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Research article

Double sequences with ideal convergence in fuzzy metric spaces

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Abstract: We show ideal convergence (I-convergence), ideal Cauchy (I-Cauchy) sequences, I^*-convergence and I^*-Cauchy sequences for double sequences in fuzzy metric spaces. We define the I-limit and I-cluster points of a double sequence in these spaces. Afterward, we provide certain fundamental properties of the aspects. Lastly, we discuss whether the phenomena should be further investigated.

Keywords: double sequences; ideal convergence; ideal Cauchy sequence; limit point; fuzzy metric space
Mathematics Subject Classification: 40A05, 40A35

1. Introduction

Statistical convergence, built on the density of natural numbers, was independently defined by Steinhaus [1] and Fast [2] in 1951. Statistical convergence as a summability method was also proposed by Schoenberg [3]. Since its introduction, statistical convergence has been applied in a great diversity of fields, including summability theory [4], locally convex sequence spaces [5], trigonometric series [6], number theory [7] and measurement theory [8].

Statistical convergence is associated with the natural density of positive integer sets, while sets with a natural density of zero represent an ideal. Building on this idea, Kostryko et al. [9] introduced "ideal convergence" in 2000, which generalizes statistical convergence.

They have also investigated the closely related concept of I^*-convergence and the condition (AP). Later, Dems [10] extended statistical Cauchy sequences [11] to ideals and proposed ideal Cauchy (I-Cauchy) sequences. Nabiyev et al. [12] further introduced I^*-Cauchy sequences and analyzed the relationship between current sequences and I-Cauchy sequences. In 2003, Mursaleen and Edely [13] studied statistical convergence for double sequences. Moreover, Triphaty and Triphaty [14] coined and characterized I-convergence and I-Cauchy sequences for double sequences. Afterward, Kumar [15] developed I and I^*-convergence for double sequences, providing a more straightforward method to prove results for I and I^*-convergence. In 2008, Das et al. [16] presented I and I^*-convergence of double sequences in a metric space by exemplifying the relationships between them. Subsequently, some results on I-convergence of double sequences were presented in [17, 18].

Fuzzy sets were first introduced by Zadeh [19] and have since been utilized by many mathematicians in topology and analysis. Fuzzy metric spaces (FMSs) extend the notion of metric spaces by introducing degrees of membership or fuzziness of points. Kramosil and Michalek [20] and Kaleva and Seikkala [21] were among the first to investigate FMSs. Building on Kramosil and Michalek's [20] work, George and Veeramani [22] redefined the concept of FMSs by utilizing a continuous t-norm and obtained the Hausdorff topology of these spaces.

Recently, Mihet [23] has studied the concept of point convergence (p-convergence), a weaker concept than ordinary convergence. Additionally, Gregori et al. [24] proposed the concept of s-convergence. Standard convergence (std-convergence) was presented by Morillas and Sapena [25]. Gregori and Miňana [26] have proposed strong convergence (st-convergence), which is a stronger concept than ordinary convergence. Statistical convergence and statistical Cauchy sequences in FMSs were proposed by Li et al. [27] and they have examined some of their basic properties. Moreover, Savaş [28] has introduced statistical convergence for double sequences in FMSs.

Inspired by previous research, we focus on ideal convergence for double sequences in FMSs. We propose I- and I^*-convergence and I- and I^*-Cauchy sequences for double sequences in FMSs and investigate some of their basic properties. We define I-limit points and I-cluster points of a double sequence in FMSs.

Our study is significant in that it provides a new approach to studying the convergence behavior of double sequences in FMSs. Ideal convergence has not been analyzed in this context, and our research contributes to filling this gap in the literature. Moreover, our results can be useful for applications in various fields.

The present paper can be summarized as follows: In Section 2 of our article, we present basic definitions and properties that are essential for the following sections. We also provide necessary background information about FMSs and the notion of convergence. In Section 3, we define I- and I^*-convergence for double sequences in FMSs, and I- and I^*-Cauchy sequences. We investigate some of their basic features, such as other types of convergence and their relations with Cauchy sequences. In Section 4, we introduce I-limit points and I-cluster points of double sequences in FMSs. Finally, in the concluding section, we summarize our findings and discuss the need for further research in this area.

2. Preliminaries

This part thoroughly introduces the fundamental concepts, definitions and properties required to understand FMSs and convergence fully.

Definition 2.1. [21] Let $\circ : [0, 1]^2 \to [0, 1]$ be a binary operation. We say that $\circ$ is a triangular norm (t-norm) if it satisfies the following conditions:

(1) $\circ$ is both associative and commutative;
(2) $t \circ 1 = t$ for all $t \in [0, 1]$;
(3) Whenever $t_1 \leq t_3$ and $t_2 \leq t_4$ for each $t_1, t_2, t_3, t_4 \in [0, 1]$, it holds that $t_1 \circ t_3 \leq t_2 \circ t_4$.

Example 2.2. [19] According to the above definition, the following operators are a t-norm:

(1) $\sigma \circ \tau = \sigma\tau$,
(2) $\sigma \circ \tau = \min\{\sigma, \tau\}$.

Definition 2.3. [20] Let ϑ be a fuzzy set on $\mathbb{X}^2 \times (0, \infty)$, where $\mathbb{X}$ is an arbitrary set and $\circ$ be a continuous t-norm. If the following requirements must be fulfilled for all an $u, v > 0$ and $x_1, x_2, x_3 \in \mathbb{X}$, then ϑ is referred to as a fuzzy metric on $\mathbb{X}$,

(1) $\vartheta(x_1, x_2, u) > 0$;
(2) $\vartheta(x_1, x_2, u) = 1 \Leftrightarrow x_1 = x_2$;
(3) $\vartheta(x_1, x_2, u) = \vartheta(x_2, x_1, u)$;
(4) $\vartheta(x_1, x_3, u + v) \geq \vartheta(x_1, x_2, u) \circ \vartheta(x_2, x_3, v)$;
(5) The function $\vartheta_{x_1x_2} : (0, \infty) \to [0, 1]$, defined by $\vartheta_{x_1x_2}(u) = \vartheta(x_1, x_2, u)$ is continuous.

The 3-tuple $(\mathbb{X}, \vartheta, \circ)$ is called fuzzy metric space.

Example 2.4. [20] Consider the set $\mathbb{X} = \mathbb{R}$ and define the binary operation $\circ$ as $\sigma \circ \tau = \sigma\tau$. Additionally, we define the fuzzy set ϑ as follows:

$$\vartheta(x_1, x_2, u) = \left[\exp\left(\frac{|x_1 - x_2|}{u}\right)\right]^{-1}, \forall x_1, x_2 \in \mathbb{X}, u > 0.$$

Therefore, we can conclude that $(\mathbb{X}, \vartheta, \circ)$ is an FMS.

Definition 2.5. [22] Let $(\mathbb{X}, \vartheta, \circ)$ be an FMS. If $a \in \mathbb{X}$, then the open ball centered at a with radius $\varepsilon, 0 < \varepsilon < 1$ is the set of points $x \in \mathbb{X}$ contained in

$$B_a^{\varepsilon}(x) = \{x \in \mathbb{X} : \vartheta(a, x, u) > 1 - \varepsilon\}, u > 0.$$

Definition 2.6. [28] A double sequence (x_{jk}) in $\mathbb{X}$ is said to be convergent to $x_0 \in \mathbb{X}$ with respect to fuzzy metric ϑ if, for all $\varepsilon \in (0, 1)$ and $u > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $j, k \geq N_\varepsilon$ implies

$$\vartheta(x_{jk}, x_0, u) > 1 - \varepsilon$$

or equivalently

$$\lim_{j,k\to\infty} \vartheta(x_{jk}, x_0, u) = 1$$

and is denoted by $\vartheta - \lim_{j,k\to\infty} x_{jk} = x_0$ or $x_{jk} \xrightarrow{\vartheta} x_0$ as $j, k \to \infty$.

Definition 2.7. [13] Let $E \subseteq \mathbb{N}^2$ and $E_{mn} = \{(j, k) \in E : j \leq m \text{ and } k \leq n\}$. The set E is said to have double natural density, denoted by $\delta_2(E)$, is defined as:

$$\delta_2(E) = \lim_{m,n\to\infty} \frac{|E_{mn}|}{mn},$$

if the limit exists. It can be observed that if the set E is finite, then $\delta_2(E) = 0$.

Definition 2.8. [28] A double sequence (x_{jk}) in $\mathbb{X}$ is referred to as statistically convergent to $x_0 \in \mathbb{X}$ with respect to fuzzy metric ϑ if, for all $\varepsilon \in (0,1)$ and $u > 0$,

$$\delta_2(\{(j,k) \in \mathbb{N}^2 : \vartheta(x_{jk}, x_0, u) \leq 1 - \varepsilon\}) = 0$$

and is denoted by $st_2 - \lim_{j,k\to\infty} x_{jk} = x_0$.

The definitions of ideal and filter are provided below to explain the main results regarding ideal convergence. Then, the concepts needed in the sequel of the study are mentioned.

Definition 2.9. [16] Let $\mathbb{X} \neq \emptyset$. An ideal on $\mathbb{X}$ is a collection of subsets of $\mathbb{X}$ such that

(a) The empty set $\emptyset$ is an element of I.

(b) If $\mathcal{U}$ and $\mathcal{V}$ are set in I, then their union $\mathcal{U} \cup \mathcal{V}$ is also an element of I.

(c) If $\mathcal{U}$ is a set in I and $\mathcal{V}$ is a subset of $\mathbb{X}$ such that $\mathcal{V} \subseteq \mathcal{U}$, then $\mathcal{V}$ is an element of I.

If $\mathbb{X} \notin I$ and $I \neq \emptyset$, then I is called a non-trivial ideal. Additionally, if I is a non-trivial ideal in $\mathbb{X}$ and

$$\{\{x\} : x \in \mathbb{X}\} \subseteq I,$$

then I is referred to as an admissible ideal.

In the current study, I_2 denotes a non-trivial admissible ideal of $\mathbb{N}^2$.

Definition 2.10. [16] A strongly admissible ideal on I_2 is a collection of subsets of $\mathbb{N}^2$ such that

(a) $\{r\} \times \mathbb{N} \in I_2$,

(b) $\mathbb{N} \times \{r\} \in I_2$.

For example, $I_2^0 = \{P \subseteq \mathbb{N}^2 : (\exists l(P) \in \mathbb{N})(j,k \geq l(P) \Rightarrow (j,k) \notin P)\}$ is a non-trivial strongly admissible ideal. Any strongly admissible ideal is an admissible ideal.

Definition 2.11. [16] Let $I_2 \subseteq 2^{\mathbb{N}^2}$ be an admissible ideal, (P_i) be a sequence of mutually disjoint sets of I_2 and (R_i) be a subset of $\mathbb{N}^2$. Then, I_2 satisfies the condition (AP2) if, for all (P_i), there is a sequence (R_i) such that, for all $i \in \mathbb{N}$, $P_i \Delta R_i \in I_2^0$ i.e., $P_i \Delta R_i$ is included in limited quantities union of rows and columns in $\mathbb{N}^2$ and $R = \bigcup_i R_i \in I_2$. Here, Δ denotes the symmetric difference. Note that $R_i \in I_2$.

Definition 2.12. [16] Let $\mathbb{X} \neq \emptyset$. A filter on $\mathbb{X}$ is a collection of subsets of $\mathbb{X}$ such that

(a) The empty set $\emptyset$ is not an element of $\mathcal{F}$.

(b) If $\mathcal{U}$ and $\mathcal{V}$ are sets in $\mathcal{F}$, then their intersection $\mathcal{U} \cap \mathcal{V}$ is also an element of $\mathcal{F}$.

(c) If $\mathcal{V}$ is a set in $\mathcal{F}$ and $\mathcal{U}$ is a subset of $\mathbb{X}$ such that $\mathcal{V} \subseteq \mathcal{U}$, then $\mathcal{U}$ is an element of $\mathcal{F}$.

Furthermore, let I_2 be a non-trivial ideal. Then the collection $\mathcal{F}(I_2) = \{\mathbb{N}^2 \setminus S : S \in I_2\}$ is a filter on $\mathbb{N}^2$ and is referred to as the filter associated with the ideal I_2.

Proposition 2.13. [17] Let (P_i) be a countable collection of subsets of $\mathbb{N}^2$ such that $(P_i) \in \mathcal{F}(I_2)$, for all i, where $\mathcal{F}(I_2)$ is a filter associated with a strongly admissible ideal I_2 with the property (AP2). Then, there exists a set P that belongs to the filter $\mathcal{F}(I_2)$ and has the property that the set of elements in P that do not belong to P_i is finite for every index i.

Definition 2.14. [16] Let $I_2 \subset 2^{\mathbb{N}^2}$ be a non-trivial ideal. A double sequence (x_{jk}) in a metric space $(\mathbb{X}, \rho)$ is called ideal convergent (I_2-convergent) to $x_0 \in \mathbb{X}$, written as $I_2 - \lim_{j,k\to\infty} x_{jk} = x_0$ or $x_{jk} \xrightarrow{I_2} x_0$ as $j,k \to \infty$ if, for all $\varepsilon > 0$,

$$A(\varepsilon) = \{(j,k) \in \mathbb{N}^2 : \rho(x_{jk}, x_0) \geq \varepsilon\} \in I_2.$$

If we choose $I_2 = \left\{S \subseteq \mathbb{N}^2 : S \text{ is of the form } (\mathbb{N} \times A) \cup (A \times \mathbb{N})\right\}$, where A is a finite subset of $\mathbb{N}$, then I_2-convergent coincides with ordinary convergence of double sequences.

If we choose $I_2 = \left\{S \subseteq \mathbb{N}^2 : \delta_2(S) = 0\right\}$, then I_2-convergent is equivalent to the statistical convergence of double sequences.

Definition 2.15. [16] A double sequence (x_{jk}) in a metric space $(\mathbb{X}, \rho)$ is referred to as I_2^*-convergent to $x_0 \in \mathbb{X}$, where $I_2 \subset 2^{\mathbb{N}^2}$ be a non-trivial ideal if exists a set

$$H = \{(j_t, k_t) \in \mathbb{N}^2 : j_1 < j_2 < \ldots < j_t < \ldots ; k_1 < k_2 < \ldots < k_t < \ldots\} \in \mathcal{F}(I_2)$$

such that

$$\lim_{j_t, k_t \to \infty} \rho(x_{j_t k_t}, x_0) = 0.$$

We abbreviate it as $I_2^* - \lim\limits_{j,k\to\infty} x_{jk} = x_0$ or $x_{jk} \xrightarrow{I_2^*} x_0$.

Definition 2.16. [10] A double sequence (x_{jk}) in a metric space $(\mathbb{X}, \rho)$ is referred to as an ideal Cauchy (I_2-Cauchy) sequence in $\mathbb{X}$, where $I_2 \subseteq 2^{\mathbb{N}^2}$ be a strongly admissible ideal if, for all $\varepsilon > 0$, there exists an $(p, q) \in \mathbb{N}^2$ such that

$$A(\varepsilon) = \{(j, k) \in \mathbb{N}^2 : \rho(x_{jk}, x_{pq}) \geq \varepsilon\} \in I_2.$$

Definition 2.17. [17] Let I_2 be a strongly admissible ideal in $\mathbb{N}^2$. A double sequence (x_{jk}) in a metric space $(\mathbb{X}, \rho)$ is called an I_2^*-Cauchy sequence in $\mathbb{X}$ if there exists a set

$$H = \{(j_t, k_t) \in \mathbb{N}^2 : j_1 < j_2 < \ldots < j_t < \ldots ; k_1 < k_2 < \ldots < k_t \ldots\} \in \mathcal{F}(I_2)$$

such that

$$\lim_{\substack{j_t, k_t, p_t, q_t \to \infty \\ (j_t, k_t), (p_t, q_t) \in H}} \rho(x_{j_t k_t}, x_{p_t q_t}) = 0.$$

Definition 2.18. [18] Let $(\mathbb{X}, \rho)$ be a metric space and (x_{jk}) be a double sequence in $\mathbb{X}$. Then, an element $x_0 \in \mathbb{X}$ is referred to as an I_2-limit point of (x_{jk}) if there is a set

$$H = \{(j_t, k_t) \in \mathbb{N}^2 : j_1 < j_2 < \ldots < j_t < \ldots ; k_1 < k_2 < \ldots < k_t < \ldots\} \notin I_2,$$

and

$$\lim_{\substack{j_t, k_t \to \infty \\ (j_t, k_t) \in H}} \rho(x_{j_t k_t}, x_0) = 0.$$

Definition 2.19. [18] Let $(\mathbb{X}, \rho)$ be a metric space and (x_{jk}) be a double sequence in $\mathbb{X}$. Then, an element $x_0 \in \mathbb{X}$ is called an I_2-cluster point of (x_{jk}) if, for all $\varepsilon > 0$, $\{(j, k) \in \mathbb{N}^2 : \rho(x_{jk}, x_0) \leq \varepsilon\} \notin I_2$.

The set of all I_2-limit points and I_2-cluster points of a double sequence x are denoted by $I_2(\Lambda_x)$ and $I_2(\Gamma_x)$, respectively.

3. $\vartheta(I_2)$-convergence and $\vartheta(I_2)$-Cauchy sequence

In this chapter, we define the notions of ideal convergence and ideal Cauchy sequences for double sequences in FMSs and discuss some of their basic properties. Throughout this chapter, for brevity, we shall often write $\mathbb{X}$ instead of "$(\mathbb{X}, \vartheta, \circ)$" and (x_{jk}) instead of a "double sequence (x_{jk})".

Definition 3.1. Let $I_2 \subseteq 2^{\mathbb{N}^2}$ be a non-trivial ideal. A sequence (x_{jk}) is referred to as $\vartheta(I_2)$-convergent to $x_0 \in \mathbb{X}$ if, for all $u > 0$ and $\varepsilon \in (0, 1)$,

$$A(u, \varepsilon) = \{(j, k) \in \mathbb{N}^2 : \vartheta(x_{jk}, x_0, u) \leq 1 - \varepsilon\} \in I_2,$$

and is denoted by $\vartheta(I_2) - \lim\limits_{j,k\to\infty} x_{jk} = x_0$ or $x_{jk} \xrightarrow{\vartheta(I_2)} x_0$ as $j, k \to \infty$. The number x_0 is called the I_2-limit of (x_{jk}).

Example 3.2. If we choose $I_2 = I_0$ and $I_2 = I_\delta = \{A \subseteq \mathbb{N}^2 : \delta_2(A) = 0\}$, then I_2-convergence is the same as ordinary convergence and statistical convergence, respectively.

The following theorem presents, well-known in ordinary convergence, which gives whether the following expressions satisfy at ideal convergence:
I. Every constant double sequence converges to yourself.
II. The limit of converged double sequences can be determined by uniquely.
III. Every subsequence of the converged double sequence is convergent and has the same limit.

Theorem 3.3. Let $I_2 \subseteq 2^{\mathbb{N}^2}$.

(1) The I_2-convergence satisfies (I) and (II).
(2) Every subsequence of an I_2-convergent sequence is not I_2-convergent if I_2 is a strongly admissible ideal and contains an infinite set.

Proof.

(1) It is clear that $\vartheta(I_2)$-convergence satisfies proposition (I). We prove that it satisfies (II) as well. Suppose that $\vartheta(I_2) - \lim\limits_{j,k\to\infty} x_{jk} = x_0$, $\vartheta(I_2) - \lim\limits_{j,k\to\infty} x_{jk} = x_1$ and $x_0 \neq x_1$. Then, by assumption and Remark 2.12, the sets

$$\mathbb{N}^2 \setminus A = \{(j, k) \in \mathbb{N}^2 : \vartheta(x_{jk}, x_0, u) > 1 - \varepsilon\}$$

and

$$\mathbb{N}^2 \setminus B = \{(j, k) \in \mathbb{N}^2 : \vartheta(x_{jk}, x_1, u) > 1 - \varepsilon\}$$

are elements of $\mathcal{F}(I_2)$. Hence, the set $K = (\mathbb{N}^2 \setminus A) \cap (\mathbb{N}^2 \setminus B)$ is an element of $\mathcal{F}(I_2)$. Choose $u > 0$ and $\varepsilon = \frac{1}{n}$, $(n = 2, 3, \ldots)$. Thus, there exists a $(t, s) \in K$ such that

$$\vartheta(x_{ts}, x_0, u) > 1 - \varepsilon \text{ and } \vartheta(x_{ts}, x_1, u) > 1 - \varepsilon.$$

From this $\vartheta(x_0, x_1, u) = 1$ which is a contradiction to $x_0 \neq x_1$.

(2) Suppose that an infinite set $A = \{(j_t, k_t) \in \mathbb{N}^2 : j_1 < j_2 < \ldots < j_t < \ldots; k_1 < k_2 < \ldots < k_t < \ldots\} \subseteq \mathbb{N}^2$ belongs to I_2. We put

$$\mathbb{N}^2 \setminus A = \{(p_t, q_t) \in \mathbb{N}^2 : p_1 < p_2 < \ldots < p_t < \ldots; q_1 < q_2 < \ldots < q_t < \ldots\}.$$

The set $\mathbb{N}^2 \setminus A$ is infinite because in the opposite case $\mathbb{N}^2$ would belong to I_2. Define the sequence (x_{jk}) as follows

$$x_{j_t k_t} = x_0, \quad x_{p_t q_t} = x_1 \quad (t = 1, 2, \ldots).$$

Obviously $\vartheta(I_2) - \lim_{j,k\to\infty} x_{jk} = x_1$. In addition, the subsequence $y = (x_{j_t k_t})$ of (x_{jk}) is stationary and thus $\vartheta(I_2) - \lim y = x_0$ (see proposition (I)). Hence, I_2-convergence does not satisfy the proposition (III).

□

Proposition 3.4. Let I_2 be a non-trivial ideal given that

$$\left\{S \subseteq \mathbb{N}^2 : r \in \mathbb{N}, S = \mathbb{N} \times \{r\} \vee S = \{r\} \times \mathbb{N}\right\} \subseteq I_2,$$

then $\lim_{j,k\to\infty} \vartheta(x_{jk}, x_0, u) = 1$ implies $\vartheta(I_2) - \lim_{j,k\to\infty} x_{jk} = x_0$.

Proof. Suppose that I_2 be a non-trivial ideal such that

$$\left\{S \subseteq \mathbb{N}^2 : r \in \mathbb{N}, S = \mathbb{N} \times \{r\} \vee S = \{r\} \times \mathbb{N}\right\} \subseteq I_2,$$

and $\lim_{j,k\to\infty} \vartheta(x_{jk}, x_0, u) = 1$. Let $u > 0$ and $\varepsilon \in (0, 1)$ be given. Since (x_{jk}) is convergent to $x_0 \in \mathbb{X}$, then there exists a $N_\varepsilon \in \mathbb{N}$ such that $\vartheta(x_{jk}, x_0, u) > 1 - \varepsilon$ whenever $j, k \geq N_\varepsilon$. Hence, there exists a set

$$P = \left\{(j, k) \in \mathbb{N}^2 : \vartheta(x_{jk}, x_0, u) \leq 1 - \varepsilon\right\} \subset \mathcal{U} \cup \mathcal{V},$$

where $\mathcal{U} = \mathbb{N} \times \{1, 2, 3 \ldots N_\varepsilon - 1\}$ and $\mathcal{V} = \{1, 2, 3 \ldots N_\varepsilon - 1\} \times \mathbb{N}$. From the hypothesis $\mathcal{U} \cup \mathcal{V} \in I_2$. Since I_2 is an ideal, then $P \in I_2$. Consequently, $\vartheta(I_2) - \lim_{j,k\to\infty} x_{jk} = x_0$. □

Definition 3.5. A sequence (x_{jk}) is referred to as Cauchy sequence in $\mathbb{X}$ if, for all $u > 0$ and $\varepsilon \in (0, 1)$, exists an integer $N_\varepsilon \in \mathbb{N}$ such that

$$\vartheta(x_{jk}, x_{pq}, u) > 1 - \varepsilon,$$

whenever $j, k, p, q \geq N_\varepsilon$ or equivalently

$$\lim_{j,k,p,q\to\infty} \vartheta(x_{jk}, x_{pq}, u) = 1.$$

Definition 3.6. A sequence (x_{jk}) is said to be $\vartheta(I_2)$-Cauchy sequence in $\mathbb{X}$, where I_2 is a strongly admissible ideal if, for all $u > 0$ and $\varepsilon \in (0, 1)$, there exists an integer $(p, q) \in \mathbb{N}^2$ such that

$$A(u, \varepsilon) = \{(j, k) \in \mathbb{N}^2 : \vartheta(x_{jk}, x_{pq}, u) \leq 1 - \varepsilon\} \in I_2.$$

Proposition 3.7. Let I_2 be a strongly admissible ideal in $\mathbb{N}^2$. If (x_{jk}) is a Cauchy sequence in $\mathbb{X}$, then it is a $\vartheta(I_2)$-Cauchy sequence.

Proof. Let $u > 0$ and $\varepsilon \in (0, 1)$ be given. Since (x_{jk}) is Cauchy sequence in $\mathbb{X}$, for all $j, k, p, q \geq N_\varepsilon$, there exists an integer $N_\varepsilon \in \mathbb{N}$ such that $\vartheta(x_{jk}, x_{pq}, u) > 1 - \varepsilon$. Hence, there exists a set

$$P = \left\{(j, k) \in \mathbb{N}^2 : \vartheta(x_{jk}, x_{pq}, u) \leq 1 - \varepsilon\right\} \subset \mathcal{U} \cup \mathcal{V},$$

where $\mathcal{U} = \mathbb{N} \times \{1, 2, 3 \ldots N_\varepsilon - 1\}$ and $\mathcal{V} = \{1, 2, 3 \ldots N_\varepsilon - 1\} \times \mathbb{N}$. Since I_2 is a strongly admissible ideal, $\mathcal{U} \cup \mathcal{V} \in I_2$. Therefore $P \in I_2$. Consequently, (x_{jk}) is a $\vartheta(I_2)$-Cauchy sequence in $\mathbb{X}$. □

Theorem 3.8. For any double sequence, $\vartheta(I_2)$-convergent implies $\vartheta(I_2)$-Cauchy sequence if I_2 is a strongly admissible ideal in $\mathbb{N}^2$.

Proof. Let $\vartheta(I_2) - \lim_{j,k\to\infty} x_{jk} = x_0$. Then, for all $u > 0$ and $\varepsilon \in (0,1)$, we have

$$A(u,\varepsilon) = \{(j,k) \in \mathbb{N}^2 : \vartheta(x_{jk}, x_0, u) \leq 1 - \varepsilon\} \in I_2.$$

Because of the definition of a strongly admissible ideal, there exists a $(p,q) \notin A(u,\varepsilon)$. Assume that

$$B = \{(j,k) \in \mathbb{N}^2 : \vartheta(x_{jk}, x_{pq}, u) \leq \delta(\varepsilon)\}.$$

Considering the following inequality

$$\vartheta(x_{jk}, x_{pq}, u) \geq \vartheta\left(x_{jk}, x_0, \frac{u}{2}\right) \circ \vartheta\left(x_{pq}, x_0, \frac{u}{2}\right),$$

we observe that if $(j,k) \in B$, then

$$\delta(\varepsilon) \geq (1-\varepsilon) \circ (1-\varepsilon) \geq \vartheta\left(x_{jk}, x_0, \frac{u}{2}\right) \circ \vartheta\left(x_{pq}, x_0, \frac{u}{2}\right).$$

Moreover, we have $\vartheta(x_{pq}, x_0, u) > 1 - \varepsilon$ because $(p,q) \notin A(u,\varepsilon)$. Hence, $\vartheta(x_{jk}, x_0, u) \leq 1 - \varepsilon$, then $(j,k) \in A(u,\varepsilon)$. In this case, for all $u > 0$ and $\varepsilon \in (0,1)$, $B \subseteq A(u,\varepsilon) \in I_2$. Consequently, (x_{jk}) is a $\vartheta(I_2)$-Cauchy sequence. □

Definition 3.9. A sequence (x_{jk}) is referred to as $\vartheta(I_2^*)$-convergent to $x_0 \in \mathbb{X}$ if there exists a set

$$H = \{(j_t, k_t) \in \mathbb{N}^2 : j_1 < j_2 < \ldots < j_t < \ldots; k_1 < k_2 < \ldots < k_t < \ldots\} \in \mathcal{F}(I_2)$$

such that

$$\lim_{j_t,k_t\to\infty} x_{j_t k_t} = x_0, \tag{3.1}$$

and is denoted by $\vartheta(I_2^*) - \lim_{j,k\to\infty} x_{jk} = x_0$ or $x_{jk} \xrightarrow{\vartheta(I_2^*)} x_0$ as $j,k \to \infty$.

Theorem 3.10. $\vartheta(I_2^*) - \lim_{j,k\to\infty} x_{jk} = x_0$ implies $\vartheta(I_2) - \lim_{j,k\to\infty} x_{jk} = x_0$ that if I_2 is a strongly admissible ideal in $\mathbb{N}^2$.

Proof. By hypothesis, there is a set $K \in I_2$ such that (3.1) holds, where

$$H = \mathbb{N}^2 \setminus K = \{(j_t, k_t) \in \mathbb{N}^2 : j_1 < j_2 < \ldots < j_t < \ldots; k_1 < k_2 < \ldots < k_t < \ldots\}.$$

Let $u > 0$ and $\varepsilon \in (0,1)$. Then, there exists an integer $n_0 \in \mathbb{N}$ such that $\vartheta(x_{j_p k_p}, x_0, u) > 1 - \varepsilon$ for $j_p, k_p > n_0$. Hence,

$$A(u,\varepsilon) = \left\{(j_t, k_t) \in \mathbb{N}^2 : \vartheta\left(x_{j_t k_t}, x_0, u\right) \leq 1 - \varepsilon\right\} \subset K \cup (H \cap ((B \times \mathbb{N}) \cup (\mathbb{N} \times B))),$$

where $B = \{1, 2, \ldots, (n_0 - 1)\}$. Since $K \cup (H \cap ((B \times \mathbb{N}) \cup (\mathbb{N} \times B))) \in I_2$, then $A(u,\varepsilon) \in I_2$. As a result, $\vartheta(I_2) - \lim_{j,k\to\infty} x_{jk} = x_0$. □

The following Example 3.11 states that the converse of Theorem 3.10 does not always hold.

Example 3.11. Let $(\mathbb{R}, |.|)$ be a metric space and $x \circ y = xy$, for all $x, y \in [0, 1]$. If, for every $x, y \in \mathbb{R}$ and $s > 0$,

$$\vartheta(x, y, u) = \frac{u}{u + |x - y|},$$

then $(\mathbb{R}, \vartheta, \circ)$ is an FMS. Let $\Delta_j = \{(m, n) : \min\{m, n\} \in K_j\}$ such that $K_j = \{2^{j-1}(2s - 1) : s = 1, 2, ...\}$ be a decomposition of $\mathbb{N}$. Besides, $\{\Delta_j\}_{j\in\mathbb{N}}$ is a decomposition of $\mathbb{N}^2$ and

$$I_2 := \{A \subseteq \mathbb{N}^2 : A \subset \Delta_1 \cup \Delta_2 \cup ... \cup \Delta_j, j = 1, 2, ...\}.$$

is a strongly admissible ideal. We define a sequence (x_{st}) by

$$x_{st} := \begin{cases} \frac{1}{j}, & (s,t) \in \Delta_j \\ 0, & \mathbb{N}^2 \backslash \Delta_j \end{cases}.$$

On the other hand,

$$A_{(\varepsilon,u)} = \{(s, t) : \vartheta(x_{st}, 0, u) \leq 1 - \varepsilon\} \in I_2.$$

Hence, $\vartheta(I_2) - \lim_{s,t\to\infty} x_{st} = 0$. However, this sequence does not $\vartheta(I_2^*)$ convergent to zero.

Theorem 3.12. Let I_2 be an admissible ideal, (x_{jk}) be a sequence in $\mathbb{X}$ and $x_0 \in \mathbb{X}$.

(1) If the I_2 ideal has the condition (AP2), then $\vartheta(I_2) - \lim_{j,k\to\infty} x_{jk} = x_0$ implies $\vartheta(I_2^*) - \lim_{j,k\to\infty} x_{jk} = x_0$.

(2) If $\mathbb{X}$ has at least one accumulation point and $\vartheta(I_2) - \lim_{j,k\to\infty} x_{jk} = x_0$ implies $\vartheta(I_2^*) - \lim_{j,k\to\infty} x_{jk} = x_0$, then I_2 has the property (AP2).

Proof.

(1) Let $x_{jk} \xrightarrow{\vartheta(I)} x_0$ and I_2 satisfy the condition (AP2). Then, for all $u > 0$ and $\varepsilon \in (0, 1)$ the set

$$A(u, \varepsilon) = \{(j, k) \in \mathbb{N}^2 : \vartheta(x_{jk}, x_0, u) \leq 1 - \varepsilon\} \in I_2.$$

Put

$$P_1 = \left\{(j, k) \in \mathbb{N}^2 : \vartheta(x_{jk}, x_0, u) \leq \frac{1}{2}\right\},$$

$$P_t = \left\{(j, k) \in \mathbb{N}^2 : \frac{t-1}{t} < \vartheta(x_{jk}, x_0, u) \leq \frac{t}{t+1}\right\} \quad t \geq 2.$$

Obviously, $P_t \cap P_s = \emptyset$ for $t \neq s$ and $P_t \in I_2$ $(t = 1, 2, \ldots)$. Since I_2 satisfies (AP2), there exists sets $R_s \subseteq \mathbb{N}^2$ such that, for all $s \in \mathbb{N}$, $P_s \Delta R_s$ is contained in limited quantities union of rows and columns in $\mathbb{N}^2$ and $R = \bigcup_{s=1}^{\infty} R_s \in I_2$.

It suffices to prove that

$$\lim_{\substack{j,k\to\infty \\ (j,k)\in H}} \vartheta(x_{jk}, x_0, u) = 1, \tag{3.2}$$

where $H = \mathbb{N}^2 \setminus R$.

Let $\eta \in (0,1)$ and $u > 0$. Choose $m \in \mathbb{N}$ such that $\frac{1}{m} < \eta$. Then,

$$\{(j,k) \in \mathbb{N}^2 : \vartheta(x_{jk}, x_0, u) \leq 1 - \eta\} \subseteq \bigcup_{s=1}^{m+1} P_s.$$

The set on the right hand side belongs to I_2 by the additivity of I_2. Since, for all $s \in \mathbb{N}$, $P_s \Delta R_s$ is included in limited quantities union of rows and columns, there is an $n_0 \in \mathbb{N}$ such that

$$\bigcup_{s=1}^{m+1} R_s \cap \{(j,k) \in \mathbb{N}^2 : j,k > n_0\} = \bigcup_{s=1}^{m+1} P_s \cap \{(j,k) \in \mathbb{N}^2 : j,k > n_0\}.$$

If $(j,k) \notin R$ and $j,k > n_0$, then $(j,k) \notin \bigcup_{s=1}^{m+1} R_s$. Hence, $(j,k) \notin \bigcup_{s=1}^{m+1} P_s$. However,

$$\vartheta(x_{jk}, x_0, u) < \frac{1}{m+1} < r.$$

Consequently, (3.2) holds.

(2) Suppose $x_0 \in \mathbb{X}$ is an accumulation point of $\mathbb{X}$. Then, there exists a sequence (y_n) of distinct elements of $\mathbb{X}$ such that $y_n \neq x_0$ for any n, $\lim_{n\to\infty} \vartheta(y_n, x_0, u) = 1$. Let $\{P_1, P_2, \ldots\}$ be a disjoint family of nonempty sets in I_2. Define a sequence (x_{jk}) in the following way: $x_{jk} = y_n$ if $(j,k) \in P_t$ and $x_{jk} = x_0$ if $(j,k) \notin P_t$, for all t. Let $\eta \in (0,1)$ and $u > 0$. Choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \eta$. Then,

$$A(u,\eta) = \left\{(j,k) \in \mathbb{N}^2 : \vartheta(x_{jk}, x_0, u) \leq 1 - \eta\right\} \subset P_1 \cup P_2 \cdots \cup P_n.$$

Hence, $A(u,\eta) \in I_2$ and $\vartheta(I_2) - \lim_{j,k\to\infty} x_{jk} = x_0$. By virtue of our assumption, we have $\vartheta(I_2^*) - \lim_{j,k\to\infty} x_{jk} = x_0$. Therefore, there exists a set $R \in I_2$ such that $H = \mathbb{N}^2 \setminus R \in \mathcal{F}(I_2)$ and

$$\lim_{\substack{j_n,k_n\to\infty \\ (j_n,k_n)\in H}} \vartheta(x_{j_n k_n}, x_0, u) = 1. \tag{3.3}$$

Put $R_t = P_t \cap R$ for $t \in \mathbb{N}$. Then, $R_t \in I_2$ for all $t \in \mathbb{N}$. Moreover, $\bigcup_{t=1}^{\infty} R_t = R \cap \bigcup_{t=1}^{\infty} P_t \subset R$ and thus $\bigcup_{t=1}^{\infty} R_t \in I_2$. Let t be a fixed element in $\mathbb{N}$. Suppose the intersection $P_t \cap H$ is not contained in the limited quantities union of rows and columns in $\mathbb{N}^2$. In that case, there must exist an infinite sequence of elements $\{(j_n, k_n)\}$ in H such that both j_n and k_n tend to infinity, and $x_{j_n k_n} = y_t \neq x_0$ for all $n \in \mathbb{N}$. This contradicts (3.3). Therefore, $P_t \cap H$ should be included in the limited quantities union of rows and columns in $\mathbb{N}^2$. Consequently, the set $P_t \Delta R_t = P_t \setminus R_t = P_t \setminus R = P_t \cap H$ is also included in the limited quantities union of rows and columns. This proves that the ideal I_2 satisfies property (AP2).

□

Theorem 3.13. Let I_2 be a strongly admissible ideal in $\mathbb{N}^2$. If $\mathbb{X}$ has no accumulation point, then $\vartheta(I_2)$-convergence coincides with $\vartheta(I_2^*)$-convergence.

Proof. Let $x_0 \in \mathbb{X}$ and $x_{jk} \xrightarrow{\vartheta(I_2)} x_0$. Thanks to Theorem 3.10, it suffices to prove that $x_{jk} \xrightarrow{\vartheta(I_2^*)} x_0$ as $j,k \to \infty$. Since $\mathbb{X}$ has no accumulation points, there exists $u > 0$ and $\varepsilon \in (0,1)$ such that

$$B_{x_0}^{\varepsilon}(x) = \{x \in \mathbb{X} : \vartheta(x, x_0, u) > 1 - \varepsilon\} = \{x_0\}.$$

From the assumption $\{(j,k) \in \mathbb{N}^2 : \vartheta(x_{jk}, x_0, u) \leq 1 - \varepsilon\} \in I_2$. Hence,

$$\{(j,k) \in \mathbb{N}^2 : \vartheta(x_{jk}, x_0, u) > 1 - \varepsilon\} = \{(j,k) \in \mathbb{N}^2 : x_{jk} = x_0\} \in \mathcal{F}(I_2)$$

and obviously $x_{jk} \xrightarrow{\vartheta(I_2^*)} x_0$. □

Theorem 3.14. Let I_2 satisfy the condition (AP2) and $x = (x_{jk})$ be a sequence in $\mathbb{X}$. Then, the assumptions below are equivalent:

(1) $\vartheta(I_2) - \lim\limits_{j,k\to\infty} x_{jk} = x_0$;

(2) There exist $y = (y_{jk})$, $z = (z_{jk})$ in $\mathbb{X}$ such that $x = y + z$, $\lim\limits_{j,k\to\infty} \vartheta(y_{jk}, x_0, u) = 1$ and $suppz \in I_2$, where $suppz = \{(j,k) \in \mathbb{N}^2 : z_{jk} \neq \theta_{\mathbb{X}}\}$.

Proof. Assume that $\vartheta(I_2) - \lim\limits_{j,k\to\infty} x_{jk} = x_0$. In that case, by Theorem 3.12, there exists a set $H \in \mathcal{F}(I_2)$, $H = \{(j_t, k_t) \in \mathbb{N}^2 : j_1 < j_2 < \ldots < j_t < \ldots ; k_1 < k_2 < \ldots < k_t < \ldots\}$ such that $\lim\limits_{j_t,k_t\to\infty} \vartheta(x_{j_t k_t}, x_0, u) = 1$. Let us define a sequence $y = (y_{jk})$ in $\mathbb{X}$ such that

$$y_{jk} := \begin{cases} x_{jk}, & n \in H, \\ x_0, & n \in \mathbb{N} \setminus H. \end{cases} \tag{3.4}$$

It is clear that $\lim\limits_{j,k\to\infty} \vartheta(y_{jk}, x_0, u) = 1$. Further, let $z_{jk} = x_{jk} - y_{jk}$, $(j,k) \in \mathbb{N}^2$. We have $\{(j_t, k_t) \in \mathbb{N}^2 : z_{j_t k_t} \neq 0\} \in I_2$, because we have

$$suppz = \{(j_t, k_t) \in \mathbb{N}^2 : x_{jk} \neq y_{jk}\} \subset \mathbb{N}^2 \setminus H \in I_2.$$

In addition, $suppz \in I_2$ and by (3.4), we write $x = y + z$.

Now, let $y = (y_{jk})$ and $z = (z_{jk})$ be two sequences in $\mathbb{X}$. This sequences satisfy $x = y + z$, $\lim\limits_{j,k\to\infty} \vartheta(y_{jk}, x_0, u) = 1$ and $suppz \in I_2$. We prove that

$$\vartheta(I_2) - \lim_{j,k\to\infty} x_{jk} = x_0. \tag{3.5}$$

Assume that $H = \{(j_t, k_t) \in \mathbb{N}^2 : z_{j_t k_t} = \theta_{\mathbb{X}}\} \subset \mathbb{N}^2$. We have $H \in \mathcal{F}(I_2)$, because

$$suppz = \{(j_t, k_t) \in \mathbb{N}^2 : z_{j_t k_t} \neq \theta_{\mathbb{X}}\} \in I_2.$$

Hence, $x_{jk} = y_{jk}$ if $(j,k) \in H$. Therefore, we achieve that there exists a set

$$H = \{(j_t, k_t) \in \mathbb{N}^2 : j_1 < j_2 < \ldots < j_t < \ldots ; k_1 < k_2 < \ldots < k_t < \ldots\} \in \mathcal{F}(I_2)$$

such that

$$\lim_{j_t,k_t\to\infty} \vartheta(x_{j_t k_t}, x_0, u) = 1.$$

By Theorem 3.12, (3.5) is hold. □

Definition 3.15. Let I_2 be a strongly admissible ideal on $\mathbb{N}^2$. If exists a set

$$H = \{(j_t, k_t) \in \mathbb{N}^2 : j_1 < j_2 < \ldots < j_t < \ldots ; k_1 < k_2 < \ldots < k_t < \ldots\} \in \mathcal{F}(I_2)$$

such that

$$\lim_{j_t, k_t, p_t, q_t \to \infty} \vartheta(x_{j_t k_t}, x_{p_t q_t}, u) = 1, \tag{3.6}$$

then a sequence (x_{jk}) in $\mathbb{X}$ is referred to as $\vartheta(I_2^*)$-Cauchy sequence.

Theorem 3.16. Let I_2 be a strongly admissible ideal on $\mathbb{N}^2$. If a sequence (x_{jk}) in $\mathbb{X}$ is a $\vartheta(I_2^*)$-Cauchy sequence, then it is a $\vartheta(I_2)$-Cauchy.

Proof. Assume that (x_{jk}) be an $\vartheta(I_2^*)$-Cauchy sequence. In that case, there exists a set

$$H = \{(j_t, k_t) \in \mathbb{N}^2 : j_1 < j_2 < \ldots < j_t < \ldots ; k_1 < k_2 < \ldots < k_t < \ldots\} \in \mathcal{F}(I_2)$$

such that $\lim_{j_t, k_t, p_t, q_t \to \infty} \vartheta(x_{j_t k_t}, x_{p_t q_t}, u) = 1$. Hence, there exists a positive integer n_0 such that for $j_t, k_t, p_t, q_t > n_0$ implies $\vartheta(x_{j_t k_t}, x_{p_t q_t}, u) > 1 - \varepsilon$, where $u > 0$ and $\varepsilon \in (0, 1)$. In other words,

$$A(u, \varepsilon) = \left\{(j_p, k_p) \in \mathbb{N}^2 : \vartheta\left(x_{j_p k_p}, x_{p_t q_t}, u\right) \leq 1 - \varepsilon\right\} \subset K \cup (H \cap ((B \times \mathbb{N}) \cup (\mathbb{N} \times B))),$$

where $B = \{1, 2, \ldots, (n_0 - 1)\}$. Since $K \cup (H \cap ((B \times \mathbb{N}) \cup (\mathbb{N} \times B))) \in I_2$, then $A(u, \varepsilon) \in I_2$. Consequently, the sequence (x_{jk}) is a $\vartheta(I_2)$-Cauchy. □

Theorem 3.17. Let I_2 be a strongly admissible ideal on $\mathbb{N}^2$. If the I_2 ideal has the condition (AP2), then $\vartheta(I_2)$-Cauchy sequence and $\vartheta(I_2^*)$-Cauchy sequence coincide.

Proof. In Theorem (3.16), it was shown that while the (x_{jk}) sequence is the $\vartheta(I_2^*)$-Cauchy sequence, it is the $\vartheta(I_2)$-Cauchy sequence without requiring the (AP2) condition. Therefore, if the I_2 ideal satisfies the (AP2) condition, showing that the $\vartheta(I_2)$-Cauchy sequence is the $\vartheta(I_2^*)$-Cauchy sequence will complete the proof. Now, assume that (x_{jk}) be a $\vartheta(I_2)$-Cauchy sequence in $\mathbb{X}$. Then, for all $\varepsilon \in (0, 1)$ and $u > 0$, there exists $(p(\varepsilon), q(\varepsilon)) \in \mathbb{N}^2$ such that

$$A(u, \varepsilon) = \left\{(j, k) \in \mathbb{N}^2 : \vartheta\left(x_{jk}, x_{pq}, u\right) \leq 1 - \varepsilon\right\} \in I_2.$$

Let

$$P_s = \left\{(j, k) \in \mathbb{N}^2 : \vartheta\left(x_{jk}, x_{p_s q_s}, u\right) > \frac{s - 1}{s}\right\}; \quad (s = 1, 2, \ldots),$$

where $p_s = p(\frac{s-1}{s}), q_s = q(\frac{s-1}{s})$. It is evident that $P_s \in \mathcal{F}(I_2)$ for $s = 1, 2, \ldots$. Since I_2 satisfy the property (AP2), according to Proposition (2.13), there exists a set P such that $P \in \mathcal{F}(I_2)$ and has the property that the set of elements in P that do not belong to P_s is a limited quantity for every index s.

Let $\varepsilon \in (0, 1)$, $u > 0$ and $m \in \mathbb{N}$ such that $m > \frac{1}{\varepsilon}$. If $(j, k), (p, q) \in P$, then $P \setminus P_s$ is a limited quantities set, implying that there exists a $n = n(m)$ such that, for all $j, k, p, q > n(m)$, $(j, k), (p, q) \in P_s$.

$$\vartheta\left(x_{jk}, x_{p_m q_m}, u\right) > \frac{m - 1}{m} \text{ and } \vartheta\left(x_{pq}, x_{p_m q_m}, u\right) > \frac{m - 1}{m}.$$

Hence, it follows that

$$\vartheta\left(x_{jk}, x_{pq}, u\right) \geq \vartheta\left(x_{jk}, x_{p_m q_m}, u\right) \circ \vartheta\left(x_{pq}, x_{p_m q_m}, u\right) > \left(\frac{m-1}{m}\right) \circ \left(\frac{m-1}{m}\right) = \delta(\varepsilon),$$

for all $j, k, p, q > n(m)$. Thus, for all $\varepsilon \in (0,1)$ and $u > 0$, there exists $n = n(\varepsilon)$ such that, for $j, k, p, q > n(\varepsilon)$ and $(j,k), (p,q) \in P$,

$$\vartheta\left(x_{jk}, x_{pq}, u\right) > 1 - \varepsilon,$$

i.e., the sequence (x_{jk}) is a $\vartheta(I_2^*)$-Cauchy sequence. □

4. $\vartheta(I_2)$-limit points and $\vartheta(I_2)$-cluster points

In the current part, we characterize $\vartheta(I_2)$-limit points and $\vartheta(I_2)$-cluster points of a double sequence in FMSs. Moreover, we analyze the connection between the concept mentioned earlier and study that the set of $\vartheta(I_2)$-cluster points are closed.

Definition 4.1. Let I_2 be a strongly admissible ideal on $\mathbb{N}^2$ and (x_{jk}) be a sequence in $\mathbb{X}$. An element $x_0 \in \mathbb{X}$ is referred to as an $\vartheta(I_2)$-limit point of sequence (x_{jk}), if there exists a set

$$H = \{(j_t, k_t) \in \mathbb{N}^2 : j_1 < j_2 < \ldots < j_t < \ldots; k_1 < k_2 < \ldots < k_t < \ldots\}$$

such that

$$H \notin I_2 \text{ and } \lim_{\substack{j_t, k_t \to \infty \\ (j_t, k_t) \in H}} \vartheta(x_{j_t k_t}, x_0, u) = 1.$$

Definition 4.2. Let I_2 be a strongly admissible ideal on $\mathbb{N}^2$ and (x_{jk}) be a sequence in $\mathbb{X}$. An element $x_0 \in \mathbb{X}$ is called an $\vartheta(I_2)$-cluster point of (x_{jk}) if, for all $u > 0$ and $\varepsilon \in (0,1)$,

$$\{(j,k) \in \mathbb{N}^2 : \vartheta(x_{jk}, x_0, u) \leq 1 - \varepsilon\} \notin I_2.$$

$\vartheta(I_2)(\Lambda_x)_2$ and $\vartheta(I_2)(\Gamma_x)_2$ denote the set of all $\vartheta(I_2)$-limit points and $\vartheta(I_2)$-cluster points of a sequence $x = (x_{jk})$, respectively.

Proposition 4.3. Let (x_{jk}) be a sequence in $\mathbb{X}$ and I_2 be a strongly admissible ideal on $\mathbb{N}^2$. Then,

$$\vartheta(I_2)(\Lambda_x)_2 \subseteq \vartheta(I_2)(\Gamma_x)_2.$$

Proof. Let $x_0 \in \vartheta(I_2)(\Lambda_x)_2$, then there exists a set

$$H = \{j_1 < j_2 < \ldots < j_t < \ldots; k_1 < k_2 < \ldots < k_t < \ldots\} \notin I_2$$

such that

$$\lim_{\substack{j_t, k_t \to \infty \\ (j_t, k_t) \in H}} \vartheta(x_{j_t k_t}, x_0, u) = 1. \tag{4.1}$$

Let $u > 0$ and $\varepsilon \in (0,1)$. According to (4.1), there exists a $N_\varepsilon \in \mathbb{N}$ such that for $j, k > N_\varepsilon$ implies $\vartheta(x_{jk}, x_0, u) > 1 - \varepsilon$. Hence,

$$H \setminus \{j_1 < j_2 < \ldots < j_{N_\varepsilon}; k_1 < k_2 < \ldots < k_{N_\varepsilon}\} \subset \{(j,k) \in \mathbb{N}^2 : \vartheta(x_{jk}, x_0, u) > 1 - \varepsilon\}$$

and thus $\{(j,k) \in \mathbb{N}^2 : \vartheta(x_{jk}, x_0, u) > 1 - \varepsilon\} \notin I_2$ which means that $x_0 \in \vartheta(I_2)(\Gamma_x)_2$. □

Theorem 4.4. Let (x_{jk}) be a sequence in $\mathbb{X}$. Then, the set $\vartheta(I_2)(\Gamma_x)_2$ is closed if I_2 is a strongly admissible ideal on $\mathbb{N}^2$.

Proof. Let $y \in \overline{\vartheta(I_2)(\Gamma_x)_2}$, $u > 0$ and $\varepsilon \in (0,1)$. Then, there exists an $x_0 \in B_y^\varepsilon(x) \cap \vartheta(I_2)(\Gamma_x)_2$, where $B_y^\varepsilon(x) = \{x \in \mathbb{X} : \vartheta(y, x, u) > 1 - \varepsilon\}$. Suppose that $\delta \in (0,1)$ such that

$$B_{x_0}^\delta(x) \subset B_y^\varepsilon(x).$$

Hence,

$$\{(j,k) \in \mathbb{N}^2 : \vartheta(x_0, x_{jk}, u) > 1 - \delta\} \subset \{(j,k) \in \mathbb{N}^2 : \vartheta(y, x_{jk}, u) > 1 - \varepsilon\}.$$

Consequently, $\{(j,k) \in \mathbb{N}^2 : \vartheta(y, x_{jk}, u) > 1 - \varepsilon\} \notin I_2$ and $y \in \vartheta(I_2)(\Gamma_x)_2$. □

5. Conclusions

We showed the ideal convergence of double sequences using the concept of fuzzy metric space in the sense of George and Veeramani [22]. Besides, we introduced the $\vartheta(I^*)$-convergent of double sequences and $\vartheta(I^*)$-Cauchy sequence with regards to fuzzy metric ϑ and discussed the relations between them. In addition, we proved that $\vartheta(I_2)$-convergence and $\vartheta(I_2^*)$-convergence are equivalent for an I_2 ideal with the condition (AP2). Lastly, we defined $\vartheta(I_2)$-limit and $\vartheta(I_2)$-cluster points of a double sequence and showed every $\vartheta(I_2)$-limit point to be a $\vartheta(I_2)$-cluster point.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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