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Research article

The forbidden set, solvability and stability of a circular system of complex Riccati type difference equations

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Abstract: In this paper, the circular system of Riccati type complex difference equations of the form

$$u_{n+1}^{(j)} = \frac{a_j u_n^{(j-1)} + b_j}{c_j u_n^{(j-1)} + d_j}, \quad n = 0, 1, 2, \cdots, \quad j = 1, 2, \cdots, k,$$

where $u_n^{(0)} := u_n^{(k)}$ for all n, is investigated. First, the forbidden set of the equation is given. Then the solvability of the system is examined and the expression of the solutions, given in terms of their initial values. Next, the asymptotic behaviour of the solutions is studied. Finally, in case of negative Riccati real numbers

$$R_j := \frac{a_j d_j - b_j c_j}{[a_j + d_j]^2}, \quad j \in \overline{1, k},$$

it is shown that there exists a unique positive fixed point which attracts all solutions starting from positive states.

Keywords: difference equations; solvability; asymptotic behaviour
Mathematics Subject Classification: 39A45, 39A05, 39A23, 39A30, 39A99

1. Introduction

In the Classroom Notes section of American Mathematical Monthly, Louis Brand [5] in 1955 presented an analysis of the so called Riccati real difference equation

$$x_{n+1} = \frac{a x_n + b}{c x_n + d}, \tag{1.1}$$

where $c \neq 0$ and $ad - bc \neq 0$. By setting

$$\alpha := \frac{a + d}{c}$$

and $\beta := \frac{d}{c^2}$, Brand transformed Eq (1.1) into

$$y_{n+1} = \alpha - \frac{\beta}{y_n}, \tag{1.2}$$

where

$$y_n := x_n + \frac{d}{c}.$$

Next, putting

$$y_n := \frac{z_{n+1}}{z_n}$$

from Eq (1.2) the linear difference equation

$$z_{n+2} - \alpha z_{n+1} + \beta z_n = 0$$

is obtained. If k_1, k_2 are the roots of its characteristic quadratic $k^2 - \alpha k + \beta = 0$, the general solution is determined and, finally, among others, it is shown the following results:

i) If $\alpha \neq 0$, $\alpha^2 > 4\beta$ and $y_1 \neq k_1$, then $y_{n+1} \to k_2$, when $|k_2| > |k_1|$.

ii) If $\alpha^2 = 4\beta$ and $k_1 = k_2 (= \frac{1}{2}\alpha)$, then $y_n \to k_1$.

iii) If $\alpha^2 < 4\beta$, then k_1, k_2 are complex conjugate and let $\theta := \cos^{-1}(\frac{\alpha}{2\sqrt{\beta}})$. If θ/π is rational, then the sequence (y_n) is periodic. If θ/π is irrational, then the set $\{y_n : \ n = 1, 2, \cdots\}$ is dense in the real line.

In the complex case, Eq (1.1) can be written in the form

$$x_{n+1} = f(x_n),$$

where $f(x) := \frac{ax+b}{cx+d}$. This is the Möbius transformation in the complex plane which include dilations, rotations, translations, and complex inversion.

We would, also, like to refer to the work done by Elaydi and Sacker [10], where for Eq (1.1), they give the same convergence results as by Brand. Moreover, discuss the equation where the coefficients are periodic having the same period. A special case which they discuss is the so called Beverton-Holt stock-recruitment equation

$$x_{n+1} = \frac{\mu K x_n}{K + (\mu - 1)x_n}, \quad x_0 > 0, \quad K > 0, \tag{1.3}$$

which has wide applications in population dynamics [4]. For $\mu \in (0, 1)$, the point 0 is globally asymptotically stable, whereas, for $\mu > 1$, the fixed point K is globally asymptotically stable. Furthermore, they investigated the existence of a periodic solution of (1.3), when the parameter K is replaced by the term K_n of a periodic sequence. Notice that this form of equation has been investigated by Cushing and Henson in [9]. See, also, [8, 25].

For $b = 0$ and $c = 1$, Eq (1.1) describes a population model and it is refereed in [12], p. 43.

In [1], the global asymptotic stability of the difference equation

$$x_{n+1} = \frac{A + Bx_{n-1}}{C + Dx_n^2}, \quad n = 0, 1, \cdots,$$

where A, B are nonnegative real numbers and $C, D > 0$. In [15] Kocić, Ladas and Rondrigues studied the $k+1$-order rational difference equation of the form

$$x_{n+1} = \frac{a + bx_n}{A + x_{n-k}}, \tag{1.4}$$

where a, A, b are nonnegative real numbers and k is a positive integer. They proved that the positive equilibrium point of Eq (1.4) is globally asymptotically stable. In addition, they showed that all positive solutions oscillate about the positive equilibrium point.

The first thing we shall discuss is the forbidden set $\mathbb{F}$. For Eq (1.1) in the real case, the forbidden set consists of all initial values x_0 which lead to a solution (x_n) such that, for some index n, the denominator $cx_n + d$ becomes zero. This means that the term x_{n+1} becomes infinity and so it is not a finite number. An interesting study of the forbidden set of Eq (1.1), in the real case, is given by Kulenović and Ladas in their informative book [16], pp. 17–24. Also, the authors describe in detail the long and short term behavior of the solutions when the initial value does not belong to the forbidden set. The book provides a very rich reference list. The same equation, or some of its variants, is studied by many authors, e.g. [2], p. 122, [20, 21, 22, 26]. See, also the references therein.

In this paper we are dealing with the circular system of Riccati difference equations of the form

$$u_{n+1}^{(j)} = \frac{a_j u_n^{(j-1)} + b_j}{c_j u_n^{(j-1)} + d_j}, \quad n = 0, 1, 2, \cdots, \quad j = 1, 2, \cdots, k, \tag{1.5}$$

with complex coefficients. Circular systems of rational real difference equations are investigated in a great number of papers, such as [3, 6, 14, 11, 17, 23, 24] and in the references therein. Notice that such systems were first presented by Laplace in 1773 ([18], p. 140), in his attempt to present the solution of a differential equation in a series form.

By the term solution of Eq (1.5) we mean a (finite, or infinite) sequence of complex numbers satisfying it by choosing suitable initial values. As we shall see, system (1.5) can be written in the form

$$y_{n+1}^{(j+1)} = 1 - \frac{R_j}{y_n^{(j)}}, \quad n = 0, 1, 2, \cdots, \quad j \in \overline{1, k}, \tag{1.6}$$

where the complex numbers R_j, $j = 1, 2, \cdots, k$ are the so called Riccati numbers for the system, defined by

$$R_j := \frac{a_j d_j - b_j c_j}{[a_j + d_j]^2}, \quad j \in \overline{1, k}.$$

In this new form the forbidden set consists of k-tuples, $\mathbb{F}_j$, $j = 1, 2, \cdots, k$ where $\mathbb{F}_j$ corresponds to the j-th coordinate. More facts about them we will see in the text. Next we show that for each index j and some values of R_j we get properties analogous to those described by Brand, as above. But notice that we shall work on the complex plane.

The paper is organized as follows: In Section 2, we describe the collection of the forbidden sets $\mathbb{F}_j$, $j = 1, 2, \cdots, k$ of system (1.5) and then we proceed to the solvability of the system. We do that by leaving all coefficients and the initial values to be complex numbers, thus the solutions are complex, too. We work further to express the solutions in terms of their initial values. Then, by using the expression of the solutions, we discuss the asymptotic behavior of them, and show that the solutions either converge to a fixed point, are periodic, or their range is dense in a line or in a circle of the complex plane. Periodicity of nonlinear difference equations is a very significant subject studied elsewhere, e.g. [13] and the references therein. In Section 3, we consider the case of negative Riccati real numbers and we derive the set where the unique positive fixed point belongs, which attracts all solutions starting from positive states.

2. The complex case

In this section, we shall study the system when all coefficients are complex numbers. Due to the fact that we have to deal with a circular system, in order to simplify our presentation, we start with the following convention: For any $m \in \mathbb{Z}$ and $l, k \in \mathbb{N}$, we shall use the symbol $\mathrm{mod}^*(k)$ as in the type

$$m = l(\mathrm{mod}^*(k)) := \begin{cases} l(\mathrm{mod}(k)), & l \in \{1, 2, \cdots, k-1\} \\ k, & if \;\; l = 0. \end{cases}$$

2.1. *The forbidden sets*

To better understand the forbidden set we can use the Riemann sphere S and the stereographic projection P of the extended complex plane onto S, when the primitive circle of the plane is the equator of the sphere. As it is known, this function is given by

$$P(z) = \Big(\frac{z+\bar{z}}{|z|^2+1}, \frac{z-\bar{z}}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\Big).$$

Obviously we have

$$P(0) = (0, 0, -1) \text{ the south pole of the sphere}$$

and

$$P(\infty) = (0, 0, 1) \text{ the north pole of the sphere.}$$

Now, consider Eq (2.2) and assume* that

$$c_j(a_j + b_j) \neq 0, \quad j \in \overline{1, k}. \tag{2.1}$$

By defining the sequence $y_n^{(j)} := \frac{1}{a_j+d_j}(c_j u_n^{(j)} + d_j), \;\; j \in \overline{1, k}$, system (1.5) is written in the form

$$y_{n+1}^{(j+1)} = 1 - \frac{R_j}{y_n^{(j)}}, \;\; n = 0, 1, 2, \cdots, \;\; j \in \overline{1, k}, \tag{2.2}$$

where $y_n^{(k+1)} := y_n^{(1)}$. The first thing we must do is to obtain the forbidden set $\mathbb{F}_j$ of each coordinate j of system (1.5), namely the set of all initial values $u_0^{(j)}$, which lead to the solution with a zero denominator $c_j u_n^{(j)} + d_j$. Actually, we are interested in the complement of the forbidden set, namely in the set of all values $u_0^{(j)}$ of the complex plane, which lead to solutions defined on the whole set of natural numbers. We call this set the active set of the system. Making the transformation from $u_n^{(j)}$ to $y_n^{(j)}$, it is enough to seek for the active sets of the new sequences, namely for the vectors $(y_0^{(1)}, y_0^{(2)}, \cdots, y_0^{(k)})$, which produce

*The special case $c_j(a_j + b_j) = 0$, for some or for all indices j, will be discussed in a forthcoming work.

solutions of the system (2.2) with no zero terms. Taking the stereographic projection P, we see that a term $y_n^{(j)}$ vanishes when $P(y_n^{(j)})$ is the south pole of the Riemann sphere. Then $P(y_{n+1}^{(j+1)})$ reaches the north pole, $P(y_{n+2}^{(j+2)})$ reaches the south pole, etc.

To continue, for any $j \in \overline{1,k}$, consider the sequence of complex numbers $q_n^{(j)},\ \ n = 1, 2, \cdots$ defined inductively as

$$q_0^{(j)} := 0,\ \ q_{n+1}^{(j)} := \frac{R_j}{1 - q_n^{(j+1)}},\ \ j \in \overline{1,k}$$

where $q_n^{(k+1)} := q_n^{(1)}$ and, for all $n \geq 0$ for which the terms of the sequence are defined. Our first result is the following.

Theorem 2.1. *Let $j \in \{1, 2, \cdots, k\}$. The forbidden set for the j_{th} coordinate of the system* (2.2) *is the set $\mathbb{F}_j = \{q_n^{(j)},\ \ n = 0, 1, 2, \cdots, N\}$, where $N \in \mathbb{N} \cup \{+\infty\}$.*

Proof. To prove the theorem, assume that $y_0^{(j)} = q_n^{(j)}$, for some $j \in \overline{1,k}$ and $n \geq 1$. Then, we have

$$y_0^{(j)} = \frac{R_j}{1 - q_{n-1}^{(j+1)}}.$$

Hence

$$y_1^{(j+1)} = 1 - \frac{R_j}{y_0^{(j)}} = 1 - \frac{R_j}{q_n^{(j)}} = q_{n-1}^{(j+1)}.$$

This relation gives

$$y_2^{(j+2)} = 1 - \frac{R_{j+1}}{y_1^{(j+1)}} = 1 - \frac{R_{j+1}}{q_{n-1}^{(j+1)}} = q_{n-2}^{(j+2)}.$$

Continue in this way, inductively, to obtain

$$y_{n-1}^{(j+n-1)} = q_1^{(j+n-1)} = R_{j+n-1}.$$

Therefore,

$$y_n^{(j+n)} = 1 - \frac{R_{j+n-1}}{y_{n-1}^{(j+n-1)}} = 0.$$

If $\hat{j}$ is the (unique) index in the interval of integers $\{1, 2, \cdots, k\}$ such that $j + n = \hat{j}(\text{mod}^*(k))$, then $y_n^{(\hat{j})} = 0$. Actually, this is the pre-image of $q_n^{(j)}$ through the Möbius transformation.

Conversely, assume that for some indices j, n it holds $y_n^{(j)} = 0$. Then, we have

$$1 - \frac{R_{j-1}}{y_{n-1}^{(j-1)}} = 0 \ \text{ and so } \ R_{j-1} = y_{n-1}^{(j-1)}.$$

This implies that

$$1 - \frac{R_{j-2}}{y_{n-2}^{(j-2)}} = y_{n-1}^{(j-1)} = R_{j-1} = q_1^{(j-1)} = 1 - \frac{R_{j-2}}{q_2^{(j-2)}},$$

which gives

$$1 - \frac{R_{j-3}}{y_{n-3}^{(j-3)}} = y_{n-2}^{(j-2)} = q_2^{(j-2)} = 1 - \frac{R_{j-3}}{q_3^{(j-3)}}.$$

The last step of this procedure is the relation $y_0^{(j-n)} = q_n^{(j-n)}$. If $\tilde{j}$ is the index in the set $\{1, 2, \cdots, k\}$ such that $j - n = \tilde{j}(\text{mod}^*(k))$, then we obtain $y_0^{(\tilde{j})} = q_n^{(\tilde{j})} \in \mathbb{F}_{\tilde{j}}$. The proof of the theorem is complete. □

Since the system is circular, if the initial value of the j_{th}- coordinate of the solution is taken from the forbidden set $\mathbb{F}_j$, then the evolution of the whole vector solution $(y_n^{(1)}, y_n^{(2)}, \cdots, y_n^{(k)})$ is completely determined.

Example 1. Consider the circular system

$$y_{n+1}^{(1)} = 1 - \frac{i}{y_n^{(2)}}, \ \ y_{n+1}^{(2)} = 1 - \frac{1+i}{y_n^{(1)}}, \ \ n = 0, 1, \cdots .$$

The first five elements of the forbidden sets for this system are the following:

$$\mathbb{F}_1 = \{0, i, -1, \frac{-1+i}{2}, -1+i, \cdots\}, \ \ \mathbb{F}_2 = \{0, 1+i, i, \frac{1+i}{2}, \frac{2(1+2i)}{5}, \cdots\}.$$

If we consider the point $\frac{1+i}{2}$ as the initial value $y_0^{(2)}$ for the 2-coordinate, we obtain $y_1^{(1)} = -1, \ y_2^{(2)} = 1+i, \ y_3^{(1)} = 0$.

Now the question which arises is what happens in case $1 - q_{n_0}^{(j_0)} = 0$, for some indices j_0, n_0. Our answer is that the result of the theorem remains in force. Indeed, all steps in the proof of the theorem are made between finite numbers among those quantities $q_n^{(j)}$ which exist. Moreover, it is easy to see that if for some indices j_0, n_0 it holds $q_{n_0}^{(j_0)} = 1$, then the quantity $q_{n_0+1}^{(j_0-1)}$ does not exist, and therefore the forbidden sets are finite.

Example 2. Consider the system

$$y_{n+1}^{(1)} = 1 - \frac{2}{y_n^{(2)}}, \ \ y_{n+1}^{(2)} = 1 - \frac{1}{y_n^{(1)}}.$$

Here, we observe that $q_1^{(2)} = 1$. Therefore the forbidden sets are finite. Indeed, they are defined by $\mathbb{F}_1 = \{0, 2\}$ and $\mathbb{F}_2 = \{0, 1, -1\}$.

2.2. Solvability

Solvability of a difference equation, in general, is a not easy problem, and it has attracted the interest of many authors. Actually, Brand [5] was the first to show how to solve a scalar difference equation, where the map is a Möbius transformation. See, also, [3, 14, 11, 19, 21, 23, 26] and the references therein. In this subsection, we shall present a way of expressing, in a closed form, the solution of the circular system of difference equations (1.5), when the initial values do no belong to the forbidden sets. An advantage of the expression of the solutions is the fact that it helps us to describe its asymptotic behaviour.

In the sequel, we assume that assumption (2.1) holds which guarantees that circular system (1.5) is transformed into (2.2).

First, our plan is to give the general expression of each coordinate of the solution, in terms of general parameters. This is the easy part of the problem. The second step is to express the general parameters in terms of the initial values $(y_0^{(j)})$, $j = 1, 2, \cdots, k$, so that the expression of the solutions are in terms of their initial values.

The central role in all the sequel will be played by the k- triples of parameters $(a_m^{(j)}, b_m^{(j)}, c_m^{(j)})$ defined inductively as follows:

$$a_0^{(j)} = 1, \ \ b_0^{(j)} = R_{j-1}, \ \ c_0^{(j)} = 0,$$

and, for any index $m = 1, 2, \cdots, k-1$ as

$$a_m^{(j)} = a_{m-1}^{(j)} - \frac{b_{m-1}^{(j)}}{1 - c_{m-1}^{(j)}}, \quad b_m^{(j)} = \frac{b_{m-1}^{(j)} R_{j-m-1}}{(1 - c_{m-1}^{(j)})^2}, \quad c_m^{(j)} = \frac{R_{j-m-1}}{1 - c_{m-1}^{(j)}},$$

where $R_0 := R_k$.

It is clear that in order, these items to be well defined, the condition

$$c_m^{(j)} \neq 1, \quad j \in \overline{1,k}, m = 1, 2, \cdots, k-2 \tag{2.3}$$

must be satisfied[†].

To continue, we are firstly going to express the term $y_{n+1}^{(j)}$ of the j_{th}- coordinate as a function of the term $y_{n+1-k}^{(j)}$ of the same coordinate. To do that, we observe that the transformed system (2.2), for all $n = 0, 1, 2, \cdots$ and $j \in \overline{1,k}$, becomes

$$y_{n+1}^{(j)} = 1 - \frac{R_{j-1}}{y_n^{(j-1)}} = a_0^{(j)} - \frac{b_0^{(j)}}{y_n^{(j-1)} - c_0^{(j)}}. \tag{2.4}$$

Then,

$$y_{n+1}^{(j)} = 1 - \frac{R_{j-1}}{1 - \frac{R_{j-2}}{y_{n-1}^{(j-2)}}} = 1 - R_{j-1} - \frac{R_{j-1}R_{j-2}}{y_{n-1}^{(j-2)} - R_{j-2}} = a_1^{(j)} - \frac{b_1^{(j)}}{y_{n-1}^{(j-2)} - c_1^{(j)}}.$$

To predict the rule, we do one more substitution and obtain that, for $n = 2, 3, \cdots$,

$$y_{n+1}^{(j)} = a_2^{(j)} - \frac{b_2^{(j)}}{y_{n-2}^{(j-3)} - c_2^{(j)}}.$$

Inductively, for $n = k-1, k, \cdots$, we obtain

$$y_{n+1}^{(j)} = a_m^{(j)} - \frac{b_m^{(j)}}{y_{n-m}^{(j-m-1)} - c_m^{(j)}}, \quad m = 0, 1, 2, \cdots, k-1. \tag{2.5}$$

The final step we arrive at is

$$y_{n+1}^{(j)} = a_{k-1}^{(j)} - \frac{b_{k-1}^{(j)}}{y_{n-k+1}^{(j-k)} - c_{k-1}^{(j)}} = a_{k-1}^{(j)} - \frac{b_{k-1}^{(j)}}{y_{n-k+1}^{(j)} - c_{k-1}^{(j)}}, \tag{2.6}$$

because $j-k = j(\text{mod}^*(k))$. Before we continue to our discussion, we must notice that the relation (2.6) gives a well defined value whenever $y_{n-k+1}^{(j)} \neq c_{k-1}^{(j)}$, for all $j \in \overline{1,k}$ and $n = k-2, k-1, \cdots$. Indeed, if this statement is not true, then due to (2.4) the value $y_0^{(j-1)}$ is equal to $c_0^{(j)} = 0$, which contradicts the fact that $y_0^{(j-1)}$ belongs to the active set.

The advantage of relation (2.6) is that it is a relation involving the j_{th}-coordinate only.

Next, we set

$$z_n^{(j)} = y_n^{(j)} - c_{k-1}^{(j)}.$$

[†]The case $c_m^{(j)} = 1$, for some indices j, m, will be discussed later at the end of this work.

Since the system has k coordinates, it is convenient to define the two-parameter sequence

$$r_n^{(l,j)} := z_{kn+l}^{(j)}, \quad l = 0, 1, 2, \cdots, k-1,$$

which, obviously, satisfies the relation

$$r_{n+1}^{(l,j)} = -c_{k-1}^{(j)} + a_{k-1}^{(j)} - \frac{b_{k-1}^{(j)}}{r_n^{(l,j)}}, \quad l = 0, 1, \cdots, k-1, \quad j \in \overline{1,k}.$$

Finally, we make one more substitution by setting $s_n^{(l,j)} := \prod_{j=0}^{n} r_j^{(l,j)}$. Then, we have

$$r_0^{(l,j)} = s_0^{(l,j)}, \quad r_n^{(l,j)} = \frac{s_n^{(l,j)}}{s_{n-1}^{(l,j)}}, \quad n \geq 1.$$

The new sequence satisfies the linear difference equation

$$s_{n+1}^{(l,j)} + (c_{k-1}^{(j)} - a_{k-1}^{(j)}) s_n^{(l,j)} + b_{k-1}^{(j)} s_{n-1}^{(l,j)} = 0, \quad n = 1, 2, \cdots, \tag{2.7}$$

for all $l = 0, 1, 2, \cdots, k-1$ and $j = 1, 2, \cdots, k$. To solve this equation, we consider its characteristic equation

$$\lambda^2 + (c_{k-1}^{(j)} - a_{k-1}^{(j)})\lambda + b_{k-1}^{(j)} = 0 \tag{2.8}$$

and obtain its (complex) characteristic roots $\lambda_1^{(j)}$, $\lambda_2^{(j)}$. We distinguish the following cases.

Case 1: Assume that the characteristic values are not equal. Then, the general solution of Eq (2.7) can be written in the form

$$s_n^{(l,j)} = \sigma_1^{(l,j)} (\lambda_1^{(j)})^n + \sigma_2^{(l,j)} (\lambda_2^{(j)})^n, \quad n = 0, 1, 2, \cdots, \quad l = 0, 1, 2, \cdots, k-1,$$

where the coefficients $\sigma_1^{(l,j)}, \sigma_2^{(l,j)}$ are complex numbers being not both equal to zero, and they depend on the initial values of the solution. So, in order to express them in terms of the initial value $Y_0 := (y_0^{(1)}, y_0^{(2)}, \cdots, y_0^{(k)})^T$ we observe that

$$r_0^{(l,j)} = s_0^{(l,j)} = \sigma_1^{(l,j)} + \sigma_2^{(l,j)}$$

and

$$r_n^{(l,j)} = \frac{s_n^{(l,j)}}{s_{n-1}^{(l,j)}} = \frac{\sigma_1^{(l,j)} (\lambda_1^{(j)})^n + \sigma_2^{(l,j)} (\lambda_2^{(j)})^n}{\sigma_1^{(l,j)} (\lambda_1^{(j)})^{n-1} + \sigma_2^{(l,j)} (\lambda_2^{(j)})^{n-1}}. \tag{2.9}$$

Since $\sigma_1^{(l,j)}, \sigma_2^{(l,j)}$ are not both equal to zero, it follows that one of the two fractions $\sigma_1^{(l,j)}/\sigma_2^{(l,j)}$, and $\sigma_2^{(l,j)}/\sigma_2^{(l,j)}$ is a finite number. If we assume that the quantity $\xi^{(l,j)} = \sigma_1^{(l,j)}/\sigma_2^{(l,j)}$ is a finite number, we can write the previous relation as

$$r_n^{(l,j)} = \frac{\xi^{(l,j)} (\lambda_1^{(j)})^n + (\lambda_2^{(j)})^n}{\xi^{(l,j)} (\lambda_1^{(j)})^{n-1} + (\lambda_2^{(j)})^{n-1}}, \quad n = 1, 2, \cdots. \tag{2.10}$$

If the fraction $\zeta^{(l,j)} := \sigma_2^{(l,j)}/\sigma_1^{(l,j}$ is a finite number, we can write it as

$$r_n^{(l,j)} = \frac{(\lambda_1^{(j)})^n + \zeta^{(l,j)} (\lambda_2^{(j)})^n}{(\lambda_1^{(j)})^{n-1} + \zeta^{(l,j)} (\lambda_2^{(j)})^{n-1}}, \quad n = 1, 2, \cdots$$

and we can work analogously.

To proceed, we keep in mind (2.10) and discuss the two cases $n = 0$ and $n \geq 1$, separately. First, we observe that

$$y_l^{(j)} = z_l^{(j)} + c_{k-1}^{(j)} = r_0^{(l,j)} + c_{k-1}^{(j)} = s_0^{(l,j)} + c_{k-1}^{(j)} = \sigma_1^{(l,j)} + \sigma_2^{(l,j)} + c_{k-1}^{(j)}, \quad l = 0, 1, \cdots, k-1 \tag{2.11}$$

and

$$y_{kn+l}^{(j)} = z_{kn+l}^{(j)} + c_{k-1}^{(j)} = r_n^{(l,j)} + c_{k-1}^{(j)} = \frac{\xi^{(l,j)}(\lambda_1^{(j)})^n + (\lambda_2^{(j)})^n}{\xi^{(l,j)}(\lambda_1^{(j)})^{n-1} + (\lambda_2^{(j)})^{n-1}} + c_{k-1}^{(j)}, \quad l = 0, 1, \cdots, k-1, \quad n \geq 1. \tag{2.12}$$

Setting $n = 1$ and $l = 0$ in relation (2.12) and taking into account (2.6), we obtain

$$\frac{\xi^{(0,j)}\lambda_1^{(j)} + \lambda_2^{(j)}}{\xi^{(0,j)} + 1} + c_{k-1}^{(j)} = y_k^{(j)} = a_{k-1}^{(j)} + \frac{b_{k-1}^{(j)}}{y_0^{(j)} - c_{k-1}^{(j)}}.$$

From this equation, we obtain

$$\xi^{(0,j)} = \frac{(y_0^{(j)} - c_{k-1}^{(j)})(a_{k-1}^{(j)} - c_{k-1}^{(j)} - \lambda_2^{(j)}) + b_{k-1}^{(j)}}{(y_0^{(j)} - c_{k-1}^{(j)})(\lambda_1^{(j)} - a_{k-1}^{(j)} + c_{k-1}^{(j)}) - b_{k-1}^{(j)}}. \tag{2.13}$$

Substitute this value in (2.12) and get the expression of the values of the sequence $(y_{kn}^{(j)})$.

Next, fix any $l \in \{1, 2, \ldots, k-1\}$. Setting $l = m$ in the place of $n + 1$ in (2.5), we get

$$a_{l-1}^{(j)} - \frac{b_{l-1}^{(j)}}{y_0^{(j-l)} - c_{l-1}^{(j)}} = y_l^{(j)} = \sigma_1^{(l,j)} + \sigma_2^{(l,j)} + c_{k-1}^{(j)}, \tag{2.14}$$

due to (2.11). Therefore,

$$\sigma_1^{(l,j)} + \sigma_2^{(l,j)} = a_{l-1}^{(j)} - c_{k-1}^{(j)} - \frac{b_{l-1}^{(j)}}{y_0^{(j-l)} - c_{l-1}^{(j)}} =: d_l^{(j)}. \tag{2.15}$$

On the other hand, from (2.12), for $n = 1$, (2.6) and (2.11) we have

$$\frac{\sigma_1^{(l,j)}\lambda_1^{(j)} + \sigma_2^{(l,j)}\lambda_2^{(j)}}{\sigma_1^{(l,j)} + \sigma_2^{(l,j)}} + c_{k-1}^{(j)} = y_{k+i}^{(j)} = y_{(k+l-1)+1}^{(j)} = a_{k-1}^{(j)} - \frac{b_{k-1}^{(j)}}{y_i^{(j)} - c_{k-1}^{(j)}} = a_{k-1}^{(j)} - \frac{b_{k-1}^{(j)}}{\sigma_1^{(l,j)} + \sigma_2^{(l,j)}},$$

namely

$$\frac{\sigma_1^{(l,j)}\lambda_1^{(j)} + \sigma_2^{(l,j)}\lambda_2^{(j)}}{d_l^{(j)}} + c_{k-1}^{(j)} = a_{k-1}^{(j)} - \frac{b_{k-1}^{(j)}}{d_l^{(j)}}.$$

Solving the system of these last two equations in terms of the initial value $y_0^{(j-i)}$, we obtain

$$\sigma_1^{(l,j)} = \frac{1}{\lambda_1^{(j)} - \lambda_2^{(j)}}\Big(- b_{k-1}^{(j)} + (a_{k-1}^{(j)} - c_{k-1}^{(j)} - \lambda_2^{(j)})d_l^{(j)}\Big) \tag{2.16}$$

and

$$\sigma_2^{(l,j)} = \frac{1}{\lambda_1^{(j)} - \lambda_2^{(j)}} \Big(b_{k-1}^{(j)} + (\lambda_1^{(j)} - a_{k-1}^{(j)} + c_{k-1}^{(j)}) d_l^{(j)} \Big) \tag{2.17}$$

from which the value of the item $\xi^{(i,j)} = \sigma_i^{(1,j)}/\sigma_2^{(i,j)}$ is obtained. Taking into account the values $\xi^{(0,j)}$ from (2.13) and relations (2.16) and (2.17), from (2.12), the expression of the solution $y_n^{(j)}$ follows.

Case 2: Next, assume that $\lambda_1^{(j)} = \lambda_2^{(j)} =: \lambda^{(j)}$. Then the solution can be written in the form

$$s_n^{(l,j)} = (\sigma_1^{(l,j)} + n\sigma_2^{(l,j)})(\lambda^{(j)})^n, \quad n = 0, 1, 2, \cdots, \quad l = 0, 1, 2, \cdots, k-1,$$

where, again, the coefficients $\sigma_1^{(l,j)}, \sigma_2^{(l,j)}$ are complex numbers and not both equal to zero.

In this case we have $r_0^{(l,j)} = s_0^{(l,j)} = \sigma_1^{(l,j)}$ and

$$r_n^{(l,j)} = \frac{s_n^{(l,j)}}{s_{n-1}^{(l,j)}} = \frac{(\sigma_1^{(l,j)} + n\sigma_2^{(l,j)})(\lambda^{(j)})^n}{(\sigma_1^{(l,j)} + (n-1)\sigma_2^{(l,j)})(\lambda^{(j)})^{n-1}} = \frac{s_n^{(l,j)}}{s_{n-1}^{(l,j)}} = \frac{(\sigma_1^{(l,j)} + n\sigma_2^{(l,j)})\lambda^{(j)}}{\sigma_1^{(l,j)} + (n-1)\sigma_2^{(l,j)}}.$$

If we assume, as previously, that the quantity

$$\xi^{(l,j)} = \frac{\sigma_1^{(l,j)}}{\sigma_2^{(l,j)}}$$

is a finite complex number, we can write the previous relation as

$$r_n^{(l,j)} = \frac{(\xi^{(l,j)} + n)\lambda^{(j)}}{\xi^{(l,j)} + (n-1)}.$$

The case where $\sigma_2^{(l,j)}/\sigma_1^{(l,j)}$ is finite, is analogous.

Now, we observe that

$$y_l^{(j)} = z_l^{(j)} + c_{k-1}^{(j)} = r_0^{(l,j)} + c_{k-1}^{(j)} = s_0^{(l,j)} + c_{k-1}^{(j)} = \sigma_1^{(l,j)} + c_{k-1}^{(j)}, \quad l = 0, 1, \cdots, k-1 \tag{2.18}$$

and

$$y_{kn+l}^{(j)} = z_{kn+l}^{(j)} + c_{k-1}^{(j)} = r_n^{(l,j)} + c_{k-1}^{(j)} = \frac{(\xi^{(l,j)} + n)\lambda^{(j)}}{\xi^{(l,j)} + (n-1)} + c_{k-1}^{(j)}, \quad l = 0, 1, 2, \cdots, k-1, \quad n \geq 1. \tag{2.19}$$

Setting $n = 1$ and $l = 0$ in relations (2.6) and (2.19) we obtain

$$\frac{(\xi^{(0,j)} + 1)\lambda^{(j)}}{\xi^{(0,j)}} + c_{k-1}^{(j)} = y_k^{(j)} = a_{k-1}^{(j)} - \frac{b_{k-1}^{(j)}}{y_0^{(j)} - c_{k-1}^{(j)}}.$$

Therefore

$$\xi^{(0,j)} = \frac{(y_0^{(j)} - c_{k-1}^{(j)})\lambda^{(j)}}{(y_0^{(j)} - c_{k-1}^{(j)})(a_{k-1}^{(j)} - c_{k-1}^{(j)} - \lambda^{(j)}) - b_{k-1}^{(j)}}. \tag{2.20}$$

Next, fix any $l \in \{1, 2, \ldots, k-1\}$. Setting in (2.5), $l = m$ in the place of $n+1$ we get

$$a_{l-1}^{(j)} - \frac{b_{l-1}^{(j)}}{y_0^{(j-l)} - c_{l-1}^{(j)}} = y_l^{(j)} = \sigma_1^{(l,j)} + c_{k-1}^{(j)},$$

due to (2.18). Therefore,

$$\sigma_1^{(l,j)} = a_{l-1}^{(j)} - c_{k-1}^{(j)} - \frac{b_{l-1}^{(j)}}{y_0^{(j-l)} - c_{l-1}^{(j)}} = d_l^{(j)}. \tag{2.21}$$

On the other hand, from (2.19), for $n = 1$, and from (2.6) and (2.18) we have

$$\lambda^{(j)} + \frac{\sigma_2^{(l,j)}}{\sigma_1^{(l,j)}}\lambda^{(j)} + c_{k-1}^{(j)} = y_{k+l}^{(j)} = y_{(k+l-1)+1}^{(j)} = a_{k-1}^{(j)} - \frac{b_{k-1}^{(j)}}{y_l^{(j)} - c_{k-1}^{(j)}} = a_{k-1}^{(j)} - \frac{b_{k-1}^{(j)}}{\sigma_1^{(l,j)}}.$$

From here, due to (2.21), we obtain

$$\sigma_2^{(l,j)} = \frac{1}{\lambda^{(j)}}\Big(- b_{k-1}^{(j)} + (a_{k-1}^{(j)} - c_{k-1}^{(j)} - \lambda^{(j)})d_l^{(j)}\Big) \tag{2.22}$$

Taking into account the values $\xi^{(0,j)}$ from (2.20) and $\xi^{(l,j)}$ from relations (2.21), (2.22), we can go to (2.19) and obtain the expression of the j_{th}-coordinate of the solution.

The results we derived so far are summarised in the following theorem:

Theorem 2.2. *Assume that* (2.3) *is satisfied and consider the algebraic equation* (2.8).

1) If Eq (2.8) *has unequal roots, the solution of the system is given by* (2.11) *and* (2.12) *where* $\xi^{(l,j)} = \sigma_1^{(l,j)}/\sigma_2^{(l,j)}$ *and* $\sigma_1^{(l,j)}, \sigma_2^{(l,j)}$ *are given in* (2.16), (2.17) *and* $d_l^{(j)}$ *is defined in* (2.15).

2) If Eq (2.8) *has equal roots, then the solution of system* (2.2) *is given by* (2.18) *and* (2.19), *where* $\sigma_1^{(l,j)}, \sigma_2^{(l,j)}$ *are given by* (2.21) *and* (2.22).

Remark. *Here and in all the sequel the reader can see that analogous results hold if we use the item* $\zeta^{(l,j)} := \sigma_2^{(l,j)}/\sigma_1^{(l,j)}$.

Before we close the subject of the solvability of the solutions, we must notice that one could ask how many initial values we need to express the solutions. Indeed, as we have seen the number of the quantities $d_l^{(j)}$ is equal to k^2, though we need only k of them. But we can observe that each $d_l^{(j)}$ depends on the initial value $y_0^{(j-l)}$. Therefore this initial value determines all parameters of the form $d_{l+m}^{(j+m)}$, for any $m = 0, 1, 2, \cdots, k-1$.

2.3. Asymptotic behaviour of the solutions

In this subsection we assume, again, that the initial values of the solutions do not belong to the forbidden sets. Then, we can use the expression of the solutions as it is obtained above. We have the following result.

Theorem 2.3. *Let* $\lambda_-^{(j)}$, $\lambda_+^{(j)}$ *be the roots of the algebraic equation* (2.8) *and let* $(y_n^{(j)})$ *be the j-th coordinate of the solution of system* (2.2) *with initial value in the active set.*

1) a) If $|\lambda_-^{(j)}| < |\lambda_+^{(j)}|$ *and* $\xi^{(l,j)} \neq 0$, *then it holds*

$$\lim_{n\to+\infty} y_{kn+l}^{(j)} = \lambda_+^{(j)} + c_{k-1}^{(j)},$$

while if $\xi^{(l,j)} = 0$, *then*

$$\lim_{n\to+\infty} y_{kn+l}^{(j)} = \lambda_-^{(j)} + c_{k-1}^{(j)}.$$

b) If $|\lambda_+^{(j)}| < |\lambda_-^{(j)}|$, *and* $\xi^{(l,j)} \neq 0$, *then*

$$\lim_{n\to+\infty} y_{kn+l}^{(j)} = \lambda_-^{(j)} + c_{k-1}^{(j)},$$

while if $\xi^{(l,j)} = 0$, *then*

$$\lim_{n\to+\infty} y_{kn+l}^{(j)} = \lambda_+^{(j)} + c_{k-1}^{(j)}.$$

2) If the roots are equal, say, to $\lambda^{(j)}$, *then* $\lim_{n\to+\infty} y_{kn+l}^{(j)} = \lambda^{(j)} + c_{k-1}^{(j)}$.

3) If the roots are of the form $\lambda_1^{(j)} = q_j e^{i\omega_j}$ *and* $\lambda_2^{(j)} = q_j e^{i\phi_j}$, *then, the* j_{th}*- coordinate of the solution has the form*

$$y_{kn+l}^{(j)} = \begin{cases} \sigma_1^{(l,j)} + \sigma_2^{(l,j)} + c_{k-1}^{(j)}, & n = 0, \\ q_j e^{i\omega_j} \frac{\xi^{(l,j)} + e^{in(\phi_j - \omega_j)}}{\xi^{(l,j)} + e^{i(n-1)(\phi_j - \omega_j)}} + c_{k-1}^{(j)}, & n \geq 1. \end{cases}$$

Hence, if $\xi^{(l,j)} = 0$, *then*

$$y_{kn+l}^{(j)} = q_j e^{i\phi_j},$$

for all l, n, *namely, it is constant.*

Assume that $\xi^{(l,j)} \neq 0$.

i) If $(\phi_j - \omega_j)/\pi$ *is a rational number, the sequence* $(y_n^{(j)})|_{n\geq k}$ *is periodic.*

ii) Let $\theta_j/\pi := (\phi_j - \omega_j)/\pi$ *be irrational. Then the terms of the sequence form a subset of the complex plane, which is dense in the straight line*

$$\zeta(t) = q_j e^{i\omega_j}\Big(t + i\frac{1-\cos(\theta_j)}{\sin(\theta_j)}t\Big) + c_{k-1}^{(j)}, \quad t \in \mathbb{R}, \tag{2.23}$$

in case $|\xi^{(l,j)}| = 1$, *dense in the circle* $q_j e^{i\omega_j} C + c_{k-1}^{(j)}$, *where* C *is the circle*

$$|z|^2 + 2\frac{\cos(\theta_j) + \sin(\theta_j) - |\xi^{(l,j)}|^2}{|\xi^{(l,j)}|^2 - 1} z + \frac{\cos(\theta_j) - \sin(\theta_j) - |\xi^{(l,j)}|^2}{|\xi^{(l,j)}|^2 - 1}\bar{z} + 1 = 0, \tag{2.24}$$

in case $|\xi^{(l,j)}| \neq 1$.

Proof. 1) a) Assume that $|\lambda_-^{(j)}| < |\lambda_+^{(j)}|$. From (2.12), we observe that for each n and l, j, it holds

$$\lim_{n\to+\infty} y_{kn+l}^{(j)} = \lim_{n\to+\infty} \frac{\xi^{(l,j)}(\lambda_+^{(j)})^n + (\lambda_-^{(j)})^n}{\xi^{(l,j)}(\lambda_+^{(j)})^{n-1} + (\lambda_-^{(j)})^{n-1}} + c_{k-1}^{(j)} = \lim_{n\to+\infty} \frac{\xi^{(l,j)}\lambda_+^{(j)} + \lambda_-^{(j)}(\frac{\lambda_-^{(j)}}{\lambda_+^{(j)}})^{n-1}}{\xi^{(l,j)} + (\frac{\lambda_-^{(j)}}{\lambda_+^{(j)}})^{n-1}} + c_{k-1}^{(j)} = \lambda_+^{(j)} + c_{k-1}^{(j)}.$$

b) If it holds $|\lambda_+^{(j)}| < |\lambda_-^{(j)}|$, then, from the previous relation, we obtain

$$\lim_{n\to+\infty} y_{kn+l}^{(j)} = \lambda_-^{(j)} + c_{k-1}^{(j)}.$$

2) If the roots are equal to $\lambda^{(j)}$, then, from (2.19), we obtain

$$\lim_{n\to+\infty} y_{kn+l}^{(j)} = \lim_{n\to+\infty} \frac{(\xi^{(l,j)} + n)\lambda^{(j)}}{\xi^{(l,j)} + (n-1)} + c_{k-1}^{(j)} = \lim_{n\to+\infty} \frac{\frac{\xi^{(l,j)}}{n} + 1}{\frac{\xi^{(l,j)}}{n} + \frac{n-1}{n}}\lambda^{(j)} + c_{k-1}^{(j)} = \lambda^{(j)} + c_{k-1}^{(j)}.$$

3) In the third case, the expression of the solutions follows from (2.11) and (2.12).

Let $\xi^{(l,j)} \neq 0$.

i) Assume that θ_j/π is a rational real number p_j/r_j, say. Define the positive integer $N_j := 2kr_j$ and consider an index $m \geq k$. Then, we can write it in the form $m = kn + l$, for some indices n, l. From the expression

$$y^{(j)}_{kn+l} = q_j e^{i\omega_j} \frac{\xi^{(l,j)} + e^{in\theta_j}}{\xi^{(l,j)} + e^{i(n-1)\theta_j}} + c^{(j)}_{k-1},$$

we obtain

$$y^{(j)}_{m+N_j} = y^{(j)}_{k(n+2r_j)+l} = y^{(j)}_{kn+l} = y^{(j)}_m,$$

due to the fact that $e^{i(n+2r_j)\theta_j} = e^{in\theta_j} e^{i2\pi p_j} = e^{in\theta_j}$. This proves that the sequence $y^{(j)}$ is periodic.

ii) Next, assume that θ_j/π is irrational. Then, it is well known (see, for instance, Prop. 4.1, p. 10 in [7]) that the set $\{e^{in\theta_j} : \ n = 1, 2, \cdots\}$ is dense in the unit circle C. Let $z \in C$. Then z is the limit of a sequence of the form $e^{in_m\theta_j}, \ \ m = 1, 2, \cdots$. First we are going to find the set of all $\zeta_j \in \mathbb{C}$ such that

$$\frac{\xi^{(l,j)} + z}{\xi^{(l,j)} + z e^{-i\theta_j}} = \zeta_j.$$

From here we obtain that

$$z = \frac{(\zeta_j - 1)\xi^{(l,j)}}{1 - \zeta_j e^{-i\theta_j}}$$

and so

$$|\zeta_j - 1||\xi^{(l,j)}| = |1 - \zeta_j e^{-i\theta_j}|. \tag{2.25}$$

This is equivalent to the relation

$$(|\xi^{(l,j)}|^2 - 1)|\zeta_j|^2 + 2\Re(\zeta_j)(\cos(\theta_j) - |\xi^{(l,j)}|^2) + 2\Im(\zeta_j)\sin(\theta_j) - 1 = 0. \tag{2.26}$$

Inversely, if ζ is a complex number satisfying Eq (2.25) then, the number $\frac{(\zeta_j-1)\xi^{(l,j)}}{1-\zeta_j e^{-i\theta_j}}$ is a point in C. Now, if $|\xi^{(l,j)}| = 1$, the first factor in (2.26) vanishes, and we get a straight line given by (2.23). Otherwise, we have the circle given by (2.24). Now, the proof follows by observing that dilations of straight lines and circles produce straight lines and circles, respectively. □

3. The real case

As we proved in the previous section, in case condition (2.3) holds, there exists a limit of all solutions of system (1.5). This limit turns out to be a fixed point of the system. In this section, we restrict ourselves to the case where all Ricatti numbers are negative real numbers and we shall try to locate the existence of a fixed point of system (1.5) in the positive orthant of the space $(\mathbb{R}^+)^k$. Moreover, this fixed point is unique. The method we use relies on the classical fixed point of Brouwer.

By setting $p_j := -R_j$, $j = 1, 2, \cdots, k$, we have $p_j > 0$ and system (2.2) is written as

$$y^{(j+1)}_{n+1} = 1 + \frac{p_j}{y^{(j)}_n}, \ \ n = 0, 1, 2, \cdots, \ \ j = 1, 2, \cdots, k. \tag{3.1}$$

We shall work on the set

$$D := [1, 1 + p_k] \times [1, 1 + p_1] \times \cdots \times [1, 1 + p_{k-1}].$$

The main result of this section refers to the existence and uniqueness of a fixed point of the system lying in D and it is the following.

Theorem 3.1. *Assume that* (2.1) *holds and the Riccati numbers* $R_j, \quad j = 1, 2, \cdots, k$ *are real negative. Then system* (2.2) *admits a unique fixed point in the set* D*, which attracts all solutions starting from positive states. It turns out that system* (1.5) *admits a fixed point in the set*

$$\times_{j=0,1,2,\cdots,k-1} \begin{cases} \left[\frac{a_j}{c_j}, \quad \frac{a_j^2+b_jc_j}{c_j(a_j+d_j)}\right], & \textit{if} \quad c_j(a_j+d_j) > 0 \\ \left[\frac{a_j^2+b_jc_j}{c_j(a_j+d_j)}, \quad \frac{a_j}{c_j}\right], & \textit{if} \quad c_j(a_j+d_j) < 0, \end{cases}$$

which attracts all solutions with initial states in

$$\times_{j=0,1,2,\cdots,k-1} \begin{cases} (\frac{d_j}{c_j}, \quad +\infty), & \textit{if} \quad c_j(a_j+d_j) > 0 \\ (-\infty, \quad \frac{d_j}{c_j}), & \textit{if} \quad c_j(a_j+d_j) < 0. \end{cases}$$

Proof. We write the system of Eq (3.1) in the form of an equation in the k-dimensional space. To do that consider the function

$$\phi(w_1, w_2, \cdots, w_k) := (1 + \frac{p_k}{w_k}, 1 + \frac{p_1}{w_1}, \cdots, 1 + \frac{p_{k-1}}{w_{k-1}})^T$$

and the sequence of vectors $Y_n := (y_n^{(1)}, y_n^{(2)}, \cdots, y_n^{(k)})^T$, where the superscript T denotes transposition. Then, the system (3.1) takes the form

$$Y_{n+1} = \phi(y_n^{(1)}, y_n^{(2)}, \cdots, y_n^{(k)}). \tag{3.2}$$

We observe that for all $z \in [1, 1 + p_j]$ it holds

$$1 < 1 + \frac{p_j}{z} < 1 + p_j, \quad j = 1, 2, \cdots, k,$$

which means that the function ϕ maps D into D. Since ϕ is a continuous function and D is a closed, bounded and convex set, applying Brouwer's fixed-point theorem, we conclude that there is a fixed point $V \in D$ of equation

$$Z^T = \phi(Z).$$

Vector $V := (v_1, v_2, \cdots, v_k)^T$ is such that

$$v_j = 1 + \frac{p_{j-1}}{v_{j-1}}, \quad j = 1, 2, \cdots, k,$$

where $v_0 := v_k$ and $p_0 := p_k$.

Next let $Y_n := (y_n^{(1)}, y_n^{(2)}, \cdots, y_n^{(k)})^T$ be a solution. From (3.1), we have

$$y_{n+1}^{(j)} = 1 + \frac{p_{j-1}}{y_n^{(j-1)}},$$

for all $j = 1, 2, \cdots, k$ and $n = 0, 1, 2, \cdots$. Here we have put $y_n^{(0)} = y_n^{(k)}$. If V is a fixed point, as above, we observe that it holds

$$\begin{aligned}
|v_j - y_{n+1}^{(j)}| &= \frac{p_{j-1}}{v_{j-1}y_n^{(j-1)}}|v_{j-1} - y_n^{(j-1)}| = \cdots = \frac{p_{j-1}}{v_{j-1}y_n^{(j-1)}}\frac{p_{j-2}}{v_{j-2}y_n^{(j-2)}}|v_{j-2} - y_n^{(j-2)}| \\
&= \frac{p_{j-1}}{v_{j-1}y_n^{(j-1)}}\frac{p_{j-2}}{v_{j-2}y_n^{(j-2)}}\cdots\frac{p_1}{v_1y_n^{(1)}}|v_1 - y_n^{(1)}| = \frac{p_{j-1}}{v_{j-1}y_n^{(j-1)}}\frac{p_{j-2}}{v_{j-2}y_n^{(j-2)}}\cdots\frac{p_1}{v_1y_n^{(1)}}\frac{p_k}{v_1y_n^{(k)}}|v_k - y_n^{(k)}| \\
&= \frac{p_{j-1}}{v_{j-1}y_n^{(j-1)}}\cdots\frac{p_1}{v_1y_n^{(1)}}\frac{p_k}{v_1y_n^{(k)}}\frac{p_{k-1}}{v_1y_n^{(k-1)}}|v_{k-1} - y_n^{(k-1)}| = \cdots \\
&= \sqrt{\frac{p_1}{v_1^2}}\sqrt{\frac{p_2}{v_2^2}}\cdots\sqrt{\frac{p_k}{v_k^2}}\sqrt{\frac{p_1}{y_n^{(1)^2}}}\sqrt{\frac{p_2}{y_n^{(2)^2}}}\cdots\sqrt{\frac{p_k}{y_n^{(k)^2}}}|v_j - y_n^{(j)}| \\
&= \sqrt{\frac{v_2 - 1}{v_1}}\sqrt{\frac{v_3 - 1}{v_2}}\cdots\sqrt{\frac{v_k - 1}{v_{k-1}}}\sqrt{\frac{v_1 - 1}{v_k}} \\
&\times \sqrt{\frac{y_n^{(2)} - 1}{y_n^{(1)}}}\sqrt{\frac{y_n^{(3)} - 1}{y_n^{(2)}}}\cdots\sqrt{\frac{y_n^{(k)} - 1}{y_n^{(k-1)}}}\sqrt{\frac{y_n^{(1)} - 1}{y_n^{(k)}}}|v_j - y_n^{(j)}| \\
&= \sqrt{\frac{v_1 - 1}{v_1}}\sqrt{\frac{v_2 - 1}{v_2}}\cdots\sqrt{\frac{v_k - 1}{v_k}}\sqrt{\frac{y_n^{(1)} - 1}{y_n^{(1)}}}\sqrt{\frac{y_n^{(2)} - 1}{y_n^{(2)}}}\cdots\sqrt{\frac{y_n^{(k)} - 1}{y_n^{(k)}}}|v_j - y_n^{(j)}|
\end{aligned}$$

and so $|v_j - y_{n+1}^{(j)}| \le \mu|v_j - y_n^{(j)}|, \quad j = 1, 2, \cdots, \quad n = 0, 1, 2, \cdots$ where

$$\mu := \sqrt{\frac{v_1 - 1}{v_1}}\sqrt{\frac{v_2 - 1}{v_2}}\cdots\sqrt{\frac{v_k - 1}{v_k}} < 1.$$

The latter implies that

$$|v_j - y_{n+1}^{(j)}| \le \mu^{n+1}|v_j - y_0^{(j)}|, \quad j = 1, 2, \cdots, k,$$

from which we conclude that $\lim_{n\to+\infty} y_n^{(j)} = v_j, \quad j = 1, 2, \cdots, k$. After this fact the proof of the theorem is complete. □

Remark 1. It is easy to see that in case $k = 2$, the fixed point has coordinates given by

$$v_1 := \frac{1 - p_1 + p_2 + (1 - p_1 + p_2)^2 + 4p_1)^{1/2}}{2}, \quad v_2 := \frac{1 + p_1 - p_2 + (1 + p_1 - p_2)^2 + 4p_2)^{1/2}}{2}.$$

In case $k = 3$,

$$v_1 = \frac{1 + p_1 - p_2 + p_3 + \sqrt{(1 + p_1 - p_2 + p_3)^2 + 4(p_1 + 1)p_2(p_3 + 1)}}{2(p_3 + 1)},$$

$$v_2 = \frac{1 + p_2 - p_3 + p_1 + \sqrt{(1 + p_2 - p_3 + p_1)^2 + 4(p_2 + 1)p_3(p_1 + 1)}}{2(p_1 + 1)},$$

$$v_3 = \frac{1 + p_3 - p_1 + p_2 + \sqrt{(1 + p_3 - p_1 + p_2)^2 + 4(p_3 + 1)p_1(p_2 + 1)}}{2(p_2 + 1)}.$$

Remark 2. Since the Riccati numbers R_j are reals and have negative signs, it is not hard to see that, the roots $\lambda_-^{(j)}, \lambda_+^{(j)}$ of the characteristic equation (2.8) are reals and they satisfy the case *1) a)* of Theorem 2.3. This means that the eigenvalues $\lambda_+^{(j)}$ (2.8) are the coordinates of the fixed point V.

A special case

The question what can we say when, for some (or for all) indices j and m, it holds $c_m^{(j)} = 1$? As we see, if $1 - c_{m-1}^{(j)} = 0$, the quantity $c_m^{(j)}$ is not defined. If $R_{j-m-1} = 1$, then $c_m^{(j)} = 1$, if $c_{m-1}^{(j)} = 0$. Since the Riccati numbers cann't be zero, we have $m = 1$. So, if $R_{j-2} = 1$, then $c_1^{(j)} = 1$, and this is the only value of the sequence of parameters c which exists, after $c_0^{(j)}$. As we shall see in this case all solutions are periodic.

To prove it, consider the circular system

$$y_{n+1}^{(j)} = 1 - \frac{1}{y_n^{(j-1)}}, \quad j \in \overline{1, k}$$

and we shall discuss the j_{th}- coordinate. Since the system is circular, without loss of generality we can assume that $j = 1$. After some manipulations and applying induction[‡], we obtain that

$$y_n^{(k)} = y_{n-3}^{(k-3)} = \cdots = \begin{cases} = y_{n-k}^{(0)} = y_{n-k}^{(k)}, & if \;\; k = 0(\text{mod}3) \\ = y_{n-k+1}^{(1)} = y_{n-2k+2}^{(2)} = y_{n-3k}^{(0)} = y_{n-3k}^{(k)}, & if \;\; k = 1(\text{mod}3) \\ = y_{n-k+2}^{(2)} = y_{n-2k+1}^{(1)} = y_{n-4k+2}^{(2)} = y_{n-3k}^{(k)}, & if \;\; k = 2(\text{mod}3), \end{cases}$$

for all $n \geq \max\{4k - 2, 3k\}$. This means that the sequence (y_n) is periodic with period $3k$.

4. Discussion

We have discussed a circular system of Riccati type complex difference equations. First, we presented the forbidden set of the system. Then we discussed the solvability of it, by expressing the solutions in terms of the initial values. The expression of the solutions helps to obtain their asymptotic behavior. In the case of negative real Riccati numbers, it is shown that there is a unique fixed point of the system, which attracts all solutions starting from positive values.

5. Conclusions

The forbidden set of the circular system (1.5) is given in terms of the elements of an auxiliary system obtained from the Riccati numbers. Also, the expression of the solutions is given by using an auxiliary finite system of sequences, which follows from these numbers. In the case, of negative real Riccati numbers, we show the convergence of the solutions (with positive initial values) to a fixed point, which is unique. In case $R_j = 1$, for all $j = 1, 2 \cdots$, it is shown that the solutions are periodic.

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[‡]We can find analogous expressions for all coordinates and not only for $j = k$.

Conflict of interest

The author declares that he does not have competing interests.

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