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Research article

Differential subordination and superordination studies involving symmetric functions using a q-analogue multiplier operator

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Abstract: The present investigation focus on applying the theories of differential subordination, differential superordination and related sandwich-type results for the study of some subclasses of symmetric functions connected through a linear extended multiplier operator, which was previously defined by involving the q-Choi-Saigo-Srivastava operator. The aim of the paper is to define a new class of analytic functions using the aforementioned linear extended multiplier operator and to obtain sharp differential subordinations and superordinations using functions from the new class. Certain subclasses are highlighted by specializing the parameters involved in the class definition, and corollaries are obtained as implementations of those new results using particular values for the parameters of the new subclasses. In order to show how the results apply to the functions from the recently introduced subclasses, numerical examples are also provided.

Keywords: q-analogue of Choi-Saigo-Srivastava operator; symmetric function; Hadamard (convolution) product; differential subordination; differential superordination; sandwich-type result
Mathematics Subject Classification: 30C45, 30C80

1. Introduction

The original results obtained in this work are connected to the geometric function theory, and they were obtained using methods based on subordination and with the help of a q-calculus operator. The main notions that define the context of the research are first presented.

Let H be the class of analytic functions in the open unit disc $\mathbb{D} := \{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$.

A notable subclass of H is denoted by $H[a, n]$ and contains functions $f \in H$ of the form

$$f(\varsigma) = a + a_n\varsigma^n + a_{n+1}\varsigma^{n+1} + \dots. \quad (\varsigma \in \mathbb{D}).$$

Another remarkable subclass of H is denoted by $A(n)$ and consists of functions $f \in H$ of the form

$$f(\varsigma) = \varsigma + \sum_{\vartheta=n+1}^{\infty} a_\vartheta \varsigma^\vartheta, \ \varsigma \in \mathbb{D}, \tag{1.1}$$

with $n \in \mathbb{N} = \{1, 2, ...\}$ and written as $A = A(1)$.

The subclass of A represented by

$$K = \left\{ f \in A : \mathrm{Re}\left(\frac{f''(\varsigma)}{f'(\varsigma)} + 1\right) > 0, \ f(0) = 0, \ f'(0) \neq 0, \ \varsigma \in \mathbb{D} \right\},$$

denotes the class of convex functions in the unit disk $\mathbb{D}$.

The notion of subordination [1–3] is characterized by the following:

If f and $\hbar$ are analytic in $\mathbb{D}$, f is said to be *subordinate* to $\hbar$, denoted by $f(\varsigma) \prec \hbar(\varsigma)$, if there exists an analytic function ϖ; with $\varpi(0) = 0$ and $|\varpi(\varsigma)| < 1$ for all $\varsigma \in \mathbb{D}$, such that $f(\varsigma) = \hbar(\varpi(\varsigma))$. Moreover, if the function $\hbar$ is univalent in $\mathbb{D}$, then the following equivalence holds:

$$f(\varsigma) \prec \hbar(\varsigma) \Leftrightarrow f(0) = \hbar(0) \quad \text{and} \quad f(\Delta) \subset \hbar(\Delta).$$

For a function $f \in A(n)$ and a function $\hbar$ of the form

$$\hbar(\varsigma) = \varsigma + \sum_{\vartheta=n+1}^{\infty} b_\vartheta \varsigma^\vartheta, \ \varsigma \in \mathbb{D},$$

the well-known *convolution product* is defined as:

$$(f * \hbar)(\varsigma) := \varsigma + \sum_{\vartheta=n+1}^{\infty} a_\vartheta b_\vartheta \varsigma^\vartheta, \ \varsigma \in \mathbb{D}.$$

Many studies involving the q-derivative and the q-integral operators described by Jackson [4, 5] have emerged in recent years due to the multiple applications of those operators in various branches of mathematics and other related fields. A comprehensive review regarding the quantum calculus apects applied in the geometric function theory was done [6], and Kanas and Răducanu [7] presented the q-analogue of the Ruscheweyh differential operator and looked into some of its features by utilizing the concept of convolution. Aldweby and Darus [8], Mahmood and Sokol [9] and others analyzed many types of analytical functions defined by the q-analogue of the Ruscheweyh differential operator. Multivalent analytic functions were investigated involving the q-difference operator [10], and bi-univalent analytic functions are investigated under a similar operator in [11]. Analytic functions were investigated in a conic domain using q-calculus [12] and applications of the subordination concept and q-calculus were given [13–15]. The Faber polynomial expansion method was applied on bi-univalent functions using a q-integral operator [16], and the q-derivative linked Gegenbauer polynomials for certain bi-univalent functions [17]. Close-to-convex functions were investigated in association with q-Srivastava-Attiya in operator [18], an extended q-analogue of multiplier transformation was used for subordination and superordination studies [19] and q-analogue of the Choi-Saigo-Srivastava operator was associated for the study presented [20].

The pleasant results recently obtained by combining the aforementioned quantum calculus components into the geometric function theory are what inspired the introduction of the new findings in this study. We were motivated to further explore the q-analogue of the multiplier transformation after reading about its applications in the definition of new subclasses of univalent functions, as well as after taking into account recent findings involving another quantum calculus operator and the classical theories of differential subordination and superordination [21–23].

The studies presented above motivated the use of the linear extended multiplier operator, which was recently defined using certain quantum calculus means [24] for the investigations applying the theories of differential subordination and superordiantion presented in this paper connected to new classes of analytic functions.

The fundamental concepts of the q-calculus, created by Jackson [4] and relevant to our research, will now be discussed. This method can also be applied to higher dimensional domains.

Jackson [4, 5] defined the q-derivative operator D_q of a function f:

$$D_q f(\varsigma) := \partial_q f(\varsigma) = \frac{f(q\varsigma) - f(\varsigma)}{(q-1)\varsigma}, \quad (0 < q < 1,\ \varsigma \neq 0).$$

As a remark, for a function f written as (1.1), it implies

$$D_q f(\varsigma) = D_q\left(\varsigma + \sum_{\vartheta=n+1}^{\infty} a_\vartheta \varsigma^\vartheta\right) = 1 + \sum_{\vartheta=n+1}^{\infty} [\vartheta]_q a_\vartheta \varsigma^{\vartheta-1}, \tag{1.2}$$

where $[\vartheta]_q$ is the *q-bracket of* ϑ, that is

$$[\vartheta]_q := \frac{1-q^\vartheta}{1-q} = 1 + \sum_{\kappa=1}^{\vartheta-1} q^\kappa,\ [0]_q := 0,$$

and

$$\lim_{q\to 1^-} [\vartheta]_q = \vartheta.$$

The definition of the q-number shift factorial for every nonnegative integer ϑ is

$$[\vartheta, q]! := \begin{cases} 1, & \text{if } \ \vartheta = 0, \\ [1,q]\,[2,q]\,[3,q]\ldots\ldots[\vartheta,q]\,, & \text{if } \ \vartheta \in \mathbb{N}. \end{cases}$$

Wang et al. [20] the notion of the q-derivative and the concept of the convolution, the q-analogue Choi-Saigo-Srivastava operator $I^q_{\alpha,\beta} : A \to A$,

$$I^q_{\alpha,\beta} f(\varsigma) := f(\varsigma) * \mathcal{F}_{q,\alpha+1,\beta}(\varsigma),\ \varsigma \in \mathbb{D} \quad (\alpha > -1,\ \beta > 0), \tag{1.3}$$

where

$$\mathcal{F}_{q,\alpha+1,\beta}(\varsigma) = \varsigma + \sum_{\vartheta=2}^{\infty} \frac{\Gamma_q(\beta+\vartheta-1)\Gamma_q(\alpha+1)}{\Gamma_q(\beta)\Gamma_q(\alpha+\vartheta)} \varsigma^\vartheta = \varsigma + \sum_{\vartheta=2}^{\infty} \frac{[\beta,q]_{\vartheta-1}}{[\alpha+1,q]_{\vartheta-1}} \varsigma^\vartheta,\ \varsigma \in \mathbb{D},$$

where $[\beta, q]_\vartheta$ is the q-generalized Pochhammer symbol for $\beta > 0$ defined by

$$[\beta, q]_\vartheta := \begin{cases} 1, & \text{if} \quad \vartheta = 0, \\ [\beta]_q \ [\beta+1]_q \dots [\beta+\vartheta-1]_q\,, & \text{if} \quad \vartheta \in \mathbb{N}. \end{cases}$$

Thus,

$$I^q_{\alpha,\beta} f(\varsigma) = \varsigma + \sum_{\vartheta=2}^{\infty} \frac{[\beta, q]_{\vartheta-1}}{[\alpha+1, q]_{\vartheta-1}} a_\vartheta \varsigma^\vartheta, \ \varsigma \in \mathbb{D}, \tag{1.4}$$

while

$$I^q_{0,2} f(\varsigma) = \varsigma D_q f(\varsigma) \quad \text{and} \quad I^q_{1,2} f(\varsigma) = f(\varsigma).$$

In [24], an extended multiplier operator was defined applying the operator $I^q_{\alpha,\beta}$ as follows:

Definition 1. *[24] For $\mu \geq 0$ and $\tau > -1$, with the aid of the operator $I^q_{\alpha,\beta}$, we will define the new linear extended multiplier q-Choi-Saigo-Srivastava operator $D^{m,q}_{\alpha,\beta}(\mu,\tau) : A \to A$ as follows:*

$$\begin{aligned} D^{0,q}_{\alpha,\beta}(\mu,\tau) f(\varsigma) =&: D^q_{\alpha,\beta}(\mu,\tau) f(\varsigma) = f(\varsigma), \\ D^{1,q}_{\alpha,\beta}(\mu,\tau) f(\varsigma) =& \left(1 - \frac{\mu}{\tau+1}\right) I^q_{\alpha,\beta} f(\varsigma) + \frac{\mu}{\tau+1} \varsigma D_q \left(I^q_{\alpha,\beta} f(\varsigma)\right) \\ =& \varsigma + \sum_{\vartheta=2}^{\infty} \left(\frac{[\beta, q]_{\vartheta-1}}{[\alpha+1, q]_{\vartheta-1}} \cdot \frac{\tau + 1 + \mu\left([\vartheta]_q - 1\right)}{\tau+1} \right) a_\vartheta \varsigma^\vartheta, \\ \dots\dots&\dots\dots\dots\dots\dots\dots\dots\dots \\ D^{m,q}_{\alpha,\beta}(\mu,\tau) f(\varsigma) =& D^q_{\alpha,\beta}(\mu,\tau) \left(D^{m-1,q}_{\alpha,\beta}(\mu,\tau) f(\varsigma)\right), \ m \geq 1, \end{aligned}$$

where $\mu \geq 0$, $\tau > -1$, $m \in \mathbb{N}_0$, $\alpha > -1$, $\beta > 0$ and $0 < q < 1$.

If $f \in A$ has the form (1.1) from (1.4) and the above definition, it follows that

$$D^{m,q}_{\alpha,\beta}(\mu,\tau) f(\varsigma) = \varsigma + \sum_{\vartheta=2}^{\infty} \left(\frac{[\beta, q]_{\vartheta-1}}{[\alpha+1, q]_{\vartheta-1}} \cdot \frac{\tau + 1 + \mu\left([\vartheta]_q - 1\right)}{\tau+1} \right)^m a_\vartheta \varsigma^\vartheta, \ \varsigma \in \mathbb{D}. \tag{1.5}$$

From (1.3) and (1.5), we find that

$$D^{m,q}_{\alpha,\beta}(\mu,\tau) f(\varsigma) = \underbrace{\left[\left(I^q_{\alpha,\beta} f(\varsigma) * \wp^q_{\mu,\tau}(\varsigma)\right) * \dots * \left(I^q_{\alpha,\beta} f(\varsigma) * \wp^q_{\mu,\tau}(\varsigma)\right)\right]}_{j-\text{times}} * f(\varsigma),$$

where

$$\wp^q_{\mu,\tau}(\varsigma) := \frac{\varsigma - \left(1 - \frac{\mu}{\tau+1}\right) q \varsigma^2}{(1-\varsigma)(1-q\varsigma)}.$$

We note that

$$\lim_{q \to 1^-} D^{m,q}_{\alpha,\beta}(\mu,\tau) f(\varsigma) = \mathcal{L}^m_{\alpha,\beta}(\mu,\tau) f(\varsigma) = \varsigma + \sum_{\vartheta=2}^{\infty} \left(\frac{(\beta)_{\vartheta-1}}{(\alpha+1)_{\vartheta-1}} \cdot \frac{\tau + 1 + \mu\,(\vartheta - 1)}{\tau+1} \right)^m a_\vartheta \varsigma^\vartheta, \ \varsigma \in \mathbb{D}. \tag{1.6}$$

Assuming that $\lambda, \hbar \in H$, suppose

$$\Phi(r, s, t; \varsigma) : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}.$$

If λ satisfies the first order differential subordination

$$\Phi(\lambda(\varsigma), \varsigma\lambda^{'}(\varsigma); \varsigma) \prec \hbar(\varsigma), \tag{1.7}$$

then λ is called to be a solution of the differential subordination in (1.7). The function $\varkappa$ is called a dominant of the solutions of the differential subordination in (1.7) if $\lambda(\varsigma) \prec \varkappa(\varsigma)$ for all the functions λ satisfying (1.7). A dominant $\widetilde{\varkappa}$ is said to be the best dominant of (1.7) if $\widetilde{\varkappa}(\varsigma) \prec \varkappa(\varsigma)$ for all the dominants $\varkappa$.

If the following first order differential superordination is met by λ,

$$\hbar(\varsigma) \prec \Phi(\lambda(\varsigma), \varsigma\lambda^{'}(\varsigma); \varsigma), \tag{1.8}$$

then λ is called to be a solution of the differential superordination in (1.8). The function $\varkappa$ is called a subordinant of the solutions of the differential superordination in (1.8) if $\varkappa(\varsigma) \prec \lambda(\varsigma)$ for all the functions λ satisfying (1.8). A subordinant $\widetilde{\varkappa}$ is said to be the best subordinant of (1.8) if $\varkappa(\varsigma) \prec \widetilde{\varkappa}(\varsigma)$ for all the subordinants $\varkappa$.

Miller and Mocanu [25] obtained sufficient conditions on the functions $\hbar$, $\varkappa$ and Φ for which the following implication holds:

$$\hbar(\varsigma) \prec \Phi(\lambda(\varsigma), \varsigma\lambda^{'}(\varsigma); \varsigma) \Rightarrow \varkappa(\varsigma) \prec \lambda(\varsigma).$$

Using the results presented [25], Bulboacă [26] investigated several classes of first-order differential superordinations and also considered superordination preserving integral operators [27]. Ali et al. [28] developed on Bulboacă's results and obtained sufficient conditions for specific normalized analytic functions $f(\varsigma)$ to satisfy

$$\varkappa_1(\varsigma) \prec \frac{\varsigma f'(\varsigma)}{f(\varsigma)} \prec \varkappa_2(\varsigma),$$

where $\varkappa_1$ and $\varkappa_2$ are univalent functions in $\mathbb{D}$ normalized with $\varkappa_1(0) = \varkappa_2(\varsigma) = 1$.

The function $f(\varsigma)$ defined by (1.1) is said to be a member of the class denoted by S_s^* of starlike functions with respect to symmetric points if it satisfies the following condition:

$$\Re\left\{\frac{\varsigma f^{'}(\varsigma)}{f(\varsigma) - f(-\varsigma)}\right\} > 0, \ \varsigma \in \mathbb{D}.$$

The class S_s^* was introduced by Sakaguchi [29] as a subclass of close-to-convex functions, and, hence, univalent in $\mathbb{D}$. It is also known that the class of convex functions and the class of odd starlike functions, with respect to the origin, are also included in S_s^* [29, 30].

Using this class as inspiration, Aouf et al. [31] developed and investigated the class $S_{s,n}^*T(1, 1)$ of functions n-starlike with respect to symmetric points, consisting of functions $f \in A$ with $a_\vartheta \le 0$ for $\vartheta \ge 2$, and satisfying the inequality

$$\Re\left\{\frac{D^{n+1} f(\varsigma)}{D^n f(\varsigma) - D^n f(-\varsigma)}\right\} > 0, \ \varsigma \in \mathbb{D},$$

where D^n is the Sălăgean operator [32].

The classes defined in [30, 31] are generalized by the following new class of functions, defined in this paper by applying the $D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma)$ operator seen in Definition 1. The new class is introduced here as follows:

Definition 2. *The function $f \in A(n)$ complying*

$$D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma) \neq 0, \qquad \varsigma \in \mathbb{D}^* = \mathbb{D}\setminus\{0\}, \tag{1.9}$$

is said to belong to the class $\aleph^{m,q,\tau}_{\alpha,\beta,\mu}(\eta,\delta,C,D)$ if the following subordination condition is satisfied:

$$\begin{aligned}
&(1+\eta)\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}\\
&-\eta\left(\frac{\varsigma(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma))'}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}\\
\prec\ &\frac{1+C\varsigma}{1+D\varsigma},\\
\eta \ \in\ &\mathbb{C}, 0<\delta<1, -1\le D<C\le 1, \mu\ge 0, \tau>-1, m\in\mathbb{N}_0, \alpha>-1, \beta>0 \text{ and } 0<q<1.
\end{aligned}$$

Using specific values for the parameters μ,τ,α,β and q the following subclasses appear:

(i) For $q \to 1^-$, the class $\Im^{m,\tau}_{\alpha,\beta,\mu}(\eta,\delta,C,D)$ is obtained as follows:

$$\begin{aligned}
\Im^{m,\tau}_{\alpha,\beta,\mu}(\eta,\delta,C,D) := \Bigg\{ f \in A(n) : (1+\eta)\left(\frac{2\varsigma}{\mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - \mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}\\
-\eta\left(\frac{\varsigma(\mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - \mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(-\varsigma))'}{\mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - \mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)\left(\frac{2\varsigma}{\mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - \mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}\\
\prec\ \frac{1+C\varsigma}{1+D\varsigma}\Bigg\},
\end{aligned}$$

with the operator $\mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(\varsigma)$ given by (1.7);

(ii) For $q \to 1^-$and $m = 0$, the class $N^{\eta,\delta}(n,C,D)$ is obtained and rectifies the class introduced by Muhammad and Marwan [33] as follows:

$$\begin{aligned}
N^{\eta,\delta}(n,C,D) \ :\ = \Bigg\{ f \in A(n) : (1+\eta)\left(\frac{2\varsigma}{f(\varsigma)-f(-\varsigma)}\right)^{\delta}\\
-\eta\left(\frac{\varsigma(f'(\varsigma)-f'(-\varsigma))}{f(\varsigma)-f(-\varsigma)}\right)\left(\frac{2\varsigma}{f(\varsigma)-f(-\varsigma)}\right)^{\delta} \prec \frac{1+C\varsigma}{1+D\varsigma}\Bigg\}.
\end{aligned}$$

The study exposed in this research tries to connect the special class of analytic functions with coefficients defined by the q-analogue operator with the differential subordination and superordination theory. As a result, certain sharp differential subordination and superordination results are investigated in the following theorems and corollaries for the functions belonging to the class $\aleph^{m,q,\tau}_{\alpha,\beta,\mu}(\eta,\delta,C,D)$.

2. Materials and methods

In order to prove the new differential subordination and superordination findings, the following known results will be used.

Definition 3. *[3] (Definition 2.2b., p. 21). Denote by $\wp$ the set of all functions $f(\varsigma)$ that are analytic and injective on $\overline{\mathbb{D}} \setminus E(f)$, where*

$$E(f) = \{\zeta : \zeta \in \partial\mathbb{D} \ \text{ and } \ \lim_{\varsigma\to\zeta} f(\varsigma) = \infty\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{D} \setminus E(f)$.

Lemma 1. *[3] (Theorem 3.1b., p. 71). Let h be a convex function in $\mathbb{D}$ with $h(0) = a$ and let $\gamma \in \mathbb{C}$ with $\mathrm{Re}(\gamma) \geq 0$. If $p \in H[a, n]$ and*

$$p(\varsigma) + \frac{\varsigma p^{'}(\varsigma)}{\gamma} \prec h(\varsigma), \tag{2.1}$$

then

$$p(\varsigma) \prec q(\varsigma) = \frac{\gamma}{n\varsigma^{(\gamma/n)}} \int_0^{\varsigma} h(t)t^{(\gamma/n)-1}dt \prec h(\varsigma).$$

The function q is convex and is the best dominant of (2.1).

Lemma 2. *[34] (Lemma 2.2., p. 3). Let q be univalent in $\mathbb{D}$ with $q(0) = 1$. Let $\xi, \varphi \in \mathbb{C}$ with $\varphi \neq 0$, and suppose that*

$$\mathrm{Re}\left(1 + \frac{\varsigma q^{''}(\varsigma)}{q'(\varsigma)}\right) > \max\left\{0; -\mathrm{Re}\frac{\xi}{\varphi}\right\}, \ \varsigma \in \mathbb{D}.$$

If λ is analytic in $\mathbb{D}$ and

$$\xi\lambda(\varsigma) + \varphi\varsigma\lambda'(\varsigma) \prec \xi q(\varsigma) + \varphi\varsigma q'(\varsigma), \tag{2.2}$$

then $\lambda(\varsigma) \prec q(\varsigma)$, and q is the best dominant of (2.2).

From [25] (Theorem 6, p. 820), we could easily obtain the following lemma:

Lemma 3. *Let q be convex in $\mathbb{D}$ and $\lambda \neq 0$, with $\mathrm{Re}(\lambda) \geq 0$. If $\breve{g} \in H[q(0), 1] \cap \wp$ such that $\breve{g}(\varsigma) + \lambda\varsigma\breve{g}'(\varsigma)$ is univalent in $\mathbb{D}$, then*

$$q(\varsigma) + \lambda\varsigma q'(\varsigma) \prec \breve{g}(\varsigma) + \lambda\varsigma\breve{g}'(\varsigma) \tag{2.3}$$

implies $q(\varsigma) \prec \breve{g}(\varsigma)$ and q is the best subordinant of (2.3).

Lemma 4. *[35]. Let $\mathcal{F}$ be analytic and convex in $\mathbb{D}$ and $0 \leq \lambda \leq 1$. If $f, g \in A$ such that $f(\varsigma) \prec \mathcal{F}(\varsigma)$ and $g(\varsigma) \prec \mathcal{F}(\varsigma)$, then*

$$\lambda f(\varsigma) + (1 - \lambda)g(\varsigma) \prec \mathcal{F}(\varsigma).$$

3. Results

The remainder of this paper assumes, unless otherwise stated, $\eta \in \mathbb{C}$, $0 < \delta < 1, -1 \leq D < C \leq 1, \mu \geq 0, \tau > -1, m \in \mathbb{N}_0, \alpha > -1, \beta > 0$, $0 < q < 1$ and all the powers are understood as principle values.

Theorem 1. *Consider* $f \in \aleph^{m,q,\tau}_{\alpha,\beta,\mu}(\eta,\delta,C,D)$ *and* $\eta \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$, $\delta > 0$ *satisfying* $\mathrm{Re}\eta \geq 0$. *Then,*

$$\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta} \prec q(\varsigma) = \frac{\delta}{\eta n}\int_0^1 \frac{1+C\varsigma u}{1+D\varsigma u} u^{(\delta/\eta n)-1} du \prec \frac{1+C\varsigma}{1+D\varsigma},$$

and q is convex $q \in H[1,n]$ *and is the best dominant.*

Proof. Define the function $\omega(\varsigma)$ by

$$\omega(\varsigma) = \left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}, \quad (\varsigma \in \mathbb{D}). \tag{3.1}$$

This function $\omega(\varsigma) \in H$ complies $\omega(0) = 1$. Differentiating (3.1) with respect to ς logarithmically, we have

$$\begin{aligned} & (1+\eta)\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta} \\ & -\eta\left(\frac{\varsigma(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma))'}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta} \\ = \;& \omega(\varsigma) + \frac{\eta}{\delta}\varsigma\omega'(\varsigma) \prec \frac{1+C\varsigma}{1+D\varsigma}. \end{aligned} \tag{3.2}$$

Since

$$D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) = \varsigma + \sum_{\vartheta=n+1}^{\infty} \chi_\vartheta \varsigma^\vartheta \qquad \text{and} \quad D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma) = -\varsigma + \sum_{\vartheta=n+1}^{\infty} \chi_\vartheta (-1)^\vartheta \varsigma^\vartheta,$$

where,

$$\chi_\vartheta = \left(\frac{[\beta,q]_{\vartheta-1}}{[\alpha+1,q]_{\vartheta-1}} \cdot \frac{\tau+1+\mu\left([\vartheta]_q - 1\right)}{\tau+1}\right)^m a_\vartheta \quad \vartheta \geq n+1,$$

we have

$$\begin{aligned} \Omega(\varsigma) &= \frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)} = \frac{2\varsigma}{2\varsigma + \sum\limits_{\vartheta=n+1}^{\infty} \chi_\vartheta[1+(-1)^\vartheta]\varsigma^\vartheta} \\ &= \frac{1}{1+\sum\limits_{s=n}^{\infty} \rho_s \varsigma^s} \end{aligned}$$

with

$$\rho_s = \frac{\chi_{s+1}[1+(-1)^s]}{2}, s \geq n.$$

Moreover,

$$U(\varsigma) = \frac{1}{1+\sum\limits_{s=n}^{\infty} \rho_s \varsigma^s} = 1 + \sum_{j=1}^{\infty} \eta_j \varsigma^j$$

with unknowns η_j, $j \geq 1$, we have

$$1 = (1 + \rho_n \varsigma^n + \rho_{n+1}\varsigma^{n+1} + ...)(1 + \eta_1 \varsigma + \eta_2 \varsigma^2 + ... + \eta_n \varsigma^n + \eta_{n+1}\varsigma^{n+1} + ...),$$

and equating the corresponding coefficients it follows that

$$\eta_1 = \eta_2 = ...\eta_{n-1} = 0, \quad \eta_n = -\rho_n, \quad \eta_{n+1} = -\rho_{n+1}, ... \ .$$

Hence

$$U(\varsigma) = 1 + \sum_{j=n}^{\infty} \eta_j \varsigma^j \in H[1,n].$$

Applying (3.1), the following can be written as

$$\omega = U^{\delta} \quad with \quad U \in H[1,n].$$

By employing the well-known binomial power expansion formula, we obtain

$$\omega = U^{\delta} \in H[1,n].$$

Now, from the subordination in (3.2) and using Lemma 1 for $\gamma = \frac{\delta}{\eta}$, the desired result is obtained. □

Using in Theorem 1 the assumption $q \to 1^-$, the following corollary emerges:

Corollary 1. *Consider* $f \in \mathfrak{I}^{m,\tau}_{\alpha,\beta,\mu}(\eta,\delta,C,D)$ *and* $\eta \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$, $\delta > 0$ *with* $\mathrm{Re}\eta \geq 0$. *Then,*

$$\left(\frac{2\varsigma}{\mathcal{L}^m_{\alpha,\beta}(\mu,\tau)f(\varsigma) - \mathcal{L}^m_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta} \prec q(\varsigma) = \frac{\delta}{\eta n}\int_0^1 \frac{1+C\varsigma u}{1+D\varsigma u} u^{(\delta/\eta n)-1} du \prec \frac{1+C\varsigma}{1+D\varsigma},$$

and q is convex $q \in H[1,n]$ *and is the best dominant.*

Remark 1. *From Theorem 1 the following inclusion relation can be written:*

$$\aleph^{m,q,\tau}_{\alpha,\beta,\mu}(\eta,\delta,C,D) \subset \aleph^{m,q,\tau}_{\alpha,\beta,\mu}(0,\delta,C,D), \quad \eta \in \mathbb{C} \ with \ \mathrm{Re}\eta \geq 0.$$

Furthermore, the following inclusion relation holds for the class $\aleph^{m,q,\tau}_{\alpha,\beta,\mu}(\eta,\delta,C,D)$:

Theorem 2. *If* $\eta_1, \eta_2 \in \mathbb{R}$ *such that* $0 \leq \eta_1 \leq \eta_2$ *and* $-1 \leq D_1 \leq D_2 < C_2 \leq C_1 \leq 1$, *then*

$$\aleph^{m,q,\tau}_{\alpha,\beta,\mu}(\eta_2,\delta,C_2,D_2) \subset \aleph^{m,q,\tau}_{\alpha,\beta,\mu}(\eta_1,\delta,C_1,D_1). \tag{3.3}$$

Proof. If $f \in \aleph^{m,q,\tau}_{\alpha,\beta,\mu}(\eta_2, \delta, C_2, D_2)$, since $-1 \leq D_1 \leq D_2 < C_2 \leq C_1 \leq 1$, it is easy to show that

$$\begin{aligned}
&(1+\eta_2)\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta} \\
&-\eta_2\left(\frac{\varsigma(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma))'}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta} \\
\prec\ &\frac{1+C_2\varsigma}{1+D_2\varsigma} \prec \frac{1+C_1\varsigma}{1+D_1\varsigma},
\end{aligned} \tag{3.4}$$

that is $f \in \aleph^{m,q,\tau}_{\alpha,\beta,\mu}(\eta_1, \delta, C_1, D_1)$, hence, the assertion in (3.3) holds for $\eta_1 = \eta_2$.

If $0 \leq \eta_1 < \eta_2$ and considering Remark 1 and (3.4), it follows $f \in \aleph^{m,q,\tau}_{\alpha,\beta,\mu}(0, \delta, C_1, D_1)$, that is

$$\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta} \prec \frac{1+C_1\varsigma}{1+D_1\varsigma}. \tag{3.5}$$

A simple computation shows that

$$\begin{aligned}
&(1+\eta_1)\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta} \\
&-\eta_1\left(\frac{\varsigma(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma))'}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta} \\
=\ &(1-\frac{\eta_1}{\eta_2})\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta} + \\
&\frac{\eta_1}{\eta_2}\Bigg[(1+\eta_2)\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta} \\
&-\eta_2\left(\frac{\varsigma(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma))'}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}\Bigg].
\end{aligned} \tag{3.6}$$

Moreover,

$$0 \leq \frac{\eta_1}{\eta_2} < 1$$

and the function $\frac{1+C_1\varsigma}{1+D_1\varsigma}$ with $-1 \leq D_1 < C_1 \leq 1$ is analytic and convex in $\mathbb{D}$. Considering relation (3.6), using the subordination results given by (3.4) and (3.5) and using Lemma 4, we deduce that

$$\begin{aligned}
&(1+\eta_1)\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta} \\
&-\eta_1\left(\frac{\varsigma(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma))'}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta} \\
\prec\ &\frac{1+C_1\varsigma}{1+D_1\varsigma},
\end{aligned}$$

that is $f \in \aleph^{m,q,\tau}_{\alpha,\beta,\mu}(\eta_1, \delta, C_1, D_1)$. □

Using in Theorem 2 the assumption $q \to 1^-$, the following corollary emerges:

Corollary 2. *If $\eta_1, \eta_2 \in \mathbb{R}$ such that $0 \le \eta_1 \le \eta_2$ and $-1 \le D_1 \le D_2 < C_2 \le C_1 \le 1$, then*

$$\mathfrak{I}^{m,\tau}_{\alpha,\beta,\mu}(\eta_2, \delta, C_2, D_2) \subset \mathfrak{I}^{m,\tau}_{\alpha,\beta,\mu}(\eta_1, \delta, C_1, D_1).$$

Example 1. *Use $C_1 = 1$ and $D_1 = -1$ in Theorem 2 and Corollary 2 . Let $\eta_1,\ \eta_2 \in \mathbb{R}$ such that $0 \le \eta_1 \le \eta_2$ and $-1 \le D_2 < C_2 \le 1$.We obtain*

(i) If $f \in \aleph^{m,q,\tau}_{\alpha,\beta,\mu}(\eta_2, \delta, C_2, D_2)$, then

$$\begin{aligned}&\operatorname{Re}\Bigg\{(1+\eta_1)\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}\\ &-\eta_1\left(\frac{\varsigma(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma))'}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}\Bigg\}\\ >\ & 0, \quad \varsigma \in \mathbb{D};\end{aligned}$$

(ii) $f \in \mathfrak{I}^{m,\tau}_{\alpha,\beta,\mu}(\eta_2, \delta, C_2, D_2)$, then

$$\begin{aligned}&\operatorname{Re}\Bigg\{(1+\eta_1)\left(\frac{2\varsigma}{\mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - \mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}\\ &-\eta_1\left(\frac{\varsigma(\mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - \mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(-\varsigma))'}{\mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - \mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)\left(\frac{2\varsigma}{\mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - \mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}\Bigg\}\\ >\ & 0, \quad \varsigma \in \mathbb{D}.\end{aligned}$$

Theorem 3. *Consider $q \in K$, with $q(0) = 1$, $\eta \in \mathbb{C}^*$ such that*

$$\operatorname{Re}\left(1 + \frac{\varsigma q''(\varsigma)}{q'(\varsigma)}\right) > \max\left\{0; -\operatorname{Re}\left(\frac{\delta}{\eta}\right)\right\}. \tag{3.7}$$

If $f \in A(n)$ complies (1.9) and the following subordination is satisfied:

$$\begin{aligned}&(1+\eta)\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}\\ &-\eta\left(\frac{\varsigma(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma))'}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}\\ \prec\ & q(\varsigma) + \frac{\eta}{\delta}\varsigma q'(\varsigma),\end{aligned} \tag{3.8}$$

then

$$\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta} \prec q(\varsigma)$$

and $q(\varsigma)$ is the best dominant of (3.8).

Proof. The function $f \in A(n)$ is assumed to comply (1.9), hence, the function defined by (3.1) satisfies $\omega \in H$, with $\omega(0) = 1$. As done for proving Theorem 1, differentiating (3.1) with respect to ς gives that (3.8) is equivalent to

$$\omega(\varsigma) + \frac{\eta}{\delta}\varsigma\omega^{'}(\varsigma) \prec q(\varsigma) + \frac{\eta}{\delta}\varsigma q^{'}(\varsigma).$$

Thus, by Lemma 2, for $\xi = 1$ and $\varphi = \frac{\eta}{\delta}$ we get $\omega(\varsigma) \prec q(\varsigma)$, *and* $q(\varsigma)$ is the best dominant of (3.8). □

Taking $q(\varsigma) = \frac{1+C\varsigma}{1+D\varsigma}$ with $-1 \le D < C \le 1$ in Theorem 3, the following corollary holds:

Corollary 3. *Let $\eta \in \mathbb{C}^*$ such that*

$$\max\left\{-1; -\frac{1+\mathrm{Re}\left(\frac{\delta}{\eta}\right)}{1-\mathrm{Re}\left(\frac{\delta}{\eta}\right)}\right\} \le D \le 0 \quad or \quad 0 \le D \le \min\left\{1; -\frac{1+\mathrm{Re}\left(\frac{\delta}{\eta}\right)}{1-\mathrm{Re}\left(\frac{\delta}{\eta}\right)}\right\}. \tag{3.9}$$

If $f \in A(n)$ complies (1.9) and the following subordination is satisfied:

$$\begin{aligned}&(1+\eta)\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}\\ &-\eta\left(\frac{\varsigma(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma))^{'}}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}\\ \prec\ &\frac{1+C\varsigma}{1+D\varsigma} + \frac{\eta}{\delta}\frac{(C-D)\varsigma}{(1+D\varsigma)^2}\end{aligned} \tag{3.10}$$

then

$$\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta} \prec \frac{1+C\varsigma}{1+D\varsigma}$$

and $\frac{1+C\varsigma}{1+D\varsigma}$ is the best dominant of (3.10).

Proof. For $q(\varsigma) = \frac{1+C\varsigma}{1+D\varsigma}$, the condition in (3.7) reduces to

$$\mathrm{Re}\frac{1-D\varsigma}{1+D\varsigma} > \max\left\{0; -\mathrm{Re}\left(\frac{\delta}{\eta}\right)\right\},\ \varsigma \in \mathbb{D}. \tag{3.11}$$

Since

$$\inf\left\{\mathrm{Re}\frac{1-D\varsigma}{1+D\varsigma} : \varsigma \in \mathbb{D}\right\} = \begin{cases} \frac{1-D\varsigma}{1+D\varsigma}, & if \quad -1 \le D \le 0\\ \frac{1-D\varsigma}{1+D\varsigma}, & if \quad 0 \le D < 1,\end{cases}$$

it is easy to check that (3.11) holds, if and only if, the assumption in (3.9) is satisfied whenever $-1 \le D < 1$. □

Using in Theorem 3 the assumption $q \to 1^-$, the next corollary is obtained:

Corollary 4. *Let $q \in K$, with $q(0) = 1$, $\eta \in \mathbb{C}^*$. Suppose that q complies (3.7). If $f \in A(n)$ satisfies the subordination*

$$(1+\eta)\left(\frac{2\varsigma}{\mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - \mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}$$

$$-\eta\left(\frac{\varsigma(\mathcal{L}^m_{\alpha,\beta}(\mu,\tau)f(\varsigma)-\mathcal{L}^m_{\alpha,\beta}(\mu,\tau)f(-\varsigma))'}{\mathcal{L}^m_{\alpha,\beta}(\mu,\tau)f(\varsigma)-\mathcal{L}^m_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)\left(\frac{2\varsigma}{\mathcal{L}^m_{\alpha,\beta}(\mu,\tau)f(\varsigma)-\mathcal{L}^m_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}$$
$$\prec \quad q(\varsigma)+\frac{\eta}{\delta}\varsigma q'(\varsigma), \tag{3.12}$$

then

$$\left(\frac{2\varsigma}{\mathcal{L}^m_{\alpha,\beta}(\mu,\tau)f(\varsigma)-\mathcal{L}^m_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta} \prec q(\varsigma)$$

and $q(\varsigma)$ is the best dominant of (3.12).

Theorem 4. *Let $\eta \in \mathbb{C}^*, \mathrm{Re}\eta \geq 0$ and $q \in K$, with $q(0) = 1$. Let $f \in A(n)$ such that*

$$\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma)-D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta} \in H[q(0),1]\cap\wp, \tag{3.13}$$

and ,consider the function

$$(1+\eta)\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma)-D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta},$$
$$-\eta\left(\frac{\varsigma(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma)-D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma))'}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma)-D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma)-D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}, \tag{3.14}$$

that is univalent in $\mathbb{D}$. If

$$q(\varsigma)+\frac{\eta}{\delta}\varsigma q'(\varsigma) \prec (1+\eta)\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma)-D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}$$
$$-\eta\left(\frac{\varsigma(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma)-D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma))'}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma)-D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma)-D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}, \tag{3.15}$$

then

$$q(\varsigma) \prec \left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma)-D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}$$

and $q(\varsigma)$ is the best dominant of (3.14).

Proof. Considering that the function ω is defined by (3.1), we know that $\omega \in H[q(0), m]$, and using (3.14) we have that $\omega \in H[q(0), 1] \cap \wp$. As in the proof of Theorem 1, differentiating (3.1) with respect to ς we get

$$q(\varsigma)+\frac{\eta}{\delta}\varsigma q'(\varsigma) \prec \omega(\varsigma)+\frac{\eta}{\delta}\varsigma\omega'(\varsigma).$$

Then, by applying Lemma 3 for $\lambda = \frac{\eta}{\delta}$, the desired result is obtained. □

Using in Theorem 4 the assumption $q(\varsigma) = \frac{1+C\varsigma}{1+D\varsigma}$ with $-1 \leq D < C \leq 1$ the next corollary can be written:

Corollary 5. *Let $\eta \in \mathbb{C}^*, \mathrm{Re}\eta \geq 0$. If $f \in A(n)$ such that the assumptions in (3.13) and (3.14) hold and satisfy the subordination*

$$\frac{1+C\varsigma}{1+D\varsigma} + \frac{\eta}{\delta}\frac{(C-D)\varsigma}{(1+D\varsigma)^2} \prec (1+\eta)\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}$$

$$-\eta\left(\frac{\varsigma(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma))'}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}, \tag{3.16}$$

then

$$\frac{1+C\varsigma}{1+D\varsigma} \prec \left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}$$

and $\frac{1+C\varsigma}{1+D\varsigma}$ is the best dominant of (3.16).

Using in Theorem 4 the assumption $q \to 1^-$, the next result can be derived:

Corollary 6. *Let $\eta \in \mathbb{C}^*$, $\mathrm{Re}\eta \geq 0$ and $q \in K$ with $q(0) = 1$. Let $f \in A(n)$ such that*

$$\left(\frac{2\varsigma}{\mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - \mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta} \in H[q(0),1] \cap \wp, \tag{3.17}$$

and consider the function

$$(1+\eta)\left(\frac{2\varsigma}{\mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - \mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}$$
$$-\eta\left(\frac{\varsigma(\mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - \mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(-\varsigma))'}{\mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - \mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)\left(\frac{2\varsigma}{\mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - \mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta} \tag{3.18}$$

that is univalent in $\mathbb{D}$. If

$$q(\varsigma) + \frac{\eta}{\delta}\varsigma q'(\varsigma) \prec (1+\eta\left(\frac{2\varsigma}{\mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - \mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}$$

$$-\eta\left(\frac{\varsigma(\mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - \mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(-\varsigma))'}{\mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - \mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)\left(\frac{2\varsigma}{\mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - \mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}, \tag{3.19}$$

then

$$q(\varsigma) \prec \left(\frac{2\varsigma}{\mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - \mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}$$

and $q(\varsigma)$ is the best dominant of (3.17).

Combining Theorems 3 and 4, the following sandwich-type theorem can be stated.

Theorem 5. *Let $q_1, q_2 \in K$, with $q_1(0) = q_2(0) = 1$, and let $\eta \in \mathbb{C}^*$, $\mathrm{Re}\eta \geq 0$. If $f \in A(n)$ such that the assumptions in (3.13) and (3.14) hold, then*

$$q_1(\varsigma) + \frac{\eta}{\delta}\varsigma q_1'(\varsigma) \prec \Upsilon(\varsigma) = (1+\eta)\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}$$

$$-\eta\left(\frac{\varsigma(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma))'}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta} \tag{3.20}$$

$$\prec \quad q_2(\varsigma) + \frac{\eta}{\delta}\varsigma q_2'(\varsigma)$$

implies that

$$q_1(\varsigma) \prec \Phi(\varsigma) = \left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta} \prec q_2(\varsigma)$$

and q_1 and *q_2 are respectively the best subordinant and best dominant of (3.20).*

Combining Corollaries 4 and 6, the following sandwich-type result is stated.

Corollary 7. *Let $q_1, q_2 \in K$ with $q_1(0) = q_2(0) = 1$, and let $\eta \in \mathbb{C}^*$, $\mathrm{Re}\eta \geq 0$. If $f \in A(n)$ such that the assumptions in (3.17) and (3.18) hold, then*

$$q_1(\varsigma) + \frac{\eta}{\delta}\varsigma q_1'(\varsigma) \prec \widetilde{\Upsilon}(\varsigma) = (1+\eta)\left(\frac{2\varsigma}{\mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - \mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}$$

$$-\eta\left(\frac{\varsigma(\mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - \mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(-\varsigma))'}{\mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - \mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)\left(\frac{2\varsigma}{\mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - \mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta} \tag{3.21}$$

$$\prec \quad q_2(\varsigma) + \frac{\eta}{\delta}\varsigma q_2'(\varsigma)$$

implies that

$$q_1(\varsigma) \prec \widetilde{\Phi}(\varsigma) = \left(\frac{2\varsigma}{\mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - \mathcal{L}^{m}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta} \prec q_2(\varsigma),$$

and q_1 and *q_2 are respectively the best subordinant and best dominant of (3.21).*

Using $q_i = e^{r_i\varsigma}$ with $0 < r_1 < r_2 \leq 1,\ i = 1,2$ in Theorem 5 and Corollary 7, the following examples are constructed:

Example 2. *(i) Let $\eta \in \mathbb{C}^*$ with* $\mathrm{Re}\eta \geq 0$. *If $f \in A(n)$ such that the assumptions in (3.13) and (3.14) hold, then*

$$\left(1 + \frac{\eta}{\delta}\varsigma\right)e^{r_1\varsigma} \prec \Upsilon(\varsigma) \prec \left(1 + \frac{\eta}{\delta}\varsigma\right)e^{r_2\varsigma} \Rightarrow e^{r_1\varsigma} \prec \Phi(\varsigma) \prec e^{r_2\varsigma}, \quad (0 < r_1 < r_2 \leq 1)$$

where Υ and Φ are given in Theorem 5, and $e^{r_1\varsigma}$ and $e^{r_2\varsigma}$are, respectively, the best subordinant and the best dominant.

(ii) If $f \in A(n)$ such that the assumptions in (3.17) and (3.18) hold, then

$$\left(1+\frac{\eta}{\delta}\varsigma\right)e^{r_1\varsigma} \prec \widetilde{\Upsilon}(\varsigma) \prec \left(1+\frac{\eta}{\delta}\varsigma\right)e^{r_2\varsigma} \Rightarrow e^{r_1\varsigma} \prec \widetilde{\Phi}(\varsigma) \prec e^{r_2\varsigma}, \quad (0 < r_1 < r_2 \leq 1)$$

where $\widetilde{\Upsilon}$ and $\widetilde{\Phi}$ are given in Corollary 7, and $e^{r_1\varsigma}$ and $e^{r_2\varsigma}$are, respectively, the best subordinant and the best dominant.

Theorem 6. *If $f \in \aleph^{m,q,\tau}_{\alpha,\beta,\mu}(0,\delta,1-2\sigma,-1)$ with $0 \leq \sigma < 1$, then $f \in \aleph^{m,q,\tau}_{\alpha,\beta,\mu}(\eta,\delta,1-2\sigma,-1)$ for $|\varsigma| < R$, where*

$$R = \left(\sqrt{\frac{|\eta|^2 m^2}{\delta^2}+1} - \frac{|\eta|\, m}{\delta}\right)^{\frac{1}{m}}. \tag{3.22}$$

Proof. For $f \in \aleph^{m,q,\tau}_{\alpha,\beta,\mu}(0,\delta,1-2\sigma,-1)$ *with* $0 \leq \sigma < 1$, let the function ω be defined by

$$\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta} = (1-\sigma)\omega(\varsigma)+\sigma, \quad \varsigma \in \mathbb{D}. \tag{3.23}$$

Then the function ω is analytic in $\mathbb{D}$ with $\omega(0) = 1$, and since $f \in \aleph^{m,q,\tau}_{\alpha,\beta,\mu}(0,\delta,1-2\sigma,-1)$ is equivalent to

$$\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta} \prec \frac{1+(1-2\sigma)\varsigma}{1-\varsigma},$$

it follows that $\mathrm{Re}\,\omega(\varsigma) > 0$.

As in the proof of Theorem 1, since $f \in \aleph^{m,q,\tau}_{\alpha,\beta,\mu}(0,\delta,1-2\sigma,-1)$ with $0 \leq \sigma < 1$, we deduce that

$$\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta} \in H[1,n],$$

and from the relation (3.23), we get $\omega \in H[1,n]$. Therefore, the following estimate holds

$$\left|\varsigma\omega'(\varsigma)\right| \leq \frac{2mr^m \mathrm{Re}\,\omega(\varsigma)}{1-r^{2m}}, \quad |\varsigma| = r < 1$$

that represents the result of Shah [36] (the inequality (6), p. 240, for $\alpha = 0$), which generalizes Lemma 2 [37].

Simple calculations demonstrate that

$$\begin{aligned}
&\frac{1}{1-\sigma}\left\{(1+\eta)\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}\right.\\
&\left.-\eta\left(\frac{\varsigma(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma))'}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)\left(\frac{2\varsigma}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) - D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta} - \sigma\right\}\\
=\ &\omega(\varsigma) + \frac{\eta}{\delta}\varsigma\omega'(\varsigma),
\end{aligned}$$

hence, we have

$$\begin{aligned}
&\mathrm{Re}\left\{\frac{1}{1-\sigma}\left[(1+\eta)\left(\frac{2\varsigma}{D_{\alpha,\beta}^{m,q}(\mu,\tau)f(\varsigma)-D_{\alpha,\beta}^{m,q}(\mu,\tau)f(-\varsigma)}\right)^{\delta}\right.\right.\\
&\left.\left.-\eta\left(\frac{\varsigma(D_{\alpha,\beta}^{m,q}(\mu,\tau)f(\varsigma)-D_{\alpha,\beta}^{m,q}(\mu,\tau)f(-\varsigma))'}{D_{\alpha,\beta}^{m,q}(\mu,\tau)f(\varsigma)-D_{\alpha,\beta}^{m,q}(\mu,\tau)f(-\varsigma)}\right)\left(\frac{2\varsigma}{D_{\alpha,\beta}^{m,q}(\mu,\tau)f(\varsigma)-D_{\alpha,\beta}^{m,q}(\mu,\tau)f(-\varsigma)}\right)^{\delta}-\sigma\right]\right\}\\
\geq\ &\mathrm{Re}\,\omega(\varsigma)\left[1-\frac{2\,|\eta|\,mr^{m}}{\delta\,(1-r^{2m})}\right],\ |\varsigma|=r<1, \qquad (3.24)
\end{aligned}$$

and the righthand side of (3.24) is positive provided that $r<R$, where R is given by (3.22). □

Theorem 7. *Suppose that* $f\in\aleph_{\alpha,\beta,\mu}^{m,q,\tau}(\eta,\delta,C,D)$*, let* $\eta\in\mathbb{C}^{*}$ *with* $\mathrm{Re}\eta\geq 0$ *and* $-1\leq D<C\leq 1$*. Then, the following inequalities hold:*

(i)

$$\begin{aligned}
\frac{\delta}{\eta n}\int_{0}^{1}\frac{1-Cu}{1-Du}u^{(\delta/\eta n)-1}du\ &<\ \mathrm{Re}\left(\frac{2\varsigma}{D_{\alpha,\beta}^{m,q}(\mu,\tau)f(\varsigma)-D_{\alpha,\beta}^{m,q}(\mu,\tau)f(-\varsigma)}\right)^{\delta}\\
&<\ \frac{\delta}{\eta n}\int_{0}^{1}\frac{1+Cu}{1+Du}u^{(\delta/\eta n)-1}du,\quad \varsigma\in\mathbb{D}. \qquad (3.25)
\end{aligned}$$

(ii) *For* $|\varsigma|=r<1$*, we have*

$$\begin{aligned}
2r\left(\frac{\delta}{\eta n}\int_{0}^{1}\frac{1+Cru}{1+Dru}u^{(\delta/\eta n)-1}du\right)^{-\frac{1}{\delta}}\ &<\ \left|D_{\alpha,\beta}^{m,q}(\mu,\tau)f(\varsigma)-D_{\alpha,\beta}^{m,q}(\mu,\tau)f(-\varsigma)\right|\\
&<\ 2r\left(\frac{\delta}{\eta n}\int_{0}^{1}\frac{1-Cru}{1-Dru}u^{(\delta/\eta n)-1}du\right)^{-\frac{1}{\delta}}. \qquad (3.26)
\end{aligned}$$

All these inequalities are the best possible.

Proof. Applying the results given by Theorem 1 for the hypothesis of this theorem, we obtain that

$$\left(\frac{2\varsigma}{D_{\alpha,\beta}^{m,q}(\mu,\tau)f(\varsigma)-D_{\alpha,\beta}^{m,q}(\mu,\tau)f(-\varsigma)}\right)^{\delta}\prec q(\varsigma)=\frac{\delta}{\eta n}\int_{0}^{1}\frac{1+C\varsigma u}{1+D\varsigma u}u^{(\delta/\eta n)-1}du, \qquad (3.27)$$

and the convex function $q\in H[1,n]$ is the best dominant.
Therefore,

$$\mathrm{Re}\left(\frac{2\varsigma}{D_{\alpha,\beta}^{m,q}(\mu,\tau)f(\varsigma)-D_{\alpha,\beta}^{m,q}(\mu,\tau)f(-\varsigma)}\right)^{\delta}\ <\ \sup_{\varsigma\in\mathbb{D}}\mathrm{Re}\left(\frac{\delta}{\eta n}\int_{0}^{1}\frac{1+C\varsigma u}{1+D\varsigma u}u^{(\delta/\eta n)-1}du\right)$$

$$
\begin{aligned}
&= \frac{\delta}{\eta n}\int_0^1 \sup_{\varsigma\in\mathbb{D}}\mathrm{Re}\left(\frac{1+C\varsigma u}{1+D\varsigma u}\right)u^{(\delta/\eta n)-1}du\\
&= \frac{\delta}{\eta n}\int_0^1 \frac{1+Cu}{1+Du}u^{(\delta/\eta n)-1}du,\quad \varsigma\in\mathbb{D},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Re}\left(\frac{2\varsigma}{D_{\alpha,\beta}^{m,q}(\mu,\tau)f(\varsigma)-D_{\alpha,\beta}^{m,q}(\mu,\tau)f(-\varsigma)}\right)^{\delta} &> \inf_{\varsigma\in\mathbb{D}}\mathrm{Re}\left(\frac{\delta}{\eta n}\int_0^1 \frac{1-C\varsigma u}{1-D\varsigma u}u^{(\delta/\eta n)-1}du\right)\\
&= \frac{\delta}{\eta n}\int_0^1 \inf_{\varsigma\in\mathbb{D}}\mathrm{Re}\left(\frac{1-C\varsigma u}{1-D\varsigma u}\right)u^{(\delta/\eta n)-1}du\\
&= \frac{\delta}{\eta n}\int_0^1 \frac{1-Cu}{1-Du}u^{(\delta/\eta n)-1}du,\quad \varsigma\in\mathbb{D}.
\end{aligned}
$$

In addition, since

$$
\begin{aligned}
\left|\frac{2\varsigma}{D_{\alpha,\beta}^{m,q}(\mu,\tau)f(\varsigma)-D_{\alpha,\beta}^{m,q}(\mu,\tau)f(-\varsigma)}\right|^{\delta} &< \sup_{\varsigma\in\mathbb{D}}\left|\frac{\delta}{\eta n}\int_0^1 \frac{1+C\varsigma u}{1+D\varsigma u}u^{(\delta/\eta n)-1}du\right|\\
&= \frac{\delta}{\eta n}\int_0^1 \sup_{\varsigma\in\mathbb{D}}\left|\frac{1+C\varsigma u}{1+D\varsigma u}u\right|^{(\delta/\eta n)-1}du\\
&= \frac{\delta}{\eta n}\int_0^1 \frac{1+Cur}{1+Dur}u^{(\delta/\eta n)-1}du,\quad |\varsigma|=r<1,
\end{aligned}
$$

we get

$$
\left|D_{\alpha,\beta}^{m,q}(\mu,\tau)f(\varsigma)-D_{\alpha,\beta}^{m,q}(\mu,\tau)f(-\varsigma)\right| > 2r\left(\frac{\delta}{\eta n}\int_0^1 \frac{1+Cru}{1+Dru}u^{(\delta/\eta n)-1}du\right)^{-\frac{1}{\delta}},
$$

while

$$
\begin{aligned}
\left|\frac{2\varsigma}{D_{\alpha,\beta}^{m,q}(\mu,\tau)f(\varsigma)-D_{\alpha,\beta}^{m,q}(\mu,\tau)f(-\varsigma)}\right|^{\delta} &> \inf_{\varsigma\in\mathbb{D}}\left|\frac{\delta}{\eta n}\int_0^1 \frac{1-C\varsigma u}{1-D\varsigma u}u^{(\delta/\eta n)-1}du\right|\\
&= \frac{\delta}{\eta n}\int_0^1 \inf_{\varsigma\in\mathbb{D}}\left|\frac{1-C\varsigma u}{1-D\varsigma u}\right|u^{(\delta/\eta n)-1}du\\
&= \frac{\delta}{\eta n}\int_0^1 \frac{1-Cur}{1-Dur}u^{(\delta/\eta n)-1}du,\quad |\varsigma|=r<1,
\end{aligned}
$$

implies that

$$\left|D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma)-D^{m,q}_{\alpha,\beta}(\mu,\tau)f(-\varsigma)\right|<2r\left(\frac{\delta}{\eta n}\int_0^1\frac{1-Cru}{1-Dru}u^{(\delta/\eta n)-1}du\right)^{-\frac{1}{\delta}}.$$

The inequalities of (3.25) and (3.26) are the best possible because the subordination in (3.27) is sharp. □

Using in Theorem 7 the assumption $q\to 1^-$, we state the corollary:

Corollary 8. *Suppose that $f\in\Im^{m,\tau}_{\alpha,\beta,\mu}(\eta,\delta,C,D)$, let $\eta\in\mathbb{C}^*$ with $\mathrm{Re}\eta\geq 0$ and $-1\leq D<C\leq 1$. Then,*

(i)

$$\begin{aligned}\frac{\delta}{\eta n}\int_0^1\frac{1-Cu}{1-Du}u^{(\delta/\eta n)-1}du &< \mathrm{Re}\left(\frac{2\varsigma}{\mathcal{L}^m_{\alpha,\beta}(\mu,\tau)f(\varsigma)-\mathcal{L}^m_{\alpha,\beta}(\mu,\tau)f(-\varsigma)}\right)^{\delta}\\ &< \frac{\delta}{\eta n}\int_0^1\frac{1+Cu}{1+Du}u^{(\delta/\eta n)-1}du,\quad \varsigma\in\mathbb{D}\end{aligned}$$

(ii) *For $|\varsigma|=r<1$, we have*

$$\begin{aligned}2r\left(\frac{\delta}{\eta n}\int_0^1\frac{1+Cru}{1+Dru}u^{(\delta/\eta n)-1}du\right)^{-\frac{1}{\delta}} &< \left|\mathcal{L}^m_{\alpha,\beta}(\mu,\tau)f(\varsigma)-\mathcal{L}^m_{\alpha,\beta}(\mu,\tau)f(-\varsigma)\right|\\ &< 2r\left(\frac{\delta}{\eta n}\int_0^1\frac{1-Cru}{1-Dru}u^{(\delta/\eta n)-1}du\right)^{-\frac{1}{\delta}}.\end{aligned}$$

All these inequalities are the best possible.

Using in Theorem 7 the assumptions $q\to 1^-$ and $m=0$, we obtain the corollary:

Corollary 9. *Suppose that $f\in N^{\eta,\delta}(n,C,D)$, let $\eta\in\mathbb{C}^*$ with $\mathrm{Re}\eta\geq 0$ and $-1\leq D<C\leq 1$. Then,*

(i)

$$\frac{\delta}{\eta n}\int_0^1\frac{1-Cu}{1-Du}u^{(\delta/\eta n)-1}du<\mathrm{Re}\left(\frac{2\varsigma}{f(\varsigma)-f(-\varsigma)}\right)^{\delta}<\frac{\delta}{\eta n}\int_0^1\frac{1+Cu}{1+Du}u^{(\delta/\eta n)-1}du,\quad \varsigma\in\mathbb{D}$$

(ii) *For $|\varsigma|=r<1$, we have*

$$2r\left(\frac{\delta}{\eta n}\int_0^1\frac{1+Cru}{1+Dru}u^{(\delta/\eta n)-1}du\right)^{-\frac{1}{\delta}}<|f(\varsigma)-f(-\varsigma)|<2r\left(\frac{\delta}{\eta n}\int_0^1\frac{1-Cru}{1-Dru}u^{(\delta/\eta n)-1}du\right)^{-\frac{1}{\delta}}.$$

All these inequalities are the best possible.

4. Conclusions

There has been a resurgence of interest in the study of q-series and q-polynomials and related topics, which has a history dating back to the 19th century as a result of the creation of quantum groups and their applications in mathematics and physics beginning in 1980. This study introduces the class $\aleph^{m,q,\tau}_{\alpha,\beta,\mu}(\eta, \delta, C, D))$ of normalized analytic functions by using the linear extended multiplier q-Choi-Saigo-Srivastava operator in the open unit disk $\mathbb{D}$ given by Definition 1. Some applications of the theory of differential subordination differential superordination, and sandwich-type results were obtained here for the class $\aleph^{m,q,\tau}_{\alpha,\beta,\mu}(\eta, \delta, C, D))$ with interesting corollaries obtained when particularizing the parameters of the defined class.

Future investigations can be done on the newly defined class considering coefficient estimates [38, 39]. The classes obtained in this paper can be investigated using the newer theories of strong and fuzzy differential subordination and superordination [19, 40]. Also, new classes of other types of functions could be investigated using the same linear extended multiplier q-Choi-Saigo-Srivastava operator like it is done for bi-univalent functions [41] or for meromorphic functions [42, 43].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

References

1. T. Bulboacă, *Differential Subordinations and Superordinations, Recent Results*, Cluj-Napoca: House of Scientific Book Publication, 2005.

2. S. S. Miller, P. T. Mocanu, Differential subordinations and univalent functions, *Michigan Math. J.*, **28** (1981), 157–171. https://doi.org/10.1307/mmj/1029002507

3. S. S. Miller, P. T. Mocanu, *Differential Subordinations: Theory and Applications*, New York, Basel: Marcel Dekker, 2000.

4. F. H. Jackson, On q-functions and a certain difference operator, *Earth Env. Sci. T. R. So.*, **46** (1909), 253–281. https://doi.org/10.1017/S0080456800002751

5. F. H. Jackson, On q-definite integrals, *Quart. J. Pure Appl. Math*, **41** (1910), 193–203.

6. H. M. Srivastava, Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis, *Iran. J. Sci. Technol. Trans. Sci.*, **44** (2020), 327–344. https://doi.org/10.1007/s40995-019-00815-0

7. S. Kanas, D. Raducanu, Some classes of analytic functions related to conic domains, *Math. Slovaca*, **64** (2014), 1183–1196. https://doi.org/10.2478/s12175-014-0268-9
8. H. Aldweby, M. Darus, Some subordination results on q-analogue of Ruscheweyh differential operator, *Abstr. Appl. Anal.*, **2014** (2014), 958563.
9. S. Mahmood, J. Sokol, New subclass of analytic functions in conical domain associated with Ruscheweyh q-differential operator, *Results Math.*, **71** (2017), 1345–1357. https://doi.org/10.1007/s00025-016-0592-1
10. E. E. Ali, T. Bulboaca, Subclasses of multivalent analytic functions associated with a q-difference operator, *Mathematics*, **8** (2020), 2184. https://doi.org/10.3390/math8122184
11. E. E. Ali, A. Y. Lashin, A. M. Albalahi, Coefficient estimates for some classes of bi-univalent function associated with Jackson q-difference Operator, *J. Funct. Spaces*, **2022** (2022), 2365918. https://doi.org/10.1155/2022/2365918
12. M. Govindaraj, S. Sivasubramanian, On a class of analytic functions related to conic domains involving q-calculus, *Anal. Math.*, **43** (2017), 475–487. https://doi.org/10.1007/s10476-017-0206-5
13. W. Y. Kota, R. M. El-Ashwah, Some application of subordination theorems associated with fractional q-calculus operator, *Math. Bohem.*, **148** (2023), 131–148. http://doi.org/10.21136/MB.2022.0047-21
14. A. Alb Lupaş, G. I. Oros, Differential sandwich theorems involving Riemann-Liouville fractional integral of q-hypergeometric function, *AIMS Mathematics*, **8** (2023), 4930–4943. https://doi.org/10.3934/math.2023246
15. A. Alb Lupaş, G. I. Oros, Sandwich-type results regarding Riemann-Liouville fractional integral of q-hypergeometric function, *Demonstr. Math.*, **56** (2023), 20220186. https://doi.org/10.1515/dema-2022-0186
16. H. M. Srivastava, S. Khan, Q. Z. Ahmad, N. Khan, S. Hussain, The Faber polynomial expansion method and its application to the general coefficient problem for some subclasses of bi-univalent functions associated with a certain q-integral operator, *Stud. Univ. Babeş-Bolyai Math.*, **63** (2018), 419–436. https://doi.org/10.24193/subbmath.2018.4.01
17. S. Kazımoglu, E. Deniz, L.-I. Cotirla, Certain subclasses of analytic and bi-univalent functions governed by the gegenbauer polynomials linked with q-derivative, *Symmetry*, **15** (2023), 1192. https://doi.org/10.3390/sym15061192
18. D. Breaz, A. A. Alahmari, L.-I. Cotirla, S. A. Shah, On generalizations of the close-to-convex functions associated with q-Srivastava–Attiya Operator, *Mathematics*, **11** (2023), 2022. https://doi.org/10.3390/math11092022
19. A. A. Lupa, F. Ghanim, Strong Differential Subordination and Superordination Results for Extended q-Analogue of Multiplier Transformation, *Symmetry*, **15** (2023), 713. https://doi.org/10.3390/sym15030713
20. Z. G. Wang, S. Hussain, M. Naeem, T. Mahmood, S. A. Khan, A subclass of univalent functions associated with $q-$analogue of Choi-Saigo-Srivastava operator, *Hacet. J. Math. Stat.*, **49** (2019), 1471–1479. https://dergipark.org.tr/en/download/article-file/970404

21. B. Khan, H. M. Srivastava, M. Tahir, M. Darus, Q. Z. Ahmad, N. Khan, Applications of a certain q-integral operator to the subclasses of analytic and bi-univalent functions, *AIMS Mathematics*, **6** (2020), 1024–1039. https://doi.org/10.3934/math.2021061

22. L. Shi, B. Ahmad, N. Khan, M. G. Khan, S. Araci, W. K. Mashwani, B. Khan, Coefficient estimates for a subclass of meromorphic multivalent q-close-to-convex functions, *Symmetry*, **13** (2021), 1840. https://doi.org/10.3390/sym13101840

23. C. Zhang, B. Khan, T. G. Shaba, J. S. Ro, S. Araci, M. G. Khan, Applications of q-Hermite polynomials to subclasses of analytic and bi-univalent Functions, *Fractal Fract.*, **6** (2022), 420. https://doi.org/10.3390/fractalfract6080420

24. E. E. Ali, G. I. Oros, S. A. Shah, A. M. Albalahi, Applications of q-calculus multiplier operators and subordination for the study of particular analytic function subclasses, *Mathematics*, **11** (2023), 2705. https://doi.org/10.3390/math11122705

25. S. S. Miller, P. T. Mocanu, Subordinations of differential superordinations, *Complex Var. Elliptic*, **48** (2003), 815–826. https://doi.org/10.1080/0278107031000l599322

26. T. Bulboacă, Classes of first-order differential subordinations, *Demonstr. Math.*, **35** (2002), 287–392.

27. T. Bulboacă, A class of superordination preserving integral operators, *Indagat. Math.*, **13** (2002), 301–311. https://doi.org/10.1016/S0019-3577(02)80013-1

28. R. M. Ali, V. Ravichandran, M. H. Khan, K. G. Subramaniam, Differential sandwich theorems for certain analytic functions, *Far East J. Math. Sci.*, **15** (2004), 87–94.

29. K. Sakaguchi, On certain univalent mapping, *JMSJ*, **11** (1959), 72–75. https://doi.org/10.2969/jmsj/01110072

30. A. Muhammad, Some differential subordination and superordination properties of symmetric functions, *Rend. Sem. Mat. Univ. Politec. Torino*, **69** (2011), 247–259.

31. M. K. Aouf, R. M. El-Ashwah, S. M. El-Deeb, Certain classes of univalent functions with negative coefficients and n-starlike with respect to certain points, *Mat. Vestn.*, **62** (2010), 215–226.

32. G. S. Sălăgean, Subclasses of univalent functions, *Complex Analysis-Fifth Romanian-Finish Seminar: Part 1 Proceedings of the Seminar held in Bucharest*, 1981, 362–372.

33. A. Muhammad, M. Marwan, Some properties of generalized two-fold symmetric non-Bazilevic analytic functions, *Le Matematiche*, **69** (2014), 223–235. https://doi.org/10.4418/2014.69.2.19

34. T. N. Shanmugam, V. Ravichandran, S. Sivasubramanian, Differential sandwich theorems for some subclasses of analytic functions, *Aust. J. Math. Anal. Appl.*, **3** (2006), 8.

35. M. S. Liu, On certain subclass of analytic functions, *J. South China Norm. Univ. Natur. Sci. Ed.*, **4** (2002), 15–20.

36. G. M. Shah, On the univalence of some analytic functions, *Pac. J. Math.*, **43** (1972), 239–250.

37. T. H. MacGregor, The radius of univalence of certain analytic functions, *Proc. Am. Math. Soc.*, **14** (1963), 514–520.

38. I. Al-Shbeil, A. Catas, H. M. Srivastava, N. Aloraini, Coefficient estimates of new families of analytic functions associated with q-Hermite Polynomials, *Axioms*, **12** (2023), 52. https://doi.org/10.3390/axioms12010052

39. H. M. Srivastava, I. Al-Shbeil, Q. Xin, F. Tchier, S. Khan, S. N. Malik, Faber polynomial coefficient estimates for bi-close-to-convex functions defined by the q-fractional derivative, *Axioms*, **12** (2023), 585. https://doi.org/10.3390/axioms12060585

40. A. Alb Lupa, G. I. Oros, Fuzzy differential subordination and superordination results involving the q-hypergeometric function and fractional calculus aspects, *Mathematics*, **10** (2022), 4121. https://doi.org/10.3390/math10214121

41. H. M. Srivastava, A. K. Wanas, R. Srivastava, Applications of the q-Srivastava-Attiya operator involving a certain family of bi-univalent functions associated with the Horadam Polynomials, *Symmetry*, **13** (2021), 1230. https://doi.org/10.3390/sym13071230

42. H. M. Srivastava, M. Arif, M. Raza, Convolution properties of meromorphically harmonic functions defined by a generalized convolution q-derivative operator, *AIMS Mathematics*, **6** (2021), 5869–5885. https://doi.org/10.3934/math.2021347

43. E. E. Ali, H. M. Srivastava, A. M. Y. Lashin, A. M. Albalahi, Applications of some subclasses of meromorphic functions associated with the q-derivatives of the q-binomials, *Mathematics*, **11** (2023), 2496. https://doi.org/10.3390/math11112496