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Research article

A simple proof of the refined sharp weighted Caffarelli-Kohn-Nirenberg inequalities

Steven Kendell, Nguyen Lam*, **Dylan Smith, Austin White and Parker Wiseman**

School of Science & Environment, Grenfell Campus, Memorial University of Newfoundland, Corner Brook, NL A2H5G4, Canada

* **Correspondence:** Email: nlam@grenfell.mun.ca.

Abstract: We provided a simple and direct proof of an improved version of the main results of the paper by Catrina and Costa (2009).

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1. Introduction

The main subject of this note is the Caffarelli-Kohn-Nirenberg inequalities of the form

$$\left(\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}} dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} dx\right)^{\frac{1}{2}} \geq C(N,a,b) \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} dx, \quad u \in C_0^\infty(\mathbb{R}^N \setminus \{0\}, \tag{1.1}$$

where $a, b \in \mathbb{R}$ are given constants. Clearly, the sharp constant in (1.1) was naturally defined by

$$C(N,a,b) := \inf_{u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})} \frac{\left(\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}} dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} dx\right)^{\frac{1}{2}}}{\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} dx}.$$

We note that (1.1) contains some important inequalities in the literature such as the Heisenberg uncertainty principle ($a = -1, \ b = 0$), the hydrogen uncertainty principle ($a = b = 0$), the Hardy inequalities ($a = 1, \ b = 0$), and more. It is also worth mentioning that (1.1) belongs to a more general family of Caffarelli-Kohn-Nirenberg inequalities that were first introduced and studied in [1,2] as tools to investigate well-posedness and regularity of solutions to certain Navier-Stokes equations. Because of their important roles in many areas of mathematics, the Caffarelli-Kohn-Nirenberg type inequalities and their applications have been investigated intensively and extensively in the literature. We refer the interested reader to [4, 5, 7, 9–21], to name just a few.

(1.1) was first investigated by Costa in [8] with $C(N,a,b) = \frac{|N-(a+b+1)|}{2}$ using the method of expanding the square. However, this constant is not optimal for certain parameter values. More precisely, if we introduce the following regions in the plane:

$$\begin{cases} \mathcal{A}_1 := \{(a,b) \mid b+1-a > 0,\ b \le (N-2)/2\}, \\ \mathcal{A}_2 := \{(a,b) \mid b+1-a < 0,\ b \ge (N-2)/2\}, \\ \mathcal{A} := \mathcal{A}_1 \cup \mathcal{A}_2, \\ \mathcal{B}_1 := \{(a,b) \mid b+1-a < 0,\ b \le (N-2)/2\}, \\ \mathcal{B}_2 := \{(a,b) \mid b+1-a > 0,\ b \ge (N-2)/2\}, \\ \mathcal{B} := \mathcal{B}_1 \cup \mathcal{B}_2, \end{cases}$$

then it is shown in [8] that in the region $\mathcal{A}$, the best constant is $C(N,a,b) = \frac{|N-(a+b+1)|}{2}$ and it is achieved by the functions $u(x) = D\exp(\frac{t|x|^{b+1-a}}{b+1-a})$, with $t < 0$ in $\mathcal{A}_1$ and $t > 0$ in $\mathcal{A}_2$, and D a nonzero constant. We note that the method in [8], which is very simple, does not lead to the optimal constant $C(N,a,b)$ in the region $\mathcal{B}$.

In [3], in order to obtain the sharpness of (1.1) in the region $\mathcal{B}$, Catrina and Costa used some advanced and technical tools such as the Emden-Fowler transformation, the spherical harmonics decomposition and the Kelvin-type transform. In particular, the main results in [3, 8] claim that

Theorem 1.1. *[3, Theorem 1], [8, Theorem 2.1] According to the location of the points (a,b) in the plane, we have:*

(a) In the region $\mathcal{A}$ the best constant is $C(N,a,b) = \frac{|N-(a+b+1)|}{2}$ and it is achieved by the functions $u(x) = D\exp(\frac{t|x|^{b+1-a}}{b+1-a})$, with $t < 0$ in $\mathcal{A}_1$ and $t > 0$ in $\mathcal{A}_2$, and D a nonzero constant.

(b) In the region $\mathcal{B}$ the best constant is $C(N,a,b) = \frac{|N-(3b-a+3)|}{2}$ and it is achieved by the functions $u(x) = D|x|^{2(b+1)-N}\exp(\frac{t|x|^{b+1-a}}{b+1-a})$, with $t > 0$ in $\mathcal{B}_1$ and $t < 0$ in $\mathcal{B}_2$.

(c) In addition, the only values of the parameters where the best constant is not achieved are those on the line $a = b+1$, where $C(N,b+1,b) = \frac{|N-2(b+1)|}{2}$.

Recently, the authors in [6] presented a direct proof to the aforementioned results using a clever way of expanding the square.

The principal goal of this note is to provide a new simple and straightforward method to derive the optimal constant of a refined version of (1.1). Our proof is based on the divergence theorem and is elementary.

2. Main result

The main result of our paper is the following sharp Caffarelli-Kohn-Nirenberg inequality:

Theorem 2.1. *Let $(a,b) \in \mathcal{A} \cup \mathcal{B}$. Then for all $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$, there holds*

$$\left(\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}} dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} dx\right)^{\frac{1}{2}} \geq \left(\int_{\mathbb{R}^N} \frac{|\frac{x}{|x|} \cdot \nabla u|^2}{|x|^{2b}} dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} dx\right)^{\frac{1}{2}}$$
$$\geq C(N,a,b) \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} dx. \tag{2.1}$$

Here

$$C(N,a,b) = \max\{\frac{|N-(a+b+1)|}{2}, \frac{|N-(3b-a+3)|}{2}\}.$$

Proof. We will first prove that for all $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$, we have

$$\int_{\mathbb{R}^N} \frac{|\frac{x}{|x|} \cdot \nabla u|^2}{|x|^{2b}} dx + \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} dx \geq 2C(N,a,b) \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} dx. \tag{2.2}$$

Indeed, let us first assume that $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ and u is real-valued. We note that $\operatorname{div}(|x|^\alpha x) = (N+\alpha)|x|^\alpha$. Therefore, by the divergence theorem and the AM-GM inequality, we have that

$$\begin{aligned}
&\pm\left[(N-a-b-1)\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} dx \ + \ \gamma(N-2b-2)\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2b+2}} dx\right] \\
&= \pm \int_{\mathbb{R}^N} \operatorname{div}\left(\frac{x}{|x|^{a+b+1}} + \gamma \frac{x}{|x|^{2b+2}}\right)|u|^2 dx \\
&= \mp \int_{\mathbb{R}^N} \left(\frac{x}{|x|^{a+b+1}} + \gamma \frac{x}{|x|^{2b+2}}\right) \cdot \nabla |u|^2 dx \\
&= \mp \int_{\mathbb{R}^N} 2u \left(\frac{1}{|x|^a} + \frac{\gamma}{|x|^{b+1}}\right)\left(\frac{x}{|x|} \cdot \nabla u\right) \frac{1}{|x|^b} dx \\
&\leq \int_{\mathbb{R}^N} \frac{|\frac{x}{|x|} \cdot \nabla u|^2}{|x|^{2b}} dx + \int_{\mathbb{R}^N} \left(\frac{1}{|x|^a} + \frac{\gamma}{|x|^{b+1}}\right)^2 |u|^2 dx \\
&= \int_{\mathbb{R}^N} \frac{|\frac{x}{|x|} \cdot \nabla u|^2}{|x|^{2b}} dx + \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} dx + 2\gamma \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} dx + \gamma^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2b+2}} dx.
\end{aligned}$$

This implies

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{|\frac{x}{|x|} \cdot \nabla u|^2}{|x|^{2b}} dx + \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} dx \\
&\geq (\pm(N-a-b-1) - 2\gamma) \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} dx + \left(\pm\gamma(N-2b-2) - \gamma^2\right) \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2b+2}} dx.
\end{aligned}$$

By choosing $\gamma^2 = \pm\gamma(N-2b-2)$, that is $\gamma = 0$ or $\gamma = \pm(N-2b-2)$, so that the last term vanishes, we obtain

$$\int_{\mathbb{R}^N} \frac{|\frac{x}{|x|} \cdot \nabla u|^2}{|x|^{2b}} dx + \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} dx \geq |N-a-b-1| \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} dx,$$

and

$$\int_{\mathbb{R}^N} \frac{|\frac{x}{|x|} \cdot \nabla u|^2}{|x|^{2b}} dx + \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} dx \geq |N-3b+a-3| \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} dx.$$

Therefore, we have

$$\int_{\mathbb{R}^N} \frac{|\frac{x}{|x|}\cdot\nabla u|^2}{|x|^{2b}}dx + \int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2a}}dx \geq 2C(N,a,b)\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{a+b+1}}dx,$$

with

$$C(N,a,b) = \max\{\frac{|N-(a+b+1)|}{2}, \frac{|N-(3b-a+3)|}{2}\}.$$

Now, if u is complex-valued, then by using the fact that $|\frac{x}{|x|}\cdot\nabla u| \geq |\frac{x}{|x|}\cdot\nabla|u||$ and noting that $|u|$ is real-valued, we get

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{|\frac{x}{|x|}\cdot\nabla u|^2}{|x|^{2b}}dx + \int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2a}}dx &\geq \int_{\mathbb{R}^N} \frac{|\frac{x}{|x|}\cdot\nabla |u||^2}{|x|^{2b}}dx + \int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2a}}dx \\ &\geq 2C(N,a,b)\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{a+b+1}}dx.\end{aligned}$$

Finally, by applying (2.2) for $u(x) \rightsquigarrow u(\lambda x)$, $u \in C_0^\infty(\mathbb{R}^N\setminus\{0\})\setminus\{0\}$, $\lambda > 0$, and making change of variables, we have

$$\lambda^{2b+2-N}\int_{\mathbb{R}^N} \frac{|\frac{x}{|x|}\cdot\nabla u|^2}{|x|^{2b}}dx + \lambda^{2a-N}\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2a}}dx \geq 2C(N,a,b)\lambda^{a+b+1-N}\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{a+b+1}}dx.$$

Equivalently,

$$\lambda^{b+1-a}\int_{\mathbb{R}^N} \frac{|\frac{x}{|x|}\cdot\nabla u|^2}{|x|^{2b}}dx + \lambda^{a-b-1}\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2a}}dx \geq 2C(N,a,b)\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{a+b+1}}dx.$$

Now, by choosing $\lambda = \left(\frac{\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2a}}dx}{\int_{\mathbb{R}^N}\frac{|\frac{x}{|x|}\cdot\nabla u|^2}{|x|^{2b}}dx}\right)^{\frac{1}{2(b+1-a)}}$, we obtain (2.1).

3. Conclusions

Caffarelli-Kohn-Nirenberg type inequalities generalize many well-known and important inequalities in analysis. Due to their important roles in many areas of mathematics, the optimal Caffarelli-Kohn-Nirenberg type inequalities and their applications have been extensively studied in many settings. In this paper, we provide a simple proof to a class of the Caffarelli-Kohn-Nirenberg type inequalities that contain the Heisenberg uncertainty principle, the hydrogen uncertainty principle, and the Hardy inequalities. Our proof is based on the divergence theorem and is elementary.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Professor Nguyen Lam is the guest editor of special issue "Functional, Geometric and Matrix Inequalities and Applications" for AIMS Mathematics. Professor Nguyen Lam was not involved in the editorial review nor the decision to publish this article.

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