \bmod p}{}' e\left(\frac{-d^2\left(a^2-1\right)\left(b^2-1\right)}{p}\right)$$

$$= \sum_{\substack{a \bmod p \\ (a^2-1)(b^2-1)\equiv 0(\bmod p)}}{}' \sum_{b \bmod p}{}' \sum_{d \bmod p}{}' e\left(\frac{-d^2\left(a^2-1\right)\left(b^2-1\right)}{p}\right)$$
$$= 4(p-1)(p-3) + 4(p-1) = 4(p-1)(p-2). \tag{3.4}$$

When $p \equiv 1(\bmod 4)$, we have

$$\sum_{\substack{a \bmod p \\ (a^4-1)(b^4-1)\equiv 0(\bmod p)}}{}' \sum_{b \bmod p}{}' \sum_{d \bmod p}{}' e\left(\frac{-d^2\left(a^2-1\right)\left(b^2-1\right)}{p}\right)$$
$$= \sum_{\substack{a \bmod p \\ (a^2-1)(b^2-1)\equiv 0(\bmod p)}}{}' \sum_{b \bmod p}{}' \sum_{d \bmod p}{}' e\left(\frac{-d^2\left(a^2-1\right)\left(b^2-1\right)}{p}\right)$$
$$+ \sum_{\substack{a \bmod p \\ (a^2+1)(b^2+1)\equiv 0(\bmod p)}}{}' \sum_{b \bmod p}{}' \sum_{d \bmod p}{}' e\left(\frac{-d^2\left(a^2-1\right)\left(b^2-1\right)}{p}\right)$$
$$= 4(p-1)(p-2) + \sum_{\substack{a=2 \\ (a^2+1)(b^2+1)\equiv 0(\bmod p)}}^{p-2} \sum_{b=2}^{p-2} \left(\frac{-\left(a^2-1\right)\left(b^2-1\right)}{p}\right)\sqrt{p} - \sum_{\substack{a=2 \\ (a^2+1)(b^2+1)\equiv 0(\bmod p)}}^{p-2} \sum_{b=2}^{p-2} 1$$
$$= 4(p-1)(p-2) + 4\sum_{b=2}^{p-2}\left(\frac{2\left(b^2-1\right)}{p}\right)\sqrt{p} - 4\left(\frac{-4}{p}\right)\sqrt{p} - 4(p-4)$$
$$= 4(p^2-4p+6) + 4\sqrt{p}\left(\left(\frac{2}{p}\right)\sum_{b=1}^{p-1}\left(\frac{\left(b^2-1\right)}{p}\right) - 1\right)$$
$$= 4(p^2-4p+6) + 4\sqrt{p}\left(-2\left(\frac{2}{p}\right) - 1\right)$$
$$= \begin{cases} 4\left(p^2-4p+6+\sqrt{p}\right), & \text{if } p \equiv 5(\bmod 8); \\ 4\left(p^2-4p+6-3\sqrt{p}\right), & \text{if } p \equiv 1(\bmod 8), \end{cases} \tag{3.5}$$

which used $\chi_2(2) = 1$, if $p \equiv 1(\bmod 8)$, and $\chi_2(2) = -1$, if $p \equiv 5(\bmod 8)$.

Then from (3.3)–(3.5),we can get

$$\frac{1}{p(p-1)} \sum_{\chi \bmod p} \sum_{m \bmod p} \left| \sum_{a \bmod p}{}' \chi(a) e\left(\frac{ma^4+a^2}{p}\right)\right|^4$$
$$= \begin{cases} 4(p-1)(p-2), & \text{if } p \equiv 3(\bmod 4); \\ 4\left(p^2-4p+6+\sqrt{p}\right), & \text{if } p \equiv 5(\bmod 8); \\ 4\left(p^2-4p+6-3\sqrt{p}\right), & \text{if } p \equiv 1(\bmod 8). \end{cases}$$

This completes the proofs of our all results.

4. Conclusions

The main result of this paper is to give two exact calculating formulae for the sixth power mean values of a generalized two-term exponential sums. One of which is

$$\frac{1}{p(p-1)} \sum_{\chi \bmod p} \sum_{m \bmod p} \left| \sum_{a \bmod p}{}' \chi(a) e\left(\frac{ma^4 + a^2}{p}\right) \right|^6 = 23p^3 - 126p^2 + 179p + 8,$$

here, $p \equiv 3(\bmod 4)$.

If $p \equiv 1(\bmod 4)$, then we do not have an identity or a nontrivial asymptotic formula for this sixth power mean yet. This is an open problem.

Of course, our result also provides some effective methods for calculating the sixth power mean of the high-*th* two-term exponential sums. We assert that these contributions will greatly advance the investigation of irrelated issues.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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