Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(11): 27725–27741.
DOI: 10.3934/math.20231419
Received: 11 July 2023
Revised: 15 September 2023
Accepted: 21 September 2023
Published: 08 October 2023

Research article

Some solutions to a third-order quaternion tensor equation

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Abstract: The paper deals with the third-order quaternion tensor equation. Based on the Qt multiplication operation, we derive solvability conditions and also get the general solution, the least-squares solution, the minimum-norm solution and the minimum-norm least-squares solution of the tensor equation $\mathcal{A} *_{\mathbb{Q}} \mathcal{X} = \mathcal{B}$. Finally, two numerical examples are presented.

Keywords: tensor equation; Qt-product; generalized inverse; quaternion matrix; least-squares solution; minimum-norm solution
Mathematics Subject Classification: 15A09, 15A24, 15A72

1. Introduction

In 1843, Hamilton extended the real number field $\mathbb{R}$ and the complex number field $\mathbb{C}$ to quaternions $\mathbb{Q}$. By now, quaternions and quaternion matrices have been widely used in many fields such as computer graphics, color image processing and signals [1–3]. Tensors, as multidimensional arrays of vectors and matrices, appear widely in applications such as chemometrics [4], image and signal processing [5–8]. For instance, Soto-Quiros [5] considered the inverse tensor problem for denoising data, which can be represented as a least-squares problem of a linear tensor equation of third-order. The author proposed a numerical method of estimating a least-squares solution to a complex tensor linear equation and used it for audio denoising and color image deconvolution. Based on t-product, Reichel et al. [6] also solved a penalized least-squares problem of a linear tensor equation of third-order by generalized Arnoldi-type and bidiagonalization solution methods. As applications, it is used on the color image and video restoration. Guide et al. [7] proposed a tensor iterative Krylov subspace method to solve large multi-linear tensor equations $\mathcal{M}(\mathcal{X}) = \mathcal{C}$, like

$\mathcal{A}\mathcal{X} = \mathcal{C}$ and $\mathcal{A}\mathcal{X}\mathcal{B} = \mathcal{C}$. While Jin et al. [9] developed an algorithm to compute the Moore-Penrose inverse of p-order tensor, and then it can deal with a linear p-order tensor equation problems. All those work are about the tensor over the real field. In [10, 11], they considered the third-order tensor over the quaternions $\mathbb{Q}$. Qin et al. [10] proposed a numerical method to compute the singular value decomposition of $\mathcal{A} \in \mathbb{Q}^{n_1\times n_2\times n_3}$ and presented it to compress the color video. Inspired by [5, 6, 8], we realized that the third-order linear tensor equation problems over $\mathbb{Q}$ always exists when doing the color video deconvolution, color video denoising, color video reconstruction, and so on. Thus, we aim to solve the classic third-order quaternion tensor equation $\mathcal{A} *_{\mathbb{Q}} \mathcal{X} = \mathcal{B}$, especially the least-squares solutions and the minimum-norm least-squares solutions. Up to now, there are some numerical methods to solve the quaternion linear tensor equations, like [12, 13]. In [13], Zhang et al. solved the generalized Sylvester quaternion p-order quaternion tensor equations by tensor form of GPBiCG algorithm. But an effective way to find the least-squares solution to the tensor equation $\mathcal{A} *_{\mathbb{Q}} \mathcal{X} = \mathcal{B}$ is to be developed. Thus, this paper will explore this problem in theoretical way.

2. Preliminary results

The set of quaternions $\mathbb{Q}$ is a linear space over $\mathbb{R}$. An element q of $\mathbb{Q}$ is of the form

$$q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}, a, b, c, d \in \mathbb{R}.$$

Here $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are three imaginary units with the following multiplication laws:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

For a quaternion, the conjugate quaternion of q is $q^* = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$. The norm of q is $\mid q \mid = \sqrt{qq^*} = \sqrt{a^2 + b^2 + c^2 + d^2}$.

A third-order tensor $\mathcal{A} = (a_{i_1 i_2 i_3}), 1 \leq i_j \leq n_j, (j = 1, 2, 3)$ is a multidimensional array with $n_1 n_2 n_3$ entries. In this paper, we use the notations $a, \mathbf{a}, A$ and $\mathcal{A}$ to denote the scalar, vector, matrix and third-order quaternion tensor, respectively. In [10], the horizontal, lateral and frontal slices of a third-order tensor are denoted by $\mathcal{A}(i, :, :), \mathcal{A}(:, i, :), \mathcal{A}(:, :, i)$, respectively, and for simplicity we denote the frontal slice of a third-order tensor by $A^{(i)} = \mathcal{A}(:, :, i)$. In the paper, we use A^* to represent the conjugate transpose of matrix A. $\mathcal{O}$ represents zero tensor with all the entries being zero. $\mathcal{I}$ denotes the identity tensor, in which the first frontal slice is an identity matrix and the other slice matrices are zero. For a positive integer n, $[n]$ stands for $\{1, 2, \ldots, n\}$. Let $\mathcal{A} \in \mathbb{C}^{n_1\times n_2\times n_3}$, the block circulant matrix $\mathbf{circ}(\mathcal{A}) \in \mathbb{C}^{n_1 n_3\times n_2 n_3}$ generated by a third-order tensor $\mathcal{A}$'s frontal slices $A^{(1)}, A^{(2)}, ..., A^{(n_3)}$ is defined as

$$\mathbf{circ}(\mathcal{A}) = \begin{bmatrix} A^{(1)} & A^{(n_3)} & ... & A^{(2)} \\ A^{(2)} & A^{(1)} & ... & A^{(3)} \\ \vdots & \vdots & \ddots & \vdots \\ A^{(n_3)} & A^{(n_3-1)} & ... & A^{(1)} \end{bmatrix}, \tag{2.1}$$

see [7]. For the quaternion $\mathcal{A} \in \mathbb{Q}^{n_1\times n_2\times n_3}$, the block circulant matrix $\mathbf{circ}(\mathcal{A}) \in \mathbb{Q}^{n_1 n_3\times n_2 n_3}$ can be defined in the same way.

The operations $\mathbf{unfold}(\mathcal{A})$, $diag(\mathcal{A})$ are as follows:

$$\mathbf{unfold}(\mathcal{A}) = \begin{bmatrix} A^{(1)} \\ A^{(2)} \\ \vdots \\ A^{(n_3)} \end{bmatrix},$$

$$diag(\mathcal{A}) = \begin{bmatrix} A^{(1)} & & & \\ & A^{(2)} & & \\ & & \ddots & \\ & & & A^{(n_3)} \end{bmatrix}.$$

The inverse operation of $\mathbf{unfold}(\bullet)$, denoted as $\mathbf{fold}(\bullet)$, turns a block tensor with the size of $n_1n_3 \times n_2$ into a tensor with the size of $n_1 \times n_2 \times n_3$, that is,

$$\mathbf{fold}(\mathbf{unfold}(\mathcal{A})) = \mathcal{A}.$$

In [10], the authors defined the Qt-product of two third-order quaternion tensors.

Definition 1. *(Qt-product) Let $\mathcal{A} \in \mathbb{Q}^{n_1\times n_2\times n_3}$ and $\mathcal{B} \in \mathbb{Q}^{n_2\times n_4\times n_3}$. Then Qt-product of $\mathcal{A}$ and $\mathcal{B}$ is defined /as*

$$\mathcal{A} *_{\mathbb{Q}} \mathcal{B} = \mathbf{fold}((\mathbf{circ}(\mathcal{A}_{\mathbf{1,i}}) + \mathbf{j}\ \mathbf{circ}(\mathcal{A}_{\mathbf{j,k}}) \cdot (P_{n_3} \otimes I_{n_2})) \cdot \mathbf{unfold}(\mathcal{B})) \in \mathbb{Q}^{n_1\times n_4\times n_3},$$

where $\mathcal{A} = \mathcal{A}_{\mathbf{1,i}} + \mathbf{j}\mathcal{A}_{\mathbf{j,k}}, \mathcal{A}_{\mathbf{1,i}}, \mathcal{A}_{\mathbf{j,k}} \in \mathbb{C}^{n_1\times n_2\times n_3}$.

The matrix $P_{n_3} = (P_{ij}) \in \mathbb{R}^{n_3\times n_3}$ is a permutation matrix where $P_{11} = P_{ij} = 1$ if $i + j = n_3 + 2, 2 \le i, j \le n_3; P_{ij} = 0$, otherwise. The notation $'\otimes'$ means the Kronecker product.

Definition 2. *(See [10]) The Discrete Fourier Transformation (DFT) of $\mathcal{A} \in \mathbb{Q}^{n_1\times n_2\times n_3}$ along the third mode is denoted as tensor $\widehat{\mathcal{A}} \in \mathbb{Q}^{n_1\times n_2\times n_3}$, where $\widehat{\mathcal{A}}(i, j, :) = \sqrt{n_3}F_{n_3}\mathcal{A}(i, j, :)$, $i \in [n_1]$, $j \in [n_2]$ and $F_{n_3} \in \mathbb{C}^{n_3\times n_3}$ is the normalized DFT matrix, with $F_{n_3}(i, j) = \frac{1}{\sqrt{n_3}}\omega^{(i-1)(j-1)}$, $i, j \in [n_3]$, $\omega = e^{-\frac{2\pi i}{n_3}}$.*

By Definition 2, $\widehat{\mathcal{A}}$ satisfying

$$\mathbf{unfold}(\widehat{\mathcal{A}}) = \begin{bmatrix} \widehat{A}^{(1)} \\ \widehat{A}^{(2)} \\ \vdots \\ \widehat{A}^{(n_3)} \end{bmatrix} = \sqrt{n_3}(F_{n_3} \otimes I_{n_1}) \cdot \mathbf{unfold}(\mathcal{A}). \tag{2.2}$$

In the paper, denoting

$$diag(\widehat{\mathcal{A}}) = \begin{bmatrix} \widehat{A}^{(1)} & & & \\ & \widehat{A}^{(2)} & & \\ & & \ddots & \\ & & & \widehat{A}^{(n_3)} \end{bmatrix},$$

with $\widehat{A}^{(1)}, \dots, \widehat{A}^{(n_3)}$ are defined by (2.2). Next, we introduce an important result that can turn third-order tensor problems into matrix problems.

Lemma 1. *(See [10]) Let $\mathcal{A} \in \mathbb{Q}^{n_1\times n_2\times n_3}$, $\mathcal{B} \in \mathbb{Q}^{n_2\times n_4\times n_3}$ and $C \in \mathbb{Q}^{n_1\times n_4\times n_3}$, $\mathcal{A}, \mathcal{B}, C$ after DFT to obtain $\widehat{\mathcal{A}}, \widehat{\mathcal{B}}, \widehat{C}$, respectively. Then $C = \mathcal{A} *_{\mathbb{Q}} \mathcal{B} \Longleftrightarrow diag(\widehat{C}) = diag(\widehat{\mathcal{A}}) \cdot diag(\widehat{\mathcal{B}})$.*

It follows from Lemma 1 that

$$\mathcal{I} *_{\mathbb{Q}} \mathcal{I} = \mathcal{I}, \mathcal{A} *_{\mathbb{Q}} \mathcal{I} = \mathcal{I} *_{\mathbb{Q}} \mathcal{A} = \mathcal{A}.$$

The conjugate transpose $\mathcal{A}^*$ of third-order complex tensor $\mathcal{A} \in \mathbb{C}^{n_1\times n_2\times n_3}$ is defined as follows: first conjugately transpose each frontal slice of $\mathcal{A}$, and then reverse the order of conjugately transposed frontal slices 2 through n_3, see [10]. For the third-order quaternion tensor $\mathcal{A} = \mathcal{A}_{\mathbf{1,i}} + \mathbf{j}\mathcal{A}_{\mathbf{j,k}}, \mathcal{A}_{\mathbf{1,i}}, \mathcal{A}_{\mathbf{j,k}} \in \mathbb{C}^{n_1\times n_2\times n_3}$, $\mathcal{A}^*$ is defined in a more generalized way. [10] defined the third-order quaternion tensor $\mathcal{A}^*$ through $\mathbf{unfold}(\mathcal{A}^*)$, which should satisfies

$$\mathbf{unfold}(\mathcal{A}^*) = \mathbf{unfold}(\mathcal{A}^*_{\mathbf{1,i}}) - (P_{n_3} \otimes I_{n_2})\mathbf{unfold}(\mathcal{A}^*_{\mathbf{j,k}})\mathbf{j}. \tag{2.3}$$

For example, for the third-order quaternion tensor $\mathcal{A} \in \mathbb{Q}^{n_1\times n_2\times 4}$, to get $\mathcal{A}^* \in \mathbb{Q}^{n_2\times n_1\times 4}$, we should first derive $\mathbf{unfold}(\mathcal{A}^*)$ by (2.3):

$$\begin{aligned}
\mathbf{unfold}(\mathcal{A}^*) &= \mathbf{unfold}(\mathcal{A}^*_{\mathbf{1,i}}) - (P_4 \otimes I_{n_2})\mathbf{unfold}(\mathcal{A}^*_{\mathbf{j,k}})\mathbf{j} \\
&= \begin{bmatrix} \left(A^{(1)}_{1,i}\right)^* \\ \left(A^{(4)}_{1,i}\right)^* \\ \left(A^{(3)}_{1,i}\right)^* \\ \left(A^{(2)}_{1,i}\right)^* \end{bmatrix} - \begin{bmatrix} I_{n_2} & & & \\ & & & I_{n_2} \\ & & I_{n_2} & \\ & I_{n_2} & & \end{bmatrix} \begin{bmatrix} \left(A^{(1)}_{j,k}\right)^* \\ \left(A^{(4)}_{j,k}\right)^* \\ \left(A^{(3)}_{j,k}\right)^* \\ \left(A^{(2)}_{j,k}\right)^* \end{bmatrix} \mathbf{j} \\
&= \begin{bmatrix} \left(A^{(1)}_{1,i}\right)^* \\ \left(A^{(4)}_{1,i}\right)^* \\ \left(A^{(3)}_{1,i}\right)^* \\ \left(A^{(2)}_{1,i}\right)^* \end{bmatrix} - \begin{bmatrix} \left(A^{(1)}_{j,k}\right)^* \\ \left(A^{(2)}_{j,k}\right)^* \\ \left(A^{(3)}_{j,k}\right)^* \\ \left(A^{(4)}_{j,k}\right)^* \end{bmatrix} \mathbf{j},
\end{aligned}$$

thus, according to $\mathbf{unfold}(\mathcal{A}^*)$, the first frontal slice of $\mathcal{A}^*$ is $\left(A^{(1)}_{1,i}\right)^* - \left(A^{(1)}_{j,k}\right)^* \mathbf{j}$, the second frontal slice of $\mathcal{A}^*$ is $\left(A^{(4)}_{1,i}\right)^* - \left(A^{(2)}_{j,k}\right)^* \mathbf{j}$, the third frontal slice of $\mathcal{A}^*$ is $\left(A^{(3)}_{1,i}\right)^* - \left(A^{(3)}_{j,k}\right)^* \mathbf{j}$, the fourth frontal slice of $\mathcal{A}^*$ is $\left(A^{(2)}_{1,i}\right)^* - \left(A^{(4)}_{j,k}\right)^* \mathbf{j}$.

From the definition of third-order quaternion tensor $\mathcal{A}^*$, we can see that it still satisfies

$$\mathbf{fold}(\mathbf{unfold}(\mathcal{A}^*)) = \mathcal{A}^*.$$

For the third-order quaternion tensor $\mathcal{A} \in \mathbb{Q}^{n_1\times n_2\times n_3}$, it should be noted that the definition of $\mathcal{A}^*$ generalizes the definition of $\mathcal{A}^*$ when $\mathcal{A}$ is a third-order complex tensor.

If $\mathcal{U}^* *_{\mathbb{Q}} \mathcal{U} = \mathcal{U} *_{\mathbb{Q}} \mathcal{U}^* = \mathcal{I}_{nnl}$, then we call $\mathcal{U}$ the $n \times n \times l$ unitary tensor.

Next, we will show some properties for $\mathcal{A}^*$.

Proposition 1. *Let $\mathcal{A} \in \mathbb{C}^{n_1\times n_2\times n_3}$, then $diag(\widehat{\mathcal{A}^*}) = (diag(\widehat{\mathcal{A}}))^*$.*

Proof. According to (2.1), we can see that $\mathbf{circ}(\mathcal{A}^*) = (\mathbf{circ}(\mathcal{A}))^*$. From [14], each complex circulant block matrix can be diagonalized by the DFT matrix, i.e.,

$$(F_{n_3} \otimes I_{n_1}) \cdot \mathbf{circ}(\mathcal{A}) \cdot (F_{n_3}^* \otimes I_{n_2}) = \begin{bmatrix} \widehat{A}^{(1)} & & & \\ & \widehat{A}^{(2)} & & \\ & & \ddots & \\ & & & \widehat{A}^{(n_3)} \end{bmatrix} = diag(\widehat{\mathcal{A}}), \tag{2.4}$$

where $\mathcal{A} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$. By applying (2.4), we have

$$\begin{aligned} diag(\widehat{\mathcal{A}^*}) &= (F_{n_3} \otimes I_{n_2}) \cdot \mathbf{circ}(\mathcal{A}^*) \cdot (F_{n_3}^* \otimes I_{n_1}) \\ &= (F_{n_3} \otimes I_{n_2}) \cdot (\mathbf{circ}(\mathcal{A}))^* \cdot (F_{n_3}^* \otimes I_{n_1}) \\ &= (diag(\widehat{\mathcal{A}}))^*. \end{aligned}$$

□

Here we show that Proposition 1 is also true over quaternion skew field.

Proposition 2. *Let $\mathcal{A} \in \mathbb{Q}^{n_1 \times n_2 \times n_3}$, then $diag(\widehat{\mathcal{A}^*}) = (diag(\widehat{\mathcal{A}}))^*$.*

Proof. Since $\widehat{\mathcal{A}^*}$ is the DFT of $\mathcal{A}^*$, $F_{n_3}F_{n_3} = P_{n_3}$ and $F_{n_3}\mathbf{j} = \mathbf{j}P_{n_3}F_{n_3}$, it follows from (2.2) and (2.3) that

$$\begin{aligned} \mathbf{unfold}(\widehat{\mathcal{A}^*}) &= \sqrt{n_3}(F_{n_3} \otimes I_{n_2})\mathbf{unfold}(\mathcal{A}^*) \\ &= \sqrt{n_3}(F_{n_3} \otimes I_{n_2})(\mathbf{unfold}(\mathcal{A}_{1,\mathbf{i}}^*) - (P_{n_3} \otimes I_{n_2})\mathbf{unfold}(\mathcal{A}_{\mathbf{j},\mathbf{k}}^*)\mathbf{j}) \\ &= \sqrt{n_3}(F_{n_3} \otimes I_{n_2}) \cdot \mathbf{unfold}(\mathcal{A}_{1,\mathbf{i}}^*) \\ &- \sqrt{n_3}(P_{n_3} \otimes I_{n_2})(F_{n_3} \otimes I_{n_2})\mathbf{unfold}(\mathcal{A}_{\mathbf{j},\mathbf{k}}^*)\mathbf{j} \\ &= \mathbf{unfold}(\widehat{\mathcal{A}_{1,\mathbf{i}}^*}) - (P_{n_3} \otimes I_{n_2})\mathbf{unfold}(\widehat{\mathcal{A}_{\mathbf{j},\mathbf{k}}^*})\mathbf{j}, \end{aligned}$$

which means

$$diag(\widehat{\mathcal{A}^*}) = diag(\widehat{\mathcal{A}_{1,\mathbf{i}}^*}) - (P_{n_3} \otimes I_{n_2})diag(\widehat{\mathcal{A}_{\mathbf{j},\mathbf{k}}^*})(P_{n_3} \otimes I_{n_1})\mathbf{j}. \tag{2.5}$$

It is shown by [10] that $diag(\widehat{\mathcal{A}}) = diag(\widehat{\mathcal{A}_{1,\mathbf{i}}}) + \mathbf{j}(P_{n_3} \otimes I_{n_1})diag(\widehat{\mathcal{A}_{\mathbf{j},\mathbf{k}}})(P_{n_3} \otimes I_{n_2}) \in \mathbb{Q}^{n_1 n_3 \times n_2 n_3}$. The conjugate transpose of it is $(diag(\widehat{\mathcal{A}}))^* = (diag(\widehat{\mathcal{A}_{1,\mathbf{i}}}))^* - (P_{n_3} \otimes I_{n_2})(diag(\widehat{\mathcal{A}_{\mathbf{j},\mathbf{k}}}))^*(P_{n_3} \otimes I_{n_1})\mathbf{j} \in \mathbb{Q}^{n_2 n_3 \times n_1 n_3}$. From Proposition 1 and Eq (2.5), we can get $diag(\widehat{\mathcal{A}^*}) = (diag(\widehat{\mathcal{A}}))^*$. □

We correct two equations in [10] (page 3, line-4 and line-6) as follows:

$$diag(\widehat{\mathcal{A}}) = diag(\widehat{\mathcal{A}_{1,\mathbf{i}}}) + \mathbf{j}(P_{n_3} \otimes I_{n_1})diag(\widehat{\mathcal{A}_{\mathbf{j},\mathbf{k}}})(P_{n_3} \otimes I_{n_2}),$$

$$unfold(\widehat{\mathcal{A}}) = unfold(\widehat{\mathcal{A}_{1,\mathbf{i}}}) + \mathbf{j}(P_{n_3} \otimes I_{n_1})unfold(\widehat{\mathcal{A}_{\mathbf{j},\mathbf{k}}}).$$

It should be noted that, for $\mathcal{A} = \mathcal{A}_{1,\mathbf{i}} + \mathbf{j}\mathcal{A}_{\mathbf{j},\mathbf{k}}, \widehat{\mathcal{A}} = \widehat{\mathcal{A}}_{1,\mathbf{i}} + \mathbf{j}\widehat{\mathcal{A}}_{\mathbf{j},\mathbf{k}}$, the following equations still hold

$$diag(\mathcal{A}) = diag(\mathcal{A}_{1,\mathbf{i}}) + \mathbf{j}diag(\mathcal{A}_{\mathbf{j},\mathbf{k}}),$$

$$diag(\widehat{\mathcal{A}}) = diag(\widehat{\mathcal{A}}_{1,\mathbf{i}}) + \mathbf{j}diag(\widehat{\mathcal{A}}_{\mathbf{j},\mathbf{k}}).$$

And in general, $\widehat{\mathcal{A}} \neq \widehat{\mathcal{A}_{1,\mathbf{i}}} + \mathbf{j}\widehat{\mathcal{A}_{\mathbf{j},\mathbf{k}}}$, or

$$diag(\widehat{\mathcal{A}}) \neq diag(\widehat{\mathcal{A}_{1,\mathbf{i}}}) + \mathbf{j}diag(\widehat{\mathcal{A}_{\mathbf{j},\mathbf{k}}}).$$

Based on Qt-product, we can find that the multiplication operation between tensors obeys an excellent law, similar to matrix multiplication.

Proposition 3. *Let* $\mathcal{A} \in \mathbb{Q}^{n_1\times n_2\times n_3}$, $\mathcal{B} \in \mathbb{Q}^{n_2\times n_4\times n_3}$, $\mathcal{C} \in \mathbb{Q}^{n_2\times n_4\times n_3}$, *and* $\mathcal{D} \in \mathbb{Q}^{n_4\times n_5\times n_3}$. *Then*

(a) $(\mathcal{A}^*)^* = \mathcal{A}$;
(b) $(\mathcal{B} + \mathcal{C})^* = \mathcal{B}^* + \mathcal{C}^*$;
(c) $(\mathcal{A} *_{\mathbb{Q}} \mathcal{B}) *_{\mathbb{Q}} \mathcal{D} = \mathcal{A} *_{\mathbb{Q}} (\mathcal{B} *_{\mathbb{Q}} \mathcal{D})$;
(d) $\mathcal{A} *_{\mathbb{Q}} (\mathcal{B} + \mathcal{C}) = \mathcal{A} *_{\mathbb{Q}} \mathcal{B} + \mathcal{A} *_{\mathbb{Q}} \mathcal{C}$;
(e) $(\mathcal{B} + \mathcal{C}) *_{\mathbb{Q}} \mathcal{D} = \mathcal{B} *_{\mathbb{Q}} \mathcal{D} + \mathcal{C} *_{\mathbb{Q}} \mathcal{D}$;
(f) $(\mathcal{A} *_{\mathbb{Q}} \mathcal{B})^* = \mathcal{B}^* *_{\mathbb{Q}} \mathcal{A}^*$.

Proof. For the simplicity, we only prove (f). Denote $\mathcal{C} = \mathcal{A} *_{\mathbb{Q}} \mathcal{B}$. By Proposition 2 and Lemma 1,

$$\begin{aligned} diag(\widehat{\mathcal{C}^*}) &= (diag(\widehat{\mathcal{C}}))^* = (diag(\widehat{\mathcal{A} *_{\mathbb{Q}} \mathcal{B}}))^* = [diag(\widehat{\mathcal{A}}) \cdot diag(\widehat{\mathcal{B}})]^* \\ &= (diag(\widehat{\mathcal{B}}))^* \cdot (diag(\widehat{\mathcal{A}}))^* = diag(\widehat{\mathcal{B}^*}) \cdot diag(\widehat{\mathcal{A}^*}). \end{aligned}$$

Moreover,

$$diag(\widehat{\mathcal{B}^*}) \cdot diag(\widehat{\mathcal{A}^*}) = diag(\widehat{\mathcal{B}^* *_{\mathbb{Q}} \mathcal{A}^*}),$$

thus

$$diag(\widehat{\mathcal{C}^*}) = diag(\widehat{\mathcal{B}^* *_{\mathbb{Q}} \mathcal{A}^*}),$$

which implies

$$(\mathcal{A} *_{\mathbb{Q}} \mathcal{B})^* = \mathcal{B}^* *_{\mathbb{Q}} \mathcal{A}^*.$$

□

The Frobenius norm of a quaternion tensor $\mathcal{A}$ is the sum of all norms of its entries, i.e.

$$\| \mathcal{A} \|_F = \sqrt{\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{k=1}^{n_3} | a_{ijk} |^2}.$$

For a tensor $\mathcal{A} = \mathcal{A}_{1,\mathbf{i}} + \mathbf{j}\mathcal{A}_{\mathbf{j},\mathbf{k}}$, $\mathcal{A}_{1,\mathbf{i}}, \mathcal{A}_{\mathbf{j},\mathbf{k}} \in \mathbb{C}^{n_1\times n_2\times n_3}$, its Frobenius norm can also be expressed as

$$\| \mathcal{A} \|_F^2 = \| \mathbf{unfold}(\mathcal{A}) \|_F^2 = \| \mathbf{unfold}(\mathcal{A}_{1,\mathbf{i}}) \|_F^2 + \| \mathbf{unfold}(\mathcal{A}_{\mathbf{j},\mathbf{k}}) \|_F^2 . \tag{2.6}$$

According to equality (2.2) and (2.6), it is easy to show that

$$\| \mathcal{A} \|_F = \frac{1}{\sqrt{n_3}} \| diag(\widehat{\mathcal{A}}) \|_F, \mathcal{A} \in \mathbb{Q}^{n_1\times n_2\times n_3}. \tag{2.7}$$

3. Generalized inverses

In this section, we will define some generalized inverses and explore their properties.

Definition 3. *For an $\mathcal{A} \in \mathbb{Q}^{n_1\times n_2\times n_3}$, if there exists a quaternion tensor $\mathcal{X} \in \mathbb{Q}^{n_2\times n_1\times n_3}$ satisfying:*

$$\begin{aligned}
&(1)\ \ \mathcal{A} *_{\mathbb{Q}} \mathcal{X} *_{\mathbb{Q}} \mathcal{A} = \mathcal{A};\\
&(2)\ \ \mathcal{X} *_{\mathbb{Q}} \mathcal{A} *_{\mathbb{Q}} \mathcal{X} = \mathcal{X};\\
&(3)\ \ (\mathcal{A} *_{\mathbb{Q}} \mathcal{X})^* = \mathcal{A} *_{\mathbb{Q}} \mathcal{X};\\
&(4)\ \ (\mathcal{X} *_{\mathbb{Q}} \mathcal{A})^* = \mathcal{X} *_{\mathbb{Q}} \mathcal{A};
\end{aligned} \tag{3.1}$$

then we call $\mathcal{X}$ the Moore-Penrose inverse of the tensor $\mathcal{A}$. Also, denote it as $\mathcal{A}^\dagger$.

In [10], based on the Qt-product between the two third-order quaternion tensors, the authors derived the SVD decomposition of $\mathcal{A} \in \mathbb{Q}^{n_1\times n_2\times n_3}$.

Lemma 2. *Let $\mathcal{A} \in \mathbb{Q}^{n_1\times n_2\times n_3}$. Then $\mathcal{A} = \mathcal{U} *_{\mathbb{Q}} \mathcal{S} *_{\mathbb{Q}} \mathcal{V}^*$ is the Qt-SVD of quaternion tensor $\mathcal{A}$, where*

$$\begin{aligned}
\mathcal{U} &\doteq \textbf{\textit{fold}}((F_{n_3}^* \otimes I_{n_1})\frac{1}{\sqrt{n_3}} diag(\widehat{\mathcal{U}})(e \otimes I_{n_1})),\\
\mathcal{S} &\doteq \textbf{\textit{fold}}((F_{n_3}^* \otimes I_{n_1})\frac{1}{\sqrt{n_3}} diag(\widehat{\mathcal{S}})(e \otimes I_{n_2})),\\
\mathcal{V} &\doteq \textbf{\textit{fold}}((F_{n_3}^* \otimes I_{n_2})\frac{1}{\sqrt{n_3}} diag(\widehat{\mathcal{V}})(e \otimes I_{n_2})),
\end{aligned}$$

e is an n_3-dimensional column vector with all elements being 1.

From Lemma 2, we can derive the SVD decomposition of $\mathcal{A}^\dagger$.

Theorem 1. *Let the Qt-SVD of quaternion tensor $\mathcal{A} \in \mathbb{Q}^{n_1\times n_2\times n_3}$ be $\mathcal{A} = \mathcal{U} *_{\mathbb{Q}} \mathcal{S} *_{\mathbb{Q}} \mathcal{V}^*$. Then tensor $\mathcal{A}$ has a unique Moore-Penrose inverse*

$$\mathcal{A}^\dagger = \mathcal{V} *_{\mathbb{Q}} \mathcal{S}^\dagger *_{\mathbb{Q}} \mathcal{U}^*,$$

where

$$\begin{aligned}
\mathcal{U} &\doteq \textbf{\textit{fold}}((F_{n_3}^* \otimes I_{n_1})\frac{1}{\sqrt{n_3}} diag(\widehat{\mathcal{U}})(e \otimes I_{n_1})),\\
\mathcal{S}^\dagger &\doteq \textbf{\textit{fold}}(\frac{1}{\sqrt{n_3}}(F_{n_3}^* \otimes I_{n_2})(diag(\widehat{\mathcal{S}}))^\dagger(e \otimes I_{n_1})),\\
\mathcal{V} &\doteq \textbf{\textit{fold}}((F_{n_3}^* \otimes I_{n_2})\frac{1}{\sqrt{n_3}} diag(\widehat{\mathcal{V}})(e \otimes I_{n_2})).
\end{aligned}$$

Proof. First of all, it can be verified that $\textbf{fold}(\frac{1}{\sqrt{n_3}}(F_{n_3}^* \otimes I_{n_2})(diag(\widehat{\mathcal{S}}))^\dagger(e \otimes I_{n_1}))$ is the Moore-Penrose inverse of $\mathcal{S}$ by substituting it into the four equations in Definition 3. Then, obviously, $\mathcal{V} *_{\mathbb{Q}} \mathcal{S}^\dagger *_{\mathbb{Q}} \mathcal{U}^*$ also satisfies the four equations in Definition 3 as $\mathcal{U}, \mathcal{V}$ are unitrary tensors. Next, by Proposition 3, using the exactly same method as proving the uniqueness of Moore-Penrose inverse of a matrix, we can show that the Moore-Penrose inverse of a tensor $\mathcal{A} \in \mathbb{Q}^{n_1\times n_2\times n_3}$ is unique. □

Next, we list some properties of the Moore-Penrose inverse of a quaternion tensor $\mathcal{A} \in \mathbb{Q}^{n_1 \times n_2 \times n_3}$.

Proposition 4. *Let $\mathcal{A} \in \mathbb{Q}^{n_1 \times n_2 \times n_3}$. Then*
(a) $(\mathcal{A}^{\dagger})^{\dagger} = \mathcal{A}$;
(b) $(\mathcal{A}^{\dagger})^{*} = (\mathcal{A}^{*})^{\dagger}$;
(c) $\mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{\dagger} = (\mathcal{A}^{*})^{\dagger} *_{\mathbb{Q}} \mathcal{A}$;
(d) $(\mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{\dagger})^{*} = \mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{\dagger}$;
(e) $\mathcal{E}_{\mathcal{A}} *_{\mathbb{Q}} \mathcal{A} = \mathcal{O}, \mathcal{A} *_{\mathbb{Q}} \mathcal{F}_{\mathcal{A}} = \mathcal{O}$, where $\mathcal{E}_{\mathcal{A}} = \mathcal{I} - \mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{\dagger}$, $\mathcal{F}_{\mathcal{A}} = \mathcal{I} - \mathcal{A}^{\dagger} *_{\mathbb{Q}} \mathcal{A}$;
(f) $\mathcal{A}^{\dagger}$ always exists and is unique.

Proof. We only prove (b) as all of those are using the same approach with the proof of matrix.
By Proposition 3, $(\mathcal{A} *_{\mathbb{Q}} \mathcal{B})^{*} = \mathcal{B}^{*} *_{\mathbb{Q}} \mathcal{A}^{*}$, thus

$$\begin{aligned}
&\mathcal{A}^{*} *_{\mathbb{Q}} (\mathcal{A}^{\dagger})^{*} *_{\mathbb{Q}} \mathcal{A}^{*} = (\mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{\dagger} *_{\mathbb{Q}} \mathcal{A})^{*} = \mathcal{A}^{*},\\
&(\mathcal{A}^{\dagger})^{*} *_{\mathbb{Q}} \mathcal{A}^{*} *_{\mathbb{Q}} (\mathcal{A}^{\dagger})^{*} = (\mathcal{A}^{\dagger} *_{\mathbb{Q}} \mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{\dagger})^{*} = (\mathcal{A}^{\dagger})^{*},\\
&\mathcal{A}^{*} *_{\mathbb{Q}} (\mathcal{A}^{\dagger})^{*} = (\mathcal{A}^{\dagger} *_{\mathbb{Q}} \mathcal{A})^{*} = \mathcal{A}^{\dagger} *_{\mathbb{Q}} \mathcal{A} = (\mathcal{A}^{*} *_{\mathbb{Q}} (\mathcal{A}^{\dagger})^{*})^{*},\\
&(\mathcal{A}^{\dagger})^{*} *_{\mathbb{Q}} \mathcal{A}^{*} = (\mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{\dagger})^{*} = \mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{\dagger} = ((\mathcal{A}^{\dagger})^{*} *_{\mathbb{Q}} \mathcal{A}^{*})^{*},
\end{aligned}$$

which means $(\mathcal{A}^{\dagger})^{*}$ satisfying the definition of the Moore-Penrose inverse of $\mathcal{A}^{*}$. □

Definition 4. *Given a tensor $\mathcal{A} \in \mathbb{Q}^{n_1 \times n_2 \times n_3}$, let $\mathcal{A}\{1,3\}$ denote the set of tensor $\mathcal{X} \in \mathbb{Q}^{n_2 \times n_1 \times n_3}$, which satisfies Eqs (1), (3) of (3.1). In this case, $\mathcal{X} \in \mathcal{A}\{1,3\}$ is called a $\{1,3\}$-inverse of $\mathcal{A}$. It can also be written as $\mathcal{A}^{(1,3)}$. $\mathcal{A}^{(1,4)}$ and $\mathcal{A}^{(1)}$ can be defined in the same way.*

By verifying the equations in (3.1), we derive the following result, which reveals that $\mathcal{A}^{\dagger}$ can also be expressed by $\mathcal{A}^{(1,3)}$ and $\mathcal{A}^{(1,4)}$.

Corollary 1. *Let $\mathcal{A} \in \mathbb{Q}^{n_1 \times n_2 \times n_3}$. Then $\mathcal{A}^{\dagger} = \mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{(1,3)}$.*

Here, we actually generalized the result in [15], i.e., for any finite matrix A of complex elements,

$$A^{(1,4)} A A^{(1,3)} = A^{\dagger}.$$

4. Solutions to third-order quaternion tensor equation $\mathcal{A} *_{\mathbb{Q}} \mathcal{X} = \mathcal{B}$

In this section, we will derive general solutions, least-squares solutions, minimum-norm solutions and minimum-norm least-squares solution of the third-order quaternion tensor equation

$$\mathcal{A} *_{\mathbb{Q}} \mathcal{X} = \mathcal{B} \tag{4.1}$$

by some generalized inverses. We first introduce a well-known result of matrix equation:

Lemma 3. *(See [16]) For the complex matrix equation*

$$AX = B, \tag{4.2}$$

then:
(a) If (4.2) is consistent, then $X = A^{(1)}B$ is the solution of (4.2), moreover, $X = A^{(1,4)}B$ is the least-norm

solution of (4.2)*, where* $A^{(1,4)}$ *is the* $(1,4)-$ *inverse of matrix A.*
(b) The matrix equation (4.2) *don't have to be consistent.* $X = A^{(1,3)}B$ *is the least-squares solution of* (4.2)*, where* $A^{(1,3)}$ *is the* $(1,3)-$ *inverse of matrix A.*
(c) The matrix equation (4.2) *don't have to be consistent.* $X = A^{\dagger}B$ *is the minimum-norm least-squares solution of* (4.2)*, where* $A^{\dagger}$ *is the Moore-Penrose inverse of matrix A.*

Remark 1. *The statement also holds when the matrix equation* (4.2) *is over quaternion skew field.*

Next, we will describe our required solutions by some generalized inverses of tensor.

Theorem 2. *Let* $\mathcal{A} \in \mathbb{Q}^{n_1\times n_2\times n_3}$*,* $\mathcal{B} \in \mathbb{Q}^{n_1\times n_4\times n_3}$*. Then* $\mathcal{X} = \mathcal{A}^{(1)} *_{\mathbb{Q}} \mathcal{B}$ *is the solution of the quaternion tensor equation (4.1), when it is consistent.*

Proof. By Lemma 1 and (2.4),

$$\mathcal{A} *_{\mathbb{Q}} \mathcal{X} = \mathcal{B} \Longleftrightarrow diag(\widehat{\mathcal{A}}) \cdot diag(\widehat{\mathcal{X}}) = diag(\widehat{\mathcal{B}}) \Longleftrightarrow \widehat{A^{(i)}}\widehat{X^{(i)}} = \widehat{B^{(i)}},$$

$i = 1, \cdots, n_3$. By (a) in Lemma 3 and its Remark 1, for the consistent quaternion matrix equation

$$\widehat{A^{(i)}}\widehat{X^{(i)}} = \widehat{B^{(i)}}. \tag{4.3}$$

If the matrix $\widehat{T^{(i)}} \in \widehat{A^{(i)}}\{1\}$, then $\widehat{X^{(i)}} = \widehat{T^{(i)}}\widehat{B^{(i)}}$ is the solution of the consistent matrix equation (4.3). Thus, if $diag(\widehat{\mathcal{T}}) \in (diag(\widehat{\mathcal{A}}))\{1\}$, then $diag(\widehat{\mathcal{X}}) = diag(\widehat{\mathcal{T}}) \cdot diag(\widehat{\mathcal{B}})$ is the solution of the consistent quaternion matrix equation $diag(\widehat{\mathcal{A}}) \cdot diag(\widehat{\mathcal{X}}) = diag(\widehat{\mathcal{B}})$. Note that

$$\mathcal{A} *_{\mathbb{Q}} \mathcal{X} = \mathcal{B} \Longleftrightarrow diag(\widehat{\mathcal{A}}) \cdot diag(\widehat{\mathcal{X}}) = diag(\widehat{\mathcal{B}}).$$

Thus, $\mathcal{X} = \mathcal{T} *_{\mathbb{Q}} \mathcal{B}$ is the solution of consistent tensor equation $\mathcal{A} *_{\mathbb{Q}} \mathcal{X} = \mathcal{B}$, where $\mathcal{T} \in \mathcal{A}\{1\}$.

□

Corollary 2. *Let* $\mathcal{A} \in \mathbb{Q}^{n_1\times n_2\times n_3}$*,* $\mathcal{B} \in \mathbb{Q}^{n_1\times n_4\times n_3}$*. The tensor equation* (4.1) *is consistent if and only if*

$$\mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{(1)} *_{\mathbb{Q}} \mathcal{B} = \mathcal{B}.$$

In this case, the general solution is given by

$$\mathcal{X} = \mathcal{A}^{(1)} *_{\mathbb{Q}} \mathcal{B} + (\mathcal{I} - \mathcal{A}^{(1)} *_{\mathbb{Q}} \mathcal{A}) *_{\mathbb{Q}} \mathcal{Y}, \tag{4.4}$$

where $\mathcal{Y} \in \mathbb{Q}^{n_2\times n_4\times n_3}$ *is arbitrary.*

Proof. If $\mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{(1)} *_{\mathbb{Q}} \mathcal{B} = \mathcal{B}$, then it can be seen that $\mathcal{A}^{(1)} *_{\mathbb{Q}} \mathcal{B}$ is a solution to (4.1). If the tensor equation (4.1) is consistent then there exists $\mathcal{X}_0$ such that $\mathcal{A} *_{\mathbb{Q}} \mathcal{X}_0 - \mathcal{B} = \mathcal{O}$. Considering the definition of $\mathcal{A}^{(1)}$, we have

$$\mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{(1)} *_{\mathbb{Q}} \mathcal{B} - \mathcal{B} = \mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{(1)} *_{\mathbb{Q}} \mathcal{A} *_{\mathbb{Q}} \mathcal{X}_0 - \mathcal{B} = \mathcal{A} *_{\mathbb{Q}} \mathcal{X}_0 - \mathcal{B} = \mathcal{O}.$$

Next, we show that (4.4) is the general expression of the solution to (4.1). Firstly, we can verify that $\mathcal{X}$ is the solution to (4.1). Secondly, we aim to show that any solution of (4.1) is in the form of (4.4). Assume that $\mathcal{X}_0$ is an arbitrary solution to (4.1). Setting $\mathcal{Y} = \mathcal{X}_0$, then

$$\mathcal{A}^{(1)} *_{\mathbb{Q}} \mathcal{B} + (\mathcal{I} - \mathcal{A}^{(1)} *_{\mathbb{Q}} \mathcal{A}) *_{\mathbb{Q}} \mathcal{X}_0 = \mathcal{A}^{(1)} *_{\mathbb{Q}} \mathcal{B} + \mathcal{X}_0 - \mathcal{A}^{(1)} *_{\mathbb{Q}} \mathcal{B} = \mathcal{X}_0.$$

□

In Corollary 2, if $\mathcal{A}^{(1)}$ is replaced by $\mathcal{A}^{\dagger}$, the following result is also true. Since the proof is almost the same with the proof of Corollary 2, thus we omit for simplicity.

Corollary 3. *Let $\mathcal{A} \in \mathbb{Q}^{n_1\times n_2\times n_3}$, $\mathcal{B} \in \mathbb{Q}^{n_1\times n_4\times n_3}$. Then quaternion tensor equation (4.1) is consistent if only if*

$$\mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{\dagger} *_{\mathbb{Q}} \mathcal{B} = \mathcal{B}. \tag{4.5}$$

When the equation is consistent, then the general solution is

$$\mathcal{X} = \mathcal{A}^{\dagger} *_{\mathbb{Q}} \mathcal{B} + (\mathcal{I} - \mathcal{A}^{\dagger} *_{\mathbb{Q}} \mathcal{A}) *_{\mathbb{Q}} \mathcal{Y},$$

where $\mathcal{Y}$ is an arbitrary quaternion tensor with appropriate size.

Next, we aim to derive the least-squares solution of the tensor equation (4.1).

We now provide a result for the Frobenius norm of the sum of two third-order tensors.

Theorem 3. *Let $\mathcal{A} \in \mathbb{Q}^{n_1\times n_2\times n_3}$, $\mathcal{B} \in \mathbb{Q}^{n_1\times n_2\times n_3}$. Then*

$$\| \mathcal{A} + \mathcal{B} \|_F^2 = \| \mathcal{A} \|_F^2 + \| \mathcal{B} \|_F^2 + \frac{2}{n_3} tr(diag(\widehat{\mathcal{A}} *_{\mathbb{Q}} \widehat{\mathcal{B}^*})).$$

Proof. By (2.7) and $\| K \|_F^2 = tr(K^* K)$, $tr(L) = tr(L^*)$, where K, L are any quaternion matrices and $tr(\bullet)$ represents the trace of a matrix, we have

$$\begin{aligned}
\| \mathcal{A} + \mathcal{B} \|_F^2 &= \frac{1}{n_3} \| diag(\widehat{\mathcal{A}}) + diag(\widehat{\mathcal{B}}) \|_F^2 = \frac{1}{n_3} tr((diag(\widehat{\mathcal{A}}) + diag(\widehat{\mathcal{B}}))^* \cdot (diag(\widehat{\mathcal{A}}) + diag(\widehat{\mathcal{B}}))) \\
&= \frac{1}{n_3}(\| diag(\widehat{\mathcal{A}}) \|_F^2 + \| diag(\widehat{\mathcal{B}}) \|_F^2 + 2tr(diag^*(\widehat{\mathcal{A}}) \cdot diag(\widehat{\mathcal{B}}))) \\
&= \| \mathcal{A} \|_F^2 + \| \mathcal{B} \|_F^2 + \frac{2}{n_3} tr(diag(\widehat{\mathcal{A}^*} *_{\mathbb{Q}} \widehat{\mathcal{B}})).
\end{aligned}$$

□

Theorem 4. *Let $\mathcal{A} \in \mathbb{Q}^{n_1\times n_2\times n_3}$, $\mathcal{B} \in \mathbb{Q}^{n_1\times n_4\times n_3}$. Then $\mathcal{X} = \mathcal{A}^{(1,3)} *_{\mathbb{Q}} \mathcal{B}$ is the least-squares solution of the tensor equation* (4.1).

Proof. It follows from Theorem 3

$$\begin{aligned}
\| \mathcal{A} *_{\mathbb{Q}} \mathcal{X}_0 - \mathcal{B} \|_F^2 &= \| (\mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{(1,3)} *_{\mathbb{Q}} \mathcal{B} - \mathcal{B}) + \mathcal{A} *_{\mathbb{Q}} (\mathcal{X}_0 - (\mathcal{A}^{(1,3)} *_{\mathbb{Q}} \mathcal{B})) \|_F^2 \\
&= \| \mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{(1,3)} *_{\mathbb{Q}} \mathcal{B} - \mathcal{B} \|_F^2 + \| \mathcal{A} *_{\mathbb{Q}} (\mathcal{X}_0 - \mathcal{A}^{(1,3)} *_{\mathbb{Q}} \mathcal{B}) \|_F^2 \\
&\quad + \frac{2}{n_3} tr(diag(\widehat{\mathcal{M}^*} *_{\mathbb{Q}} \widehat{\mathcal{N}})),
\end{aligned} \tag{4.6}$$

where $\mathcal{M} = \mathcal{A} *_{\mathbb{Q}} (\mathcal{X}_0 - \mathcal{A}^{(1,3)} *_{\mathbb{Q}} \mathcal{B})$, $\mathcal{N} = \mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{(1,3)} *_{\mathbb{Q}} \mathcal{B} - \mathcal{B}$ and $\mathcal{X}_0 \in \mathbb{Q}^{n_2\times n_4\times n_3}$ is arbitrary.

The property of generalized inverse $\mathcal{A}^{(1,3)}$ gives us

$$\mathcal{A}^* *_{\mathbb{Q}} (\mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{(1,3)} - \mathcal{I}) = \mathcal{O}.$$

Thus

$$\begin{aligned}
&(\mathcal{A} *_{\mathbb{Q}} (\mathcal{X}_0 - \mathcal{A}^{(1,3)} *_{\mathbb{Q}} \mathcal{B}))^* *_{\mathbb{Q}} (\mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{(1,3)} *_{\mathbb{Q}} \mathcal{B} - \mathcal{B}) \\
&= (\mathcal{X}_0 - \mathcal{A}^{(1,3)} *_{\mathbb{Q}} \mathcal{B})^* *_{\mathbb{Q}} \mathcal{A}^* *_{\mathbb{Q}} (\mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{(1,3)} - \mathcal{I}) *_{\mathbb{Q}} \mathcal{B} = \mathcal{O}.
\end{aligned} \tag{4.7}$$

So $diag(\widehat{\mathcal{M}^*} *_{\mathbb{Q}} \widehat{\mathcal{N}}) = O$.

Then we can get

$$\begin{aligned} \| \mathcal{A} *_{\mathbb{Q}} \mathcal{X}_0 - \mathcal{B} \|_F^2 &= \| \mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{(1,3)} *_{\mathbb{Q}} \mathcal{B} - \mathcal{B} \|_F^2 + \| \mathcal{A} *_{\mathbb{Q}} (\mathcal{X}_0 - \mathcal{A}^{(1,3)} *_{\mathbb{Q}} \mathcal{B}) \|_F^2 \\ &\geq \| \mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{(1,3)} *_{\mathbb{Q}} \mathcal{B} - \mathcal{B} \|_F^2, \end{aligned}$$

which means that $\mathcal{X} = \mathcal{A}^{(1,3)} *_{\mathbb{Q}} \mathcal{B}$ is the least-squares solution of Eq (4.1). □

In solving practical applications, we sometimes need to find solutions for which the norm is minimal. The next theorem provides the mimimum-norm solution of tensor equation (4.1).

Theorem 5. *Let $\mathcal{A} \in \mathbb{Q}^{n_1 \times n_2 \times n_3}$, $\mathcal{B} \in \mathbb{Q}^{n_1 \times n_4 \times n_3}$. Then $\mathcal{X} = \mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{B}$ is the minimum-norm solution of the tensor equation* (4.1)*, when it is consistent.*

Proof. Since $\mathcal{A}^{(1,4)} \in \mathcal{A}\{1\}$, by Theorem 2, $\mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{B}$ is a solution of (4.1), if the tensor equation (4.1) is consistent. Next, we prove $\mathcal{X} = \mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{B}$ is the minimum-norm solution. By Corollary 2, for any solution $\mathcal{X}_0$ to (4.1), $\mathcal{X}_0$ can be written in the form of $\mathcal{X}_0 = \mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{B} + (\mathcal{I} - \mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{A}) *_{\mathbb{Q}} \mathcal{Y}$, with $\mathcal{Y} \in \mathbb{Q}^{n_2 \times n_4 \times n_3}$. Thus, by Theorem 3,

$$\begin{aligned} \| \mathcal{X}_0 \|_F^2 &= \| \mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{B} + (\mathcal{I} - \mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{A}) *_{\mathbb{Q}} \mathcal{Y} \|_F^2 \\ &= \| \mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{B} \|_F^2 + \| (\mathcal{I} - \mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{A}) *_{\mathbb{Q}} \mathcal{Y} \|_F^2 + \frac{2}{n_3} tr(diag(\widehat{\mathcal{M}^*} *_{\mathbb{Q}} \widehat{\mathcal{N}})), \end{aligned}$$

where $\mathcal{M} = \mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{B}$, $\mathcal{N} = (\mathcal{I} - \mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{A}) *_{\mathbb{Q}} \mathcal{Y}$.

By the property of generalized inverse $\mathcal{A}^{(1,4)}$, we have

$$\begin{aligned} \widehat{\mathcal{M}^*} *_{\mathbb{Q}} \widehat{\mathcal{N}} &= (\mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{B})^* *_{\mathbb{Q}} (\mathcal{I} - \mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{A}) *_{\mathbb{Q}} \mathcal{Y} \\ &= (\mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{A} *_{\mathbb{Q}} \mathcal{X}_0)^* *_{\mathbb{Q}} (\mathcal{I} - \mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{A}) *_{\mathbb{Q}} \mathcal{Y} \\ &= \mathcal{X}_0^* *_{\mathbb{Q}} (\mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{A})^* *_{\mathbb{Q}} (\mathcal{I} - \mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{A}) *_{\mathbb{Q}} \mathcal{Y} \\ &= \mathcal{X}_0^* *_{\mathbb{Q}} \mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{A} *_{\mathbb{Q}} (\mathcal{I} - \mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{A}) *_{\mathbb{Q}} \mathcal{Y} \\ &= \mathcal{X}_0^* *_{\mathbb{Q}} (\mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{A} - \mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{A}) *_{\mathbb{Q}} \mathcal{Y} \\ &= \mathcal{X}_0^* *_{\mathbb{Q}} (\mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{A} - \mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{A}) *_{\mathbb{Q}} \mathcal{Y} = O. \end{aligned}$$

We conclude that

$$\| \mathcal{X}_0 \|_F^2 = \| \mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{B} \|_F^2 + \| (\mathcal{I} - \mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{A}) *_{\mathbb{Q}} \mathcal{Y} \|_F^2 \geq \| \mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{B} \|_F^2 = \| \mathcal{X} \|_F^2 .$$

□

Remark 2. *In the assumption, the tensor equation is consistent means it has a general solution. The solution don't have to be a minimum-norm solution. So, According to Corollary 3, Theorem 5 can be rewritten as follows: Let $\mathcal{A} \in \mathbb{Q}^{n_1 \times n_2 \times n_3}$, $\mathcal{B} \in \mathbb{Q}^{n_1 \times n_4 \times n_3}$. If $\mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{\dagger} *_{\mathbb{Q}} \mathcal{B} = \mathcal{B}$. Then $\mathcal{X} = \mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{B}$ is the minimum-norm solution of the tensor equation* (4.1).

It is well known that the least-squares solution of an equation is not unique, neither is the minimum-norm solution. Then we consider the minimum-norm least-squares solution to this problem.

Theorem 6. *Let $\mathcal{A} \in \mathbb{Q}^{n_1\times n_2\times n_3}$, $\mathcal{B} \in \mathbb{Q}^{n_1\times n_4\times n_3}$, and $\mathcal{X}_0 \in \mathbb{Q}^{n_2\times n_4\times n_3}$. The tensor $\mathcal{X}_0$ is the least-squares solutions of* (4.1) *if and only if $\mathcal{X}_0$ is the solution of the consistent tensor equation*

$$\mathcal{A} *_{\mathbb{Q}} \mathcal{X} = \mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{(1,3)} *_{\mathbb{Q}} \mathcal{B}. \tag{4.8}$$

Proof. "$\Rightarrow$" Assuming $\mathcal{X}_0$ is a least-squares solution of the tensor equation (4.1), from Theorem 4, we have

$$\| \mathcal{A} *_{\mathbb{Q}} \mathcal{X}_0 - \mathcal{B} \|_F = \| \mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{(1,3)} *_{\mathbb{Q}} \mathcal{B} - \mathcal{B} \|_F = \min_{\mathcal{X}\in\mathbb{Q}^{n_1\times n_2\times n_3}} \| \mathcal{A} *_{\mathbb{Q}} \mathcal{X} - \mathcal{B} \|_F . \tag{4.9}$$

From Theorem 3, we have

$$\| \mathcal{A} *_{\mathbb{Q}} \mathcal{X}_0 - \mathcal{B} \|_F^2 - \| \mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{(1,3)} *_{\mathbb{Q}} \mathcal{B} - \mathcal{B} \|_F^2 = \| \mathcal{A} *_{\mathbb{Q}} (\mathcal{X}_0 - \mathcal{A}^{(1,3)} *_{\mathbb{Q}} \mathcal{B}) \|_F^2 + \frac{2}{n_3} tr(diag(\widehat{\mathcal{M}^* *_{\mathbb{Q}} \mathcal{N}})),$$

where $\mathcal{M} = \mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{(1,3)} *_{\mathbb{Q}} \mathcal{B} - \mathcal{B}$, $\mathcal{N} = \mathcal{A} *_{\mathbb{Q}} (\mathcal{X}_0 - \mathcal{A}^{(1,3)} *_{\mathbb{Q}} \mathcal{B})$.

Based on (4.7) and (4.9), $\| \mathcal{A} *_{\mathbb{Q}} (\mathcal{X}_0 - \mathcal{A}^{(1,3)} *_{\mathbb{Q}} \mathcal{B}) \|_F = 0$, thus $\mathcal{A} *_{\mathbb{Q}} (\mathcal{X}_0 - \mathcal{A}^{(1,3)} *_{\mathbb{Q}} \mathcal{B}) = \mathcal{O}$, which means that $\mathcal{X}_0$ is a solution of the tensor equation (4.8).

"$\Leftarrow$" If $\mathcal{X}_0$ is a solution of the tensor equation (4.8), note that $\mathcal{A}^* *_{\mathbb{Q}} \mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{(1,3)} = \mathcal{A}^* *_{\mathbb{Q}} (\mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{(1,3)})^* = (\mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{(1,3)} *_{\mathbb{Q}} \mathcal{A})^* = \mathcal{A}^*$, then $\mathcal{A}^* *_{\mathbb{Q}} \mathcal{A} *_{\mathbb{Q}} \mathcal{X}_0 = \mathcal{A}^* *_{\mathbb{Q}} \mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{(1,3)} *_{\mathbb{Q}} \mathcal{B} = \mathcal{A}^* *_{\mathbb{Q}} \mathcal{B}$.

By Lemma 1 and Proposition 2,

$$\mathcal{A}^* *_{\mathbb{Q}} \mathcal{A} *_{\mathbb{Q}} \mathcal{X}_0 = \mathcal{A}^*\mathcal{B} \Longleftrightarrow (diag(\widehat{\mathcal{A}}))^* \cdot diag(\widehat{\mathcal{A}}) \cdot diag(\widehat{\mathcal{X}_0}) = (diag(\widehat{\mathcal{A}}))^* \cdot diag(\widehat{\mathcal{B}}),$$

which is equivalent to

$$\widehat{A^{(i)}}^*\widehat{A^{(i)}}\widehat{X_0^{(i)}} = \widehat{A^{(i)}}^*\widehat{B^{(i)}}, i = 1, \cdots, n_3.$$

In other words, $\widehat{X_0^{(i)}}$ is the least-squares solution to

$$\widehat{A^{(i)}}\widehat{X^{(i)}} = \widehat{B^{(i)}}, i = 1, \cdots, n_3.$$

Or, $diag(\widehat{\mathcal{X}_0})$ is the least-squares solution to

$$diag(\widehat{\mathcal{A}}) \cdot diag(\widehat{\mathcal{X}}) = diag(\widehat{\mathcal{B}}).$$

Since

$$\| \mathcal{A} *_{\mathbb{Q}} \mathcal{X} - \mathcal{B} \|_F = \frac{1}{\sqrt{n_3}} \| diag(\widehat{\mathcal{A}}) \cdot diag(\widehat{\mathcal{X}}) - diag(\widehat{\mathcal{B}}) \|_F .$$

Then, we can see that $\mathcal{X}_0$ is the least-squares solution of (4.1). □

Remark 3. *The tensor equation* (4.1) *always has a least-squares solution, thus by Theorem 6, the tensor equation* (4.8) *is always consistent.*

Theorem 7. *Let $\mathcal{A} \in \mathbb{Q}^{n_1\times n_2\times n_3}$, $\mathcal{B} \in \mathbb{Q}^{n_1\times n_4\times n_3}$. The tensor $\mathcal{X}_0 = \mathcal{T} *_{\mathbb{Q}} \mathcal{B}$ is the minimum-norm least-squares solution of the tensor equation* (4.1) *if and only if $\mathcal{T} = \mathcal{A}^\dagger$.*

Proof. "⇒" If $\mathcal{X}_0 = \mathcal{T} *_{\mathbb{Q}} \mathcal{B}$ is the minimum-norm least-squares solution of the tensor equation (4.1), by Theorem 6, $\mathcal{X}_0$ is the minimum-norm solution of Eq (4.8). Then, by Corollary 1 and Theorem 5, we have

$$\mathcal{X}_0 = \mathcal{A}^{(1,4)} *_{\mathbb{Q}} \mathcal{A} *_{\mathbb{Q}} \mathcal{A}^{(1,3)} *_{\mathbb{Q}} \mathcal{B} = \mathcal{A}^{\dagger} *_{\mathbb{Q}} \mathcal{B},$$

which means that $\mathcal{T} = \mathcal{A}^{\dagger}$.

"⇐" If $\mathcal{T} = \mathcal{A}^{\dagger}$, since $\mathcal{A}^{\dagger} \in \mathcal{A}\{1,2,3,4\}$, it satisfies both the properties of a least-squares solution and minimum-norm solution to the Eq (4.1) by Theorem 4 and Theorem 5. □

5. Numerical examples

In this section, we give two numerical examples.

Example 1. *Consider the third-order tensor equation $\mathcal{A} *_{\mathbb{Q}} \mathcal{X} = \mathcal{B}$, where $\mathcal{A}$ is a $2 \times 2 \times 3$ quaternion tensor with frontal slices $A^{(1)}$, $A^{(2)}$, $A^{(3)}$ which are given by*

$$\boldsymbol{unfold}(\mathcal{A}) = \begin{bmatrix} A^{(1)} \\ A^{(2)} \\ A^{(3)} \end{bmatrix} = \begin{bmatrix} 5 - 2\boldsymbol{i} + 5\boldsymbol{j} + 2\boldsymbol{k} & 8 - 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} \\ 8 + 3\boldsymbol{j} - \boldsymbol{k} & -2 + 3\boldsymbol{i} + 3\boldsymbol{j} + 6\boldsymbol{k} \\ 2\boldsymbol{i} + 6\boldsymbol{j} + 5\boldsymbol{k} & 3 + 3\boldsymbol{i} + 7\boldsymbol{j} \\ 12 + \boldsymbol{i} + 3\boldsymbol{k} & 8 - \boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k} \\ 5 + 10\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k} & 2\boldsymbol{i} - 4\boldsymbol{j} - \boldsymbol{k} \\ 6 - \boldsymbol{i} + 5\boldsymbol{k} & 4\boldsymbol{i} + 9\boldsymbol{j} \end{bmatrix},$$

and the $2\times2\times3$ quaternion tensor $\mathcal{B}$, with frontal slices $B^{(1)}$, $B^{(2)}$, $B^{(3)}$, which are given by the following ***unfold*** *form*

$$\boldsymbol{unfold}(\mathcal{B}) = \begin{bmatrix} B^{(1)} \\ B^{(2)} \\ B^{(3)} \end{bmatrix} = \begin{bmatrix} 1 - 6\boldsymbol{i} + 3\boldsymbol{j} - 9\boldsymbol{k} & 30 - \boldsymbol{j} - 2\boldsymbol{k} \\ 9\boldsymbol{i} - 12\boldsymbol{j} + 6\boldsymbol{k} & -6\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k} \\ 10 - 2\boldsymbol{i} + 14\boldsymbol{k} & 11\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} \\ 8 - \boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k} & \boldsymbol{i} + 18\boldsymbol{j} + \boldsymbol{k} \\ 2\boldsymbol{i} - 9\boldsymbol{j} + 19\boldsymbol{k} & 1 - 3\boldsymbol{j} + 5\boldsymbol{k} \\ -5\boldsymbol{i} + 16\boldsymbol{j} - 2\boldsymbol{k} & -6 + 6\boldsymbol{j} + 2\boldsymbol{k} \end{bmatrix}.$$

To investigate whether the tensor equation is consistent or not, we check the solvability condition (4.5) *in Corollary 3. First, by Theorem 1, we get $\mathcal{A}^{\dagger}$ via MATLAB as follows:*

$$\boldsymbol{unfold}(\mathcal{A}^{\dagger}) = \begin{bmatrix} 0.0087 + 0.0233i + 0.0105j + 0.0144k & 0.0134 + 0.0326i - 0.0141j + 0.0114k \\ 0.0292 + 0.0206i + 0.0010j - 0.0092k & -0.0156 - 0.0136i - 0.0167j - 0.0318k \\ -0.0082 - 0.0292i - 0.0221j + 0.0096k & -0.0215 - 0.0010i + 0.0138j - 0.0129k \\ -0.0070 - 0.0035i + 0.0050j + 0.0025k & -0.0074 - 0.0064i - 0.0174j + 0.0012k \\ -0.0031 - 0.0137i - 0.0075j - 0.0408k & 0.0262 - 0.0269i + 0.0078j + 0.0067k \\ 0.0185 - 0.0104i + 0.0063j + 0.0228k & 0.0142 + 0.0127i + 0.0027j + 0.0146k \end{bmatrix}.$$

Then, we get

$$\boldsymbol{unfold}(\mathcal{E}_{\mathcal{A}} *_{\mathbb{Q}} \mathcal{B}) = \begin{bmatrix} 0.0002 - 0.0010i + 0.0005j - 0.0016k & 0.0052 + 0.0000i - 0.0002j - 0.0003k \\ 0.0000 + 0.0016i - 0.0021j + 0.0010k & -0.0000 - 0.0010i - 0.0007j + 0.0002k \\ 0.0009 - 0.0000i - 0.0008j + 0.0028k & 0.0001 + 0.0009i - 0.0002j + 0.0003k \\ 0.0007 - 0.0005i + 0.0016j + 0.0000k & -0.0005 + 0.0001i + 0.0021j + 0.0003k \\ 0.0009 - 0.0000i - 0.0008j + 0.0028k & 0.0001 + 0.0009i - 0.0002j + 0.0003k \\ 0.0007 - 0.0005i + 0.0016j + 0.0000k & -0.0005 + 0.0001i + 0.0021j + 0.0003k \end{bmatrix},$$

Clearly, $\mathcal{E}_{\mathcal{A}} *_{\mathbb{Q}} \mathcal{B}$ is close to O. Therefore, it is almost consistent. By Theorem 7, the minimum-norm solution is $\mathcal{X} = \mathcal{A}^{\dagger} *_{\mathbb{Q}} \mathcal{B}$, and it is given by the following ***unfold*** *form*

$$\boldsymbol{unfold}(\mathcal{X}) = \begin{bmatrix} -0.2586 - 1.2710i + 0.6598j - 0.3004k & 0.5098 + 0.7523i + 0.1790j - 0.1183k \\ -0.2554 - 0.7382i + 0.4967j + 0.4155k & 0.8895 + 0.3581i + 0.9405j - 0.3567k \\ 1.0649 - 0.4286i - 0.0780j - 0.2524k & -0.2410 - 0.9762i + 0.0943j + 0.7470k \\ 0.0320 + 0.4399i - 0.0515j + 0.8712k & 0.0811 + 0.0845i - 0.1269j - 0.0926k \\ -0.6175 + 1.0406i - 0.0296j + 0.4714k & -0.4104 - 0.6945i - 0.7305j - 0.7121k \\ 0.6650 + 0.3582i - 1.1311j - 0.2368k & 0.9305 + 0.5687i - 0.2718j + 0.6352k \end{bmatrix}.$$

And we can check that $\boldsymbol{unfold}(\mathcal{A} *_{\mathbb{Q}} \mathcal{X} - \mathcal{B}) = 1.0e^{-13}Z$, *where*

$$Z = \begin{bmatrix} -0.1410 - 0.0977i - 0.0118j - 0.0474k & -0.0592 - 0.1007i - 0.0089j - 0.0452k \\ -0.0374 - 0.0755i - 0.0355j - 0.0563k & 0.0829 - 0.0118i - 0.1303j - 0.0444k \\ 0.1133 + 0.0109i + 0.0473j - 0.0530k & -0.0059 - 0.0194i - 0.0502j + 0.1227k \\ -0.0208 - 0.0059i - 0.0689j + 0.0015k & 0.0011 - 0.0052i - 0.0975j + 0.0574k \\ -0.0611 + 0.0602i + 0.0268j + 0.0649k & -0.0059 - 0.0399i - 0.0386j + 0.0202k \\ 0.0049 - 0.0296i - 0.0022j + 0.0015k & -0.0040 + 0.0615i - 0.0565j - 0.0041k \end{bmatrix}.$$

Example 2. *For an original color video $\mathcal{X}$, we take the first four frames of the original color video (see Original Frames in Table 1, the video data is from Densely Annotation Video Segmentation dataset (DAVIS)). Then we noise the color video $\mathcal{X}$ by $\mathcal{N}$ and get the color video with noises $\mathcal{X} + \mathcal{N}$ (see the Frames With Noise in Table 1). Now, we disturbed the color video with noise by the tensor $\mathcal{A}$ and get C. Now, we aim to restore the original color video $\mathcal{X}$. In color video processing, we generated the disturbing tensor $\mathcal{A}$ randomly with its elements in* $[-30, 30]$. *To restore the color video $\mathcal{X}$, we have to find the minimum-norm least-squares solution to the tensor equation*

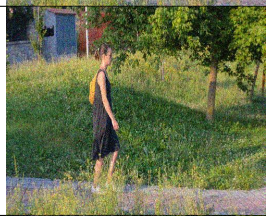
$$\mathcal{A} *_{\mathbb{Q}} \mathcal{X} = C, \tag{5.1}$$

where $\mathcal{A}, C \in \mathbb{Q}^{400\times500\times4}$, $\mathcal{X}, \mathcal{N} \in \mathbb{Q}^{500\times400\times4}$ ($\mathcal{N}$ is a white noise with a mean of 0 *and a standard deviation of* 0.01*). By Theorem 7, our required minimum-norm least-squares solution is $\mathcal{X}_0 = \mathcal{A}^{\dagger}C$. By computation, we can get the restored color video (see the Restored Frames in Table 1), with*

$$\| \mathcal{X} - \mathcal{X}_0 \|_F = 2.2870e - 08.$$

We can see from the Table 1 that our restored color video achieves a good accuracy and has a satisfied result.

Table 1. Restored frames.

	Original Frames	Frames With Noise	Restored Frames
Frame 1			
Frame 2			
Frame 3			
Frame 4			

6. Conclusions

In this paper, by utilizing the Qt-product and generalized inverses of third-order quaternion tensors, we derive solvability conditions of the third-order quaternion tensor equation $\mathcal{A} *_{\mathbb{Q}} \mathcal{X} = \mathcal{B}$. Also we get the general solution, the least-squares solution, the minimum-norm solution and the minimum-norm least-squares solution of the tensor equation. Finally, two examples demonstrate the theoretical results of the paper.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research was funded by Macao Science and Technology Development Fund (No. 0013/2021/ITP), The Joint Research and Development Fund of Wuyi University, Hong Kong and Macao (2019WGALH20), and The MUST Faculty Research Grants (FRG-22-073-FIE).

Conflict of interest

The authors declare no competing interests.

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