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Application of aggregated control functions for approximating $\mathscr{C}$-Hilfer fractional differential equations

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Abstract: The main issue we are studying in this paper is that of aggregation maps, which refers to the process of combining various input values into a single output. We apply aggregated special maps on Mittag-Leffler-type functions in one parameter to get diverse approximation errors for fractional-order systems in Hilfer sense using an optimal method. Indeed, making use of various well-known special functions that are initially chosen, we establish a new class of matrix-valued fuzzy controllers to evaluate maximal stability and minimal error. An example is given to illustrate the numerical results by charts and tables.

Keywords: aggregation maps; special functions; stability
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1. Introduction and preliminaries

Consider the $\mathscr{C}$-Hilfer fractional system below:

$$\begin{cases} {}^{\mathscr{H}}D_{0^+}^{\mathtt{D}_1,\mathtt{D}_2;\mathscr{C}} \pitchfork(\mathscr{V}) = \varrho(\mathscr{V}, \pitchfork(\mathscr{V}), \pitchfork(\rho(\mathscr{V}))), & \mathscr{V} \in (0, \mathtt{T}], \\ I_{0^+}^{1-\mathtt{D}_3;\mathscr{C}} \pitchfork(0^+) = f_0, & f_0 \in \mathbb{R}, \end{cases} \tag{1.1}$$

in which ${}^{\mathscr{H}}D_{0^+}^{\mathtt{D}_1,\mathtt{D}_2;\mathscr{C}}(.)$ is the $\mathscr{C}$-Hilfer fractional derivative of order $0 < \mathtt{D}_1 \leq 1$ and type $0 \leq \mathtt{D}_2 \leq 1$, $I_{0^+}^{1-\mathtt{D}_3;\mathscr{C}}(.)$ is a fractional-order integral with $\mathtt{D}_3 = \mathtt{D}_1 + \mathtt{D}_2(1 - \mathtt{D}_1)$, in regard to the function $\mathscr{C}$, and $\varrho : (0, \mathtt{T}] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a function.

There are many authors who have studied stability results for the fractional-order system (1.1). Aderyani, Saadati and Feckan [1] investigated the existence, uniqueness and Gauss hypergeometric stability of $\mathscr{C}$-Hilfer fractional differential system (1.1) defined on compact domains via the Cadariu-Radu method derived from the Diaz-Margolis theorem. Also, they proved the major results for unbounded domains. In [2], the authors introduced a class of fuzzy matrix valued control functions and applied the Radu-Mihet method derived from an alternative fixed point theorem to investigate the Ulam-Hyers-Mittag-Leffler stability for a class of $\mathscr{C}$-Hilfer fractional differential system (1.1) in matrix valued fuzzy Banach spaces. Aderyani et al. [3] investigated the approximation of the fractional system (1.1) using an alternative theorem and in comparison to the Picard method, they proved that the fixed point method has a better error estimate and economic solution.

In the present paper, we study a novel concept of Ulam type stability with the applications of Mittag-Leffler type functions of one variable and aggregation maps. This stability allows us to get the best approximation error estimates for the above fractional-order system. In addition, we will be able to obtain maximal stability with minimal error which leads us to calculate the optimal solution.

1.1. Weighted spaces

Consider $[\mathfrak{L}_1, \mathfrak{L}_2]$ $(0 < \mathfrak{L}_1 < \mathfrak{L}_2 < \infty)$, and

$$C[\mathfrak{L}_1, \mathfrak{L}_2] = \{\rho : [\mathfrak{L}_1, \mathfrak{L}_2] \longrightarrow \mathbb{R} \ : \ \rho \ \text{ is continuous}\},$$

with the following norm

$$\|\rho\|_{C[\mathfrak{L}_1,\mathfrak{L}_2]} = \max_{\mathfrak{L}_1 \le \mathscr{V} \le \mathfrak{L}_2} |\rho(\tau)|.$$

The weighted space $C_{1-\mathtt{D}_3;\mathscr{C}}[\mathfrak{L}_1, \mathfrak{L}_2]$ of continuous functions ρ on $(\mathfrak{L}_1, \mathfrak{L}_2]$ is defined by

$$C_{1-\mathtt{D}_3;\mathscr{C}}[\mathfrak{L}_1, \mathfrak{L}_2] = \Big\{\rho : (\mathfrak{L}_1, \mathfrak{L}_2] \longrightarrow \mathbb{R}; (\mathscr{C}(\mathscr{V}) - \mathscr{C}(\mathfrak{L}_1))^{1-\mathtt{D}_3}\rho(\mathscr{V}) \in C[\mathfrak{L}_1, \mathfrak{L}_2]\Big\}, \quad 0 \le \mathtt{D}_3 < 1$$

with norm

$$\|\rho\|_{C_{1-\mathtt{D}_3;\mathscr{C}}[\mathfrak{L}_1,\mathfrak{L}_2]} = \max_{\mathscr{V} \in [\mathfrak{L}_1,\mathfrak{L}_2]} \Big|(\mathscr{C}(\mathscr{V}) - \mathscr{C}(\mathfrak{L}_1))^{1-\mathtt{D}_3}\rho(\mathscr{V})\Big|.$$

1.2. Fractional calculus

Here, we present the $\mathscr{C}$-Hilfer fractional derivative.

Definition 1.1. *[4] Suppose the real interval* $(\mathfrak{L}_1, \mathfrak{L}_2)$*, and* $\mathtt{D}_1 > 0$*. Suppose* $\mathscr{C}(\lambda)$ *is an increasing and positive monotone function on* $(\mathfrak{L}_1, \mathfrak{L}_2]$ *with continuous derivative* $\mathscr{C}'(\lambda)$ *on* $(\mathfrak{L}_1, \mathfrak{L}_2)$*. We define the fractional integral of a function A with respect to* $\mathscr{C}$*, on* $[\mathfrak{L}_1, \mathfrak{L}_2]$ *as follows:*

$$I^{\mathtt{D}_1;\mathscr{C}}_{\mathfrak{L}_1^+} A(\lambda) = \frac{1}{\Gamma(\mathtt{D}_1)} \int_{\mathfrak{L}_1}^{\mathscr{V}} \mathscr{C}'(\mathscr{V})(\mathscr{C}(\lambda) - \mathscr{C}(\mathscr{V}))^{\mathtt{D}_1-1} A(\mathscr{V}) d\mathscr{V}.$$

Definition 1.2. *[4] Suppose* $\mathtt{D}_1 \in (E-1, E)$ *with* $E \in \mathbb{N}$*, and* $A, \mathscr{C} \in C^E[\mathfrak{L}_1, \mathfrak{L}_2]$ *are two functions s.t.* $\mathscr{C}$ *is increasing and* $\mathscr{C}'(\lambda) \neq 0$*, for all* $\lambda \in [\mathfrak{L}_1, \mathfrak{L}_2]$*. The fractional-order derivative* ${}^{\mathscr{H}}D^{\mathtt{D}_1,\mathtt{D}_2;\mathscr{C}}_{\mathfrak{L}_1^+}(.)$ *in Hilfer sense of order* $\mathtt{D}_1$ *and type* $\mathtt{D}_2 \in [0, 1]$ *with respect to function* $\mathscr{C}$ *is defined by*

$${}^{\mathscr{H}}D^{\mathtt{D}_1,\mathtt{D}_2;\mathscr{C}}_{\mathfrak{L}_1^+} A(\lambda) = I^{\mathtt{D}_2(E-\mathtt{D}_1);\mathscr{C}}_{\mathfrak{L}_1^+} \Big(\frac{1}{\mathscr{C}'(\lambda)} \frac{d}{d\lambda}\Big)^E I^{(1-\mathtt{D}_2)(E-\mathtt{D}_1);\mathscr{C}}_{\mathfrak{L}_1^+} A(\lambda).$$

Theorem 1.3. *[4] Suppose* $A \in C^1[\mathfrak{L}_1, \mathfrak{L}_2]$, $\mathtt{D}_1 \in (0,1)$ *and* $\mathtt{D}_2 \in [0,1]$. *Then*

$$^{\mathscr{H}}D_{\mathfrak{L}_1^+}^{\mathtt{D}_1,\mathtt{D}_2;\mathscr{C}} I_{\mathfrak{L}_1^+}^{\mathtt{D}_1;\mathscr{C}} A(\lambda) = A(\lambda).$$

Theorem 1.4. *[4] Suppose* $A \in C^1[\mathfrak{L}_1, \mathfrak{L}_2]$, $\mathtt{D}_1 \in (0,1)$ *and* $\mathtt{D}_2 \in [0,1]$. *Then,*

$$I_{\mathfrak{L}_1^+}^{\mathtt{D}_1;\mathscr{C}}\, {}^{\mathscr{H}}D_{\mathfrak{L}_1^+}^{\mathtt{D}_1,\mathtt{D}_2;\mathscr{C}} A(\lambda) = A(\lambda) - \frac{(\mathscr{C}(\lambda) - \mathscr{C}'(\mathfrak{L}_1))^{\mathtt{D}_3 - 1}}{\Gamma(\mathtt{D}_3)} I_{\mathfrak{L}_1^+}^{(1-\mathtt{D}_2)(1-\mathtt{D}_1);\mathscr{C}} A(\mathfrak{L}_1).$$

Lemma 1.5. *[5] Let* $\rho_1, \rho_2 > 0$. *If* $A(\mathscr{V}) = (\mathscr{C}(\mathscr{V}) - \mathscr{C}(\mathfrak{L}_1))^{\rho_2 - 1}$, *then*

$$I_{\mathfrak{L}_1^+}^{\rho_1;\mathscr{C}} A(\mathscr{V}) = \frac{\Gamma(\rho_2)}{\Gamma(\rho_2 + \rho_1)} (\mathscr{C}(\mathscr{V}) - \mathscr{C}(\mathfrak{L}_1))^{\rho_1 + \rho_2 - 1}. \tag{1.2}$$

1.3. Generalized fuzzy spaces

Let $\mathfrak{Q} := [0,1]$ and

$$\widehat{\text{Diagonal}}A_n(\mathfrak{Q}) = \left\{ \begin{bmatrix} E_1 & & \\ & \ddots & \\ & & E_n \end{bmatrix} = \widehat{\text{Diagonal}}[E_1, \cdots, E_n], \underbrace{E_i}_{i=1,\cdots,n} \in \mathfrak{Q} \right\},$$

where $\widehat{\text{Diagonal}}A_n(\mathfrak{Q})$ is equipped with the following relation:

$$\begin{aligned} &\mathbf{E} := \widehat{\text{Diagonal}}[E_1, \cdots, E_n], \mathbf{D} := \widehat{\text{Diagonal}}[D_1, \cdots, D_n] \in \widehat{\text{Diagonal}}A_n(\mathfrak{Q}), \\ &\mathbf{E} \leq \mathbf{D} \Longleftrightarrow E_i \leq D_i, \quad \forall i \in \mathbb{N}. \end{aligned}$$

Plus, $\mathbf{E} \prec \mathbf{D}$ shows that $\mathbf{E} \leq \mathbf{D}$ and $\mathbf{E} \neq \mathbf{D}$ and $\mathbf{E_i} < \mathbf{D_i}$, for all $i \in \mathbb{N}$. We define $\boldsymbol{\varrho} := \widehat{\text{Diagonal}}[\varrho, \cdots, \varrho]$ in $\widehat{\text{Diagonal}}A_n(\mathfrak{Q})$ in which $\varrho \in \mathfrak{Q}$. For example, $\mathbf{0} = \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix}$ and $\mathbf{1} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$.

Here, we generalize the t-norm $\bigotimes_{\mathtt{TN}}$ on $\widehat{\text{Diagonal}}A_n(\mathfrak{Q})$.

Definition 1.6. *[6] A generalized triangular norm (GTN) on* $\widehat{\text{Diagonal}}A_n(\mathfrak{Q})$ *is an operation* $\bigotimes_{\mathtt{GTN}}$: $\widehat{\text{Diagonal}}A_n(\mathfrak{Q}) \times \widehat{\text{Diagonal}}A_n(\mathfrak{Q}) \to \widehat{\text{Diagonal}}A_n(\mathfrak{Q})$ *s.t.,*

(1) $(\forall \mathbf{E} \in \widehat{\text{Diagonal}}A_n(\mathfrak{Q}))(\mathbf{E} \bigotimes_{\mathtt{GTN}} \mathbf{1}) = \mathbf{E})$ (boundary condition);

(2) $(\forall (\mathbf{E}, \mathbf{D}) \in (\widehat{\text{Diagonal}}A_n(\mathfrak{Q}))^2)(\mathbf{E} \bigotimes_{\mathtt{GTN}} \mathbf{D} = \mathbf{D} \bigotimes_{\mathtt{GTN}} \mathbf{E})$ (commutativity);

(3) $(\forall (\mathbf{E}, \mathbf{D}, \mathscr{K}) \in (\widehat{\text{Diagonal}}A_n(\mathfrak{Q})^3)(\mathbf{E} \bigotimes_{\mathtt{GTN}} (\mathbf{D} \bigotimes_{\mathtt{GTN}} \mathscr{K}) = (\mathbf{E} \bigotimes_{\mathtt{GTN}} \mathbf{D}) \bigotimes_{\mathtt{GTN}} \mathscr{K})$ (associativity);

(4) $(\forall (\mathbf{E}, \mathbf{E}', \mathbf{D}, \mathbf{D}') \in (\widehat{\text{Diagonal}}A_n(\mathfrak{Q}^4)(\mathbf{E} \leq \mathbf{E}'$ and $\mathbf{D} \leq \mathbf{D}' \Longrightarrow \mathbf{E} \bigotimes_{\mathtt{GTN}} \mathbf{D} \leq \mathbf{E}' \bigotimes_{\mathtt{GTN}} \mathbf{D}'$ (monotonicity).

Let $n, k \in \mathbb{N}$, $\mathbf{E} := \text{diag}[E_1, \cdots, E_n]$, $\mathbf{D} := \text{diag}[D_1, \cdots, D_n]$, $E_k := \text{diag}[E_{1k}, \cdots, E_{nk}]$, and $D_k := \text{diag}[D_{1k}, \cdots, D_{nk}]$. For all $\mathbf{E}, \mathbf{D} \in \widehat{\text{Diagonal}}A_n(\mathfrak{Q})$ and all sequences $\{\mathbf{D}_k\}$ and $\{\mathbf{E}_k\}$ converging to $\mathbf{D}$ and $\mathbf{E}$, suppose we have

$$\lim_k (\mathbf{E}_k \bigotimes_{\mathtt{GTN}} \mathbf{D}_k) = \mathbf{E} \bigotimes_{\mathtt{GTN}} \mathbf{D},$$

then, $\bigotimes_{\mathtt{GTN}}$ on $\widehat{\text{Diagonal}}A_n(\mathfrak{Q})$ is a continuous generalized triangular norm (CGTN). Consider the following examples of CGTNs.

(1) Define $\bigotimes_{\mathtt{GTN}}^{P} : \widehat{\text{Diagonal}}A_n(\mathfrak{Q}) \times \widehat{\text{Diagonal}}A_n(\mathfrak{Q}) \to \widehat{\text{Diagonal}}A_n(\mathfrak{Q})$, so that,

$$\begin{aligned}\lim_k \left(E_k \bigotimes_{\mathtt{GTN}}^{P} D_k\right) &= \lim_k \left(\widehat{\text{Diagonal}}[E_{1k}, \cdots, E_{nk}] \bigotimes_{\mathtt{GTN}}^{P} \widehat{\text{Diagonal}}[D_{1k}, \cdots, D_{nk}]\right) \\ &= \widehat{\text{Diagonal}}[E_1.D_1, \cdots, E_n.D_n],\end{aligned}$$

hence, $\bigotimes_{\mathtt{GTN}}^{P}$ is a CGTN.

(2) Define $\bigotimes_{\mathtt{GTN}}^{M} : \widehat{\text{Diagonal}}A_n(\mathfrak{Q}) \times \widehat{\text{Diagonal}}A_n(\mathfrak{Q}) \to \widehat{\text{Diagonal}}A_n(\mathfrak{Q})$, so that,

$$\begin{aligned}\lim_k \left(E_k \bigotimes_{\mathtt{GTN}}^{M} D_k\right) &= \lim_k \left(\widehat{\text{Diagonal}}[E_{1k}, \cdots, E_{nk}] \bigotimes_{\mathtt{GTN}}^{M} \widehat{\text{Diagonal}}[D_{1k}, \cdots, D_{nk}]\right) \\ &= \widehat{\text{Diagonal}}[\min\{E_1, D_1\}, \cdots, \min\{E_n, D_n\}],\end{aligned}$$

hence, $\bigotimes_{\mathtt{GTN}}^{M}$ is a CGTN.

(3) Define $\bigotimes_{\mathtt{GTN}}^{L} : \widehat{\text{Diagonal}}A_n(\mathfrak{Q}) \times \widehat{\text{Diagonal}}A_n(\mathfrak{Q}) \to \widehat{\text{Diagonal}}A_n(\mathfrak{Q})$, so that,

$$\begin{aligned}\lim_k \left(E_k \bigotimes_{\mathtt{GTN}}^{L} D_k\right) &= \lim_k \left(\widehat{\text{Diagonal}}[E_{1k}, \cdots, E_{nk}] \bigotimes_{\mathtt{GTN}}^{L} \widehat{\text{Diagonal}}[D_{1k}, \cdots, D_{nk}]\right) \\ &= \widehat{\text{Diagonal}}[\max\{E_1 + D_1 - 1, 0\}, \cdots, \max\{E_n + D_n - 1, 0\}],\end{aligned}$$

hence, $\bigotimes_{\mathtt{GTN}}^{L}$ is a CGTN.

Here, we present some numerical instances and compare the results,

$$\widehat{\text{Diagonal}}\left[\frac{1}{2}, 0.2, 1\right] \bigotimes_{\mathtt{GTN}}^{M} \widehat{\text{Diagonal}}\left[\frac{3}{10}, 0.7, 0\right] = \begin{bmatrix}\frac{1}{2} & & \\ & 0.2 & \\ & & 1\end{bmatrix} \bigotimes_{\mathtt{GTN}}^{M} \begin{bmatrix}\frac{3}{10} & & \\ & 0.7 & \\ & & 0\end{bmatrix} = \begin{bmatrix}\frac{3}{10} & & \\ & 0.2 & \\ & & 0\end{bmatrix},$$

$$\widehat{\text{Diagonal}}\left[\frac{1}{2}, 0.2, 1\right] \bigotimes_{\mathtt{GTN}}^{P} \widehat{\text{Diagonal}}\left[\frac{3}{10}, 0.7, 0\right] = \begin{bmatrix}\frac{1}{2} & & \\ & 0.2 & \\ & & 1\end{bmatrix} \bigotimes_{\mathtt{GTN}}^{P} \begin{bmatrix}\frac{3}{10} & & \\ & 0.7 & \\ & & 0\end{bmatrix} = \begin{bmatrix}\frac{3}{20} & & \\ & \frac{7}{50} & \\ & & 0\end{bmatrix},$$

$$\widehat{\text{Diagonal}}\left[\frac{1}{2}, 0.2, 1\right] \bigotimes_{\mathtt{GTN}}^{L} \widehat{\text{Diagonal}}\left[\frac{3}{10}, 0.7, 0\right] = \begin{bmatrix}\frac{1}{2} & & \\ & 0.2 & \\ & & 1\end{bmatrix} \bigotimes_{\mathtt{GTN}}^{L} \begin{bmatrix}\frac{3}{10} & & \\ & 0.7 & \\ & & 0\end{bmatrix} = \begin{bmatrix}0 & & \\ & 0 & \\ & & 0\end{bmatrix}.$$

Further, since

$$\widehat{\text{Diagonal}}\,[0.3, 0.2, 0] \geq \widehat{\text{Diagonal}}\left[\frac{3}{20}, \frac{7}{50}, 0\right] \geq \widehat{\text{Diagonal}}\,[0, 0, 0]\,,$$

we get

$$\begin{aligned} & \widehat{\text{Diagonal}}\left[\frac{1}{2}, 0.2, 1\right] \bigotimes_{\mathtt{GTN}}^{M} \widehat{\text{Diagonal}}\left[\frac{3}{10}, 0.7, 0\right] \\ \geq\ & \widehat{\text{Diagonal}}\left[\frac{1}{2}, 0.2, 1\right] \bigotimes_{\mathtt{GTN}}^{P} \widehat{\text{Diagonal}}\left[\frac{3}{10}, 0.7, 0\right] \\ \geq\ & \widehat{\text{Diagonal}}\left[\frac{1}{2}, 0.2, 1\right] \bigotimes_{\mathtt{GTN}}^{L} \widehat{\text{Diagonal}}\left[\frac{3}{10}, 0.7, 0\right]. \end{aligned}$$

Suppose $\mathcal{T} > 0$ and ξ is a vector space. We denote the collection of matrix fuzzy sets by Λ^*. Now, $\mathcal{N} \in \Lambda^*$ denotes $\mathcal{N} : \xi \times (0, +\infty) \longrightarrow \widehat{\text{Diagonal}} A_n(\mathfrak{Q})$ s.t.,

- $\mathcal{N}$ is continuous;
- $\mathcal{N}(\zeta, .)$ is non-decreasing (here $\zeta \in \xi$);
- $\lim_{\mathcal{T}\to+\infty}(\zeta, \mathcal{T}) = \mathbf{1}$ (here $\zeta \in \xi$).

In Λ^*, we denote " $\leq$ " as follows:

$$\mathcal{N} \leq \mathcal{N}_{\circ} \iff \mathcal{N}(\zeta, \mathcal{T}) \leq \mathcal{N}_{\circ}(\zeta, \mathcal{T}'), \quad \forall \mathcal{T}', \mathcal{T} > 0, \text{ and } \zeta \in \xi.$$

Definition 1.7. *[6] Consider the matrix valued fuzzy set* $\mathcal{N} : \xi \times (0, +\infty) \longrightarrow \widehat{\text{Diagonal}} A_n(\mathfrak{Q})$*, a vector space* ξ *and the CGTN* $\bigotimes_{\mathtt{GTN}}$*. In this case, we consider a matrix fuzzy normed space (MFN-space)* $(\xi, \mathcal{N}, \bigotimes_{\mathtt{GTN}})$ *as,*

- ♠ $\mathcal{N}(\zeta, \mathcal{T}) = \mathbf{1}$ *for any* $\mathcal{T} > 0$ *if and only if* $\zeta = 0$*;*
- ♠ $\mathcal{N}(\jmath\zeta, \mathcal{T}) = \mathcal{N}(\zeta, \frac{\mathcal{T}}{|\jmath|})$ *for any* $\zeta \in \xi, \mathcal{T} > 0$*, and* $0 \neq \jmath \in \mathbb{C}$*;*
- ♠ $\mathcal{N}(\zeta + \zeta', \mathcal{T} + \mathcal{T}') \geq \mathcal{N}(\zeta, \mathcal{T}) \bigotimes_{\mathtt{GTN}} \mathcal{N}(\zeta', \mathcal{T}')$ *for all* $\zeta, \zeta' \in \xi$ *and* $\mathcal{T}, \mathcal{T}' \geq 0$*;*
- ♠ $\lim_{\mathcal{T}\to+\infty} \mathcal{N}(\zeta, \mathcal{T}) = \mathbf{1}$*, for all* $\zeta \in \xi$*.*

For example, the matrix valued fuzzy set $\mathcal{N}$

$$\mathcal{N}(\zeta, \mathcal{T}) = \widehat{\text{Diagonal}}\left[\exp(-\frac{\|\zeta\|}{\mathcal{T}}), \frac{\mathcal{T}}{\mathcal{T} + \|\zeta\|}\right],$$

is an MFN, where $\mathcal{T} > 0$ and $(\xi, \mathcal{N}, \bigotimes_{\mathtt{GTN}}^{M})$ is an MFN-space and $(\xi, \|\cdot\|)$ is a linear normed space.

A complete MFN-space is named a matrix fuzzy Banach space (in short, MFB-space).

1.4. Generalized metric spaces and fixed point theory

Note 1.8. Define $\hbar := (\hbar_1, \ldots, \hbar_m)$ and $\Bbbk := (\Bbbk_1, \ldots, \Bbbk_m), m \in \mathbb{N}$. We have

$$\hbar \leq \Bbbk \iff \hbar_j \leq \Bbbk_j, \quad j = 1, \cdots, m;$$

and also

$$\hbar \to 0 \iff \hbar_j \to 0, \quad j = 1, \cdots, m.$$

Definition 1.9. *[7] Let the set $\emptyset \neq \top$ and a given mapping $\hbar : \top^2 \to [0, +\infty]^m, m \in \mathbb{N}$. A generalized metric $\hbar$ on $\top$ is a function s.t.,*

♠ *for all $(\aleph, \wp) \in \top^2$, we get*

$$\hbar(\aleph, \wp) = \mathbf{0} = \underbrace{(0, \cdots, 0)}_{m} \Longleftrightarrow \aleph = \wp;$$

♠ *for all $(\aleph, \wp) \in \top^2$, we get*

$$\hbar(\wp, \aleph) = \hbar(\aleph, \wp) \Longleftrightarrow \aleph = \wp;$$

♠ *for all $\aleph, \wp, \iota \in \top$, we get*

$$\hbar(\aleph, \iota) + \hbar(\iota, \wp) \geq \hbar(\wp, \aleph).$$

Theorem 1.10. *[7] Suppose $m \in \mathbb{N}$ and a function $\hbar : \top^2 \to [0, +\infty]^m$, and a complete generalized metric space $(\top, \hbar)$, and a contraction mappings $\Gamma : \top \to \top$ with Lipschitz constant $\vee < 1$. Therefore, for any $\vartheta \in \top$, either*

$$\hbar(\Gamma^n\vartheta, \Gamma^{n+1}\vartheta) = \underbrace{(+\infty, \cdots, +\infty)}_{m}$$

for all $n \in \mathbb{N} \cup \{0\}$ or there is an $n_0 \in \mathbb{N}$ s.t.

♠ $\hbar(\Gamma^n\vartheta, \Gamma^{n+1}\vartheta) \leq \underbrace{(+\infty, \cdots, +\infty)}_{m}, \qquad \forall n \geq n_0;$

♠ *The fixed point κ^* of Γ is a convergence point of sequence $\{\Gamma^n\vartheta\}$. and is unique in the set $\top' = \{\kappa \in \top \mid \hbar(\Gamma^{n_0}\vartheta, \kappa) \leq \underbrace{(+\infty, \cdots, +\infty)}_{m}\}$;*

♠ $\hbar(\kappa, \kappa^*) \leq \frac{1}{1-\vee}\hbar(\kappa, \Gamma\kappa)$ *for every $\kappa \in \top'$.*

1.5. On aggregate functions

Suppose $[n] := \{1, \cdots, n\}$, with $n \in \mathbb{N}$. We apply the bold symbol $\mathbf{Y}$ to show the n-tuple $\widehat{\text{Diagonal}}[y_1, \cdots, y_n]_{n^2}$.

Definition 1.11. *[8] A mapping $\lambda^{(n)} : \widehat{Diagonal}[\mathfrak{Q}, \cdots, \mathfrak{Q}]_{n^2} \longrightarrow \mathfrak{Q}$ is called an aggregation map if it is increasing in all variables and also, it fulfills the boundary conditions*

$$\inf_{\mathbf{Y} \in \mathfrak{Q}^n} \lambda^{(n)}(\mathbf{Y}) = \inf \mathfrak{Q}, \quad and \quad \sup_{\mathbf{Y} \in \mathfrak{Q}^n} \lambda^{(n)}(\mathbf{Y}) = \sup \mathfrak{Q}. \tag{1.3}$$

Note that $n \in \mathbb{N}$ displays the arity of the aggregation function and we will use symbol λ instead of $\lambda^{(n)}$.

Now, we propose some classical aggregation maps, as follows:

- The arithmetic mean and the geometric mean maps

$$\mathtt{AG}_1, \mathtt{AG}_2 : \widehat{\text{Diagonal}}[\mathfrak{Q}, \cdots, \mathfrak{Q}]_{n \times n} \longrightarrow \mathfrak{Q}$$

  are given by

$$\mathtt{AG}_1(\mathbf{Y}) := \frac{1}{n}\sum_{i=1}^{n} y_i, \tag{1.4}$$

$$\mathtt{AG}_2(\mathbf{Y}) := (\prod_{i=1}^{n} y_i)^{\frac{1}{n}}. \tag{1.5}$$

- The projection and the order statistic maps

$$\mathtt{AG}_3, \mathtt{AG}_4 : \widehat{\text{Diagonal}}[\mathfrak{Q}, \cdots, \mathfrak{Q}]_{n\times n} \longrightarrow \mathfrak{Q}$$

related to the $\Bbbk^{th}$ argument with $\Bbbk \in [n]$, are separately defined as

$$\mathtt{AG}_3(\mathbf{Y}) := y_{\Bbbk}, \tag{1.6}$$

$$\mathtt{AG}_4(\mathbf{Y}) := (y)_{\Bbbk}, \tag{1.7}$$

in which $(y)_{\Bbbk}$ is the $\Bbbk^{th}$ lowest coordinate of y, that is,

$$y_{(1)} \leqslant \cdots \leqslant y_{(k)} \leqslant \cdots y_{(n)}.$$

The projections onto the first and the last coordinates are defined by

$$\mathtt{AG}_5(\mathbf{Y}) := y_1, \tag{1.8}$$

$$\mathtt{AG}_6(\mathbf{Y}) := y_n. \tag{1.9}$$

Pluse, the extreme order statistics y_n and y_1 are the maximum and the minimum maps which are defined by

$$\mathtt{AG}_7(\mathbf{Y}) := \max\{y_1, \cdots, y_n\}, \tag{1.10}$$

$$\mathtt{AG}_8(\mathbf{Y}) := \min\{y_1, \cdots, y_n\}. \tag{1.11}$$

- The partial minimum and the partial maximum

$$\mathtt{AG}_9, \mathtt{AG}_{10} : \widehat{\text{Diagonal}}[\mathfrak{Q}, \cdots, \mathfrak{Q}]_{n\times n} \longrightarrow \mathfrak{Q}$$

related to K with $\emptyset \neq K \subseteq [n]$, are separately given by

$$\mathtt{AG}_9(\mathbf{Y}) := \min_{i\in K} y_i, \tag{1.12}$$

$$\mathtt{AG}_{10}(\mathbf{Y}) := \max_{i\in K} y_i. \tag{1.13}$$

- The sum and product functions $\mathtt{AG}_{11}, \mathtt{AG}_{12} : \widehat{\text{Diagonal}}[\mathfrak{Q}, \cdots, \mathfrak{Q}]_{n\times n} \longrightarrow \mathfrak{Q}$ are defined by

$$\mathtt{AG}_{11}(\mathbf{Y}) := \sum_{i=1}^{n} y_i, \tag{1.14}$$

$$\mathtt{AG}_{12}(\mathbf{Y}) := \prod_{i=1}^{n} y_i. \tag{1.15}$$

1.6. On special functions

Consider the one parameter Mittag-Leffler-type functions [9] below for every $\lambda, Y \in \mathbb{C}, i \in \mathbb{N}$, and $\Re(\lambda) > 0$.

$$\ni_1 (Y) := \nabla_\lambda(Y) = \sum_{i=0}^{\infty} \frac{Y^i}{\Gamma(i\lambda + 1)}, \tag{1.16}$$

$$
\begin{aligned}
\ni_2(Y) &:= precosh_\lambda(Y)\\
&= 0.5\Big(\nabla_\lambda(Y)+\nabla_\lambda(-Y)\Big)\\
&= \sum_{i=0}^{\infty}\frac{Y^{2i}}{\Gamma((2i)\lambda+1)},
\end{aligned}
$$

$$
\begin{aligned}
\ni_3(Y) &:= precos_\lambda(Y)\\
&= \frac{1}{2}\Big(\nabla_\lambda(iY)+\nabla_\lambda(-iY)\Big)\\
&= \sum_{i=0}^{\infty}\frac{(-1)^iY^{2i}}{\Gamma((2i)\lambda+1)},
\end{aligned}
$$

$$
\begin{aligned}
\ni_4(Y) &:= presinh_\lambda(Y)\\
&= \frac{1}{2}\Big(\nabla_\lambda(Y)-\nabla_\lambda(-Y)\Big)\\
&= \sum_{i=0}^{\infty}\frac{Y^{2i+1}}{\Gamma((2i+1)\lambda+1)},
\end{aligned}
$$

$$
\begin{aligned}
\ni_5(Y) &:= presin_\lambda(Y)\\
&= \frac{1}{2i}\Big(\nabla_\lambda(iY)-\nabla_\lambda(-iY)\Big)\\
&= \sum_{i=0}^{\infty}\frac{(-1)^iY^{2i+1}}{\Gamma((2i+1)\lambda+1)}.
\end{aligned}
$$

We now introduce the matrix valued controller $\mathfrak{P}$, as follows:

$$\mathfrak{P}(Y)=\widehat{\text{Diagonal}}[\ni_1(Y),\cdots,\ni_5(Y)].$$

Let $n\in\mathbb{N}$. For any $\underbrace{\mathcal{T}_i}_{i=1,\cdots,n}\in(0,+\infty)$, we have the inequalities below:

$$
\begin{aligned}
&\mathscr{N}\Big(\int_0^{\mathscr{V}}\mathscr{C}'(\mathscr{V}_\circ)(\mathscr{C}(\mathscr{V})-\mathscr{C}(\mathscr{V}_\circ))^{\mathtt{D}_1-1}\nabla_{\mathtt{D}_1}((\mathscr{C}(\mathscr{V}_\circ)-\mathscr{C}(0))^{\mathtt{D}_1})d\mathscr{V}_\circ,(\mathcal{T}_1,\cdots,\mathcal{T}_n)\Big)\\
\geq\ &\mathscr{N}\Big(\frac{\theta}{\Gamma(\mathtt{D}_1)}\int_0^{\mathscr{V}}\mathscr{C}'(\mathscr{V}_\circ)(\mathscr{C}(\mathscr{V})-\mathscr{C}(\mathscr{V}_\circ))^{\mathtt{D}_1-1}\sum_{k=0}^{\infty}\frac{(\mathscr{C}(\mathscr{V}_\circ)-\mathscr{C}(0))^{k\mathtt{D}_1}}{\Gamma(k\mathtt{D}_1+1)}d\mathscr{V}_\circ,(\mathcal{T}_1,\cdots,\mathcal{T}_n)\Big)\\
\geq\ &\mathscr{N}\Big(\frac{\theta}{\Gamma(\mathtt{D}_1)}\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\mathtt{D}_1+1)}\\
&\qquad\times\int_0^{\mathscr{V}}(\mathscr{C}(\mathscr{V})-\mathscr{C}(\mathscr{V}_\circ))^{\mathtt{D}_1-1}(\mathscr{C}(\mathscr{V}_\circ)-\mathscr{C}(0))^{k\mathtt{D}_1}d\mathscr{C}(\mathscr{V}_\circ),(\mathcal{T}_1,\cdots,\mathcal{T}_n)\Big)\\
\geq\ &\mathscr{N}\Big(\frac{\theta}{\Gamma(\mathtt{D}_1)}\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\mathtt{D}_1+1)}\int_0^{\mathscr{C}(\mathscr{V})-\mathscr{C}(0)}(\mathscr{C}(\mathscr{V})-\mathscr{C}(0)-\mathcal{E})^{\mathtt{D}_1-1}\mathcal{E}^{k\mathtt{D}_1}d\mathcal{E},(\mathcal{T}_1,\cdots,\mathcal{T}_n)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\text{(let } \mathcal{E} = \mathscr{C}(\mathscr{V}_\circ) - \mathscr{C}(0))\\
\geq\ & \mathscr{N}\Big(\frac{\theta}{\Gamma(\mathtt{D}_1)}\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\mathtt{D}_1+1)}(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1-1}\\
&\qquad\times\int_0^{\mathscr{C}(\mathscr{V})-\mathscr{C}(0)}\Big(1-\frac{\mathcal{E}}{\mathscr{C}(\mathscr{V})-\mathscr{C}(0)}\Big)^{\mathtt{D}_1-1}\mathcal{E}^{k\mathtt{D}_1}d\mathcal{E},(\mathcal{T}_1,\cdots,\mathcal{T}_n)\Big)\\
\geq\ & \mathscr{N}\Big(\frac{\theta}{\Gamma(\mathtt{D}_1)}\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\mathtt{D}_1+1)}(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{(k+1)\mathtt{D}_1}\int_0^1(1-\mathcal{S})^{\mathtt{D}_1-1}\mathcal{S}^{k\mathtt{D}_1}d\mathcal{S},(\mathcal{T}_1,\cdots,\mathcal{T}_n)\Big)\\
&\Big(\text{let } \mathcal{S} = \frac{\mathcal{E}}{\mathscr{C}(\mathscr{V})-\mathscr{C}(0))}\Big)\\
\geq\ & \mathscr{N}\Big(\frac{\theta}{\Gamma(\mathtt{D}_1)}\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\mathtt{D}_1+1)}(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{(k+1)\mathtt{D}_1}\frac{\Gamma(k\mathtt{D}_1+1)\Gamma(\mathtt{D}_1)}{\Gamma((k+1)\mathtt{D}_1+1)},(\mathcal{T}_1,\cdots,\mathcal{T}_n)\Big)\\
\geq\ & \mathscr{N}\Big(\theta\sum_{n=0}^{\infty}\frac{(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{n\mathtt{D}_1}}{\Gamma(n\mathtt{D}_1+1)},(\mathcal{T}_1,\cdots,\mathcal{T}_n)\Big)\\
\geq\ & \widehat{\text{Diagonal}}\Big[\nabla_{\mathtt{D}_1}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\mathcal{T}_1}\Big),\cdots,\nabla_{\mathtt{D}_1}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\mathcal{T}_n}\Big)\Big].
\end{aligned}
$$

In a similar way, we have:

$$
\begin{aligned}
&\mathscr{N}\Big(\int_0^{\mathscr{V}}\mathscr{C}'(\mathscr{V}_\circ)(\mathscr{C}(\mathscr{V})-\mathscr{C}(\mathscr{V}_\circ))^{\mathtt{D}_1-1}\underbrace{\ni_i}_{i=2,\cdots,5}((\mathscr{C}(\mathscr{V}_\circ)-\mathscr{C}(0))^{\mathtt{D}_1})d\mathscr{V}_\circ,(\mathcal{T}_1,\cdots,\mathcal{T}_n)\Big)\\
\geq\ & \widehat{\text{Diagonal}}\Big[\underbrace{\ni_i}_{i=2,\cdots,5}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\mathcal{T}_1}\Big),\cdots,\underbrace{\ni_i}_{i=2,\cdots,5}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\mathcal{T}_n}\Big)\Big],
\end{aligned}
$$

for any $\underbrace{\mathcal{T}_i}_{i=1,\cdots,n} \in (0,+\infty)$.

2. Multi stability results

For $\varrho \in C((0,\mathtt{T}]\times\mathbb{R}^2,\mathbb{R})$ and $\theta > 0$, suppose

$$
{}^{\mathscr{H}}D_{0^+}^{\mathtt{D}_1,\mathtt{D}_2;\mathscr{C}} \pitchfork(\mathscr{V}) = \varrho(\mathscr{V},\pitchfork(\mathscr{V}),\pitchfork(\rho(\mathscr{V}))), \qquad \mathscr{V}\in(0,\mathtt{T}], \tag{2.1}
$$

$$
I_{0^+}^{1-\mathtt{D}_3;\mathscr{C}} \pitchfork(0^+) = f_0, \qquad f_0\in\mathbb{R}, \tag{2.2}
$$

and

$$
\begin{aligned}
&\mathscr{N}\Big({}^{\mathscr{H}}D_{0^+}^{\mathtt{D}_1,\mathtt{D}_2;\mathscr{C}} \pitchfork(\mathscr{V}) - \varrho(\mathscr{V},\pitchfork(\mathscr{V}),\pitchfork(\rho(\mathscr{V}))),(\mathcal{T}_1,\cdots,\mathcal{T}_{12})\Big) \qquad (2.3)\\
\geq\ & \widehat{\text{Diagonal}}\Big[\mathtt{AG}_1\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\mathcal{T}_1}\Big)\Big),\cdots,\mathtt{AG}_{12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\mathcal{T}_{12}}\Big)\Big)\Big],
\end{aligned}
$$

in which $\mathscr{V}\in(0,\mathtt{T}]$, $\underbrace{\mathcal{T}_i}_{i=1,\cdots,12} \in (0,+\infty)$.

Definition 2.1. *Equations* (2.1) *and* (2.2) *are multi stable w.r.t,*

$$\widehat{Diagonal}\Big[\mathtt{AG}_1\Big(\mathfrak{P}\Big((\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}\Big)\Big),\cdots,\mathtt{AG}_{12}\Big(\mathfrak{P}\Big((\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}\Big)\Big)\Big]$$

if there is a $\eth > 0$ *s.t., for any* $\theta \in (0,+\infty)$, $I_{0^+}^{1-\mathtt{D}_3;\mathscr{C}}\psi(0^+) = f_0 \in \mathbb{R}$, *and any solution* $\psi \in C_{1-\mathtt{D}_3;\mathscr{C}}(0,\mathtt{T}]$ *to (2.3), there is a solution* $\pitchfork \in C_{1-\mathtt{D}_3;\mathscr{C}}(0,\mathtt{T}]$ *to* (2.1) *and* (2.2) *with*

$$\begin{aligned}&\mathscr{N}\Big(\psi(\mathscr{V})-\pitchfork(\mathscr{V}),(\mathcal{T}_1,\cdots,\mathcal{T}_{12})\Big)\\ \geq\;& \widehat{Diagonal}\Big[\mathtt{AG}_1\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\eth\theta\mathcal{T}_1}\Big)\Big),\cdots,\mathtt{AG}_{12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\eth\theta\mathcal{T}_{12}}\Big)\Big)\Big],\end{aligned}$$

for any $\mathscr{V} \in (0,\mathtt{T}]$ *and* $\underbrace{\mathcal{T}_i}_{i=1,\cdots,12} \in (0,+\infty)$.

Through an Alternative theorem, we present the existence, uniqueness and the multi-stability of the fractional system (1.1) in a MFB-space $(\xi,\mathscr{N},\bigotimes_{\mathtt{GTN}})$ (see [10–12]).

Lemma 2.2. *[13] Consider a continuous function* $\varrho : (0,\mathtt{T}]\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$. *Then,* (2.1) *and* (2.2) *are equivalent to*

$$\begin{aligned}\pitchfork(\mathscr{V}) \;=\;& \frac{(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_3-1}}{\Gamma(\mathtt{D}_3)}f_0\\ &+\frac{1}{\Gamma(\mathtt{D}_1)}\int_0^{\mathscr{V}}\mathscr{C}'(\mathscr{V}_\circ)(\mathscr{C}(\mathscr{V})-\mathscr{C}(\mathscr{V}_\circ))^{\mathtt{D}_1-1}\varrho(\mathscr{V}_\circ,\pitchfork(\mathscr{V}_\circ),\pitchfork(\rho(\mathscr{V}_\circ)))d\mathscr{V}_\circ.\end{aligned}$$

Remark 2.3. Suppose we have a solution $\psi \in C_{1-\mathtt{D}_3;\mathscr{C}}(0,\mathtt{T}]$ of the inequality below

$$\begin{aligned}&\mathscr{N}\Big({}^{\mathscr{H}}D_{0^+}^{\mathtt{D}_1,\mathtt{D}_2;\mathscr{C}}\pitchfork(\mathscr{V})-\varrho(\mathscr{V},\pitchfork(\mathscr{V}),\pitchfork(\rho(\mathscr{V}))),(\mathcal{T}_1,\cdots,\mathcal{T}_{12})\Big)\\ \geq\;& \widehat{\mathrm{Diagonal}}\Big[\mathtt{AG}_1\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\mathcal{T}_1}\Big)\Big),\cdots,\mathtt{AG}_{12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\mathcal{T}_{12}}\Big)\Big)\Big],\end{aligned}$$

in which $\mathscr{V} \in (0,\mathtt{T}]$, $\underbrace{\mathcal{T}_i}_{i=1,\cdots,12} \in (0,+\infty)$, and $I_{0^+}^{1-\mathtt{D}_3;\mathscr{C}}\psi(0^+) = f_0 \in \mathbb{R}$. Then, ψ is a solution of:

$$\begin{aligned}&\mathscr{N}\Big(\psi(\mathscr{V})-\frac{(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_3-1}}{\Gamma(\mathtt{D}_3)}f_0\\ &-\frac{1}{\Gamma(\mathtt{D}_1)}\int_0^{\mathscr{V}}\mathscr{C}'(\mathscr{V}_\circ)(\mathscr{C}(\mathscr{V})-\mathscr{C}(\mathscr{V}_\circ))^{\mathtt{D}_1-1}\varrho(\mathscr{V}_\circ,\psi(\mathscr{V}_\circ),\psi(\rho(\mathscr{V}_\circ)))d\mathscr{V}_\circ,(\mathcal{T}_1,\cdots,\mathcal{T}_{12})\Big)\\ \geq\;& \widehat{\mathrm{Diagonal}}\Big[\mathtt{AG}_1\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\mathcal{T}_1}\Big)\Big),\cdots,\mathtt{AG}_{12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\mathcal{T}_{12}}\Big)\Big)\Big],\end{aligned}$$

for each $\underbrace{\mathcal{T}_i}_{i=1,\cdots,12} \in (0,+\infty)$.

Let us consider the following assumptions:

$(\mathcal{A}_1)$ $\varrho \in C((0,\mathsf{T}]\times\mathbb{R}^2,\mathbb{R}), \rho \in C([0,\mathsf{T}],[0,\mathsf{T}])$ and $\psi, ⫛ \in C([0,\mathsf{T}],\mathbb{R})$,

$$\begin{aligned}&\mathscr{N}\Big(⫛ (\rho(\mathscr{V}))-\psi(\rho(\mathscr{V})),(\mathcal{T}_1,\cdots,\mathcal{T}_{12})\Big)\\ \geq\ &\mathscr{N}\Big(⫛ (\mathscr{V})-\psi(\mathscr{V}),(\mathcal{T}_1,\cdots,\mathcal{T}_{12})\Big),\quad \text{for any } \underbrace{\mathcal{T}_i}_{i=1,\cdots,12}\in(0,+\infty).\end{aligned}$$

$(\mathcal{A}_2)$ There is a $\Delta>0$ s.t.

$$\begin{aligned}&\mathscr{N}\Big(\varrho(\mathscr{V},\vartheta_1,\vartheta_2)-\varrho(\mathscr{V},\kappa_1,\kappa_2),(\mathcal{T}_1,\cdots,\mathcal{T}_{12})\Big)\\ \geq\ &\mathscr{N}\Big(\Delta\sum_{j=1}^{2}(\vartheta_j-\kappa_j),(\mathcal{T}_1,\cdots,\mathcal{T}_{12})\Big),\end{aligned}$$

for each $\underbrace{\mathcal{T}_i}_{i=1,\cdots,12}\in(0,+\infty)$, $\mathscr{V}\in(0,\mathsf{T}]$, $\vartheta_j,\kappa_j\in\mathbb{R}$, and $j=1,2$.

$(\mathcal{A}_3)$ There are $0<\underbrace{\mathcal{M}_i}_{i=1,\cdots,12}<1$ s.t.

$$\begin{aligned}&I_{0^+}^{\mathtt{D}_1;\mathscr{C}}\underbrace{\mathtt{AG}_i}_{i=1,\cdots,12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\underbrace{\mathcal{T}_i}_{i=1,\cdots,12}}\Big)\Big)\\ =\ &\frac{1}{\Gamma(\mathtt{D}_1)}\int_{0^+}^{\mathscr{V}}\mathscr{C}'(\mathscr{V}_\circ)(\mathscr{C}(\mathscr{V})-\mathscr{C}(\mathscr{V}_\circ))^{\mathtt{D}_1-1}\underbrace{\mathtt{AG}_i}_{i=1,\cdots,12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\underbrace{\mathcal{T}_i}_{i=1,\cdots,12}}\Big)\Big)\\ \geq\ &\underbrace{\mathtt{AG}_i}_{i=1,\cdots,12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\underbrace{\mathcal{M}_i\mathcal{T}_i}_{i=1,\cdots,12}}\Big)\Big),\end{aligned}$$

in which $\mathscr{V}\in(0,\mathsf{T}]$ and $\underbrace{\mathcal{T}_i}_{i=1,\cdots,12}\in(0,+\infty)$.

$(\mathcal{A}_4)$ We have $(2\Delta\mathcal{M}_1,\cdots,2\Delta\mathcal{M}_{12})<\underbrace{(1,\cdots,1)}_{12}$.

Proposition 2.4. *Consider two integrable functions α and β, s.t. for each $\underbrace{\mathcal{T}_i}_{i=1,\cdots,12}\in(0,+\infty)$,*

$$\mathscr{N}\Big(\alpha,(\mathcal{T}_1,\cdots,\mathcal{T}_{12})\Big)\geq\mathscr{N}\Big(\beta,(\mathcal{T}_1,\cdots,\mathcal{T}_{12})\Big).$$

Then for each $\underbrace{\mathcal{T}_i}_{i=1,\cdots,12}\in(0,+\infty)$,

$$\mathscr{N}\Big(I_{0^+}^{\mathtt{D}_1,\mathscr{C}}\alpha,(\mathcal{T}_1,\cdots,\mathcal{T}_{12})\Big)\geq\mathscr{N}\Big(I_{0^+}^{\mathtt{D}_1,\mathscr{C}}\beta,(\mathcal{T}_1,\cdots,\mathcal{T}_{12})\Big).$$

Proof. We get $\alpha \leq \beta$, therefore for each $\underbrace{\mathcal{T}_i}_{i=1,\cdots,12} \in (0,+\infty)$,

$$\begin{aligned} \mathscr{N}\Big(I_{0^+}^{\mathtt{D}_1,\mathscr{C}}\alpha,(\mathcal{T}_1,\cdots,\mathcal{T}_{12})\Big) &= \mathscr{N}\Big(1,\frac{(\mathcal{T}_1,\cdots,\mathcal{T}_{12})}{I_{0^+}^{\mathtt{D}_1,\mathscr{C}}\alpha}\Big) \\ &\geq \mathscr{N}\Big(1,\frac{(\mathcal{T}_1,\cdots,\mathcal{T}_{12})}{I_{0^+}^{\mathtt{D}_1,\mathscr{C}}\beta}\Big) \\ &= \mathscr{N}\Big(I_{0^+}^{\mathtt{D}_1,\mathscr{C}}\beta,(\mathcal{T}_1,\cdots,\mathcal{T}_{12})\Big). \end{aligned}$$

□

Theorem 2.5. *Suppose* $(\mathcal{A}_1),(\mathcal{A}_2),(\mathcal{A}_3)$ *and* $(\mathcal{A}_4)$ *are satisfied, and* $I_{0^+}^{1-\mathtt{D}_3;\mathscr{C}} \pitchfork (0^+) = f_0 \in \mathbb{R}$ *and also,* $\pitchfork \in C_{1-\mathtt{D}_3;\mathscr{C}}[0,\mathsf{T}]$ *satisfies* (2.3)*. Then,* (2.1) *and* (2.2) *have a unique solution* $\Upsilon \in C_{1-\mathtt{D}_3;\mathscr{C}}[0,\mathsf{T}]$ *s.t.*

$$\begin{aligned} \Upsilon(\mathscr{V}) = {} & \frac{(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_3-1}}{\Gamma(\mathtt{D}_3)}f_0 \\ & +\frac{1}{\Gamma(\mathtt{D}_1)}\int_0^{\mathscr{V}}\mathscr{C}'(\mathscr{V}_\circ)(\mathscr{C}(\mathscr{V})-\mathscr{C}(\mathscr{V}_\circ))^{\mathtt{D}_1-1}\varrho(\mathscr{V}_\circ,\Upsilon(\mathscr{V}_\circ),\Upsilon(\rho(\mathscr{V}_\circ)))d\mathscr{V}_\circ,\ \ \mathscr{V}\in(0,\mathsf{T}], \end{aligned} \tag{2.4}$$

in which $I_{0^+}^{1-\mathtt{D}_3;\mathscr{C}}\Upsilon(0^+) = f_0 \in \mathbb{R}$*, and for each* $\mathscr{V} \in (0,\mathsf{T}]$ *and* $\underbrace{\mathcal{T}_i}_{i=1,\cdots,12} \in (0,+\infty)$,

$$\begin{aligned} & \mathscr{N}\Big(\pitchfork(\mathscr{V})-\Upsilon(\mathscr{V}),(\mathcal{T}_1,\cdots,\mathcal{T}_{12})\Big) \\ \geq\ & \widehat{Diagonal}\Big[\mathtt{AG}_1\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\mathcal{D}_1\,\theta\mathcal{T}_1}\Big)\Big),\cdots,\mathtt{AG}_{12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\mathcal{D}_{12}\,\theta\mathcal{T}_{12}}\Big)\Big)\Big], \end{aligned} \tag{2.5}$$

in which

$$\underbrace{\mathcal{D}_i}_{i=1,\cdots,12} := \frac{1}{1-2\Delta\underbrace{\mathcal{M}_i}_{i=1,\cdots,12}}.$$

Proof. Define $\curlyvee := C_{1-\mathtt{D}_3;\mathscr{C}}(0,\mathsf{T}]$, and a mapping $E:\curlyvee\times\curlyvee\to[0,+\infty]$ by

$$\begin{aligned} E(\varpi,\varpi') = {} & \inf\Big\{(\complement_1,\cdots,\complement_{12})\in(0,+\infty)^{12}:\mathscr{N}\Big(\varpi(\mathscr{V})-\varpi'(\mathscr{V}),(\mathcal{T}_1,\cdots,\mathcal{T}_{12})\Big) \\ & \geq \widehat{\text{Diagonal}}\Big[\mathtt{AG}_1\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\frac{\mathcal{T}_1}{\complement_1}}\Big)\Big),\cdots, \\ & \mathtt{AG}_{12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\frac{\mathcal{T}_{12}}{\complement_{12}}}\Big)\Big)\Big], \\ & \forall\varpi,\varpi'\in\curlyvee,\ \mathscr{V}\in(0,\mathsf{T}],\underbrace{\mathcal{T}_i}_{i=1,\cdots,12}\in(0,+\infty)\Big\}. \end{aligned} \tag{2.6}$$

First of all, we prove $(\curlyvee, E)$ is a $[0,+\infty]^{12}$-valued metric space.

We show $E(\varpi,\varpi') = \underbrace{(0,\cdots,0)}_{12}$ iff $\varpi=\varpi'$. Suppose $E(\varpi,\varpi') = \underbrace{(0,\cdots,0)}_{12}$. We get

$$\begin{aligned}&\inf\Big\{(\mathsf{C}_1,\cdots,\mathsf{C}_{12})\in(0,+\infty)^{12}:\mathcal{N}\Big(\varpi(\mathscr{V})-\varpi'(\mathscr{V}),(\mathcal{T}_1,\cdots,\mathcal{T}_{12})\Big)\\&\geq \widehat{\text{Diagonal}}\Big[\mathsf{AG}_1\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathsf{D}_1}|}{\theta\frac{\mathcal{T}_1}{\mathsf{C}_1}}\Big)\Big),\cdots,\\&\mathsf{AG}_{12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathsf{D}_1}|}{\theta\frac{\mathcal{T}_{12}}{\mathsf{C}_{12}}}\Big)\Big)\Big],\\&\forall\varpi,\varpi'\in\curlyvee,\ \mathscr{V}\in(0,\mathsf{T}],\ \underbrace{\mathcal{T}_i}_{i=1,\cdots,12}\in(0,+\infty)\Big\}=0,\end{aligned}$$

and so

$$\begin{aligned}&\mathcal{N}\Big(\varpi(\mathscr{V})-\varpi'(\mathscr{V}),(\mathcal{T}_1,\cdots,\mathcal{T}_{12})\Big)\\\geq\ &\widehat{\text{Diagonal}}\Big[\mathsf{AG}_1\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathsf{D}_1}|}{\theta\frac{\mathcal{T}_1}{\mathsf{C}_1}}\Big)\Big),\cdots,\mathsf{AG}_{12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathsf{D}_1}|}{\theta\frac{\mathcal{T}_{12}}{\mathsf{C}_{12}}}\Big)\Big)\Big],\end{aligned}$$

for each $\underbrace{\mathsf{C}_i}_{i=1,\cdots,12}\in(0,+\infty)$. Let $\underbrace{\mathsf{C}_i}_{i=1,\cdots,12}$ tend to zero in the above inequality, and we have

$$\mathcal{N}\Big(\varpi(\mathscr{V})-\varpi'(\mathscr{V}),(\mathcal{T}_1,\cdots,\mathcal{T}_{12})\Big)=\mathbf{1},$$

therefore $\varpi(\mathscr{V})=\varpi'(\mathscr{V})$ for each $\mathscr{V}\in[0,\mathsf{T}]$, and vice versa. It is simple to prove $E(\varpi,\varpi') = E(\varpi',\varpi)$ for each $\varpi,\varpi'\in\curlyvee$. Let $E(\varpi,\eth)=(\ell_1,\cdots,\ell_{12})\in(0,+\infty)^{12}$ and $E(\eth,\varpi')=(\jmath_1,\cdots,\jmath_{12})\in(0,+\infty)^{12}$. Then, we get

$$\begin{aligned}&\mathcal{N}\Big(\varpi(\mathscr{V})-\eth(\mathscr{V}),(\mathcal{T}_1,\cdots,\mathcal{T}_{12})\Big)\\\geq\ &\widehat{\text{Diagonal}}\Big[\mathsf{AG}_1\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathsf{D}_1}|}{\theta\frac{\mathcal{T}_1}{\ell_1}}\Big)\Big),\cdots,\mathsf{AG}_{12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathsf{D}_1}|}{\theta\frac{\mathcal{T}_{12}}{\ell_{12}}}\Big)\Big)\Big],\end{aligned}$$

and

$$\begin{aligned}&\mathcal{N}\Big(\eth(\mathscr{V})-\varpi'(\mathscr{V}),(\mathcal{T}_1,\cdots,\mathcal{T}_{12})\Big)\\\geq\ &\widehat{\text{Diagonal}}\Big[\mathsf{AG}_1\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathsf{D}_1}|}{\theta\frac{\mathcal{T}_1}{\jmath_1}}\Big)\Big),\cdots,\mathsf{AG}_{12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathsf{D}_1}|}{\theta\frac{\mathcal{T}_{12}}{\jmath_{12}}}\Big)\Big)\Big],\end{aligned}$$

for each $\underbrace{\mathcal{T}_i}_{i=1,\cdots,12} \in (0,+\infty)$, and so we obtain

$$\begin{aligned}
&\mathscr{N}\Big(\varpi(\mathscr{V})-\varpi'(\mathscr{V}),\Big((\ell_1+\jmath_1)\mathcal{T}_1,\cdots,(\ell_{12}+\jmath_{12})\mathcal{T}_{12}\Big)\Big)\\
\geq\ &\mathscr{N}\Big(\varpi(\mathscr{V})-\eth(\mathscr{V}),(\ell_1\mathcal{T}_1,\cdots,\ell_{12}\mathcal{T}_{12})\Big)\bigotimes_{\mathtt{GTN}}\mathscr{N}\Big(\eth(\mathscr{V})-\varpi'(\mathscr{V}),(\jmath_1\mathcal{T}_1,\cdots,\jmath_{12}\mathcal{T}_{12})\Big)\\
\geq\ &\widehat{\text{Diagonal}}\Big[\mathsf{AG}_1\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\mathcal{T}_1}\Big)\Big),\cdots,\mathsf{AG}_{12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\mathcal{T}_{12}}\Big)\Big)\Big]\\
&\bigotimes_{\mathtt{GTN}}\widehat{\text{Diagonal}}\Big[\mathsf{AG}_1\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\mathcal{T}_1}\Big)\Big),\cdots,\mathsf{AG}_{12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\mathcal{T}_{12}}\Big)\Big)\Big]\\
\geq\ &\widehat{\text{Diagonal}}\Big[\mathsf{AG}_1\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\mathcal{T}_1}\Big)\Big),\cdots,\mathsf{AG}_{12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\mathcal{T}_{12}}\Big)\Big)\Big],
\end{aligned}$$

and so $E(\varpi,\varpi')\leq(\ell_1+\jmath_1,\cdots,\ell_{12}+\jmath_{12})$. Therefore, $E(\varpi,\varpi')\leq E(\varpi,\eth)+E(\eth,\varpi')$.

Now, we will prove $(\curlyvee,E)$ is complete. Suppose $\{\varpi_k\}_k$ is a Cauchy sequence in $(\curlyvee,E)$, $\mathscr{V}\in[0,\mathtt{T}]$ is fixed, $\underbrace{\sigma_i}_{i=1,\cdots,12}\in(0,+\infty)$ and $\underbrace{\ni_i}_{i=1,\cdots,12}\in(0,1)$, $k\in\mathbb{N}$, and $\underbrace{\mathcal{T}_i}_{i=1,\cdots,12}\in(0,+\infty)$ s.t.

$$\begin{aligned}
&\widehat{\text{Diagonal}}\Big[\mathsf{AG}_1\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\mathcal{T}_1}\Big)\Big),\cdots,\mathsf{AG}_{12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\mathcal{T}_{12}}\Big)\Big)\Big]\\
\succ\ &\widehat{\text{Diagonal}}\Big[1-\ni_1,\cdots,1-\ni_{12}\Big].
\end{aligned}$$

For $\underbrace{\varepsilon_i\mathcal{T}_i}_{i=1,\cdots,12} < \underbrace{\sigma_i}_{i=1,\cdots,12}$ choose $k''\in\mathbb{N}$ s.t.

$$E(\varpi_k,\varpi_{k'})<(\varepsilon_1,\cdots,\varepsilon_{12}),\quad\forall\, k,k'\geq k''.$$

Hence,

$$\begin{aligned}
&\mathscr{N}\Big(\varpi_k(\mathscr{V})-\varpi'_{k'}(\mathscr{V}),(\sigma_1,\cdots,\sigma_n)\Big)\\
\geq\ &\mathscr{N}\Big(\varpi_k(\mathscr{V})-\varpi'_{k'}(\mathscr{V}),(\varepsilon_1\mathcal{T}_1,\cdots,\varepsilon_n\mathcal{T}_n)\Big)\\
\geq\ &\widehat{\text{Diagonal}}\Big[\mathsf{AG}_1\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\mathcal{T}_1}\Big)\Big),\cdots,\mathsf{AG}_{12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\mathcal{T}_{12}}\Big)\Big)\Big]\\
\succ\ &\widehat{\text{Diagonal}}\underbrace{[1-\ni_1,\cdots,1-\ni_{12}]}_{12}.
\end{aligned}$$

Thus,

$$\mathscr{N}\Big(\varpi_k(\mathscr{V})-\varpi_{k'}(\mathscr{V}),(\sigma_1,\cdots,\sigma_{12})\Big)\succ\widehat{\text{Diagonal}}\underbrace{[1-\ni_1,\cdots,1-\ni_{12}]}_{12}.$$

Thus, i.e., the sequence $\{\varpi_k(\mathscr{V})\}_k$ is Cauchy in the complete space $(\xi,\mathscr{N},\bigotimes_{\mathtt{GTN}})$ on the compact set $[0,\mathtt{T}]$, so uniformly convergent to the mapping $\varpi\in C_{1-\mathtt{D}_3;\mathscr{C}}(0,\mathtt{T}]$. Thus, $(\curlyvee,E)$ is complete.

From Lemma 2.2 we get (2.1) and (2.2) are equivalent to the system below:

$$\begin{aligned}\psi(\mathscr{V}) = {} & \frac{(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_3-1}}{\Gamma(\mathtt{D}_3)} f_0 \\ & + \frac{1}{\Gamma(\mathtt{D}_1)} \int_0^{\mathscr{V}} \mathscr{C}'(\mathscr{V}_\circ)(\mathscr{C}(\mathscr{V})-\mathscr{C}(\mathscr{V}_\circ))^{\mathtt{D}_1-1} \varrho(\mathscr{V}_\circ, \psi(\mathscr{V}_\circ), \psi(\rho(\mathscr{V}_\circ))) d\mathscr{V}_\circ, \quad \mathscr{V} \in (0, \mathtt{T}]. \end{aligned} \tag{2.7}$$

Taking $I_{0^+}^{\mathtt{D}_1;\mathscr{C}}(.)$ on both sides of (1.1) and using Theorem 1.4, we obtain (2.7). Also, if ψ satisfies (2.7), then ψ satisfies (1.1). However, taking ${}^{\mathscr{H}}D_{0^+}^{\mathtt{D}_1,\mathtt{D}_2;\mathscr{C}}(.)$ on both sides of (2.7), we get

$${}^{\mathscr{H}}D_{0^+}^{\mathtt{D}_1,\mathtt{D}_2;\mathscr{C}} \psi(\mathscr{V}) = {}^{\mathscr{H}}D_{0^+}^{\mathtt{D}_1,\mathtt{D}_1;\mathscr{C}} \Big[\frac{(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_3-1}}{\Gamma(\mathtt{D}_3)} f_0 \Big] + {}^{\mathscr{H}}D_{0^+}^{\mathtt{D}_1,\mathtt{D}_2;\mathscr{C}} I_{0^+}^{\mathtt{D}_1;\mathscr{C}} \varrho(\mathscr{V}, \psi(\mathscr{V}), \psi(\rho(\mathscr{V}))).$$

From Theorem 1.3 and

$${}^{\mathscr{H}}D_{0^+}^{\mathtt{D}_1,\mathtt{D}_2;\mathscr{C}} (\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_3-1} = 0, \quad 0 < \mathtt{D}_3 < 1,$$

we infer that, $\psi(\mathscr{V})$ satisfies the problem Eq (1.1) iff, $\psi(\mathscr{V})$ satisfies (2.7).

Suppose $\leftthreetimes := \curlyvee \longrightarrow \curlyvee$ s.t.

$$\begin{aligned}\leftthreetimes(\pitchfork(\mathscr{V})) = {} & \frac{(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_3-1}}{\Gamma(\mathtt{D}_3)} f_0 \\ & + \frac{1}{\Gamma(\mathtt{D}_1)} \int_0^{\mathscr{V}} \mathscr{C}'(\mathscr{V}_\circ)(\mathscr{C}(\mathscr{V})-\mathscr{C}(\mathscr{V}_\circ))^{\mathtt{D}_1-1} \varrho(\mathscr{V}_\circ, \psi(\mathscr{V}_\circ), \psi(\rho(\mathscr{V}_\circ))) d\mathscr{V}_\circ, \quad \mathscr{V} \in (0, \mathtt{T}]. \end{aligned} \tag{2.8}$$

Note that if $\pitchfork \in C_{1-\mathtt{D}_3;\mathscr{C}}[0,\mathtt{T}]$, thus $\leftthreetimes \pitchfork \in C_{1-\mathtt{D}_3;\mathscr{C}}[0,\mathtt{T}]$. Indeed,

$$\begin{aligned} & \mathscr{N}\Big(\leftthreetimes(\pitchfork(\mathscr{V})) - \leftthreetimes(\pitchfork(\mathscr{V}_0)), (\mathscr{T}_1, \cdots, \mathscr{T}_{12}) \Big) \\ = {} & \mathscr{N}\Big(\frac{(\boldsymbol{\varpi}(\mathscr{V})-\boldsymbol{\varpi}(0))^{\mathtt{D}_3-1}}{\boldsymbol{\Gamma}(\mathtt{D}_3)} f_0 \\ & + \frac{1}{\Gamma(\mathtt{D}_1)} \int_0^{\mathscr{V}} \psi'(\mathscr{V}_\circ)(\psi(\mathscr{V})-\psi(\mathscr{V}_\circ))^{\mathtt{D}_1-1} \varrho(\mathscr{V}_\circ, \psi(\mathscr{V}_\circ), \psi(\rho(\mathscr{V}_\circ))) d\mathscr{V}_\circ \\ & - \frac{(\boldsymbol{\varpi}(\mathscr{V}_0)-\boldsymbol{\varpi}(0))^{\mathtt{D}_3-1}}{\Gamma(\mathtt{D}_3)} f_0 \\ & - \frac{1}{\Gamma(\mathtt{D}_1)} \int_0^{\mathscr{V}} \psi'(\mathscr{V}_\circ)(\psi(\mathscr{V}_0)-\psi(\mathscr{V}_\circ))^{\mathtt{D}_1-1} \varrho(\mathscr{V}_\circ, \psi(\mathscr{V}_\circ), \psi(\rho(\mathscr{V}_\circ))) d\mathscr{V}_\circ, (\mathscr{T}_1, \cdots, \mathscr{T}_{12}) \Big) \\ \longrightarrow {} & \mathbf{1} \end{aligned}$$

as $\mathscr{V} \longrightarrow \mathscr{V}_0$. We now prove the self-mapping $\leftthreetimes$ is a contraction on $\curlyvee$. Consider $\leftthreetimes : \curlyvee \longrightarrow \curlyvee$ given in (2.8), $\pitchfork, \psi \in C[0,\mathtt{T}]$, $\underbrace{\Bbbk_i}_{i=1,\cdots,12} \in [0, +\infty]$, and $E(\pitchfork(\mathscr{V}), \psi(\mathscr{V})) \leq (\Bbbk_1, \cdots, \Bbbk_{12})$. Thus for each $\mathscr{V} \in [0,\mathtt{T}]$,

$$\begin{aligned} & \mathscr{N}\Big(\pitchfork(\mathscr{V}) - \psi(\mathscr{V}), (\mathscr{T}_1, \cdots, \mathscr{T}_{12}) \Big) \\ \geq {} & \widehat{\text{Diagonal}}\Big[\mathtt{AG}_1\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\theta \frac{\mathscr{T}_1}{\Bbbk_1}} \Big)\Big), \cdots, \mathtt{AG}_{12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\theta \frac{\mathscr{T}_{12}}{\Bbbk_{12}}} \Big)\Big) \Big]. \end{aligned}$$

For each $\mathscr{V} \in (0, \mathtt{T}]$ and $\underbrace{\mathcal{T}_i}_{i=1,\cdots,12} \in (0, +\infty)$, we get

$$
\begin{aligned}
&\mathscr{N}\Big(\leftthreetimes (\pitchfork (\mathscr{V})) - \leftthreetimes(\psi(\mathscr{V})), (\mathcal{T}_1, \cdots, \mathcal{T}_{12})\Big) \\
\geq\ & \mathscr{N}\Big(\frac{1}{\Gamma(\mathtt{D}_1)} \int_0^{\mathscr{V}} \mathscr{C}'(\mathscr{V}_\circ)(\mathscr{C}(\mathscr{V}) - \mathscr{C}(\mathscr{V}_\circ))^{\mathtt{D}_1-1}\Big(\varrho(\mathscr{V}_\circ, \pitchfork (\mathscr{V}_\circ), \pitchfork (\rho(\mathscr{V}_\circ))) \\
& -\varrho(\mathscr{V}_\circ, \psi(\mathscr{V}_\circ), \psi(\rho(\mathscr{V}_\circ)))\Big) d\mathscr{V}_\circ, (\mathcal{T}_1, \cdots, \mathcal{T}_{12})\Big) \\
\geq\ & \mathscr{N}\Big(\frac{\Delta}{\Gamma(\mathtt{D}_1)} \int_0^{\mathscr{V}} \mathscr{C}'(\mathscr{V}_\circ)(\mathscr{C}(\mathscr{V}) - \mathscr{C}(\mathscr{V}_\circ))^{\mathtt{D}_1-1} \\
& \Big[\Big(\pitchfork (\mathscr{V}_\circ) - \psi(\mathscr{V}_\circ)\Big) + \Big(\pitchfork (\rho(\mathscr{V}_\circ)) - \psi(\rho(\mathscr{V}_\circ))\Big)\Big] d\mathscr{V}_\circ, (\mathcal{T}_1, \cdots, \mathcal{T}_{12})\Big) \\
\geq\ & \mathscr{N}\Big(\frac{\Delta}{\Gamma(\mathtt{D}_1)} \int_0^{\mathscr{V}} \mathscr{C}'(\mathscr{V}_\circ)(\mathscr{C}(\mathscr{V}) - \mathscr{C}(\mathscr{V}_\circ))^{\mathtt{D}_1-1}\Big(\pitchfork (\mathscr{V}_\circ) - \psi(\mathscr{V}_\circ)\Big) d\mathscr{V}_\circ, \frac{(\mathcal{T}_1, \cdots, \mathcal{T}_{12})}{2}\Big) \\
& \bigotimes_{\mathtt{GTN}} \mathscr{N}\Big(\frac{\Delta}{\Gamma(\mathtt{D}_1)} \int_0^{\mathscr{V}} \mathscr{C}'(\mathscr{V}_\circ)(\mathscr{C}(\mathscr{V}) - \mathscr{C}(\mathscr{V}_\circ))^{\mathtt{D}_1-1}\Big(\pitchfork (\rho(\mathscr{V}_\circ)) - \psi(\rho(\mathscr{V}_\circ))\Big) d\mathscr{V}_\circ, \frac{(\mathcal{T}_1, \cdots, \mathcal{T}_{12})}{2}\Big) \\
\geq\ & \mathscr{N}\Big(\frac{\Delta}{\Gamma(\mathtt{D}_1)} \int_0^{\mathscr{V}} \mathscr{C}'(\mathscr{V}_\circ)(\mathscr{C}(\mathscr{V}) - \mathscr{C}(\mathscr{V}_\circ))^{\mathtt{D}_1-1}\Big(\pitchfork (\mathscr{V}_\circ) - \psi(\mathscr{V}_\circ)\Big) d\mathscr{V}_\circ, \frac{(\mathcal{T}_1, \cdots, \mathcal{T}_{12})}{2}\Big) \\
& \bigotimes_{\mathtt{GTN}} \mathscr{N}\Big(\frac{\Delta}{\Gamma(\mathtt{D}_1)} \int_0^{\mathscr{V}} \mathscr{C}'(\mathscr{V}_\circ)(\mathscr{C}(\mathscr{V}) - \mathscr{C}(\mathscr{V}_\circ))^{\mathtt{D}_1-1}\Big(\pitchfork (\mathscr{V}_\circ) - \psi(\mathscr{V}_\circ)\Big) d\mathscr{V}_\circ, \frac{(\mathcal{T}_1, \cdots, \mathcal{T}_{12})}{2}\Big) \\
\geq\ & \mathscr{N}\Big(\frac{\Delta}{\Gamma(\mathtt{D}_1)} \int_0^{\mathscr{V}} \mathscr{C}'(\mathscr{V}_\circ)(\mathscr{C}(\mathscr{V}) - \mathscr{C}(\mathscr{V}_\circ))^{\mathtt{D}_1-1}\Big(\pitchfork (\mathscr{V}_\circ) - \psi(\mathscr{V}_\circ)\Big) d\mathscr{V}_\circ, \frac{(\mathcal{T}_1, \cdots, \mathcal{T}_{12})}{2}\Big) \\
\geq\ & \frac{1}{\Gamma(\mathtt{D}_1)} \int_0^{\mathscr{V}} \mathscr{C}'(\mathscr{V}_\circ)(\mathscr{C}(\mathscr{V}) - \mathscr{C}(\mathscr{V}_\circ))^{\mathtt{D}_1-1} \\
& \times \widehat{\text{Diagonal}}\Big[\mathtt{AG}_1\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V}) - \mathscr{C}(0))^{\mathtt{D}_1}|}{2\Delta\theta\frac{\mathcal{T}_1}{\Bbbk_1}}\Big)\Big), \cdots, \mathtt{AG}_{12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V}) - \mathscr{C}(0))^{\mathtt{D}_1}|}{2\Delta\theta\frac{\mathcal{T}_{12}}{\Bbbk_{12}}}\Big)\Big)\Big] d\mathscr{V}_\circ \\
\geq\ & \widehat{\text{Diagonal}}\Big[\mathtt{AG}_1\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V}) - \mathscr{C}(0))^{\mathtt{D}_1}|}{2\Delta\mathcal{M}_1\theta\frac{\mathcal{T}_1}{\Bbbk_1}}\Big)\Big), \cdots, \mathtt{AG}_{12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V}) - \mathscr{C}(0))^{\mathtt{D}_1}|}{2\Delta\mathcal{M}_{12}\theta\frac{\mathcal{T}_{12}}{\Bbbk_{12}}}\Big)\Big)\Big],
\end{aligned}
$$

so we conclude that

$$
E\Big(\leftthreetimes (\pitchfork (\mathscr{V})) - \leftthreetimes(\psi(\mathscr{V})), (\mathcal{T}_1, \cdots, \mathcal{T}_{12})\Big) \ \leq\ \Big(\frac{\Bbbk_1}{2\Delta\mathcal{M}_1}, \cdots, \frac{\Bbbk_{12}}{2\Delta\mathcal{M}_{12}}\Big),
$$

and so

$$
\begin{aligned}
& E\Big(\leftthreetimes (\pitchfork (\mathscr{V})) - \leftthreetimes(\psi(\mathscr{V})), (\mathcal{T}_1, \cdots, \mathcal{T}_{12})\Big) \\
\leq\ & \Big(\frac{1}{2\Delta\mathcal{M}_1}, \cdots, \frac{1}{2\Delta\mathcal{M}_{12}}\Big) E\Big(\pitchfork (\mathscr{V}) - \psi(\mathscr{V}), (\mathcal{T}_1, \cdots, \mathcal{T}_{12})\Big),
\end{aligned}
$$

which gives the contractively property of $\leftthreetimes$, since $2\Delta \underbrace{\mathcal{M}_i}_{i=1,\cdots,12} < 1$.

Suppose $\upsilon \in \curlyvee$. We now prove $E(\leftthreetimes\upsilon, \upsilon) < \underbrace{(\infty, \cdots, \infty)}_{12}$. Using (2.3) and Remark 2.3, we have

$$\begin{aligned}&\mathscr{N}\Big(\upsilon(\mathscr{V}) - \frac{(\mathscr{C}(\mathscr{V}) - \mathscr{C}(0))^{\mathtt{D}_3-1}}{\Gamma(\mathtt{D}_3)} f_0 \\ &-\frac{1}{\Gamma(\mathtt{D}_1)}\int_0^{\mathscr{V}} \mathscr{C}'(\mathscr{V}_\circ)(\mathscr{C}(\mathscr{V}) - \mathscr{C}(\mathscr{V}_\circ))^{\mathtt{D}_1-1}\varrho(\mathscr{V}_\circ, \upsilon(\mathscr{V}_\circ), \upsilon(\rho(\mathscr{V}_\circ)))d\mathscr{V}_\circ, (\mathcal{T}_1, \cdots, \mathcal{T}_n)\Big) \\ \geq\ & \widehat{\text{Diagonal}}\Big[\mathtt{AG}_1\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V}) - \mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\mathcal{T}_1}\Big)\Big), \cdots, \mathtt{AG}_{12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V}) - \mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\mathcal{T}_{12}}\Big)\Big)\Big],\end{aligned}$$

for each $\underbrace{\mathcal{T}_i}_{i=1,\cdots,12} \in (0, +\infty)$ and $\mathscr{V} \in (0, \mathtt{T}]$, in which $I_{0^+}^{1-\mathtt{D}_3;\mathscr{C}}\upsilon(0^+) = f_0 \in \mathbb{R}$. Therefore, we get $E(\leftthreetimes\upsilon, \upsilon) < \underbrace{(1, \cdots, 1)}_{12}$.

Making use of Theorem 1.10, we obtain an element $\Upsilon \in \curlyvee$ which satisfies the following:

♠ Υ is a fixed point of $\leftthreetimes$, i.e.,

$$\begin{aligned}\leftthreetimes(\Upsilon(\mathscr{V})) = &\frac{(\mathscr{C}(\mathscr{V}) - \mathscr{C}(0))^{\mathtt{D}_3-1}}{\Gamma(\mathtt{D}_3)} f_0 \\ &+\frac{1}{\Gamma(\mathtt{D}_1)}\int_0^{\mathscr{V}} \mathscr{C}'(\mathscr{V}_\circ)(\mathscr{C}(\mathscr{V}) - \mathscr{C}(\mathscr{V}_\circ))^{\mathtt{D}_1-1}\varrho(\mathscr{V}_\circ, \Upsilon(\mathscr{V}_\circ), \Upsilon(\rho(\mathscr{V}_\circ)))d\mathscr{V}_\circ, \quad \mathscr{V} \in (0, \mathtt{T}],\end{aligned}$$

which is unique in the set

$$\curlyvee^* = \{\phi \in \curlyvee : E(\leftthreetimes\upsilon, \phi) < \underbrace{(\infty, \cdots, \infty)}_{12}\},$$

in which $I_{0^+}^{1-\mathtt{D}_3;\mathscr{C}}\Upsilon(0^+) = f_0 \in \mathbb{R}$.

♠ $E(\leftthreetimes^n(\upsilon), \Upsilon) \to \underbrace{(0, \cdots, 0)}_{12}$ as $n \to \infty$;

♠ $E(\upsilon, \Upsilon) \leq \underbrace{(\frac{1}{1-2\Delta\mathcal{M}_1}, \cdots, \frac{1}{1-2\Delta\mathcal{M}_{12}})}_{12} E(\leftthreetimes\upsilon, \upsilon) \leq \underbrace{(\frac{1}{1-2\Delta\mathcal{M}_1}, \cdots, \frac{1}{1-2\Delta\mathcal{M}_{12}})}_{12}$, which gives

$$\begin{aligned}&\mathscr{N}\Big(\upsilon(\mathscr{V}) - \Upsilon(\mathscr{V}), (\mathcal{T}_1, \cdots, \mathcal{T}_{12})\Big) \\ \geq\ & \widehat{\text{Diagonal}}\Big[\mathtt{AG}_1\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V}) - \mathscr{C}(0))^{\mathtt{D}_1}|}{\mathcal{D}_1\,\theta\mathcal{T}_1}\Big)\Big), \cdots, \mathtt{AG}_{12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V}) - \mathscr{C}(0))^{\mathtt{D}_1}|}{\mathcal{D}_{12}\,\theta\mathcal{T}_{12}}\Big)\Big)\Big],\end{aligned}$$

in which

$$\underbrace{\mathcal{D}_i}_{i=1,\cdots,12} := \frac{1}{1-2\Delta\underbrace{\mathcal{M}_i}_{i=1,\cdots,12}},$$

for each $\mathscr{V} \in (0, \mathtt{T}]$ and $\underbrace{\mathcal{T}_i}_{i=1,\cdots,12} \in (0, +\infty)$.

We prove the fixed point in $\curlyvee^*$ is unique. Suppose $\wp$ is an element of $\curlyvee$ that satisfies (2.4) and (2.5). We now show $\wp = \Upsilon$ and $\wp \in \curlyvee^*$. In view of (2.4) we have

$$\begin{aligned}\wp(\mathscr{V}) = &\frac{(\mathscr{C}(\mathscr{V}) - \mathscr{C}(0))^{\mathtt{D}_3-1}}{\Gamma(\mathtt{D}_3)} f_0 \\ &+\frac{1}{\Gamma(\mathtt{D}_1)} \int_0^{\mathscr{V}} \mathscr{C}'(\mathscr{V}_\circ)(\mathscr{C}(\mathscr{V}) - \mathscr{C}(\mathscr{V}_\circ))^{\mathtt{D}_1-1} \varrho(\mathscr{V}_\circ, \wp(\mathscr{V}_\circ), \wp(\rho(\mathscr{V}_\circ)))d\mathscr{V}_\circ, \quad \mathscr{V} \in (0, \mathtt{T}],\end{aligned} \tag{2.9}$$

and so

$$\begin{aligned}\leftthreetimes(\wp(\mathscr{V})) = &\frac{(\mathscr{C}(\mathscr{V}) - \mathscr{C}(0))^{\mathtt{D}_3-1}}{\Gamma(\mathtt{D}_3)} f_0 \\ &+\frac{1}{\Gamma(\mathtt{D}_1)} \int_0^{\mathscr{V}} \mathscr{C}'(\mathscr{V}_\circ)(\mathscr{C}(\mathscr{V}) - \mathscr{C}(\mathscr{V}_\circ))^{\mathtt{D}_1-1} \varrho(\mathscr{V}_\circ, \wp(\mathscr{V}_\circ), \wp(\rho(\mathscr{V}_\circ)))d\mathscr{V}_\circ, \quad \mathscr{V} \in (0, \mathtt{T}],\end{aligned} \tag{2.10}$$

where $\varrho \in C((0, \mathtt{T}] \times \mathbb{R}^2, \mathbb{R}), \rho \in C([0, \mathtt{T}], [0, \mathtt{T}])$, and $I_{0^+}^{1-\mathtt{D}_3;\mathscr{C}} \wp(0^+) = f_0 \in \mathbb{R}$.

We prove

$$\wp \in \{\phi \in \curlyvee : E(\leftthreetimes(\upsilon), \phi) < \underbrace{(\infty, \cdots, \infty)}_{12}\},$$

i.e., $E(\leftthreetimes(\upsilon), \wp) < \underbrace{(\infty, \cdots, \infty)}_{12}$. From (2.5) we get

$$\begin{aligned}&\mathscr{N}\Big(\upsilon(\mathscr{V}) - \wp(\mathscr{V}), (\mathcal{T}_1, \cdots, \mathcal{T}_{12})\Big) \\ \geq\ &\widehat{\text{Diagonal}}\Big[\mathtt{AG}_1\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V}) - \mathscr{C}(0))^{\mathtt{D}_1}|}{\mathcal{D}_1\, \theta\mathcal{T}_1}\Big)\Big), \cdots, \mathtt{AG}_{12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V}) - \mathscr{C}(0))^{\mathtt{D}_1}|}{\mathcal{D}_{12}\, \theta\mathcal{T}_{12}}\Big)\Big)\Big],\end{aligned} \tag{2.11}$$

in which

$$\underbrace{\mathcal{D}_i}_{i=1,\cdots,12} := \frac{1}{1 - 2\Delta \underbrace{\mathcal{M}_i}_{i=1,\cdots,12}}.$$

for each $\mathscr{V} \in (0, \mathtt{T}]$ and $\underbrace{\mathcal{T}_i}_{i=1,\cdots,12} \in (0, +\infty)$.

From the triangle inequality, (2.10), (2.11), (2.3) and Remark 2.3, we get

$$\begin{aligned}&\mathscr{N}\Big(\leftthreetimes(\upsilon(\mathscr{V})) - \wp(\mathscr{V}), (\mathcal{T}_1, \cdots, \mathcal{T}_{12})\Big) \\ =\ &\mathscr{N}\Big(\frac{(\mathscr{C}(\mathscr{V}) - \mathscr{C}(0))^{\mathtt{D}_3-1}}{\Gamma(\mathtt{D}_3)} f_0 \\ &+\frac{1}{\Gamma(\varsigma)} \int_0^{\mathscr{V}} \mathscr{C}'(\mathscr{V}_\circ)(\mathscr{C}(\mathscr{V}) - \mathscr{C}(\mathscr{V}_\circ))^{\mathtt{D}_1-1} \varrho(\mathscr{V}_\circ, \upsilon(\mathscr{V}_\circ), \upsilon(\rho(\mathscr{V}_\circ)))d\mathscr{V}_\circ \\ &+\upsilon(\mathscr{V}) - \upsilon(\mathscr{V}) - \wp(\mathscr{V}), (\mathcal{T}_1, \cdots, \mathcal{T}_{12})\Big) \\ \geq\ &\mathscr{N}\Big(\frac{(\mathscr{C}(\mathscr{V}) - \mathscr{C}(0))^{\mathtt{D}_3-1}}{\Gamma(\mathtt{D}_3)} f_0\end{aligned}$$

$$+\frac{1}{\Gamma(\mathtt{D}_3)}\int_0^{\mathscr{V}}\mathscr{C}'(\mathscr{V}_\circ)(\mathscr{C}(\mathscr{V})-\mathscr{C}(\mathscr{V}_\circ))^{\mathtt{D}_1-1}\varrho(\mathscr{V}_\circ,\upsilon(\mathscr{V}_\circ),\upsilon(\rho(\mathscr{V}_\circ)))d\mathscr{V}_\circ-\upsilon(\mathscr{V}),\frac{(\mathcal{T}_1,\cdots,\mathcal{T}_{12})}{2}\Big)$$
$$\bigotimes_{\mathtt{GTN}}\mathscr{N}\Big(\upsilon(\mathscr{V})-\wp(\mathscr{V}),\frac{(\mathcal{T}_1,\cdots,\mathcal{T}_{12})}{2}\Big)$$
$$\geq \widehat{\text{Diagonal}}\Big[\mathtt{AG}_1\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{2\theta\mathcal{T}_1}\Big)\Big),\cdots,\mathtt{AG}_{12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{2\theta\mathcal{T}_{12}}\Big)\Big)\Big]$$
$$\bigotimes_{\mathtt{GTN}}\widehat{\text{Diagonal}}\Big[\mathtt{AG}_1\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{2\mathcal{D}_1\,\theta\mathcal{T}_1}\Big)\Big),\cdots,\mathtt{AG}_{12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{2\mathcal{D}_{12}\,\theta\mathcal{T}_{12}}\Big)\Big)\Big]$$
$$\geq \widehat{\text{Diagonal}}\Big[\mathtt{AG}_1\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{2\max\{1,\mathcal{D}_1\}\theta\mathcal{T}_1}\Big)\Big),\cdots,\mathtt{AG}_{12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{2\max\{1,\mathcal{D}_{12}\}\theta\mathcal{T}_{12}}\Big)\Big)\Big],$$

for each $\mathscr{V}\in(0,\mathtt{T}]$ and $\underbrace{\mathcal{T}_i}_{i=1,\cdots,12}\in(0,+\infty)$.

We conclude $E(\curlywedge\upsilon,\wp)\leq\underbrace{(2\max\{1,\mathcal{D}_1\},\cdots,2\max\{1,\mathcal{D}_{12}\})}_{12}<\underbrace{(\infty,\cdots,\infty)}_{12}$, then $\wp\in\curlyvee^*$. □

3. Application

Example 3.1. *Suppose* $(\mathbb{R},\mathscr{N},\bigotimes_{\mathtt{GTN}})$ *is an MFB-space, and*

$$\begin{cases} {}^{\mathscr{H}}D_{0^+}^{\mathtt{D}_1,\mathtt{D}_2;\mathscr{C}}\pitchfork(\mathscr{V})=\sin^2(\pitchfork(\mathscr{V}))+\cos(\pitchfork(\frac{\mathscr{V}}{2}))+1, & \mathscr{V}\in(0,\mathtt{T}],\\ I_{0^+}^{1-\mathtt{D}_3;\mathscr{C}}\pitchfork(0^+)=f_0, & f_0\in\mathbb{R},\end{cases}$$

in which ${}^{\mathscr{H}}D_{0^+}^{\mathtt{D}_1,\mathtt{D}_2;\mathscr{C}}(.)$ *is the* $\mathscr{C}$*-Hilfer fractional derivative of order* $0<\mathtt{D}_1\leq1$ *and type* $0\leq\mathtt{D}_2\leq1$, $I_{0^+}^{1-\mathtt{D}_3;\mathscr{C}}(.)$ *is the fractional integral of order* $1-\mathtt{D}_3$, $\mathtt{D}_3=\mathtt{D}_1+\mathtt{D}_2(1-\mathtt{D}_1)$ *w.r.t the function* $\mathscr{C}$*, and* $\psi,\pitchfork\in C_{1-\mathtt{D}_3;\mathscr{C}}[0,\mathtt{T}]$ *are functions, s.t. for each* $\mathscr{V}\in[0,\mathtt{T}]$ *and* $\underbrace{\mathcal{T}_i}_{i=1,\cdots,12}\in(0,+\infty)$,

$$\mathscr{N}\Big(\pitchfork(\frac{1}{2}\mathscr{V})-\psi(\frac{1}{2}\mathscr{V}),(\mathcal{T}_1,\cdots,\mathcal{T}_{12})\Big)$$
$$\geq\mathscr{N}\Big(\pitchfork(\mathscr{V})-\psi(\mathscr{V}),(\mathcal{T}_1,\cdots,\mathcal{T}_{12})\Big),$$

and

$$\mathscr{N}\Big(\pitchfork(\mathscr{V})-\psi(\mathscr{V}),(\mathcal{T}_1,\cdots,\mathcal{T}_{12})\Big)$$
$$\geq\widehat{\mathit{Diagonal}}\Big[\mathtt{AG}_1\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\mathcal{T}_1}\Big)\Big),\cdots,\mathtt{AG}_{12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\mathcal{T}_{12}}\Big)\Big)\Big].$$

For each $\mathscr{V}\in[0,\mathtt{T}]$ *and* $\underbrace{\mathcal{T}_i}_{i=1,\cdots,12}\in(0,+\infty)$ *we get,*

$$\mathscr{N}\Big(\sin^2(\pitchfork(\mathscr{V}))-\sin^2(\psi(\mathscr{V})),(\mathcal{T}_1,\cdots,\mathcal{T}_{12})\Big)$$

$$
\begin{aligned}
&= \mathscr{N}\Big([\sin(\pitchfork(\mathscr{V})) - \sin(\psi(\mathscr{V}))][\sin(\pitchfork(\mathscr{V})) + \sin(\psi(\mathscr{V}))], (\mathcal{T}_1, \cdots, \mathcal{T}_{12})\Big)\\
&\geq \mathscr{N}\Big((\pitchfork(\mathscr{V}) - \psi(\mathscr{V}))\Big[2\max\Big(\sin(\pitchfork(\mathscr{V})), \sin(\psi(\mathscr{V}))\Big)\Big], (\mathcal{T}_1, \cdots, \mathcal{T}_{12})\Big)\\
&\geq \mathscr{N}\Big(\Delta_1(\pitchfork(\mathscr{V}) - \psi(\mathscr{V})), (\mathcal{T}_1, \cdots, \mathcal{T}_{12})\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathscr{N}\Big(\cos(\pitchfork(\frac{1}{2}\mathscr{V})) - \cos(\psi(\frac{1}{2}\mathscr{V})), (\mathcal{T}_1, \cdots, \mathcal{T}_{12})\Big)\\
&= \mathscr{N}\Big(2\sin\Big(\frac{\pitchfork(\frac{1}{2}\mathscr{V}) + \psi(\frac{1}{2}\mathscr{V})}{2}\Big)\sin\Big(\frac{\pitchfork(\frac{1}{2}\mathscr{V}) - \psi(\frac{1}{2}\mathscr{V})}{2}\Big), (\mathcal{T}_1, \cdots, \mathcal{T}_{12})\Big)\\
&\geq \mathscr{N}\Big(2\Delta_2\sin\Big(\frac{\pitchfork(\frac{1}{2}\mathscr{V}) - \psi(\frac{1}{2}\mathscr{V})}{2}\Big), (\mathcal{T}_1, \cdots, \mathcal{T}_{12})\Big)\\
&\geq \mathscr{N}\Big(2\Delta_2\frac{(\pitchfork(\frac{1}{2}\mathscr{V}) - \psi(\frac{1}{2}\mathscr{V}))}{2}, (\mathcal{T}_1, \cdots, \mathcal{T}_{12})\Big)\\
&\geq \mathscr{N}\Big(\Delta_2(\pitchfork(\frac{1}{2}\mathscr{V}) - \psi(\frac{1}{2}\mathscr{V})), (\mathcal{T}_1, \cdots, \mathcal{T}_{12})\Big)\\
&\geq \mathscr{N}\Big(\Delta_2(\pitchfork(\mathscr{V}) - \psi(\mathscr{V})), (\mathcal{T}_1, \cdots, \mathcal{T}_{12})\Big),
\end{aligned}
$$

in which $\Delta_1, \Delta_2 \in (0, +\infty)$.

Hence, for each $\mathscr{V} \in [0, \mathsf{T}]$ *and* $\underbrace{\mathcal{T}_i}_{i=1,\cdots,12} \in (0, +\infty)$ *we get,*

$$
\begin{aligned}
&\mathscr{N}\Big(\Big[\sin^2(\pitchfork(\mathscr{V})) + \cos(\pitchfork(\frac{1}{2}\mathscr{V})) + 1\Big]\\
&-\Big[\sin^2(\psi(\mathscr{V})) + \cos(\psi(\frac{1}{2}\mathscr{V})) + 1\Big], (\mathcal{T}_1, \cdots, \mathcal{T}_{12})\Big)\\
&\geq \mathscr{N}\Big(\Delta_1(\pitchfork(\mathscr{V}) - \psi(\mathscr{V})), \frac{(\mathcal{T}_1, \cdots, \mathcal{T}_{12})}{2}\Big)\\
&\bigotimes_{\mathtt{GTN}} \mathscr{N}\Big(\Delta_2(\pitchfork(\frac{1}{2}\mathscr{V}) - \psi(\frac{1}{2}\mathscr{V})), \frac{(\mathcal{T}_1, \cdots, \mathcal{T}_{12})}{2}\Big)\\
&\geq \mathscr{N}\Big(\Delta_0(\pitchfork(\mathscr{V}) - \psi(\mathscr{V})), (\mathcal{T}_1, \cdots, \mathcal{T}_{12})\Big),
\end{aligned}
$$

in which $\Delta_0 \in (0, +\infty)$.

Suppose $\Upsilon \in C_{1-\mathtt{D}_3;\mathscr{C}}[0, \mathsf{T}]$ *satisfies*

$$
\begin{aligned}
&\mathscr{N}\Big({}^{\mathscr{H}}D_{0^+}^{\mathtt{D}_1,\mathtt{D}_2;\mathscr{C}}\Upsilon(\mathscr{V}) - \sin^2(\Upsilon(\mathscr{V})) - \cos(\Upsilon(\frac{1}{2}\mathscr{V})) - 1, (\mathcal{T}_1, \cdots, \mathcal{T}_{12})\Big)\\
&\geq \widehat{Diagonal}\Big[\mathtt{AG}_1\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V}) - \mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\mathcal{T}_1}\Big)\Big), \cdots, \mathtt{AG}_{12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V}) - \mathscr{C}(0))^{\mathtt{D}_1}|}{\theta\mathcal{T}_{12}}\Big)\Big)\Big],
\end{aligned}
$$

for each $\mathscr{V} \in [0, \mathsf{T}]$ *and* $\underbrace{\mathcal{T}_i}_{i=1,\cdots,12} \in (0, +\infty)$. *Theorem 1.10 infers that, if* $2\Delta \underbrace{\mathcal{M}_i}_{i=1,\cdots,12} < 1$, *we obtain a unique function* $\wp \in C_{1-\mathsf{D}_3;\mathscr{C}}[0, \mathsf{T}]$ *s.t.*

$$ {}^{\mathscr{H}}D_{0^+}^{\mathsf{D}_1,\mathsf{D}_2;\mathscr{C}} \wp(\mathscr{V}) = \sin^2(\wp(\mathscr{V})) + \cos(\wp(\frac{1}{2}\mathscr{V})) + 1, \qquad \mathscr{V} \in [0, \mathsf{T}], $$

and

$$ \begin{aligned} &\mathscr{N}\Big(\wp(\mathscr{V}) - \Upsilon(\mathscr{V}), (\mathcal{T}_1, \cdots, \mathcal{T}_{12})\Big) \\ \geq\ & \widehat{Diagonal}\Big[\mathtt{AG}_1\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V}) - \mathscr{C}(0))^{\mathsf{D}_1}|}{\mathcal{D}_1\ \theta\mathcal{T}_1}\Big)\Big), \cdots, \mathtt{AG}_{12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V}) - \mathscr{C}(0))^{\mathsf{D}_1}|}{\mathcal{D}_{12}\ \theta\mathcal{T}_{12}}\Big)\Big)\Big], \end{aligned} $$

in which

$$ \underbrace{\mathcal{D}_i}_{i=1,\cdots,n} := \frac{1}{1 - 2\Delta \underbrace{\mathcal{M}_i}_{i=1,\cdots,12}}, $$

for each $\mathscr{V} \in [0, \mathsf{T}]$ *and* $\underbrace{\mathcal{T}_i}_{i=1,\cdots,12} \in (0, +\infty)$. *Making use of Figure 1, we get*

$$ \begin{aligned} & \widehat{Diagonal}\Big[\mathtt{AG}_1\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V}) - \mathscr{C}(0))^{\mathsf{D}_1}|}{\mathcal{D}_1\ \theta\mathcal{T}_1}\Big)\Big), \cdots, \mathtt{AG}_{12}\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V}) - \mathscr{C}(0))^{\mathsf{D}_1}|}{\mathcal{D}_{12}\ \theta\mathcal{T}_{12}}\Big)\Big)\Big] \\ \geq\ & \widehat{Diagonal}\Big[\mathtt{AG}_8\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V}) - \mathscr{C}(0))^{\mathsf{D}_1}|}{\mathcal{D}_1\ \theta\mathcal{T}_1}\Big)\Big), \cdots, \mathtt{AG}_8\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V}) - \mathscr{C}(0))^{\mathsf{D}_1}|}{\mathcal{D}_{12}\ \theta\mathcal{T}_{12}}\Big)\Big)\Big]. \end{aligned} $$

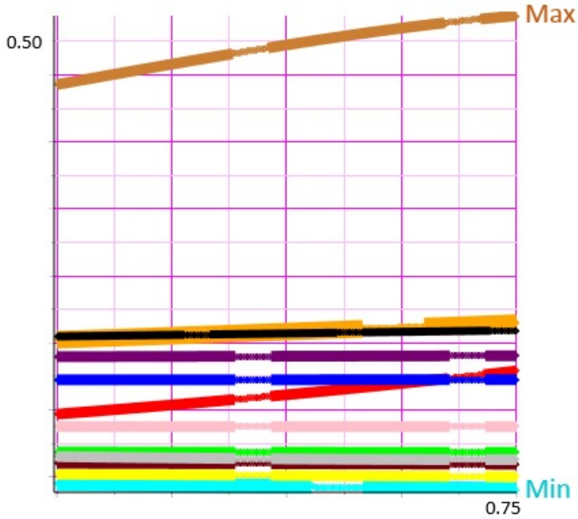

Figure 1. The plots of $\underbrace{\mathtt{AG}_i}_{1\leq i\leq 12}(\mathfrak{P})$ ($\mathtt{AG}_7(\mathfrak{P})$ and $\mathtt{AG}_8(\mathfrak{P})$ are shown in brown and cyan colors, respectively, and the rest are in between).

Now, based on Table 1, we obtain the following Mittag-Leffler stability result:

$$\begin{aligned} & \widehat{Diagonal}\Big[\mathtt{AG}_8\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\mathcal{D}_1\,\theta\mathcal{T}_1}\Big)\Big),\cdots,\mathtt{AG}_8\Big(\mathfrak{P}\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\mathcal{D}_{12}\,\theta\mathcal{T}_{12}}\Big)\Big)\Big] \\ \geq\; & \widehat{Diagonal}\Big[\ni_1\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\mathcal{D}_1\,\theta\mathcal{T}_1}\Big),\cdots,\ni_1\Big(\frac{-|(\mathscr{C}(\mathscr{V})-\mathscr{C}(0))^{\mathtt{D}_1}|}{\mathcal{D}_{12}\,\theta\mathcal{T}_{12}}\Big)\Big]. \end{aligned}$$

Table 1. The numerical results of the aggregation maps $\mathtt{AG}_7$ and $\mathtt{AG}_8$, on special functions $\ni_i,\ i=1,\ldots,5$.

	$\ni_1$	$\ni_2$	$\ni_3$	$\ni_4$	$\ni_5$
Min	0.00945	0.01492	0.03976	0.01831	0.04010
Max	0.42136	0.69421	0.89340	0.78392	0.90023

4. Conclusions

Our main goal of this article is to provide a new interpretation of Ulam type stability with the application of classical, well-known special functions and aggregation maps. This new notion of stability not only covers the previous notions but also considers the optimization of the problem. For the future research directions, we hope to replace Mittag-Leffler type functions with other classical special functions as the inputs of the n-ary aggregation maps.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare no conflict of interest.

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