AIMS Mathematics, 8(11): 27742–27761.
DOI: 10.3934/math.20231420
Received: 31 July 2023
Revised: 13 September 2023
Accepted: 21 September 2023
Published: 08 October 2023

AIMS *Mathematics*

http://www.aimspress.com/journal/Math

Research article

Quasi-periodic solutions of three-component Burgers hierarchy

Wei Liu[1,2], **Xianguo Geng**[2] **and Bo Xue**[2,*]

[1] Department of Mathematics and Physics, Shijiazhuang Tiedao University, Shijiazhuang 050043, China

[2] School of Mathematics and Statistics, Zhengzhou University, Zhengzhou 450001, China

* **Correspondence:** Email: xuebo@zzu.edu.cn.

Abstract: Starting from a 3×3 matrix spectral problem and the characteristic polynomial of the Lax matrix, we propose a trigonal curve, the associated meromorphic functions and three kinds of Abelian differentials. By discussing the asymptotic properties for the Baker-Akhiezer functions and their Riemann theta function expressions, we get quasi-periodic solutions of the three-component Burgers hierarchy. Finally, we straighten out the three-component Burgers flows.

Keywords: three-component Burgers hierarchy; quasi-periodic solutions; Baker-Akhiezer function; trigonal curve
Mathematics Subject Classification: 37K10, 37K20

1. Introduction

It is well known that quasi-periodic solutions of integrable dynamical systems can not only describe periodic nonlinear behavior, but they can also show characteristics of Liouville integrability [1–5]. Therefore, constructing quasi-periodic solutions for integrable nonlinear equations is a hot topic in the field of modern mathematical and theoretical physics. In the past decades, several systematic approaches have been used to study quasi-periodic solutions for nonlinear integrable models [6–12]. The successful idea is to use the hyperelliptic curves, and finite-genus solutions of nonlinear equations related to 2×2 matrix spectral problems have been derived [2,3,6–17]. But, this method is invalid to solve higher-order matrix spectral problems. In Refs. [18, 19], a unified work was given to study the algebro-geometric solutions to the Boussinesq equations associated with a third-order spectral problem. After that, according to the characteristic polynomial of the Lax matrix, Geng and colleagues [20, 21] developed a general approach to handle the case of the 3×3 matrix spectral problem by introducing trigonal curves. Using this method, quasi-periodic solutions for some famous equations, such as the Manakov, the cmKdV, the modified Boussinesq, the coupled Sasa-Satsuma, the Dym-type equations and so on [22–26], were successfully generalized.

Recently, in Ref. [27], three-component Burgers hierarchy was derived, and its bi-Hamiltonian structures were discussed. The first member in the whole hierarchy is

$$\begin{aligned} u_t &= 3w_{xx} - uw_x - u_x w + 2v_x, \\ v_t &= 2w_{xxx} - 2uw_{xx} + v_{xx} - v_x w - 2vw_x - uv_x, \\ w_t &= -w_{xx} - uw_x - u_x w - 4ww_x, \end{aligned} \tag{1.1}$$

which can be reduced to the well-known Burgers equation [28]

$$u_t = u_{xx} - uu_x.$$

During the past few decades, there have been some remarkable works on the Burgers equation due to its prominent mathematical structures and physical properties. The Darboux transformation for the generalized Burgers equation was constructed based on the Lax pairs [29]. Quasi-periodic solutions for the 2+1 dimensional discrete Burgers equation were proposed in view of the Jacobi inversion [30]. The Cole-Hopf transformation were used to generate the multiple-front solutions for the coupled Burgers equation [31].

The purpose of this paper is to study quasi-periodic solutions of three-component Burgers flows. By means of the asymptotic expansions and Abelian differentials, we deduce the theta function expressions of the potential functions. In Section 2, we first list the recursion relations of the three-component Burgers equation; then, we define a trigonal curve $\mathcal{K}_{m-2}$ based on the characteristic polynomial of the Lax matrix. By adding two different infinite points, the curve $\mathcal{K}_{m-2}$ is compactified. In Section 3, we discuss the asymptotic behaviors of the meromorphic functions ϕ_2 and ϕ_3. Section 4 is devoted to discussing the divisors of ϕ_2, ϕ_3, which can reveal essential singularities. In Section 5, we derive the theta function solutions of ψ_1, ϕ_2, ϕ_3 by using the Abelian differentials inferred in Section 3. Especially, we generate the quasi-periodic solutions of the three-component Burgers hierarchy.

2. The trigonal curve

The main idea of this section is to define a trigonal curve $\mathcal{K}_{m-2}$ related to the three-component Burgers hierarchy. We first list some necessary results from Ref. [27] to make this paper easier to read. We define the Lenard recurrence relations

$$K\hat{g}_j = J\hat{g}_{j+1}, \quad K\check{g}_j = J\check{g}_{j+1}, \quad j \geq 0, \tag{2.1}$$

and K, J are two 3×3 operators

$$K = \begin{pmatrix} \partial u - 3\partial^2 & 2\partial v + 2u\partial^2 + 2u_x\partial - 2\partial^3 & 0 \\ K_{21} & K_{22} & 0 \\ \partial w + 2w\partial & K_{32} & \partial^3 + \partial u\partial - v\partial \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -2\partial & -3\partial \\ -2\partial & J_{22} & 2u\partial \\ 0 & 0 & \partial \end{pmatrix}, \tag{2.2}$$

$$\hat{g}_0 = (-w, 1, 0)^T, \quad \check{g}_0 = (1, 0, 0)^T, \tag{2.3}$$

where

$$\begin{aligned} K_{21} &= v_x + 2v\partial + 2u\partial^2 - 2\partial^3, \\ K_{22} &= \partial^2 u\partial + u\partial^3 - u\partial u\partial + \partial^2 v - uv\partial - uv_x - \partial^4, \\ K_{32} &= w\partial^2 - 2uw\partial - (uw)_x - \partial^2 w, \\ J_{22} &= -\partial^2 - w\partial - 2\partial w + u\partial. \end{aligned}$$

The recursion relations given by (2.1) can be uniquely solved. So, we have

$$\hat{g}_1 = \left(\hat{g}_1^{(1)}, 3w^2 + 2uw - v, -(w_x + uw + 2w^2)\right)^T, \quad \check{g}_1 = \left(\check{g}_1^{(1)}, -\tfrac{1}{2}u - \tfrac{3}{2}w, w\right)^T, \tag{2.4}$$

with

$$\hat{g}_1^{(1)} = -w_{xx} - uw_x - u_x w + 2vw - 3uw^2 - 3ww_x - 4w^3,$$
$$\check{g}_1^{(1)} = \frac{3}{4}uw - \frac{1}{2}v + \frac{3}{4}w_x + \frac{1}{4}u_x - \frac{1}{8}u^2 + \frac{15}{8}w^2.$$

Now, we discuss the zero-curvature equation

$$V_x - [U, V] = 0, \tag{2.5}$$

where

$$U = \begin{pmatrix} 0 & 1 & 0 \\ \lambda + v & u & \lambda \\ w & 0 & 0 \end{pmatrix}, \quad V = (V_{ij})_{3\times 3} = \begin{pmatrix} V_{11} & V_{12} & \lambda V_{13} \\ V_{21} & V_{22} & \lambda V_{23} \\ V_{31} & V_{32} & \lambda V_{33} \end{pmatrix},$$

with λ as a constant spectral parameter; also, $u(x,t)$, $v(x,t)$ and $w(x,t)$ are three potentials. Each entry $V_{ij}(a,b,c)$ is defined as follows:

$$\begin{aligned} &V_{11} = -2\partial a + (v + u\partial - \partial^2)b + \lambda(b + c), \quad V_{12} = a, \quad V_{13} = b, \\ &V_{21} = (v - 2\partial^2)a + (\partial v + \partial u\partial - \partial^3)b + \lambda(a + wb + \partial b + \partial c), \\ &V_{22} = (u - \partial)a + (v + u\partial - \partial^2)b + \lambda(b + c), \quad V_{23} = a + \partial b, \\ &V_{31} = wa - (uw + \partial w)b + (u\partial + \partial^2)c, \quad V_{32} = wb - \partial c, \quad V_{33} = c. \end{aligned} \tag{2.6}$$

From (2.5) and (2.6), we obtain

$$KG = \lambda JG, \quad G = (a, b, c)^T. \tag{2.7}$$

Set

$$G = \sum_{j\geq 0} G_j \lambda^{-j}, \quad G_j = (a_j, b_j, c_j)^T; \tag{2.8}$$

then,

$$G_j = \alpha_0 \hat{g}_j + \beta_0 \check{g}_j + \alpha_1 \hat{g}_{j-1} + \beta_1 \check{g}_{j-1} + \cdots + \alpha_j \hat{g}_0 + \beta_j \check{g}_0 + \gamma_j \bar{g}_0, \quad j \geq 0, \tag{2.9}$$

satisfies

$$KG_j = JG_{j+1}, \quad JG_0 = 0, \tag{2.10}$$

for $\bar{g}_0 = (0,0,1)^T \in \ker K \cap \ker J$; also, α_j, β_j and γ_j are arbitrary constants.

In order to generate the three-component Burgers hierarchy, we set

$$\widetilde{G}_j = \tilde{\alpha}_0 \hat{g}_j + \tilde{\beta}_0 \check{g}_j + \tilde{\alpha}_1 \hat{g}_{j-1} + \tilde{\beta}_1 \check{g}_{j-1} + \cdots + \tilde{\alpha}_j \hat{g}_0 + \tilde{\beta}_j \check{g}_0 \triangleq (\tilde{a}_j, \tilde{b}_j, \tilde{c}_j)^T, \quad j \geq 0,$$

and give the following Lax pairs

$$\begin{cases} \psi_x = U\psi, \quad \psi = (\psi_1, \psi_2, \psi_3)^T, \\ \psi_{t_r} = (V_{ij}(\tilde{a}^{(r)}, \tilde{b}^{(r)}, \tilde{c}^{(r)}))_{3\times 3}\psi, \end{cases} \tag{2.11}$$

where

$$\tilde{a}^{(r)}=\sum_{j=0}^{r}\tilde{a}_j\lambda^{r-j},\quad \tilde{b}^{(r)}=\sum_{j=0}^{r}\tilde{b}_j\lambda^{r-j},\quad \tilde{c}^{(r)}=\sum_{j=0}^{r}\tilde{c}_j\lambda^{r-j}, \tag{2.12}$$

with $\tilde{\alpha}_j,\tilde{\beta}_j$ as arbitrary constants independent of α_j,β_j. The compatible condition of (2.11) implies that $U_{t_r}-\widetilde{V}^{(r)}_x+[U,\widetilde{V}^{(r)}]=0$ is equivalent to the following three-component Burgers equations:

$$(u_{t_r},v_{t_r},w_{t_r})^{\mathrm{T}}=K\widetilde{G}_r=J\widetilde{G}_{r+1},\qquad r\geq 0. \tag{2.13}$$

Next, we define a Lax matrix $V^{(n)}=(\lambda^nV)_+=(V^{(n)}_{ij}(a_j,b_j,c_j))_{3\times3}$, and

$$V^{(n)}_x-[U,V^{(n)}]=0, \tag{2.14}$$

$$V^{(n)}_{t_r}-[\widetilde{V}^{(r)},V^{(n)}]=0. \tag{2.15}$$

So, the characteristic polynomial of V^n is independent of (x,t_r). Moreover, we have

$$\det\left(yI-V^{(n)}\right)=y^3-y^2R_m(\lambda)+yS_m(\lambda)-T_m(\lambda), \tag{2.16}$$

where R_m,S_m,T_m are polynomials of λ:

$$\begin{aligned}R_m&=V^{(n)}_{11}+V^{(n)}_{22}+\lambda V^{(n)}_{33}=(2\alpha_0+3\gamma_0)\lambda^{n+1}+(2\alpha_1+3\gamma_1)\lambda^n+\cdots,\\ S_m&=\begin{vmatrix}V^{(n)}_{11}&V^{(n)}_{12}\\V^{(n)}_{21}&V^{(n)}_{22}\end{vmatrix}+\begin{vmatrix}V^{(n)}_{11}&\lambda V^{(n)}_{13}\\V^{(n)}_{31}&\lambda V^{(n)}_{33}\end{vmatrix}+\begin{vmatrix}V^{(n)}_{22}&\lambda V^{(n)}_{23}\\V^{(n)}_{32}&\lambda V^{(n)}_{33}\end{vmatrix}\\ &=(\alpha_0^2+4\alpha_0\gamma_0+3\gamma_0^2)\lambda^{2n+2}+\cdots,\\ T_m&=\det(V^{(n)})=\lambda[(\alpha_0+\gamma_0)^2\gamma_0\lambda^{3n+2}+\cdots].\end{aligned} \tag{2.17}$$

Hence, (2.16) leads to a trigonal curve $\mathcal{K}_{m-2}$:

$$\mathcal{K}_{m-2}:\quad \mathscr{F}_m(\lambda,y)=y^3-y^2R_m+yS_m-T_m=0. \tag{2.18}$$

The discriminant of (2.18) is

$$\Delta(\lambda)=4S_m^3-R_m^2S_m^2-18R_mS_mT_m+27T_m^2+4R_m^3T_m=-4\alpha_0^4\beta_0^2\lambda^{6n+5}+\cdots;$$

here, we assume that $\alpha_0\beta_0\gamma_0(\alpha_0+\gamma_0)\neq 0$ and $\mathcal{K}_{m-2}$ is nonsingular. So $\mathcal{K}_{m-2}$ can be compactified by adding the infinite point P_{∞_1} and the double branch point P_{∞_2}. In our paper, the compactification of $\mathcal{K}_{m-2}$ is still expressed by this symbol. Then, $\mathcal{K}_{m-2}$ becomes a three-sheeted Riemann surface, and its genus is $3n+1$. Each point $P=(\lambda,y)\in\mathcal{K}_{m-2}$ satisfies (2.18), along with P_{∞_1} and P_{∞_2}.

We asumme that P,P^*,P^{**} are three points on three different sheets of $\mathcal{K}_{m-2}$. Set $y_i(\lambda)$, $i=0,1,2$ to satisfy

$$(y-y_0(\lambda))(y-y_1(\lambda))(y-y_2(\lambda))=y^3-y^2R_m+yS_m-T_m=0. \tag{2.19}$$

Using (2.19), we obtain

$$\begin{aligned}&y_0+y_1+y_2=R_m,\\ &y_0y_1+y_1y_2+y_0y_2=S_m,\\ &y_0^2+y_1^2+y_2^2=R_m^2-2S_m,\\ &y_0y_1y_2=T_m,\\ &y_0^3+y_1^3+y_2^3=R_m^3-3R_mS_m+3T_m,\\ &y_0^2y_1^2+y_0^2y_2^2+y_1^2y_2^2=S_m^2-2R_mT_m.\end{aligned} \tag{2.20}$$

3. Asymptotic expansions

We shall discuss the asymptotic expansions of two associated meromorphic functions. Meanwhile, we introduce Abelian differentials to represent quasi-periodic solutions of the three-component Burgers hierarchy. First, the Baker-Akhiezer function $\psi(P, x, x_0, t_r, t_{0,r})$ satisfies

$$\begin{aligned}
&V^{(n)}(u(x,t_r), v(x,t_r), w(x,t_r); \lambda(P))\psi(P, x, x_0, t_r, t_{0,r}) = y(P)\psi(P, x, x_0, t_r, t_{0,r}),\\
&\psi_x(P, x, x_0, t_r, t_{0,r}) = U(u(x,t_r), v(x,t_r), w(x,t_r); \lambda(P))\psi(P, x, x_0, t_r, t_{0,r}),\\
&\psi_{t_r}(P, x, x_0, t_r, t_{0,r}) = \widetilde{V}^{(r)}(u(x,t_r), v(x,t_r), w(x,t_r); \lambda(P))\psi(P, x, x_0, t_r, t_{0,r}),\\
&\psi_1(P, x_0, x_0, t_{0,r}, t_{0,r}) = 1, \qquad x, t_r \in \mathbb{C}.
\end{aligned} \tag{3.1}$$

And the meromorphic functions $\phi_2(P, x, t_r)$,$\phi_3(P, x, t_r)$ on $\mathcal{K}_{m-2}$ are respectively introduced as follows:

$$\phi_2(P, x, t_r) = \frac{\psi_2(P, x, x_0, t_r, t_{0,r})}{\psi_1(P, x, x_0, t_r, t_{0,r})}, \quad P \in \mathcal{K}_{m-2}, \tag{3.2}$$

$$\phi_3(P, x, t_r) = \frac{\psi_3(P, x, x_0, t_r, t_{0,r})}{\psi_1(P, x, x_0, t_r, t_{0,r})}, \quad P \in \mathcal{K}_{m-2}. \tag{3.3}$$

Expressions (3.1)–(3.3) imply that ϕ_2 and ϕ_3 satisfy the following Riccati-type equations

$$\phi_{2,x} + \phi_2^2 - u\phi_2 - \lambda\phi_3 - v - \lambda = 0, \tag{3.4}$$

$$\phi_{3,x} + \phi_2\phi_3 - w = 0. \tag{3.5}$$

So, we can get the following lemma.

Lemma 3.1. *Let $P \in \mathcal{K}_{m-2} \setminus \{P_{\infty_1}, P_{\infty_2}\}$; then,*

$$\phi_2(P, x, t_r) \underset{\zeta\to 0}{=} \begin{cases} -w + \kappa_{1,1}\zeta + O(\zeta^2), & P \to P_{\infty_1}, \ \zeta = \lambda^{-1},\\ \zeta^{-1} + \kappa_{2,0} + \kappa_{2,1}\zeta + \kappa_{2,2}\zeta^2 + O(\zeta^3), & P \to P_{\infty_2}, \ \zeta = \lambda^{-\frac{1}{2}}, \end{cases} \tag{3.6}$$

$$\phi_3(P, x, t_r) \underset{\zeta\to 0}{=} \begin{cases} -1 + \chi_{1,1}\zeta + O(\zeta^2), & P \to P_{\infty_1}, \ \zeta = \lambda^{-1},\\ w\zeta + \chi_{2,2}\zeta^2 + \chi_{2,3}\zeta^3 + O(\zeta^4), & P \to P_{\infty_2}, \ \zeta = \lambda^{-\frac{1}{2}}, \end{cases} \tag{3.7}$$

where

$$\begin{aligned}
\kappa_{1,1} &= 3ww_x + (uw)_x - w_{xx} - v_x - w^3 - uw^2 + vw,\\
\kappa_{2,0} &= \frac{u+w}{2},\\
\kappa_{2,1} &= \frac{v}{2} - \frac{u_x}{4} - \frac{3w_x}{4} + \frac{(u+w)(u-3w)}{8},\\
\kappa_{2,2} &= \frac{u_{xx}}{8} - \frac{v_x}{4} + \frac{7w_{xx}}{8} - \frac{uu_x}{8} + \frac{5(uw)_x}{8} + \frac{15ww_x}{8} - \frac{wv}{2} + \frac{w^2(u+w)}{2},\\
\chi_{1,1} &= w^2 + uw - v - w_x,\\
\chi_{2,2} &= -w_x - \frac{uw + w^2}{2},\\
\chi_{2,3} &= w_{xx} + uw_x + \frac{3u_x w}{4} + \frac{9ww_x}{4} - \frac{wv}{2} + \frac{w(u+w)(u+5w)}{8}.
\end{aligned}$$

Proof. Substituting the hypotheses

$$
\begin{aligned}
&(1)\ \ \phi_2 \underset{\zeta\to 0}{=} \kappa_{1,0} + \kappa_{1,1}\zeta + O(\zeta^2), \quad \phi_3 \underset{\zeta\to 0}{=} \chi_{1,0} + \chi_{1,1}\zeta + O(\zeta^2);\\
&(2)\ \ \phi_2 \underset{\zeta\to 0}{=} \kappa_{1,-1}\zeta^{-1} + \kappa_{2,0} + \kappa_{2,1}\zeta + \kappa_{2,2}\zeta^2 + O(\zeta^3),\\
&\quad\ \ \phi_3 \underset{\zeta\to 0}{=} \chi_{2,1}\zeta + \chi_{2,2}\zeta^2 + \chi_{2,3}\zeta^3 + O(\zeta^4);
\end{aligned}
$$

into (3.4) and (3.5), and by analyzing the coefficients of ζ, we prove the lemma. □

Next, we study the asymptotic properties of ψ_1 near P_{∞_1} and P_{∞_2}. By means of the first two expressions of (3.1), we obtain

$$
\psi_1(P, x, x_0, t_r, t_{0,r}) = \exp\Big(\int_{x_0}^{x} \phi_2(P, x', t_r)dx' + \int_{t_{0,r}}^{t_r} I_r(P, x_0, t')dt'\Big), \tag{3.8}
$$

with

$$
I_r(P, x, t_r) = \widetilde{V}_{11}^{(r)} + \widetilde{V}_{12}^{(r)}\phi_2 + \lambda\widetilde{V}_{13}^{(r)}\phi_3, \tag{3.9}
$$

which can imply the essential singularities of ψ_1. We use the following symbols to represent the corresponding homogeneous case of $\widetilde{V}_{ij}^{(r)}$, that is to say,

$$
\bar{\bar{V}}_{ij}^{(r,1)} = \widetilde{V}_{ij}^{(r)}\ |_{\tilde{\alpha}_0=1,\tilde{\alpha}_1=\cdots=\tilde{\alpha}_r=\tilde{\beta}_0=\cdots=\tilde{\beta}_r=0}, \tag{3.10}
$$

$$
\bar{\bar{V}}_{ij}^{(r,2)} = \widetilde{V}_{ij}^{(r)}\ |_{\tilde{\beta}_0=1,\tilde{\alpha}_0=\cdots=\tilde{\alpha}_r=\tilde{\beta}_1=\cdots=\tilde{\beta}_r=0}\ . \tag{3.11}
$$

So, we have

$$
\bar{I}_r^{(\epsilon)}(P, x, t_r) = \bar{\bar{V}}_{11}^{(r,\epsilon)} + \bar{\bar{V}}_{12}^{(r,\epsilon)}\phi_2 + \lambda\bar{\bar{V}}_{13}^{(r,\epsilon)}\phi_3, \quad \epsilon = 1, 2. \tag{3.12}
$$

Especially,

$$
\bar{I}_r^{(1)}(P, x, t_r) = \begin{cases} -\hat{c}_{r+1} + O(\zeta), & P \to P_{\infty_1},\\ \zeta^{-2r-2} - \hat{b}_{r+1} - \hat{c}_{r+1} + O(\zeta), & P \to P_{\infty_2}. \end{cases} \tag{3.13}
$$

$$
\bar{I}_r^{(2)}(P, x, t_r) = \begin{cases} -\check{c}_{r+1} + O(\zeta), & P \to P_{\infty_1},\\ \zeta^{-2r-1} - \check{b}_{r+1} - \check{c}_{r+1} + O(\zeta), & P \to P_{\infty_2}. \end{cases} \tag{3.14}
$$

In fact, consider that $(r, \epsilon) = (0, 1)$, $\bar{\bar{a}}^{(0,1)} = -w$, $\bar{\bar{b}}^{(0,1)} = 1$, $\bar{\bar{c}}^{(0,1)} = 0$, $\lambda = \zeta^{-1}$ denotes the local coordinate near P_{∞_1} and $\lambda = \zeta^{-2}$ denotes the local coordinate near P_{∞_2}. When $P \to P_{\infty_1}$,

$$
\begin{aligned}
\bar{I}_0^{(1)}(P, x, t_r) &= -2\bar{\bar{a}}_x^{(0,1)} - \bar{\bar{b}}_{xx}^{(0,1)} + u\bar{\bar{b}}_x^{(0,1)} + v\bar{\bar{b}}^{(0,1)} + \zeta^{-1}(\bar{\bar{b}}^{(0,1)} + \bar{\bar{c}}^{(0,1)})\\
&\quad + \bar{\bar{a}}^{(0,1)}\phi_2 + \zeta^{-1}\bar{\bar{b}}^{(0,1)}\phi_3\\
&= w_x + 2w^2 + uw + O(\zeta)\\
&= -\hat{c}_1 + O(\zeta),
\end{aligned} \tag{3.15}
$$

and, when $P \to P_{\infty_2}$,

$$
\begin{aligned}
\bar{I}_0^{(1)}(P, x, t_r) &= -2\bar{\bar{a}}_x^{(0,1)} - \bar{\bar{b}}_{xx}^{(0,1)} + u\bar{\bar{b}}_x^{(0,1)} + v\bar{\bar{b}}^{(0,1)} + \zeta^{-2}(\bar{\bar{b}}^{(0,1)} + \bar{\bar{c}}^{(0,1)})\\
&\quad + \bar{\bar{a}}^{(0,1)}\phi_2 + \zeta^{-2}\bar{\bar{b}}^{(0,1)}\phi_3\\
&= \xi^{-2} + w_x + v - uw - w^2 + O(\zeta)\\
&= \xi^{-2} - \hat{b}_1 - \hat{c}_1 + O(\zeta).
\end{aligned} \tag{3.16}
$$

So, as $P \to P_{\infty_1}$, one infers the following:

$$\bar{I}_r^{(1)}(P,x,t_r) = \sum_{j=0}^{\infty} \sigma_j \zeta^j; \tag{3.17}$$

as $P \to P_{\infty_1}$, we assume

$$\bar{I}_r^{(1)}(P,x,t_r) = \zeta^{-2r-2} + \sum_{j=0}^{\infty} \delta_j \zeta^j, \tag{3.18}$$

and $\{\sigma_j(x,t_r)\}, \{\delta_j(x,t_r)\}$, $j \in \mathbb{N}$ are some coefficients. Using (3.1), we obtain

$$\frac{\psi_{1,x}}{\psi_1} = \phi_2, \tag{3.19}$$

$$\frac{\psi_{1,t_r}}{\psi_1} = \widetilde{V}_{11}^{(r)} + \widetilde{V}_{22}^{(r)}\phi_2 + \lambda \widetilde{V}_{13}^{(r)}\phi_3. \tag{3.20}$$

Furthermore, we have

$$(\phi_2)_{t_r} = (\widetilde{V}_{11}^{(r)} + \widetilde{V}_{22}^{(r)}\phi_2 + \lambda \widetilde{V}_{13}^{(r)}\phi_3)_x. \tag{3.21}$$

When $\tilde{\alpha}_0 = 1, \tilde{\alpha}_1 = \cdots = \tilde{\alpha}_r = 0$ and $\tilde{\beta}_0 = \cdots = \tilde{\beta}_r = 0$, we arrive at

$$\begin{aligned} &\sigma_{0,x} = (\kappa_{1,0})_{t_r}, \\ &\sigma_{j,x} = (\kappa_{1,j})_{t_r}, \\ &\delta_{0,x} = (\kappa_{2,0})_{t_r}, \\ &\delta_{j,x} = (\kappa_{2,j})_{t_r}, \quad j = 1,2,3\ldots. \end{aligned} \tag{3.22}$$

Combining (2.5), (2.13) and Lemma 3.1, one gets

$$\begin{aligned} \sigma_{0,x} =\ & (-\hat{c}_{r+1})_x, \\ \sigma_{1,x} =\ & (w\hat{a}_{r+1} + 2\hat{a}_{r+1,x} + \hat{b}_{r+1,xx} + w_x\hat{b}_{r+1} - uw\hat{b}_{r+1} - w^2\hat{b}_{r+1} - \hat{c}_{r+2} - u\hat{b}_{r+1,x})_x, \\ \delta_{0,x} =\ & -\hat{b}_{r+1,x} - \hat{c}_{r+1,x}, \\ \delta_{1,x} =\ & -\hat{a}_{r+1,x} - w_x\hat{b}_{r+1} - w\hat{b}_{r+1,x}, \\ \delta_{2,x} =\ & (\frac{1}{2}\hat{a}_{r+1,x} - \frac{1}{2}w\hat{a}_{r+1} + w_x\hat{b}_{r+1} + \frac{1}{2}uw\hat{b}_{r+1} + \frac{1}{2}w^2\hat{b}_{r+1} + \frac{1}{2}\hat{c}_{r+2})_x, \end{aligned} \tag{3.23}$$

from which it can be inferred that

$$\begin{aligned} \sigma_0 =\ & -\hat{c}_{r+1}, \\ \sigma_1 =\ & w\hat{a}_{r+1} + 2\hat{a}_{r+1,x} + \hat{b}_{r+1,xx} + w_x\hat{b}_{r+1} - u\hat{b}_{r+1,x} - uw\hat{b}_{r+1} - w^2\hat{b}_{r+1} - \hat{c}_{r+2}, \\ \delta_0 =\ & -\hat{b}_{r+1} - \hat{c}_{r+1}, \\ \delta_1 =\ & -\hat{a}_{r+1} - w\hat{b}_{r+1}, \\ \delta_2 =\ & \frac{1}{2}\hat{a}_{r+1,x} - \frac{1}{2}w\hat{a}_{r+1} + w_x\hat{b}_{r+1} + \frac{1}{2}uw\hat{b}_{r+1} + \frac{1}{2}w^2\hat{b}_{r+1} + \frac{1}{2}\hat{c}_{r+2}. \end{aligned} \tag{3.24}$$

Furthermore,

$$
\begin{aligned}
\overline{I}^{(1)}_{r+1} &= \overline{\overline{V}}^{(r+1,1)}_{11} + \overline{\overline{V}}^{(r+1,1)}_{12}\phi_2 + \lambda\overline{\overline{V}}^{(r+1,1)}_{13}\phi_3 \\
&= \zeta^{-1}\overline{I}_r - 2\hat{a}_{r+1,x} - \hat{b}_{r+1,xx} + u\hat{b}_{r+1,x} + v\hat{b}_{r+1} + \zeta^{-1}(\hat{b}_{r+1} + \hat{c}_{r+1}) \\
&\quad + \hat{a}_{r+1}\phi_2 + \zeta^{-1}\hat{b}_{r+1}\phi_3 \\
&= -\hat{c}_{r+2} + O(\zeta), \qquad\qquad P \to P_{\infty_1}, \\
\overline{I}^{(1)}_{r+1} &= \overline{\overline{V}}^{(r+1,1)}_{11} + \overline{\overline{V}}^{(r+1,1)}_{12}\phi_2 + \lambda\overline{\overline{V}}^{(r+1,1)}_{13}\phi_3 \\
&= \zeta^{-2}\overline{I}_r - 2\hat{a}_{r+1,x} - \hat{b}_{r+1,xx} + u\hat{b}_{r+1,x} + v\hat{b}_{r+1} + \zeta^{-2}(\hat{b}_{r+1} + \hat{c}_{r+1}) \\
&\quad + \hat{a}_{r+1}\phi_2 + \zeta^{-2}\hat{b}_{r+1}\phi_3 \\
&= \zeta^{-2r-4} - \hat{b}_{r+2} - \hat{c}_{r+2} + O(\zeta), \qquad P \to P_{\infty_2},
\end{aligned}
\tag{3.25}
$$

so (3.13) holds. Equation (3.14) can be proved by using the same method.

By considering (3.13), (3.14) and (2.12), one infers the following:

$$
I_r(P, x, t_r) = \begin{cases} -\tilde{c}_{r+1} + O(\zeta), & P \to P_{\infty_1}, \\ \sum\limits_{l=0}^{r}(\tilde{\alpha}_{r-l}\zeta^{-2l-2} + \tilde{\beta}_{r-l}\zeta^{-2l-1}) - \tilde{b}_{r+1} - \tilde{c}_{r+1} + \tilde{\alpha}_{r+1} + O(\zeta), & P \to P_{\infty_2}. \end{cases}
\tag{3.26}
$$

Given those above preparations, the asymptotic expansions of ψ_1 near P_{∞_1} and P_{∞_2} read as follows.

Lemma 3.2. *Let $P \in \mathcal{K}_{m-2} \setminus \{P_{\infty_1}, P_{\infty_2}\}$ and let $(x, x_0, t_r, t_{0,r}) \in \mathbb{C}^4$; so,*

$$
\psi_1(P, x, x_0, t_r, t_{0,r}) \underset{\zeta\to 0}{=} \begin{cases} \exp\left(\Delta_1 + O(\zeta)\right), & P \to P_{\infty_1}, \\ \exp\Big(\zeta^{-1}(x - x_0) + (\sum\limits_{l=0}^{r}(\tilde{\alpha}_{r-l}\zeta^{-2(1+l)} + \tilde{\beta}_{r-l}\zeta^{-(1+2l)}) + \tilde{\alpha}_{r+1}) & \\ \qquad \times(t_r - t_{0,r}) - \dfrac{1}{2}(\Delta_1 + \Delta_2) + O(\zeta)\Big), & \\ & P \to P_{\infty_2}, \end{cases}
\tag{3.27}
$$

where

$$
\Delta_1 = \partial^{-1}w(x_0, t_{0,r}) - \partial^{-1}w(x, t_r), \quad \Delta_2 = \partial^{-1}u(x_0, t_{0,r}) - \partial^{-1}u(x, t_r),
$$

are two functions independent of variable x.

Regarding $\mathcal{K}_{m-2}$, one can choose the basis $\{\mathbb{a}_j, \mathbb{b}_j\}_{j=1}^{m-2}$ with the intersection numbers

$$
\mathbb{a}_j \circ \mathbb{a}_k = 0, \quad \mathbb{b}_j \circ \mathbb{b}_k = 0, \quad \mathbb{a}_j \circ \mathbb{b}_k = \delta_{j,k}, \quad j, k = 1, \ldots, m-2,
$$

Our basis is as follows

$$
\varpi_l(P) = \frac{1}{3y^2 - 2yR_m + S_m}\begin{cases} \lambda^{l-1}d\lambda, & 1 \le l \le 1 + 2n, \\ (y - \frac{1}{3}R_m)\lambda^{l-2n-2}d\lambda, & 2 + 2n \le l \le m - 2. \end{cases}
\tag{3.28}
$$

Let us construct the matrices $A = (A_{jk})$ and $B = (B_{jk})$ by respectively applying the following:

$$
A_{jk} = \int_{\mathbb{a}_k} \varpi_j, \qquad B_{jk} = \int_{\mathbb{b}_k} \varpi_j.
\tag{3.29}
$$

Then, $\tau = A^{-1}B$ is a symmetric matrix [32–34] and we denote $C = A^{-1}$. If $\varpi_l(P)$ takes the normalized basis $\underline{\omega} = (\omega_1, \ldots, \omega_{m-2})$ with

$$\omega_j = \sum_{l=1}^{m-2} C_{jl}\varpi_l, \tag{3.30}$$

then $\int_{\mathbb{a}_k} \omega_j = \delta_{jk}$ and $\int_{\mathbb{b}_k} \omega_j = \tau_{jk}, \quad j,k = 1,2,\ldots,m-2.$

Utilizing (3.1) and (3.6)–(3.7), one gets the asymptotic properties of $y(P)$:

$$y(P) \underset{\zeta\to 0}{=} \begin{cases} \zeta^{-n-1}\Big(\gamma_0 + \gamma_1\zeta + O(\zeta^2)\Big), & \text{as } P \to P_{\infty_1}, \\ \zeta^{-2n-2}\Big(\alpha_0 + \gamma_0 + \beta_0\zeta + (\alpha_1 + \gamma_1)\zeta^2 + O(\zeta^3)\Big), & \text{as } P \to P_{\infty_2}. \end{cases} \tag{3.31}$$

So,

$$\omega_j \underset{\zeta\to 0}{=} \begin{cases} (d_{j,0}^{(\infty_1)} + O(\zeta))d\zeta, & P \to P_{\infty_1}, \\ (d_{j,0}^{(\infty_2)} + O(\zeta))d\zeta, & P \to P_{\infty_2}, \end{cases} \tag{3.32}$$

where

$$d_{j,0}^{(\infty_1)} = -\frac{1}{\alpha_0^2}C_{j,2n+1} - \frac{\gamma_0}{\alpha_0^2}C_{j,m-2}, \quad d_{j,0}^{(\infty_2)} = -\frac{1}{\alpha_0\beta_0}C_{j,2n+1} - \frac{\alpha_0+\gamma_0}{\alpha_0\beta_0}C_{j,m-2}.$$

Moreover, ω_j can be rewritten as

$$\omega_j = \sum_{l=0}^{\infty} \varrho_{j,l}(P_{\infty_s})\zeta^l d\zeta, \quad P \to P_{\infty_s}, s = 1,2, \tag{3.33}$$

where $\varrho_{j,l}(P_{\infty_s})$ represents constants.

Let $\omega_{P_{\infty_2},j}^{(2)}(P)$ $(j \geq 2)$ be the normalized differential of the second kind, satisfying

$$\omega_{P_{\infty_2},j}^{(2)}(P) \underset{\zeta\to 0}{=} (\zeta^{-j} + O(1))d\zeta, \quad P \to P_{\infty_2}, \tag{3.34}$$

with

$$\int_{\mathbb{a}_k} \omega_{P_{\infty_2},j}^{(2)}(P) = 0, \quad k = 1,2,\ldots,m-2. \tag{3.35}$$

We introduce

$$\Omega_2^{(2)}(P) = \omega_{P_{\infty_2},2}^{(2)}(P), \tag{3.36}$$

$$\widetilde{\Omega}_{2r+3}^{(2)}(P) = \sum_{l=0}^{r}(2+2l)\tilde{\alpha}_{r-l}\omega_{P_{\infty_2},3+2l}^{(2)}(P) + \sum_{l=0}^{r}(2l+1)\tilde{\beta}_{r-l}\omega_{P_{\infty_2},2+2l}^{(2)}(P). \tag{3.37}$$

Then, we have

$$\int_{Q_0}^{P} \Omega_2^{(2)}(P) = \begin{cases} e_1^{(2)}(Q_0) + O(\zeta), & P \to P_{\infty_1}, \\ -\zeta^{-1} + e_2^{(2)}(Q_0) + O(\zeta), & P \to P_{\infty_2}, \end{cases} \tag{3.38}$$

$$\int_{Q_0}^{P} \widetilde{\Omega}_{2r+3}^{(2)}(P) = \begin{cases} \tilde{e}_1^{(2)}(Q_0) + O(\zeta), & P \to P_{\infty_1}, \\ -\sum_{l=0}^{r}\tilde{\alpha}_{r-l}\zeta^{-2(1+l)} - \sum_{l=0}^{r}\tilde{\beta}_{r-l}\zeta^{-(1+2l)} + \tilde{e}_2^{(2)}(Q_0) + O(\zeta), & \\ & P \to P_{\infty_2}, \end{cases} \tag{3.39}$$

for some constants $e_1^{(2)}, e_2^{(2)}, \tilde{e}_1^{(2)}, \tilde{e}_2^{(2)}$ depending on the appropriately chosen point Q_0. The associated $\mathbb{b}$-periods are defined by

$$\underline{U}_2^{(2)} = (\frac{1}{2\pi i}\int_{\mathbb{b}_1}\Omega_2^{(2)}(P),\ldots,\frac{1}{2\pi i}\int_{\mathbb{b}_{m-2}}\Omega_2^{(2)}(P)), \tag{3.40}$$

$$\underline{\widetilde{U}}_{2r+3}^{(2)} = (\frac{1}{2\pi i}\int_{\mathbb{b}_1}\widetilde{\Omega}_{2r+3}^{(2)}(P),\ldots,\frac{1}{2\pi i}\int_{\mathbb{b}_{m-2}}\widetilde{\Omega}_{2r+3}^{(2)}(P)). \tag{3.41}$$

By means of the relations between the second kind of differential and holomorphic differential $\underline{\omega}$, we can respectively express the members $U_{2,k}^{(2)}$ of $U_2^{(2)}$ and $\widetilde{U}_{2r+3,k}^{(2)}$ of $\widetilde{U}_{2r+3}^{(2)}$ as follows:

$$\begin{aligned} &U_{2,k}^{(2)} = \varrho_{k,0}(P_{\infty_2}),\\ &\widetilde{U}_{2r+3,k}^{(2)} = \sum_{l=0}^{r}\tilde{\alpha}_{r-l}\varrho_{k,2l+1}(P_{\infty_2}) + \sum_{l=0}^{r}\tilde{\beta}_{r-l}\varrho_{k,2l}(P_{\infty_2}), \qquad k = 1,\ldots,m-2. \end{aligned} \tag{3.42}$$

We define the Abelian differential of the third kind, $\omega_{Q_1,Q_2}^{(3)}$, on $\mathcal{K}_{m-2} \setminus \{Q_1, Q_2\}$, i.e.,

$$\int_{\mathbb{a}_k}\omega_{Q_1,Q_2}^{(3)} = 0, \quad \int_{\mathbb{b}_k}\omega_{Q_1,Q_2}^{(3)} = 2\pi i\int_{Q_2}^{Q_1}\omega_k. \tag{3.43}$$

In particular,

$$\int_{Q_0}^{P}\omega_{P_{\infty_2},\hat{v}_0}^{(3)} \underset{\zeta\to 0}{=} \begin{cases} e_{1,\infty_1}(Q_0) + O(\zeta), & P \to P_{\infty_1},\\ -\ln\zeta + e_{1,\hat{v}_0}(Q_0) + \omega_0^{\hat{v}_0}\zeta + O(\zeta^2), & P \to \hat{v}_0,\\ \ln\zeta + e_{1,\infty_2}(Q_0) + O(\zeta), & P \to P_{\infty_2}, \end{cases} \tag{3.44}$$

$$\int_{Q_0}^{P}\omega_{P_0,P_{\infty_2}}^{(3)} \underset{\zeta\to 0}{=} \begin{cases} e_{2,\infty_1}(Q_0) + \omega_0^{\infty_1}\zeta + O(\zeta^2), & P \to P_{\infty_1},\\ \ln\zeta + e_{2,P_0}(Q_0) + O(\zeta), & P \to P_0,\\ -\ln\zeta + e_{2,\infty_2}(Q_0) + \omega_0^{\infty_2}\zeta + O(\zeta^2), & P \to P_{\infty_2}, \end{cases} \tag{3.45}$$

with integration constants $e_{1,\infty_s}(Q_0), e_{2,\infty_s}(Q_0), e_{1,\hat{v}_0}(Q_0), e_{2,P_0}(Q_0)$ and $s = 1, 2$.

4. Divisors of ϕ_2 and ϕ_3

We will propose the divisors of two meromorphic functions ϕ_2 and ϕ_3. The definitions of the two meromorphic functions in (3.2) and (3.3) respectively imply that

$$\begin{aligned} \phi_2 = \frac{yV_{23}^{(n)} + C_m}{yV_{13}^{(n)} + A_m} &= \frac{y^2V_{13}^{(n)} - y(A_m + V_{13}^{(n)}R_m) + B_m}{E_{m-2}}\\ &= \frac{F_{m-2}}{y^2V_{23}^{(n)} - y(C_m + V_{23}^{(n)}R_m) + D_m}, \end{aligned} \tag{4.1}$$

$$\begin{aligned} \phi_3 = \frac{yV_{32}^{(n)} + C_m}{yV_{12}^{(n)} + \mathcal{A}_m} &= \frac{y^2V_{12}^{(n)} - y(\mathcal{A}_m + V_{12}^{(n)}R_m) + \mathcal{B}_m}{-\lambda E_{m-2}}\\ &= \frac{\mathcal{F}_{m-2}}{y^2V_{32}^{(n)} - y(C_m + V_{32}^{(n)}R_m) + \mathcal{D}_m}, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned}A_m &= -V_{13}^{(n)}V_{22}^{(n)}+V_{12}^{(n)}V_{23}^{(n)},\\ B_m &= V_{12}^{(n)}(V_{23}^{(n)}V_{11}^{(n)}-V_{13}^{(n)}V_{21}^{(n)})+\lambda V_{13}^{(n)}(V_{33}^{(n)}V_{11}^{(n)}-V_{13}^{(n)}V_{31}^{(n)}),\\ C_m &= -V_{11}^{(n)}V_{23}^{(n)}+V_{13}^{(n)}V_{21}^{(n)},\\ D_m &= \lambda V_{23}^{(n)}(V_{22}^{(n)}V_{33}^{(n)}-V_{23}^{(n)}V_{32}^{(n)})+V_{21}^{(n)}(V_{13}^{(n)}V_{22}^{(n)}-V_{23}^{(n)}V_{12}^{(n)}),\end{aligned}\tag{4.3}$$

$$\begin{aligned}\mathcal{A}_m &= \lambda(V_{13}^{(n)}V_{32}^{(n)}-V_{12}^{(n)}V_{33}^{(n)}),\\ \mathcal{B}_m &= V_{12}^{(n)}(V_{22}^{(n)}V_{11}^{(n)}-V_{12}^{(n)}V_{21}^{(n)})+\lambda V_{13}^{(n)}(V_{11}^{(n)}V_{32}^{(n)}-V_{12}^{(n)}V_{31}^{(n)}),\\ \mathcal{C}_m &= V_{12}^{(n)}V_{31}^{(n)}-V_{11}^{(n)}V_{32}^{(n)},\\ \mathcal{D}_m &= \lambda V_{32}^{(n)}(V_{22}^{(n)}V_{33}^{(n)}-V_{32}^{(n)}V_{23}^{(n)})+\lambda V_{31}^{(n)}(V_{33}^{(n)}V_{12}^{(n)}-V_{13}^{(n)}V_{32}^{(n)}),\end{aligned}\tag{4.4}$$

$$\begin{aligned}E_{m-2} &= \lambda V_{32}^{(n)}(V_{13}^{(n)})^2-(V_{12}^{(n)})^2V_{23}^{(n)}+V_{12}^{(n)}V_{13}^{(n)}(V_{22}^{(n)}-\lambda V_{33}^{(n)}),\\ F_{m-2} &= \lambda(V_{23}^{(n)})^2V_{31}^{(n)}+(V_{11}^{(n)}-\lambda V_{33}^{(n)})V_{23}^{(n)}V_{21}^{(n)}-V_{13}^{(n)}(V_{21}^{(n)})^2,\\ \mathcal{F}_{m-2} &= V_{21}^{(n)}(V_{32}^{(n)})^2+(V_{11}^{(n)}-V_{22}^{(n)})V_{31}^{(n)}V_{32}^{(n)}-V_{12}^{(n)}(V_{31}^{(n)})^2.\end{aligned}\tag{4.5}$$

By complex computation, we obtain

$$\begin{aligned}&V_{13}^{(n)}F_{m-2}=V_{23}^{(n)}D_m-(V_{23}^{(n)})^2S_m-C_m(V_{23}^{(n)}R_m+C_m),\\ &A_mF_{m-2}=(V_{23}^{(n)})^2T_m+C_mD_m,\end{aligned}\tag{4.6}$$

$$\begin{aligned}&V_{23}^{(n)}E_{m-2}=V_{13}^{(n)}B_m-(V_{13}^{(n)})^2S_m-A_m(V_{13}^{(n)}R_m+A_m),\\ &C_mE_{m-2}=(V_{13}^{(n)})^2T_m+A_mB_m,\end{aligned}\tag{4.7}$$

$$\begin{aligned}&V_{23}^{(n)}B_m+V_{13}^{(n)}D_m-V_{13}^{(n)}V_{23}^{(n)}S_m+A_mC_m=0,\\ &V_{13}^{(n)}V_{23}^{(n)}T_m+(V_{23}^{(n)}S_m-D_m)(V_{13}^{(n)}R_m+A_m)+V_{13}^{(n)}C_mS_m\\ &\qquad\qquad -B_m(C_m+V_{23}^{(n)}R_m)=0,\\ &V_{23}^{(n)}T_m(A_m+V_{13}^{(n)}R_m)+V_{13}^{(n)}C_mT_m+E_{m-2}F_{m-2}-B_mD_m=0,\end{aligned}\tag{4.8}$$

$$\begin{aligned}&V_{12}^{(n)}\mathcal{F}_{m-2}=V_{32}^{(n)}\mathcal{D}_m-(V_{32}^{(n)})^2S_m-\mathcal{C}_m(V_{32}R_m+\mathcal{C}_m),\\ &\mathcal{A}_m\mathcal{F}_{m-2}=(V_{32}^{(n)})^2T_m+\mathcal{C}_m\mathcal{D}_m,\end{aligned}\tag{4.9}$$

$$\begin{aligned}&-\lambda V_{32}^{(n)}E_{m-2}=V_{12}^{(n)}\mathcal{B}_m-(V_{12}^{(n)})^2S_m-\mathcal{A}_m(V_{12}R_m+\mathcal{A}_m),\\ &-\lambda\mathcal{C}_mE_{m-2}=(V_{12}^{(n)})^2T_m+\mathcal{A}_m\mathcal{B}_m,\end{aligned}\tag{4.10}$$

$$\begin{aligned}&V_{32}^{(n)}\mathcal{B}_m+V_{12}^{(n)}\mathcal{D}_m-V_{12}^{(n)}V_{32}^{(n)}S_m+\mathcal{A}_m\mathcal{C}_m=0,\\ &V_{12}^{(n)}V_{32}^{(n)}T_m+(V_{32}^{(n)}S_m-\mathcal{D}_m)(\mathcal{A}_m+V_{12}R_m)+V_{12}^{(n)}\mathcal{C}_mS_m=\mathcal{B}_m(V_{32}R_m+\mathcal{C}_m),\\ &V_{32}^{(n)}T_m(\mathcal{A}_m+V_{12}R_m)+V_{12}^{(n)}\mathcal{C}_mT_m-\lambda E_{m-2}\mathcal{F}_{m-2}-\mathcal{B}_m\mathcal{D}_m=0,\end{aligned}\tag{4.11}$$

$$\begin{aligned}E_{m-2,x} &= -uE_{m-2}+(3B_m-2V_{13}^{(n)}S_m-R_mA_m),\\ V_{23}^{(n)}F_{m-2,x} &= ((\lambda+v)V_{23}^{(n)}-V_{21}^{(n)})(3D_m-2V_{23}^{(n)}S_m-R_mC_m)\\ &\quad +(2uV_{23}^{(n)}+R_m-3V_{22}^{(n)})F_{m-2},\\ \mathcal{F}_{m-2,x} &= -u\mathcal{F}_{m-2}+w(3\mathcal{D}_m-2V_{32}^{(n)}S_m-\mathcal{R}_m\mathcal{C}_m).\end{aligned}\tag{4.12}$$

Due to the observation of (4.5), one infers that E_{m-2}, F_{m-2} and $\mathcal{F}_{m-2}$ are polynomials of λ:

$$E_{m-2}=\alpha_0^2\beta_0\prod_{j=1}^{m-2}(\lambda-\mu_j(x,t_r)),\tag{4.13}$$

$$F_{m-2} = -\alpha_0^2\beta_0 w \prod_{j=0}^{m-2}(\lambda - \nu_j(x,t_r)), \tag{4.14}$$

$$\mathcal{F}_{m-2} = \alpha_0^2\beta_0 w^2 \prod_{j=1}^{m-2}(\lambda - \xi_j(x,t_r)). \tag{4.15}$$

As $\lambda = \mu_j(x,t_r)$, we get

$$\begin{aligned} E_{m-2} &= (\lambda(V_{13}^{(n)})^2 V_{32}^{(n)} + V_{13}^{(n)} V_{12}^{(n)}(V_{22}^{(n)} - \lambda V_{33}^{(n)}) - V_{23}^{(n)}(V_{12}^{(n)})^2) \\ &= (V_{13}^{(n)}\mathcal{A}_m - V_{12}^{(n)} A_m) = 0, \end{aligned}$$

so we can define

$$\begin{aligned} \hat{\mu}_j(x,t_r) &= \left(\mu_j(x,t_r), -\frac{\mathcal{A}_m(\mu_j(x,t_r),x,t_r)}{V_{12}^{(n)}(\mu_j(x,t_r),x,t_r)}\right), \\ \hat{\xi}_j(x,t_r) &= \left(\xi_j(x,t_r), -\frac{C_m(\xi_j(x,t_r),x,t_r)}{V_{32}^{(n)}(\xi_j(x,t_r),x,t_r)}\right), \quad j = 1,\ldots,m-2. \\ \hat{\nu}_j(x,t_r) &= \left(\nu_j(x,t_r), -\frac{C_m(\nu_j(x,t_r),x,t_r)}{V_{23}^{(n)}(\nu_j(x,t_r),x,t_r)}\right), \end{aligned} \tag{4.16}$$

In fact, for $\lambda = \mu_j(x,t_r)$, combining (4.7) and (4.13), we have

$$\begin{aligned} 0 &= V_{13}^{(n)} B_m - (V_{13}^{(n)})^2 S_m - A_m(V_{13}^{(n)} R_m + A_m), \\ 0 &= (V_{13}^{(n)})^2 T_m + A_m B_m, \end{aligned}$$

that is,

$$\left(-\frac{A_m}{V_{13}^{(n)}}\right)^3 - \left(-\frac{A_m}{V_{13}^{(n)}}\right)^2 R_m + \left(-\frac{A_m}{V_{13}^{(n)}}\right) S_m - T_m = 0,$$

which means that

$$(y^3 - y^2 R_m + y S_m - T_m)\,\Big|_{(\lambda,y)=(\mu_j(x,t_r), -\frac{\mathcal{A}_m(\mu_j(x,t_r),x,t_r)}{V_{12}^{(n)}(\mu_j(x,t_r),x,t_r)})} = 0;$$

so, the first definition of (4.16) is reasonable. Similarly, we can prove the others.

From (3.6), (3.7), (4.1) and (4.2), one infers that the divisors of ϕ_2 and ϕ_3 have the following respective forms:

$$(\phi_2(P,x,t_r)) = \mathcal{D}_{\hat{\nu}_0(x,t_r),\hat{\nu}_1(x,t_r),\ldots,\hat{\nu}_{m-2}(x,t_r)}(P) - \mathcal{D}_{P_{\infty_2},\hat{\mu}_1(x,t_r),\ldots,\hat{\mu}_{m-2}(x,t_r)}(P), \tag{4.17}$$

$$(\phi_3(P,x,t_r)) = \mathcal{D}_{P_{\infty_2},\hat{\xi}_1(x,t_r),\ldots,\hat{\xi}_{m-2}(x,t_r)}(P) - \mathcal{D}_{P_0,\hat{\mu}_1(x,t_r),\ldots,\hat{\mu}_{m-2}(x,t_r)}(P). \tag{4.18}$$

Next, our main purpose is to discuss the poles and zeros of ψ_1 on $\mathcal{K}_{m-2}\setminus\{P_{\infty_1}, P_{\infty_2}\}$. Observing (4.1) and (4.2), one gets

$$\phi_{2,t_r}(P,x,t_r) = \widetilde{V}_{21}^{(r)} + (\widetilde{V}_{22}^{(r)} - \widetilde{V}_{11}^{(r)})\phi_2 + \lambda\widetilde{V}_{23}^{(r)}\phi_3 - \widetilde{V}_{12}^{(r)}\phi_2^2 - \lambda\widetilde{V}_{13}^{(r)}\phi_2\phi_3, \tag{4.19}$$

$$\phi_{3,t_r}(P,x,t_r) = \widetilde{V}_{31}^{(r)} + (\lambda\widetilde{V}_{33}^{(r)} - \widetilde{V}_{11}^{(r)})\phi_3 + \widetilde{V}_{32}^{(r)}\phi_2 - \lambda\widetilde{V}_{13}^{(r)}\phi_3^2 - \widetilde{V}_{12}^{(r)}\phi_2\phi_3. \tag{4.20}$$

Lemma 4.1. *We suppose that (3.1) holds. Let* $(\lambda, x, t_r) \in \mathbb{C}^3$*; then,*

$$\begin{aligned}
E_{m-2,t_r}(\lambda,x,t_r) &= \widetilde{V}_{12}^{(r)}E_{m-2,x} + (3\widetilde{V}_{11}^{(r)} + u\widetilde{V}_{12}^{(r)} - \partial^{-1}u_{t_r})E_{m-2} - \widetilde{V}_{13}^{(r)}(3\mathcal{B}_m - 2V_{12}^{(n)}S_m - R_m\mathcal{A}_m),\\
F_{m-2,t_r}(\lambda,x,t_r) &= \frac{\widetilde{V}_{21}^{(r)}V_{23}^{(n)} - \lambda\widetilde{V}_{23}^{(r)}V_{21}^{(n)}}{(\lambda+v)V_{23}^{(n)} - vV_{21}^{(n)}}F_{m-2,x} + (3V_{22}^{(n)} - \partial^{-1}u_{t_r})F_{m-2}\\
&\quad + \frac{2u(\lambda\widetilde{V}_{23}^{(r)}V_{21}^{(n)} - \widetilde{V}_{21}^{(r)}V_{23}^{(n)}) + (R_m - 3V_{22}^{(n)})((\lambda+v)\lambda V_{23}^{(r)} - V_{21}^{(r)})}{(\lambda+v)V_{23}^{(n)} - V_{21}^{(n)}}F_{m-2},\\
\mathcal{F}_{m-2,t_r}(\lambda,x,t_r) &= \frac{\widetilde{V}_{31}^{(r)}V_{32}^{(n)} - \widetilde{V}_{32}^{(r)}V_{31}^{(n)}}{wV_{32}^{(n)}}\mathcal{F}_{m-2,x} - \partial^{-1}u_{t_r}\mathcal{F}_{m-2}\\
&\quad + \left[\frac{3\lambda\widetilde{V}_{33}^{(r)}V_{32}^{(n)} - 3\widetilde{V}_{32}^{(r)}V_{33}^{(n)} + R_m\widetilde{V}_{32}^{(r)}}{V_{32}^{(n)}} + u\cdot\frac{\widetilde{V}_{31}^{(r)}V_{32}^{(n)} - \widetilde{V}_{32}^{(r)}V_{31}^{(n)}}{wV_{32}^{(n)}}\right]\mathcal{F}_{m-2}.
\end{aligned} \tag{4.21}$$

Proof. Observing the compatibility condition given by (3.21), we have

$$\begin{aligned}
\left(\frac{E_{m-2,x}}{E_{m-2}}\right)_{t_r} &= \partial_{t_r}\partial_x(\ln E_{m-2}) = \left(\phi_2 + \phi_2^* + \phi_2^{**} - u\right)_{t_r}\\
&= \left[3\widetilde{V}_{11}^{(r)} + \widetilde{V}_{12}^{(r)}(\phi_2 + \phi_2^* + \phi_2^{**}) + \lambda\widetilde{V}_{13}^{(r)}(\phi_3 + \phi_3^* + \phi_3^{**})\right]_x - u_{t_r}.
\end{aligned}$$

That is to say,

$$\begin{aligned}
&\partial_{t_r}(\ln E_{m-2}) = 3\widetilde{V}_{11}^{(r)} + \widetilde{V}_{12}^{(r)}(\phi_2 + \phi_2^* + \phi_2^{**}) + \lambda\widetilde{V}_{13}^{(r)}(\phi_3 + \phi_3^* + \phi_3^{**}) - \partial_x^{-1}u_{t_r}\\
&= 3\widetilde{V}_{11}^{(r)} + \widetilde{V}_{12}^{(r)} \times \frac{3B_m - 2V_{13}^{(n)}S_m - R_mA_m}{E_{m-2}} + \lambda\widetilde{V}_{13}^{(r)} \times \frac{3\mathcal{B}_m - 2V_{12}^{(n)}S_m - R_m\mathcal{A}_m}{-\lambda E_{m-2}} - \partial_x^{-1}u_{t_r},
\end{aligned}$$

so we know that the first expression in (4.21) holds.

Furthermore, since

$$\phi_2\phi_2^*\phi_2^{**} = -\frac{F_{m-2}}{E_{m-2}},$$

differentiating the above equation with respect to the variable t_r, one gets

$$\begin{aligned}
\left(-\frac{F_{m-2}}{E_{m-2}}\right)_{t_r} &= \phi_2\phi_2^*\phi_2^{**}\left(\frac{\phi_{2,t_r}}{\phi_2} + \frac{\phi_{2,t_r}^*}{\phi_2^*} + \frac{\phi_{2,t_r}^{**}}{\phi_2^{**}}\right)\\
&= -\frac{F_{m-2}}{E_{m-2}}\Big[3(\widetilde{V}_{22}^{(r)} - \widetilde{V}_{11}^{(r)}) - \widetilde{V}_{12}^{(r)}(\phi_2 + \phi_2^* + \phi_2^{**}) - \lambda\widetilde{V}_{13}^{(r)}(\phi_3 + \phi_3^* + \phi_3^{**})\\
&+ \widetilde{V}_{21}^{(r)}(\tfrac{1}{\phi_2} + \tfrac{1}{\phi_2^*} + \tfrac{1}{\phi_2^{**}}) + \lambda\widetilde{V}_{23}^{(r)}(\tfrac{\phi_3}{\phi_2} + \tfrac{\phi_3^*}{\phi_2^*} + \tfrac{\phi_3^{**}}{\phi_2^{**}})\Big]\\
&= -\frac{F_{m-2}}{E_{m-2}}\Big[3(\widetilde{V}_{22}^{(r)} - \widetilde{V}_{11}^{(r)}) - \widetilde{V}_{12}^{(r)} \times \frac{3B_m - 2V_{13}^{(n)}S_m - R_mA_m}{E_{m-2}}\\
&- \lambda\widetilde{V}_{13}^{(r)} \times \frac{3\mathcal{B}_m - 2V_{12}^{(n)}S_m - R_m\mathcal{A}_m}{-\lambda E_{m-2}} + \widetilde{V}_{21}^{(r)} \times \frac{-(2uV_{23}^{(n)} + R_m - 3V_{22}^{(n)})F_{m-2} + V_{23}^{(n)}F_{m-2,x}}{[(\lambda+v)V_{23}^{(n)} - V_{21}^{(n)}]F_{m-2}}\\
&+ \lambda\widetilde{V}_{23}^{(r)} \times \frac{(2uV_{21}^{(n)} + (\lambda+v)(R_m - 3V_{22}^{(n)}))F_{m-2} - V_{21}^{(n)}F_{m-2,x}}{((\lambda+v)V_{23}^{(n)} - V_{21}^{(n)})F_{m-2}},
\end{aligned} \tag{4.22}$$

which can yield the second expression in (4.21). By the same method, the third expression can be obtained. □

By using (4.1), (4.2), (4.12), (4.16) and (4.21), one can compute

$$\begin{aligned}\phi_2(P,x,t_r) &= \frac{y^2V_{13}^{(n)}-y(A_m+V_{13}^{(n)}R_m)+B_m}{E_{m-2}}\\ &= \frac{V_{13}^{(n)}y^2-y(A_m+V_{13}^{(n)}R_m)+\frac{1}{3}(E_{m-2,x}+uE_{m-2}+2V_{13}^{(n)}S_m+R_mA_m)}{E_{m-2}}\\ &= \frac{E_{m-2,x}}{3E_{m-2}}+\frac{2V_{13}^{(n)}(3y^2-2yR_m+S_m)}{3E_{m-2}}+\frac{u}{3}\\ &\underset{\lambda\to\mu_j}{=} -\frac{\mu_{j,x}}{\lambda-\mu_j}+O(1)\\ &\underset{\lambda\to\mu_j}{=} \partial_x\ln(\lambda-\mu_j)+O(1).\end{aligned} \tag{4.23}$$

On the other hand,

$$\begin{aligned}&\widetilde{V}_{11}^{(r)}+\widetilde{V}_{12}^{(r)}\phi_2+\lambda\widetilde{V}_{13}^{(r)}\phi_3\\ &= \frac{1}{E_{m-2}}\Big[\frac{1}{3}E_{m-2,t_r}+(\widetilde{V}_{12}^{(r)}V_{13}^{(n)}-\widetilde{V}_{13}^{(r)}V_{12}^{(n)})(y^2-yR_m+\tfrac{2}{3}S_m)\\ &\qquad +\frac{1}{3}\partial^{-1}u_{t_r}E_{m-2}-(\widetilde{V}_{12}^{(r)}A_m-\widetilde{V}_{13}^{(r)}\mathcal{A}_m)(y-\tfrac{1}{3}R_m)\Big]\\ &= -\frac{\mu_{j,t_r}}{\lambda-\mu_j}+O(1)\\ &\underset{\lambda\to\mu_j}{=} \partial_{t_r}\ln(\lambda-\mu_j)+O(1).\end{aligned} \tag{4.24}$$

Consequently,

$$\begin{aligned}&\psi_1(P,x,x_0,t_r,t_{0,r})\\ &= \exp\Big(\int_{x_0}^{x}[\phi_2(P,x',t_r)]dx'+\int_{t_{0,r}}^{t_r}I_r(P,x_0,t')dt'\Big)\\ &= \frac{\lambda-\mu_j(x,t_r)}{\lambda-\mu_j(x_0,t_{0,r})}O(1)\\ &= \begin{cases}(\lambda-\mu_j(x_0,t_{0,r}))^{-1}O(1), & P\to\hat{\mu}_j(x_0,t_{0,r})\neq\hat{\mu}_j(x,t_r),\\ (\lambda-\mu_j(x,t_r))O(1), & P\to\hat{\mu}_j(x,t_r)\neq\hat{\mu}_j(x_0,t_{0,r}),\\ O(1), & P\to\hat{\mu}_j(x,t_r)=\hat{\mu}_j(x_0,t_{0,r}),\end{cases}\end{aligned} \tag{4.25}$$

where $O(1)\neq 0$. So $\hat{\mu}_1(x,t_r),\ldots,\hat{\mu}_{m-2}(x,t_r)$ are $m-2$ zeros of $\psi_1(P,x,x_0,t_r,t_{0,r})$, and $\hat{\mu}_1(x_0,t_{0,r})$, $\ldots,\hat{\mu}_{m-2}(x_0,t_{0,r})$ are $m-2$ poles of ψ_1 on $\mathcal{K}_{m-2}$.

5. Quasi-periodic solutions

We will study the solutions for the three-component Burgers hierarchy in this section. The period lattice $\mathcal{T}_{m-2}=\{\underline{z}\in\mathbb{C}^{m-2}\mid \underline{z}=\underline{N}+\underline{M}\tau,\quad \underline{N},\underline{M}\in\mathbb{Z}^{m-2}\}$. The Jacobian variety $\mathcal{J}_{m-2}$ of $\mathcal{K}_{m-2}$ is defined by $\mathbb{C}^{m-2}/\mathcal{T}_{m-2}$. The Abel map $\underline{\mathcal{A}}:\mathcal{K}_{m-2}\to\mathcal{J}_{m-2}$ is as follows:

$$\underline{\mathcal{A}}(P)=\Big(\int_{Q_0}^{P}\omega_1,\ldots,\int_{Q_0}^{P}\omega_{m-2}\Big)\pmod{\mathcal{T}_{m-2}}, \tag{5.1}$$

and it can be extended to $\mathrm{Div}(\mathcal{K}_{m-2})$:

$$\underline{\mathcal{A}}\Big(\sum n_k P_k\Big) = \sum n_k \underline{\mathcal{A}}(P_k). \tag{5.2}$$

Define

$$\underline{\rho}^{(1)}(x,t_r) = \underline{\mathcal{A}}(\sum_{k=1}^{m-2} \hat{\mu}_k(x,t_r)) = \sum_{k=1}^{m-2} \int_{Q_0}^{\hat{\mu}_k(x,t_r)} \underline{\omega},$$
$$\underline{\rho}^{(2)}(x,t_r) = \underline{\mathcal{A}}(\sum_{k=1}^{m-2} \hat{\nu}_k(x,t_r)) = \sum_{k=1}^{m-2} \int_{Q_0}^{\hat{\nu}_k(x,t_r)} \underline{\omega},$$
$$\underline{\rho}^{(3)}(x,t_r) = \underline{\mathcal{A}}(\sum_{k=1}^{m-2} \hat{\xi}_k(x,t_r)) = \sum_{k=1}^{m-2} \int_{Q_0}^{\hat{\xi}_k(x,t_r)} \underline{\omega}.$$

Then, the Riemann theta function

$$\theta(\underline{z}) = \sum_{\underline{N}\in\mathbb{Z}^{m-2}} \exp\Big\{\pi i\langle \underline{N}\tau, \underline{N}\rangle + 2\pi i \langle \underline{N}, \underline{z}\rangle\Big\}, \tag{5.3}$$

where $\langle\cdot,\cdot\rangle$ is the Euclidean scalar product and $\underline{z} = (z_1,\ldots,z_{m-2}) \in \mathbb{C}^{m-2}$.

For simplicity, we introduce a function $\underline{z} : \mathcal{K}_{m-2} \times \sigma^{m-2}\mathcal{K}_{m-2} \to \mathbb{C}^{m-2}$,

$$\underline{z}(P,\underline{Q}) = \underline{\Lambda} - \underline{\mathcal{A}}(P) + \sum_{Q'\in\underline{Q}} \mathscr{D}(Q')\underline{\mathcal{A}}(Q'), \quad P\in\mathcal{K}_{m-2},\ \ \underline{Q}\in\sigma^{m-2}\mathcal{K}_{m-2},$$

in which the vector of the Riemann constant $\underline{\Lambda} = (\Lambda_1,\ldots,\Lambda_{m-2})$ only depends on Q_0, and

$$\Lambda_j = \frac{1}{2}(1+\tau_{jj}) - \sum_{\substack{l=1\\ l\neq j}}^{m-2} \int_{\mathfrak{a}_l} \omega_l(P)\int_{Q_0}^{P}\omega_j, \quad j=1,2,\ldots,m-2;$$

then,

$$\theta(\underline{z}(P,\underline{Q})) = \theta(\underline{\Lambda} - \underline{\mathcal{A}}(P) + \underline{\mathcal{A}}(\underline{Q})),\ \ P\in\mathcal{K}_{m-2}.$$

Using the above preparations, we can give the solutions for the three-component Burgers hierarchy.

Theorem 5.1. *Suppose that $\mathcal{K}_{m-2}$ is nonsingular and $\Omega_\mu \subseteq \mathbb{C}^2$ is connected and open. Let $(x,t_r),(x_0,t_{0,r})\in\Omega_\mu$ and $P\in\mathcal{K}_{m-2}\setminus\{P_{\infty_1},P_{\infty_2}\}$. If $\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}$, $\mathcal{D}_{\underline{\hat{\xi}}(x,t_r)}$ or $\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}$ is nonspecial, then*

(i)

$$\phi_2(P,x,t_r) = \frac{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\mu}}(x,t_r)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(x,t_r)))}\frac{\theta(\underline{z}(P,\underline{\hat{\nu}}(x,t_r)))}{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\nu}}(x,t_r)))}\exp\left(e_{1,\infty_2}(Q_0) - \int_{Q_0}^{P}\omega^{(3)}_{P_{\infty_2},\hat{\nu}_0}\right), \tag{5.4}$$

$$\phi_3(P,x,t_r) = -\frac{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}(x,t_r)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(x,t_r)))}\frac{\theta(\underline{z}(P,\underline{\hat{\xi}}(x,t_r)))}{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\xi}}(x,t_r)))}\exp\left(e_{2,\infty_1}(Q_0) - \int_{Q_0}^{P}\omega^{(3)}_{P_0,P_{\infty_2}}\right); \tag{5.5}$$

(ii)

$$u(x,t_r) = 2\sum_{j=1}^{m-2} d^{(\infty_2)}_{j,0}\frac{\partial}{\partial z_j}\ln\frac{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\xi}}(x,t_r)))}{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\mu}}(x,t_r)))} + 2\omega_0^{\infty_2} - w - \frac{2w_x}{w}, \tag{5.6}$$

$$v(x,t_r) = w^2+uw-w_x-\omega_0^{\infty_1}-\sum_{j=1}^{m-2} d_{j,0}^{(\infty_1)}\frac{\partial}{\partial z_j}\ln\frac{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\xi}}(x,t_r)))}{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}(x,t_r)))}, \tag{5.7}$$

$$w(x,t_r) = -\frac{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}(x,t_r)))}{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\xi}}(x,t_r)))}\frac{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\xi}}(x,t_r)))}{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\mu}}(x,t_r)))}\exp\left(e_{2,\infty_1}(Q_0)-e_{2,\infty_2}(Q_0)\right); \tag{5.8}$$

(iii)

$$\begin{aligned}\psi_1(P,x,x_0,t_r,t_{0,r}) = {} & \frac{\theta(\underline{z}(P,\underline{\hat{\mu}}(x,t_r)))}{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\mu}}(x,t_r)))}\frac{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\mu}}(x_0,t_{0,r})))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(x_0,t_{0,r})))}\\ & \times\exp\Big(\Big(e_2^{(2)}(Q_0)-\int_{Q_0}^{P}\Omega_2^{(2)}(P)\Big)(x-x_0)-\frac{1}{2}(\Delta_1+\Delta_2)\\ & +(t_r-t_{0,r})\Big(\tilde{e}_2^{(2)}(Q_0)-\int_{Q_0}^{P}\widetilde{\Omega}_{2r+3}^{(2)}(P)+\tilde{\alpha}_{r+1}\Big)\Big).\end{aligned} \tag{5.9}$$

Proof. (i) From (4.17), we can know that $\hat{\mu}_1,\ldots,\hat{\mu}_{m-2}$, P_{∞_2} are simple poles of ϕ_2 and $\hat{\nu}_0,\hat{\nu}_1,\ldots,\hat{\nu}_{m-2}$ are simple poles of ϕ_2. So, one infers the following:

$$\phi_2(P,x,t_r) = N(x,t_r)\frac{\theta(\underline{z}(P,\underline{\hat{\nu}}(x,t_r)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(x,t_r)))}\exp\Big(-\int_{Q_0}^{P}\omega_{P_{\infty_2},\hat{\nu}_0}^{(3)}(P)\Big); \tag{5.10}$$

then, we need to determine the expression of $N(x,t_r)$.

From (3.6), combining the expressions of ϕ_2 near P_{∞_2}, we have

$$N(x,t_r) = \frac{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\mu}}(x,t_r)))}{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\nu}}(x,t_r)))}\exp(e_{1,\infty_2}(Q_0)), \tag{5.11}$$

so (5.4) holds. Equation (5.5) can be proved by using a similar method.

(ii) By (5.5), as $P\to P_{\infty_1}$,

$$\frac{\theta(\underline{z}(P,\underline{\hat{\xi}}(x,t_r)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(x,t_r)))} \underset{\zeta\to 0}{=} \frac{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\xi}}(x,t_r)))}{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}(x,t_r)))}\left(1-\sum_{j=1}^{m-2} d_{j,0}^{(\infty_1)}\frac{\partial}{\partial z_j}\ln\frac{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\xi}}(x,t_r)))}{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}(x,t_r)))}\zeta+O(\zeta^2)\right); \tag{5.12}$$

as $P\to P_{\infty_2}$,

$$\frac{\theta(\underline{z}(P,\underline{\hat{\xi}}(x,t_r)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(x,t_r)))} \underset{\zeta\to 0}{=} \frac{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\xi}}(x,t_r)))}{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\mu}}(x,t_r)))}\left(1-\sum_{j=1}^{m-2} d_{j,0}^{(\infty_2)}\frac{\partial}{\partial z_j}\ln\frac{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\xi}}(x,t_r)))}{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\mu}}(x,t_r)))}\zeta+O(\zeta^2)\right). \tag{5.13}$$

Hence, for $P\to P_{\infty_1}$,

$$\phi_3(P,x,t_r) = -(1-\omega_0^{\infty_1}\zeta+O(\zeta^2))\left(1-\sum_{j=1}^{m-2} d_{j,0}^{(\infty_1)}\frac{\partial}{\partial z_j}\ln\frac{\theta_{13}}{\theta_{11}}\zeta+O(\zeta^2)\right); \tag{5.14}$$

for $P\to P_{\infty_2}$,

$$\phi_3(P,x,t_r) = -\zeta(1-\omega_0^{\infty_2}\zeta+O(\zeta^2))\exp(e_{2,\infty_1}(Q_0)-e_{2,\infty_2}(Q_0))\times\frac{\theta_{11}\theta_{23}}{\theta_{13}\theta_{21}}\left(1-\sum_{j=1}^{m-2} d_{j,0}^{(\infty_2)}\frac{\partial}{\partial z_j}\ln\frac{\theta_{23}}{\theta_{21}}\zeta+O(\zeta^2)\right), \tag{5.15}$$

with $\theta_{s1} = \theta(\underline{z}(P_{\infty_s}, \underline{\hat{\mu}}(x,t_r)))$ and $\theta_{s3} = \theta(\underline{z}(P_{\infty_s}, \underline{\hat{\xi}}(x,t_r)))$, $s = 1,2$. Combining (3.6), (3.7), (5.14) and (5.15), we can get (5.6)–(5.8).

(iii) Let Ψ_1 be the right hand of (5.9). We will prove that $\psi_1 = \Psi_1$. Applying Proposition 4.2, we find that ψ_1 and Ψ_1 have the same zeros and poles. Based on the Riemann-Roch theorem, we infer that $\dfrac{\Psi_1}{\psi_1} = \gamma$ for some constant γ. From (3.27), we get

$$\begin{aligned}
&\frac{\Psi_1(P,x,x_0,t_r,t_{0,r})}{\psi_1(P,x,x_0,t_r,t_{0,r})} \underset{\zeta\to 0}{=} \frac{\exp(\zeta^{-1}(x-x_0))}{\exp(\zeta^{-1}(x-x_0))}\\
&\times\frac{\exp\{[\sum_{l=0}^{r}(\tilde{\alpha}_{r-l}\zeta^{-2(l+1)}+\tilde{\beta}_{r-l}\zeta^{-(2l+1)})+\tilde{\alpha}_{r+1}](t_r-t_{0,r})-\frac{1}{2}(\Delta_1+\Delta_2)+O(\zeta)\}(1+O(\zeta))}{\exp\{[\sum_{l=0}^{r}(\tilde{\alpha}_{r-l}\zeta^{-2(l+1)}+\tilde{\beta}_{r-l}\zeta^{-(2l+1)})+\tilde{\alpha}_{r+1}](t_r-t_{0,r})-\frac{1}{2}(\Delta_1+\Delta_2)+O(\zeta)\}}\\
&\underset{\zeta\to 0}{=} 1+O(\zeta), \qquad\qquad P\to P_{\infty_2}.
\end{aligned} \tag{5.16}$$

Hence, $\gamma = 1$, and (5.9) holds. □

Theorem 5.2. *Let* $(x_0,t_{0,r}),(x,t_r)\in\mathbb{C}^2$. *Then*

$$\underline{\rho}^{(1)}(x,t_r) = \underline{\rho}^{(1)}(x_0,t_{0,r}) + (x-x_0)\underline{U}_2^{(2)} + (t_r-t_{0,r})\underline{\widetilde{U}}_{2r+3}^{(2)} \pmod{\mathcal{T}_{\mathrm{m}-2}}, \tag{5.17}$$

$$\begin{aligned}\underline{\rho}^{(2)}(x,t_r) = {}&-\underline{\mathcal{A}}(\hat{\nu}_0(x,t_r)) + \underline{\mathcal{A}}(\hat{\nu}_0(x_0,t_{0,r})) + \underline{\rho}^{(2)}(x_0,t_{0,r})\\ &+ (x-x_0)\underline{U}_2^{(2)} + (t_r-t_{0,r})\underline{\widetilde{U}}_{2r+3}^{(2)} \pmod{\mathcal{T}_{\mathrm{m}-2}},\end{aligned} \tag{5.18}$$

$$\underline{\rho}^{(3)}(x,t_r) = \underline{\rho}^{(3)}(x_0,t_{0,r}) + (x-x_0)\underline{U}_2^{(2)} + (t_r-t_{0,r})\underline{\widetilde{U}}_{2r+3}^{(2)} \pmod{\mathcal{T}_{\mathrm{m}-2}}. \tag{5.19}$$

Proof. Set

$$\Omega(x,x_0,t_r,t_{0,r}) = \frac{\partial}{\partial\lambda}\ln(\psi_1(P,x,x_0,t_r,t_{0,r}))d\lambda. \tag{5.20}$$

By (5.9), we have

$$\Omega(x,x_0,t_r,t_{0,r}) = \widetilde{\omega} - (x-x_0)\Omega_2^{(2)} - (t_r-t_{0,r})\widetilde{\Omega}_{2r+3}^{(2)} + \sum_{j=1}^{m-2}\omega^{(3)}_{\hat{\mu}_j(x,t_r),\hat{\mu}_j(x_0,t_{0,r})}, \tag{5.21}$$

where for some $e_j\in\mathbb{C}$, $\widetilde{\omega} = \sum_{j=1}^{m-2} e_j\omega_j$. Since ψ_1 is single-valued function, all $\mathbb{a}$- and $\mathbb{b}$-periods of Ω are integer multiples of $2\pi i$; so, for some $M_k, N_k\in\mathbb{Z}$,

$$2\pi i M_k = \int_{\mathbb{a}_k}\Omega(x,x_0,t_r,t_{0,r}) = \int_{\mathbb{a}_k}\widetilde{\omega} = e_k,\ \ k=1,\ldots,m-2. \tag{5.22}$$

$$\begin{aligned}
2\pi i N_k &= \int_{\mathbb{b}_k}\Omega(x,x_0,t_r,t_{0,r})\\
&= \int_{\mathbb{b}_k}\widetilde{\omega} - (x-x_0)\int_{\mathbb{b}_k}\Omega_2^{(2)} - (t_r-t_{0,r})\int_{\mathbb{b}_k}\widetilde{\Omega}_{2r+3}^{(2)} + \sum_{j=1}^{m-2}\int_{\mathbb{b}_k}\omega^{(3)}_{\hat{\mu}_j(x,t_r),\hat{\mu}_j(x_0,t_{0,r})}\\
&= 2\pi i\sum_{j=1}^{m-2}M_j\int_{\mathbb{b}_k}\omega_j - (x-x_0)\int_{\mathbb{b}_k}\Omega_2^{(2)} - (t_r-t_{0,r})\int_{\mathbb{b}_k}\widetilde{\Omega}_{2r+3}^{(2)} + 2\pi i\sum_{j=1}^{m-2}\int_{\hat{\mu}_j(x_0,t_{0,r})}^{\hat{\mu}_j(x,t_r)}\omega_k.
\end{aligned} \tag{5.23}$$

So,

$$\underline{N} = \underline{M}\tau - (x - x_0)\underline{U}_2^{(2)} - (t_r - t_{0,r})\underline{\widetilde{U}}_{2r+3}^{(2)} + \sum_{j=1}^{m-2} \int_{Q_0}^{\hat{\mu}_j(x,t_r)} \underline{\omega} - \sum_{j=1}^{m-2} \int_{Q_0}^{\hat{\mu}_j(x_0,t_{0,r})} \underline{\omega}, \tag{5.24}$$

where $\underline{N} = (N_1, \ldots, N_{m-2})$ and $\underline{M} = (M_1, \ldots, M_{m-2}) \in \mathbb{Z}^{m-2}$.

Thus, (5.24) has the equivalent form of (5.17). The other two equations in (5.18) and (5.19) can be obtained in the same way. □

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant Nos. 12101418, 11931017, 11871440, 11971442).

Conflict of interest

All authors declare no conflict of interest that may affect the publication of this paper.

References

1. M. J. Ablowitz, P. A. Clarkson, *Solitons, Nonlinear Evolution Equation and Inverse Scattering*, Cambridge: Cambridge University Press, 1991. https://doi.org/10.1017/CBO9780511623998
2. E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its, V. B. Matveev, *Algebro-Geometric Approach to Nonlinear Integrable Equations*, Berlin: Springer, 1994.
3. S. P. Novikov, S. V. Manakov, L. P. Pitaevskii, V. E. Zakharov, *Theory of Solitons, the Inverse Scattering Method*, New York: Consultants Bureau, 1984.
4. Y. Kosmann-Schwarzbach, B. Grammaticos, K. M. Tamizhmani, *Integrability of Nonlinear Systems*, Berlin: Springer, 1997.
5. L. A. Dickey, *Soliton Equations and Hamiltonian Systems*, Singapore: World Scientific, 2003. http://dx.doi.org/10.1142/5108
6. B. A. Dubrovin, Theta functions and non-linear equations, *Russian Math. Surveys*, **36** (1981), 11–92. http://dx.doi.org/10.1070/RM1981v036n02ABEH002596
7. E. Date, S. Tanaka, Periodic multi-soliton solutions of Korteweg-de Vries equation and Toda lattice, *Progr. Theoret. Phys. Suppl.*, **59** (1976), 107–125. http://dx.doi.org/10.1143/PTPS.59.107
8. E. Date, S. Tanaka, Analogue of inverse scattering theory for the discrete Hill's equation and exact solutions for the periodic Toda lattice, *Prog. Theor. Phys.*, **55** (1976), 457–465. http://dx.doi.org/10.1143/ptp.55.457

9. J. S. Geronimo, F. Gesztesy, H. Holden, Algebro-geometric solutions of the Baxter-Szegő difference equation, *Commun. Math. Phys.*, **258** (2005), 149–177. http://dx.doi.org/10.1007/s00220-005-1305-x
10. C. W. Cao, Y. T. Wu, X. G. Geng, Relation between the Kadometsev-Petviashvili equation and the confocal involutive system, *J. Math. Phys.*, **40** (1999), 3948–3970. http://dx.doi.org/10.1063/1.532936
11. X. G. Geng, C. W. Cao, Decomposition of the (2+1)-dimensional Gardner equation and its quasi-periodic solutions, *Nonlinearity*, **14** (2001), 1433–1452. http://dx.doi.org/10.1088/0951-7715/14/6/302
12. F. Gesztesy, H. Holden, Algebro-geometric solutions of the Camassa-Holm hierarchy, *Rev. Math. Iberoam.*, **19** (2003), 73–142.
13. H. Lundmark, J. Szmigielski, An inverse spectral problem related to the Geng-Xue two-component peakon equation, *Mem. Am. Math. Soc.*, **244** (2016), 1155. http://dx.doi.org/10.1090/memo/1155
14. F. Gesztesy, H. Holden, *Soliton Equations and Their Algebro-Geometric Solutions*, Cambridge: Cambridge University Press, 2003.
15. H. Lundmark, J. Szmigielski, Dynamics of interlacing peakons (and shockpeakons) in the Geng-Xue equation, *J. Integr. Sys.*, **2** (2017), xyw014. http://dx.doi.org/10.1093/INTEGR/XYW014
16. A. R. Its, V. B. Matveev, Schrödinger operators with finite-gap spectrum and N-soliton solutions of the Korteweg-de Vries equation, *Theoret. Math. Phys.*, **23** (1975), 343–355. http://dx.doi.org/10.1007/BF01038218
17. X. G. Geng, H. Liu, The nonlinear steepest descent method to long-time asymptotics of the coupled nonlinear Schrödinger equation, *J. Nonlinear Sci.*, **28** (2018), 739–763. http://dx.doi.org/10.1007/s00332-017-9426-x
18. R. Dickson, F. Gesztesy, K. Unterkofler, A new approach to the Boussinesq hierarchy, *Math. Nachr.*, **198** (1999), 51–108. http://dx.doi.org/10.1002/mana.19991980105
19. R. Dickson, F. Gesztesy, K. Unterkofler, Algebro-geometric solutions of the Boussinesq hierarchy, *Rev. Math. Phys.*, **11** (1999), 823–879. http://dx.doi.org/10.1142/S0129055X9900026X
20. X. G. Geng, L. H. Wu, G. L. He, Algebro-geometric constructions of the modified Boussinesq flows and quasi-periodic solutions, *Phys. D*, **240** (2011), 1262–1288. http://dx.doi.org/10.1016/j.physd.2011.04.020
21. G. L. He, L. H. Wu, X. G. Geng, Finite genus solutions to the mixed Boussinesq equation, *Sci. Sin. Math.*, **42** (2012), 711–734. http://dx.doi.org/10.1360/012011-848
22. X. G. Geng, Y. Y. Zhai, H. H. Dai, Algebro-geometric solutions of the coupled modified Korteweg-de Vries hierarchy, *Adv. Math.*, **263** (2014), 123–153. http://dx.doi.org/10.1016/j.aim.2014.06.013
23. M. X. Jia, X. G. Geng, J. Wei, Y. Y. Zhai, H. Liu, Coupled discrete Sawada-Kotera equations and their explicit quasi-periodic solutions, *Anal. Math. Phys.*, **11** (2021), 140. http://dx.doi.org/10.1007/s13324-021-00577-2
24. L. H. Wu, X. G. Geng, G. L. He, Algebro-geometric solutions to the Manakov hierarchy, *Appl. Anal.*, **95** (2016), 769–800. http://dx.doi.org/10.1080/00036811.2015.1031220

25. X. G. Geng, J. Wei, Three-sheeted Riemann surface and solutions of the Itoh-Narita-Bogoyavlensky lattice hierarchy, *Rev. Math. Phys.*, **34** (2022), 2250009. http://dx.doi.org/10.1142/S0129055X2250009X

26. X. G. Geng, H. Wang, Algebro-geometric constructions of quasi-periodic flows of the Newell hierarchy and applications, *IMA J. Appl. Math.*, **82** (2017), 97–130. http://dx.doi.org/10.1093/imamat/hxw008

27. W. Liu, X. G. Geng, B. Xue, Three-component generalisation of Burgers equation and its bi-Hamiltonian structures, *Z. Naturforsch. A*, **72** (2017), 469–475. http://dx.doi.org/10.1142/S0217984917502992

28. J. M. Burgers, Mathematical examples illustrating relations occurring in the theory of turbulent fluid motion, *Verh. Nederl. Akad. Wetensch. Afd. Natuurk. Sect. 1.*, **17** (1939).

29. T. Su, Explicit solutions for a modified 2+1-dimensional coupled Burgers equation by using Darboux transformation, *Appl. Math. Lett.*, **69** (2017), 15–21. http://dx.doi.org/10.1016/j.aml.2017.01.014

30. C. W. Cao, X. G. Geng, H. Y. Wang, Algebro-geometric solution of the 2+1 dimensional Burgers equation with a discrete variable, *J. Math. Phys.*, **43** (2002), 621–643. http://dx.doi.org/10.1063/1.1415427

31. A. M. Wazwaz, Multiple-front solutions for the Burgers equation and the coupled Burgers equations, *Appl. Math. Comput.*, **190** (2007), 1198–1206. http://dx.doi.org/10.1016/j.amc.2007.02.003

32. P. Griffiths, J. Harris, *Principles of Algebraic Geometry*, New York: Wiley, 1994. http://dx.doi.org/10.1002/9781118032527

33. D. Mumford, *Tata Lectures on Theta II*, Boston, Mass: Birkhäuser, 1984.

34. H. M. Farkas, I. Kra, *Riemann Surfaces*, New York: Springer, 1992. http://dx.doi.org/10.1007/978-1-4612-2034-3

AIMS Press

© 2023 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)