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Research article

Global existence of strong solutions to compressible Navier-Stokes-Korteweg equations with external potential force

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Abstract: In this article, we consider a three dimensional compressible Navier-Stokes-Korteweg equations with the effect of external potential force. Under the smallness assumptions on both the external potential force and the initial perturbation of the stationary solution in $H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$, we prove the global existence and regularity of strong solutions for the Navier-Stokes-Korteweg equations.

Keywords: Navier-Stokes-Korteweg equations; stationary solution; global existence
Mathematics Subject Classification: 34B40, 35Q35, 93D20

1. Introduction

The theory of capillarity with diffuse interfaces was first introduced by Korteweg and derived rigorously by Dunn and Serrin [7]. The Navier-Stokes-Korteweg (NSK) equations can be used to describe the motion of a compressible fluid with capillarity effect (see [2, 5, 10]). In this work, we consider the following compressible Navier-Stokes-Korteweg equations in three dimensional (3D) space:

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ \rho[u_t + (u \cdot \nabla)u] + \nabla P(\rho) = \mu \Delta u + (\mu + \nu)\nabla(\nabla \cdot u) + \kappa\rho\nabla\triangle\rho + \rho F(x), \end{cases} \tag{1.1}$$

where $\rho > 0, u$ and $P(\rho)$ represent the density, velocity and pressure, respectively. The constants μ and ν are the viscosity coefficients satisfying $\mu > 0$ and $2\mu + 3\nu \geq 0$. In addition, $\kappa > 0$ is the capillary coefficient. $F(x) = (F_1(x), F_2(x), F_3(x))$ is a given external force.

Due to the important role of Navier-Stokes-Korteweg (NSK) equations in the field of applied and computational mathematics, there is much literature on the mathematical theory of the NSK model. In particular, the local existence and global existence of smooth solutions in Sobolev space without external force was proved by Hattori and Li [11, 12]. The existence and uniqueness results of suitably smooth solutions in critical Besov spaces was obtained by Danchin and Desjardins [8]. The existence and stability of time-periodic solution was verified by Tsuda [26]. The global existence of weak solutions has been investigated by Bresch, Desjardins and Lin [3] in 2D and 3D periodic domain and by Haspot [13] in 2D space. Kotschote proved the local existence of strong solutions in [16]. Li [17] investigated the global existence and L^2-decay rate of smooth solutions for the compressible NSK equations with small initial data and small external potential force. Tan, Wang and Xu [24] established the global existence and optimal L^2-decay rate for the strong solutions to the compressible NSK equations without external force. More mathematical theories about NSK model can be found in [4, 6, 15, 18, 25, 27, 28], and other theories of related or similar models can be found in [9, 14, 19, 21–23, 29, 30] etc.

In this paper, we consider the global existence of the solutions to the compressible Navier-Stokes-Korteweg equations with only external potential force, i.e., $F = -\nabla\phi$ and we consider the following initial value problem in three dimensional space:

$$\begin{cases} \rho_t + \nabla\cdot(\rho u) = 0, \\ u_t + (u\cdot\nabla)u + \dfrac{\nabla P(\rho)}{\rho} = \dfrac{\mu}{\rho}\Delta u + \dfrac{(\mu+\nu)}{\rho}\nabla(\nabla\cdot u) + \kappa\nabla\triangle\rho - \nabla\phi, \\ (\rho, u)(x,0) = (\rho_0, u_0)(x) \to (\rho_\infty, 0) \quad \text{as } |x|\to\infty,\ \rho_\infty > 0. \end{cases} \tag{1.2}$$

The corresponding steady-state problem can be expressed as follows:

$$\begin{cases} \nabla\cdot(\tilde\rho\tilde u) = 0, \\ \tilde\rho(\tilde u\cdot\nabla)\tilde u + \nabla P(\tilde\rho) - \mu\Delta\tilde u - (\mu+\nu)\nabla(\nabla\cdot\tilde u) - \kappa\tilde\rho\nabla\triangle\tilde\rho + \tilde\rho\nabla\phi = 0, \\ (\tilde\rho, \tilde u) \to (\rho_\infty, 0) \quad \text{as } |x|\to\infty,\ \rho_\infty > 0. \end{cases} \tag{1.3}$$

Note that the existence of the solution to problem (1.3) has been established in [17], which is the following proposition.

Proposition 1.1. *Let $P(\cdot)$ be smooth (at least C^2) in a neighborhood of ρ_∞ with $P'(\cdot) > 0$, if $\|\phi\|_3 \le \epsilon_0$ with ε_0 be a small positive constant, then the problem (1.3) has a unique solution $(\tilde\rho, \tilde u)(x)$ satisfying*

$$\tilde\rho - \rho_\infty \in H^5(R^3), \quad \tilde u = 0,$$

and

$$\frac{1}{2}\rho_\infty \le \tilde\rho(x) \le 2\rho_\infty, \quad \|\tilde\rho - \rho_\infty\|_5 \le C\epsilon_0. \tag{1.4}$$

We mention that the global existence, regularity and time decay rates of the solution (ρ, u) to the steady state $(\tilde\rho, 0)$ have been established in [17, 28] when the initial perturbations $(\rho_0 - \tilde\rho, u_0)(x)$ are small in $H^4(R^3)\times H^3(R^3)$ and $H^3(R^3)\times H^2(R^3)$, respectively. In the absence of external force, the work [24] proved the global existence of solutions when the initial perturbation $\|\rho_0 - \tilde\rho\|_2 + \|u_0\|_1$ is small and $\tilde\rho$ is a positive constant. However, there is no result on the existence of the global solutions to (1.2) with external force when $\|\rho_0 - \tilde\rho\|_2 + \|u_0\|_1$ is small and $\tilde\rho$ is not a constant. In this paper, a promising answer to this question is given. The major results are stated in the following theorem.

Theorem 1.1. *Let $P(\cdot)$ be smooth (at least C^2) in a neighborhood of ρ_∞ with $P'(\cdot) > 0$ and assume that $(\rho_0 - \tilde{\rho}, u_0)(x) \in H^2(R^3) \times H^1(R^3)$, $\|\rho_0 - \tilde{\rho}\|_2 + \|u_0\|_1 \leq \varepsilon_1$ and $\|\phi(x)\| \leq \varepsilon_1$ for some small constant $\varepsilon_1 > 0$, then the Cauchy problem (1.2) admits a unique global solution $(\rho, u)(x, t)$ satisfying*

$$\|\rho - \tilde{\rho}\|_2^2 + \|u\|_1^2 + \int_0^t (\|\nabla(\rho - \tilde{\rho})\|_2^2 + \|\nabla u\|_1^2) \mathrm{d}\tau \leq C_0(\|\rho_0 - \tilde{\rho}\|_2^2 + \|u_0\|_1^2), \ \forall\, t \geq 0, \tag{1.5}$$

where C_0 is a positive constant.

The idea of the proof is outlined as follows. First, we recall the existence and uniqueness of the stationary solution. Then, combining the local existence and global a-prior estimates derived by the elaborate energy method, we apply the continuity argument to establish the global existence of solutions for the nonlinear problem.

The rest of this article is organized as follows. In Section 2, we make some preliminaries and Section 3 is devoted to establishing the existence and regularity of global strong solutions for the initial value problem (1.2).

2. Preliminaries

In this section, we first introduce some notations and function spaces, and then recall some important inequalities.

Throughout this paper, we denote the usual Lebesgue space and Sobolev space on $\mathbb{R}^3$ by $L^p(\mathbb{R}^3)$ and $W^{m,p}(\mathbb{R}^3)$ endowed with norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{m,p}$, respectively. Especially, we denote $H^m(\Omega) := W^{m,2}(\Omega)$ with norm $\|\cdot\|_m$. For a multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}, |\alpha| = \sum_{i=1}^{3} \alpha_i$. C represents a generic positive constant. In addition, let

$L^p(I; X)$:= space of strongly measurable functions on the closed interval I,
with values in the Banach space X, endowed with norm

$$\|\varphi\|_{L^p(I;X)} := \left(\int_I \|\varphi\|_X^p \mathrm{d}t\right)^{1/p}, \ \ 1 \leq p < \infty,$$

$C^k(I; X)$:= space of the k-times continuously differentiable functions on the
interval I, with values in the space X, endowed with the usual norm.

Next, we recall some important inequalities as follows.

Lemma 2.1. *(see [1])*
(i) If $\varphi(x) \in H^1(\mathbb{R}^3)$, then the following inequalities hold:

$$\|\varphi\|_{L^6} \leq C\|\nabla\varphi\|, \quad \|\varphi\|_{L^q} \leq C(\|\varphi\| + \|\varphi\|_{L^6}) \leq C\|\varphi\|_1, \ \ 2 \leq q \leq 6.$$

(ii) Assume $\varphi(x) \in H^2(\mathbb{R}^3)$, then

$$\|\varphi\|_{L^\infty} \leq C\|\nabla\varphi\|_1.$$

3. Existence of global solutions

In this section, we concentrate on establishing the existence and stability of global-in-time solutions to the problem (1.2).

Since $\tilde{u} = 0$, it follows from (1.3) that the stationary solution $\tilde{\rho}$ satisfies

$$\begin{cases} \nabla P(\tilde{\rho}) - \kappa\tilde{\rho}\nabla\triangle\tilde{\rho} + \tilde{\rho}\nabla\phi = 0, \\ \tilde{\rho} \to \rho_\infty \quad \text{as } |x| \to \infty. \end{cases} \tag{3.1}$$

Let $(n, u) = (\rho - \tilde{\rho}, u)$, then problem (1.2) can be transformed into the following problem

$$\begin{cases} n_t + \nabla\cdot((n+\tilde{\rho})u) = 0, \\ u_t - \dfrac{\mu}{n+\tilde{\rho}}\Delta u - \dfrac{\mu+\nu}{n+\tilde{\rho}}\nabla(\nabla\cdot u) + \dfrac{P'(\rho_\infty)}{\rho_\infty}\nabla n = \kappa\nabla\triangle n + f, \\ (n,u)(x,0) = (n_0,u_0)(x) = (\rho_0 - \tilde{\rho}, u_0)(x) \to (0,0) \quad \text{as } |x| \to \infty, \end{cases} \tag{3.2}$$

where

$$f = -(u\cdot\nabla)u - \Big(\frac{P'(n+\tilde{\rho})}{n+\tilde{\rho}} - \frac{P'(\tilde{\rho})}{\tilde{\rho}}\Big)\nabla\tilde{\rho} - \Big(\frac{P'(n+\tilde{\rho})}{n+\tilde{\rho}} - \frac{P'(\rho_\infty)}{\rho_\infty}\Big)\nabla n. \tag{3.3}$$

Now, we define a function space

$$\begin{aligned} X(0,T) := \{(n,u) \,\big|\, & n \in C^0(0,T;H^2(\mathbb{R}^3)) \cap C^1(0,T;H^1(\mathbb{R}^3)), \\ & u \in C^0(0,T;H^1(\mathbb{R}^3)) \cap C^1(0,T;L^2(\mathbb{R}^3)), \\ & \nabla n \in L^2(0,T;H^2(\mathbb{R}^3)), \quad \nabla u \in L^2(0,T;H^1(\mathbb{R}^3))\}, \end{aligned}$$

and for any $T \geq 0$, let

$$N(0,T)^2 := \sup_{0\leq t\leq T}\{\|n(\cdot,t)\|_2^2 + \|u(\cdot,t)\|_1^2\} + \int_0^T (\|\nabla n(\cdot,t)\|_2^2 + \|\nabla u(\cdot,t)\|_1^2)\mathrm{d}t.$$

Before proving the existence of global solutions, we first give the results about the existence of local solutions as follows.

Proposition 3.1. *(Local existence) Assume that $(n_0,u_0)(x) \in H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ and $\|\phi(x)\| \leq \varepsilon_1$ with the positive constant ε_1 small enough. Then there exists a positive constant $T_1 > 0$ depending on n_0 and u_0, such that the initial value problem (3.2) has a unique solution $(n,u) \in X(0,T_1)$ satisfying $N(0,T_1) \leq 2N(0,0)$.*

Note that the conclusions can be proved using a similar method to that in [16,20]. Since the method is standard, we omit it here.

Next, to obtain the global existence of the solution $(n,u)(x,t)$ of system (3.2), based on standard continuity argument, some a-priori estimates need to be established first. To this end, we assume that, for $T > 0$,

$$E(T) := \sup_{0\leq t\leq T}(\|n(\cdot,t)\|_2 + \|u(\cdot,t)\|_1) \leq \delta \ll 1. \tag{3.4}$$

By the above assumption (3.4) and the Sobolev's inequality, we have

$$\|n(\cdot,t)\|_{L^\infty} \le C\delta. \tag{3.5}$$

In addition, under the conditions of Theorem 1.1, it follows from Proposition 1.1 and Lemma 2.1 that

$$\|\tilde{\rho}(\cdot) - \rho_\infty\|_{L^\infty\cap H^5} \le C\varepsilon_1. \tag{3.6}$$

Therefore,

$$\frac{1}{4}\rho_\infty \le \|n+\tilde{\rho}\|_{L^\infty} \le 4\rho_\infty. \tag{3.7}$$

In what follows, we concentrate on establishing some important a-priori estimates.

Lemma 3.1. *Assume that (3.4) hold and let* $(n,u)(x,t)$ *be a solution of system (3.2) in* $[0,T]$*, then, under the conditions of Theorem 1.1, we have the estimate*

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}\int_{\mathbb{R}^3}\Big(\tfrac{P'(\rho_\infty)}{\rho_\infty}n^2 + (n+\tilde{\rho})u^2 + \kappa(\nabla n)^2\Big)\mathrm{d}x + \mu\int_{\mathbb{R}^3}(\nabla u)^2\mathrm{d}x + (\mu+\nu)\int_{\mathbb{R}^3}(\nabla\cdot u)^2\mathrm{d}x\\ \le C\rho_\infty(\delta+\varepsilon_1)(\|\nabla n\|^2 + \|\nabla u\|^2).\end{aligned} \tag{3.8}$$

Proof. Multiplying $(3.2)_1$ and $(3.2)_2$ by $\dfrac{P'(\rho_\infty)}{\rho_\infty}n$ and $(n+\tilde{\rho})u$, respectively, then integrating over $\mathbb{R}^3$ and summing the resultant equalities, we obtain

$$\begin{aligned}\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\Big(\tfrac{P'(\rho_\infty)}{\rho_\infty}n^2 + (n+\tilde{\rho})u^2\Big)\mathrm{d}x + \mu\int_{\mathbb{R}^3}(\nabla u)^2\mathrm{d}x + (\mu+\nu)\int_{\mathbb{R}^3}(\nabla\cdot u)^2\mathrm{d}x - \kappa\int_{\mathbb{R}^3}(n+\tilde{\rho})u\nabla\triangle n\mathrm{d}x\\ = \tfrac{1}{2}\int_{\mathbb{R}^3} n_t u^2\mathrm{d}x + \int_{\mathbb{R}^3}(n+\tilde{\rho})uf\mathrm{d}x.\end{aligned} \tag{3.9}$$

According to $(3.2)_1$, it holds that

$$-\kappa\int_{\mathbb{R}^3}(n+\tilde{\rho})u\nabla\triangle n\mathrm{d}x = \kappa\int_{\mathbb{R}^3}\triangle n\nabla\cdot((n+\tilde{\rho})u)\mathrm{d}x = -\kappa\int_{\mathbb{R}^3}\triangle n n_t\mathrm{d}x = \frac{\kappa}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(\nabla n)^2\mathrm{d}x. \tag{3.10}$$

Observe that f has the following equivalent properties:

$$f \sim -(u\cdot\nabla)u - n\nabla\tilde{\rho} - n\nabla n - (\tilde{\rho}-\rho_\infty)\nabla n, \tag{3.11}$$

then it follows from Hölder inequality, Lemma 2.1, (3.4), (3.6) and (3.7) that

$$\begin{aligned}\int_{\mathbb{R}^3}(n+\tilde{\rho})uf\mathrm{d}x \sim -\int_{\mathbb{R}^3}(n+\tilde{\rho})u\Big((u\cdot\nabla)u + n\nabla\tilde{\rho} + n\nabla n + (\tilde{\rho}-\rho_\infty)\nabla n\Big)\mathrm{d}x\\ \le \|n+\tilde{\rho}\|_{L^\infty}\Big(\|u\|_{L^6}\|\nabla u\|\|u\|_{L^3} + \|n\|_{L^6}\|\nabla(\tilde{\rho}-\rho_\infty)\|_{L^3}\|u\| + \|\nabla n\|\|n\|_{L^3}\|u\|_{L^6} + \|\tilde{\rho}-\rho_\infty\|_{L^3}\|u\|_{L^6}\|\nabla n\|\Big)\\ \le C\rho_\infty(\delta+\varepsilon_1)(\|\nabla n\|^2 + \|\nabla u\|^2).\end{aligned} \tag{3.12}$$

Meanwhile, from Hölder inequality, Lemma 2.1, $(3.2)_1$ and (3.7), the following inequalities can be derived as well

$$\begin{aligned}\tfrac{1}{2}\int_{\mathbb{R}^3} n_t u^2\mathrm{d}x = -\tfrac{1}{2}\int_{\mathbb{R}^3} u^2\nabla\cdot((n+\tilde{\rho})u)\mathrm{d}x = \int_{\mathbb{R}^3} u\nabla u(n+\tilde{\rho})u\mathrm{d}xx\\ \le \|n+\tilde{\rho}\|_{L^\infty}\|\nabla u\|\|u\|_{L^6}\|u\|_{L^3} \le C\rho_\infty(\|u\| + \|\nabla u\|)\|\nabla u\|^2 \le C\rho_\infty\delta\|\nabla u\|^2.\end{aligned} \tag{3.13}$$

Finally, substituting (3.10), (3.12) and (3.13) into (3.9) gives (3.8). The proof is complete. □

Lemma 3.2. *Under the conditions of Lemma 3.1, it holds that*

$$\frac{1}{2}\frac{d}{\mathrm{d}t}\int_{\mathbb{R}^3}\Big(\frac{P'(\rho_\infty)}{\rho_\infty}(\nabla n)^2+(n+\tilde{\rho})(\nabla u)^2+\kappa(\nabla^2 n)^2\Big)\mathrm{d}x+\mu\int_{\mathbb{R}^3}(\nabla^2 u)^2\mathrm{d}x+(\mu+\nu)\int_{\mathbb{R}^3}(\mathrm{div}\nabla u)^2\mathrm{d}x$$
$$\leq C(\delta+\varepsilon_1)(\|\nabla n\|_1^2+\|\nabla\triangle n\|^2+\|\nabla u\|_1^2). \tag{3.14}$$

Proof. First applying ∂_x^α to (3.2) with $|\alpha|=1$, we obtain

$$\partial_x^\alpha n_t+\mathrm{div}((n+\tilde{\rho})\partial_x^\alpha u)=-\mathrm{div}(\partial_x^\alpha(n+\tilde{\rho})u) \tag{3.15}$$

and

$$\partial_x^\alpha u_t-\frac{\mu}{n+\tilde{\rho}}\Delta\partial_x^\alpha u-\frac{\mu+\nu}{n+\tilde{\rho}}\nabla(\mathrm{div}\partial_x^\alpha u)+\frac{P'(\rho_\infty)}{\rho_\infty}\nabla\partial_x^\alpha n$$
$$=\kappa\nabla\triangle\partial_x^\alpha n+\partial_x^\alpha f+\partial_x^\alpha\Big(\frac{\mu}{n+\tilde{\rho}}\Big)\Delta u+\partial_x^\alpha\Big(\frac{\mu+\nu}{n+\tilde{\rho}}\Big)\nabla(\mathrm{div}u). \tag{3.16}$$

Multiplying (3.15) and (3.16) by $\dfrac{P'(\rho_\infty)}{\rho_\infty}\partial_x^\alpha n$ and $(n+\tilde{\rho})\partial_x^\alpha u$, respectively, then integrating over $\mathbb{R}^3$ and summing the resultant equalities, we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\Big(\frac{P'(\rho_\infty)}{\rho_\infty}(\partial_x^\alpha n)^2+(n+\tilde{\rho})(\partial_x^\alpha u)^2\Big)\mathrm{d}x+\mu\int_{\mathbb{R}^3}(\nabla\partial_x^\alpha u)^2\mathrm{d}x+(\mu+\nu)\int_{\mathbb{R}^3}(\mathrm{div}\partial_x^\alpha u)^2\mathrm{d}x$$
$$=\frac{1}{2}\int_{\mathbb{R}^3}n_t(\partial_x^\alpha u)^2\mathrm{d}x-\frac{P'(\rho_\infty)}{\rho_\infty}\int_{\mathbb{R}^3}\mathrm{div}\Big[\partial_x^\alpha(n+\tilde{\rho})u\Big]\partial_x^\alpha n\mathrm{d}x$$
$$+\int_{\mathbb{R}^3}\partial_x^\alpha\Big(\frac{\mu}{n+\tilde{\rho}}\Big)\Delta u(n+\tilde{\rho})\partial_x^\alpha u\mathrm{d}x+\int_{\mathbb{R}^3}\partial_x^\alpha\Big(\frac{\mu+\nu}{n+\tilde{\rho}}\Big)\nabla(\mathrm{div}u)(n+\tilde{\rho})\partial_x^\alpha u\mathrm{d}x$$
$$+\kappa\int_{\mathbb{R}^3}\nabla\triangle\partial_x^\alpha n(n+\tilde{\rho})\partial_x^\alpha u\mathrm{d}x+\int_{\mathbb{R}^3}\partial_x^\alpha f(n+\tilde{\rho})\partial_x^\alpha u\mathrm{d}x$$
$$:=\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3+\mathcal{J}_4+\mathcal{J}_5+\mathcal{J}_6. \tag{3.17}$$

In the following, we focus on establishing the estimates of $\mathcal{J}_i(i=1,2,3,4,5,6)$. Noticing that $|\alpha|=1$, by Hölder inequality, Young inequality and Lemma 2.1, it follows from $(3.2)_1$, (3.4), (3.6) and (3.7) that

$$\begin{aligned}\mathcal{J}_1&=-\frac{1}{2}\int_{\mathbb{R}^3}(\partial_x^\alpha u)^2\nabla\cdot((n+\tilde{\rho})u)\mathrm{d}x=\int_{\mathbb{R}^3}\partial_x^\alpha u\cdot\nabla\partial_x^\alpha u\cdot((n+\tilde{\rho})u)\mathrm{d}x\\&\leq\|n+\tilde{\rho}\|_{L^\infty}\|u\|_{L^6}\|\partial_x^\alpha u\|_{L^3}\|\nabla\partial_x^\alpha u\|\leq C\rho_\infty\delta\|\partial_x^\alpha u\|_1^2,\end{aligned} \tag{3.18}$$

$$\begin{aligned}\mathcal{J}_2&=-\frac{P'(\rho_\infty)}{\rho_\infty}\int_{\mathbb{R}^3}\Big(\partial_x^\alpha(n+\tilde{\rho})\nabla\cdot u\partial_x^\alpha n+u\nabla\partial_x^\alpha(n+\tilde{\rho})\partial_x^\alpha n\Big)\mathrm{d}x\\&\leq\int_{\mathbb{R}^3}\Big((\partial_x^\alpha n+\partial_x^\alpha\tilde{\rho})\nabla\cdot u\partial_x^\alpha n+u(\nabla\partial_x^\alpha n+\nabla\partial_x^\alpha\tilde{\rho})\partial_x^\alpha n\Big)\mathrm{d}x\\&\leq\|\partial_x^\alpha n\|_{L^6}\|\nabla\cdot u\|\|\partial_x^\alpha n\|_{L^3}+\|\partial_x^\alpha\tilde{\rho}\|_{L^\infty}\|\nabla\cdot u\|\|\partial_x^\alpha n\|\\&+\|\nabla\partial_x^\alpha n\|\|u\|_{L^6}\|\partial_x^\alpha n\|_{L^3}+\|\nabla\partial_x^\alpha\tilde{\rho}\|\|u\|_{L^6}\|\partial_x^\alpha n\|_{L^3}\\&\leq C(\delta+\varepsilon_1)(\|\nabla u\|^2+\|\nabla n\|_1^2),\end{aligned} \tag{3.19}$$

$$\begin{aligned}
\mathcal{J}_5 =& -\kappa\int_{\mathbb{R}^3}\triangle\partial_x^\alpha n\mathrm{div}((n+\tilde{\rho})\partial_x^\alpha u)\mathrm{d}x=\kappa\int_{\mathbb{R}^3}\triangle\partial_x^\alpha n\Big[\partial_x^\alpha n_t+\mathrm{div}(\partial_x^\alpha(n+\tilde{\rho})u)\Big]\mathrm{d}x\\
=& -\frac{\kappa}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(\nabla\partial_x^\alpha n)^2\mathrm{d}x+\kappa\int_{\mathbb{R}^3}\triangle\partial_x^\alpha n\Big[\nabla\partial_x^\alpha(n+\tilde{\rho})u+\partial_x^\alpha(n+\tilde{\rho})\mathrm{div}u\Big]\mathrm{d}x\\
\leq& -\frac{\kappa}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(\nabla\partial_x^\alpha n)^2dx+\kappa\|\triangle\partial_x^\alpha n\|\Big(\|\nabla\partial_x^\alpha n\|_{L^3}\|u\|_{L^6}\\
&+\|\nabla\partial_x^\alpha\tilde{\rho}\|_{L^3}\|u\|_{L^6}+\|\partial_x^\alpha n\|_{L^6}\|\nabla\cdot u\|_{L^3}+\|\partial_x^\alpha\tilde{\rho}\|_{L^6}\|\nabla\cdot u\|_{L^3}\Big)\\
\leq& -\frac{\kappa}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(\nabla\partial_x^\alpha n)^2\mathrm{d}x+C\kappa(\delta+\varepsilon_1)(\|\Delta\nabla n\|^2+\|\nabla u\|_1^2),
\end{aligned}\tag{3.20}$$

and

$$\begin{aligned}
\mathcal{J}_3 =& -\textstyle\int_{\mathbb{R}^3}\frac{\mu}{n+\tilde{\rho}}\partial_x^\alpha(n+\tilde{\rho})\triangle u\partial_x^\alpha u\mathrm{d}x\leq\|\frac{\mu}{n+\tilde{\rho}}\|_{L^\infty}\|\partial_x^\alpha(n+\tilde{\rho})\|_{L^6}\|\triangle u\|\|\partial_x^\alpha u\|_{L^3}\\
&\leq C\rho_\infty(\|\nabla^2 n\|+\|\nabla^2\tilde{\rho}\|)\|\nabla u\|_1^2\leq C\rho_\infty(\delta+\varepsilon_1)\|\nabla u\|_1^2.
\end{aligned}\tag{3.21}$$

Similarly, we can show

$$\mathcal{J}_4\leq C\rho_\infty(\delta+\varepsilon_1)\|\nabla u\|_1^2.\tag{3.22}$$

In addition, since (3.11), it holds that

$$\begin{aligned}
\mathcal{J}_6\sim& -\textstyle\int_{\mathbb{R}^3}(n+\tilde{\rho})\partial_x^\alpha u\partial_x^\alpha(u\cdot\nabla u)\mathrm{d}x-\int_{\mathbb{R}^3}(n+\tilde{\rho})\partial_x^\alpha u\partial_x^\alpha(n\nabla\tilde{\rho})\mathrm{d}x\\
&-\textstyle\int_{\mathbb{R}^3}(n+\tilde{\rho})\partial_x^\alpha u\partial_x^\alpha(n\nabla n)\mathrm{d}x-\int_{\mathbb{R}^3}(n+\tilde{\rho})\partial_x^\alpha u\partial_x^\alpha((\tilde{\rho}-\rho_\infty)\nabla n)\mathrm{d}x\\
&:=\mathcal{J}_{61}+\mathcal{J}_{62}+\mathcal{J}_{63}+\mathcal{J}_{64}.
\end{aligned}\tag{3.23}$$

Similarly, based on Lemma 2.1 and (3.4)–(3.7), the following estimates can be derived

$$\begin{aligned}
\mathcal{J}_{61} =& -\int_{\mathbb{R}^3}(n+\tilde{\rho})\partial_x^\alpha u\Big(\partial_x^\alpha u\cdot\nabla u+u\partial_x^\alpha\nabla u\Big)\mathrm{d}x\\
&\leq\|n+\tilde{\rho}\|_{L^\infty}\|\partial_x^\alpha u\|_{L^3}\Big(\|\partial_x^\alpha u\|_{L^6}\|\nabla u\|+\|\partial_x^\alpha\nabla u\|\|u\|_{L^6}\Big)C\rho_\infty\delta\|\nabla u\|_1^2,
\end{aligned}\tag{3.24}$$

$$\begin{aligned}
\mathcal{J}_{62} =& -\int_{\mathbb{R}^3}(n+\tilde{\rho})\partial_x^\alpha u\Big(\partial_x^\alpha n\nabla\tilde{\rho}+n\partial_x^\alpha\nabla\tilde{\rho}\Big)\mathrm{d}x\\
&\leq\|n+\tilde{\rho}\|_{L^\infty}\|\partial_x^\alpha u\|\Big(\|\partial_x^\alpha n\|\|\nabla\tilde{\rho}\|_{L^\infty}+\|n\|_{L^6}\|\partial_x^\alpha\nabla\tilde{\rho}\|_{L^3}\Big)\\
&\leq C\rho_\infty\varepsilon_1(\|\nabla u\|^2+\|\nabla n\|^2),
\end{aligned}\tag{3.25}$$

$$\begin{aligned}
\mathcal{J}_{63} =& -\int_{\mathbb{R}^3}(n+\tilde{\rho})\partial_x^\alpha u\Big(\partial_x^\alpha n\nabla n+n\partial_x^\alpha\nabla n\Big)\mathrm{d}x\\
&\leq\|n+\tilde{\rho}\|_{L^\infty}\|\partial_x^\alpha u\|\Big(\|\partial_x^\alpha n\|_{L^3}\|\nabla n\|_{L^6}+\|n\|_{L^\infty}\|\partial_x^\alpha\nabla n\|\Big)\\
&\leq C\rho_\infty\delta(\|\nabla u\|^2+\|\nabla^2 n\|^2),
\end{aligned}\tag{3.26}$$

$$\begin{aligned}\mathcal{J}_{64} &= -\int_{\mathbb{R}^3}(n+\tilde{\rho})\partial_x^\alpha u\Big(\partial_x^\alpha\tilde{\rho}\nabla n + (\tilde{\rho}-\rho_\infty)\partial_x^\alpha\nabla n\Big)\mathrm{d}x \\ &\le \|n+\tilde{\rho}\|_{L^\infty}\|\partial_x^\alpha u\|\Big(\|\partial_x^\alpha\tilde{\rho}\|_{L^\infty}\|\nabla n\| + \|\tilde{\rho}-\rho_\infty\|_{L^\infty}\|\partial_x^\alpha\nabla n\|\Big) \\ &\le C\rho_\infty\varepsilon_1(\|\nabla u\|^2 + \|\nabla n\|_1^2).\end{aligned} \tag{3.27}$$

Therefore,

$$\mathcal{J}_6 \sim \mathcal{J}_{61} + \mathcal{J}_{62} + \mathcal{J}_{63} + \mathcal{J}_{64} \le C\rho_\infty(\varepsilon_1+\delta)(\|\nabla u\|_1^2 + \|\nabla n\|_1^2). \tag{3.28}$$

Then substituting (3.18)–(3.22) and (3.28) into (3.17), we can deduce the estimate (3.14). The proof is complete. □

Lemma 3.3. *Under the conditions of Lemma 3.1, we have*

$$\begin{aligned}&\tfrac{d}{dt}\int_{\mathbb{R}^3} u\nabla n\mathrm{d}x + \tfrac{P'(\rho_\infty)}{2\rho_\infty}\int_{\mathbb{R}^3}(\nabla n)^2\mathrm{d}x + \kappa\int_{\mathbb{R}^3}(\triangle n)^2\mathrm{d}x \\ &\le C(\delta+\varepsilon_1)(\|\nabla n\|_1^2 + \|\nabla u\|_1^2) + C\rho_\infty\|\nabla u\|^2 + \gamma_1\|\nabla^2 u\|^2\end{aligned} \tag{3.29}$$

and

$$\begin{aligned}&\tfrac{d}{dt}\int_{\mathbb{R}^3} \nabla u\nabla\nabla n\mathrm{d}x + \tfrac{P'(\rho_\infty)}{\rho_\infty}\int_{\mathbb{R}^3}(\nabla\nabla n)^2\mathrm{d}x + \tfrac{\kappa}{2}\int_{\mathbb{R}^3}(\nabla\triangle n)^2\mathrm{d}x \\ &\le C(\delta+\varepsilon_1)(\|\nabla n\|_1^2 + \|\nabla u\|_1^2) + C\rho_\infty\|\nabla u\|_1^2 + \gamma_2\|\nabla^2 u\|^2.\end{aligned} \tag{3.30}$$

Proof. First from $(3.2)_2$, it is easy to see that

$$\frac{P'(\rho_\infty)}{\rho_\infty}\nabla n - \kappa\nabla\triangle n = -u_t + \frac{\mu}{n+\tilde{\rho}}\Delta u + \frac{\mu+\nu}{n+\tilde{\rho}}\nabla(\nabla\cdot u) + f. \tag{3.31}$$

Taking inner product of (3.31) and ∇n over $\mathbb{R}^3$, and then integrating by parts, we have

$$\begin{aligned}&\tfrac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} u\nabla n\mathrm{d}x + \int_{\mathbb{R}^3}\tfrac{P'(\rho_\infty)}{\rho_\infty}(\nabla n)^2\mathrm{d}x + \kappa\int_{\mathbb{R}^3}(\triangle n)^2\mathrm{d}x \\ &= \int_{\mathbb{R}^3} u\nabla n_t\mathrm{d}x + \int_{\mathbb{R}^3}\Big(\tfrac{\mu}{n+\tilde{\rho}}\Delta u + \tfrac{\mu+\nu}{n+\tilde{\rho}}\nabla(\nabla\cdot u)\Big)\cdot\nabla n\mathrm{d}x + \int_{\mathbb{R}^3} f\cdot\nabla n\mathrm{d}x \\ &= -\int_{\mathbb{R}^3}\nabla\cdot u\nabla(n+\tilde{\rho})u\mathrm{d}x - \int_{\mathbb{R}^3}\nabla\cdot u(n+\tilde{\rho})\nabla\cdot u\mathrm{d}x + \int_{\mathbb{R}^3}\tfrac{\mu}{n+\tilde{\rho}}\Delta u\cdot\nabla n\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3}\tfrac{\mu+\nu}{n+\tilde{\rho}}\nabla(\nabla\cdot u)\cdot\nabla n\mathrm{d}x + \int_{\mathbb{R}^3} f\cdot\nabla n\mathrm{d}x \\ &:= \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3 + \mathcal{G}_4 + \mathcal{G}_5,\end{aligned} \tag{3.32}$$

in which we used $(3.2)_1$. For $\mathcal{G}_i(i=1,2,3,4,5)$, by using the Hölder inequality, Lemma 2.1, (3.4) and (3.6), we can deduce

$$\begin{aligned}\mathcal{G}_1 + \mathcal{G}_2 &\le \|\nabla\cdot u\|(\|\nabla(n+\tilde{\rho})\|_{L^3}\|u\|_{L^6} + \|n+\tilde{\rho}\|_{L^\infty}\|\nabla\cdot u\|) \\ &\le C\|\nabla\cdot u\|(\|\nabla(n+\tilde{\rho})\|_1\|\nabla u\| + \|n+\tilde{\rho}\|_{L^\infty}\|\nabla\cdot u\|) \\ &\le C(\delta+\varepsilon_1+\rho_\infty)\|\nabla u\|^2.\end{aligned} \tag{3.33}$$

By using the Young inequality, we can conclude there exists a positive constant γ_1 such that

$$\mathcal{G}_3 + \mathcal{G}_4 \le \frac{P'(\rho_\infty)}{2\rho_\infty}\|\nabla n\|^2 + \gamma_1\|\nabla^2 u\|^2. \tag{3.34}$$

In addition, similar to the estimation of (3.12), it holds that

$$
\begin{aligned}
\mathcal{G}_5 =& \int_{\mathbb{R}^3} f \cdot \nabla n \mathrm{d}x \\
\sim& - \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot \nabla n \mathrm{d}x - \int_{\mathbb{R}^3} n \nabla \tilde{\rho} \cdot \nabla n \mathrm{d}x - \int_{\mathbb{R}^3} n \nabla n \cdot \nabla n \mathrm{d}x \\
&+ \int_{\mathbb{R}^3} (\tilde{\rho} - \rho_\infty) \triangle n n \mathrm{d}x + \int_{\mathbb{R}^3} \nabla(\tilde{\rho} - \rho_\infty) \cdot \nabla n n \mathrm{d}x \\
\le& \|u\|_{L^6} \|\nabla u\| \|\nabla n\|_{L^3} + \|\nabla \tilde{\rho}\|_{L^3} \|n\|_{L^6} \|\nabla n\| \\
&+ \|n\|_{L^3} \|\nabla n\| \|\nabla n\|_{L^6} + \|\tilde{\rho} - \rho_\infty\|_{L^3} \|n\|_{L^6} \|\triangle n\| + \|\nabla(\tilde{\rho} - \rho_\infty)\|_{L^3} \|\nabla n\| \|n\|_{L^6} \\
\le& C(\delta + \varepsilon_1)(\|\nabla u\|^2 + \|\nabla n\|_1^2).
\end{aligned} \tag{3.35}
$$

Then substituting (3.33)–(3.35) into (3.32) yields (3.29).

Next applying $\partial_x^\alpha (|\alpha| = 1)$ to (3.31), then multiplying it by $\partial_x^\alpha \nabla n$ and integrating the resultant equation over $\mathbb{R}^3$, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \partial_x^\alpha u \cdot \partial_x^\alpha \nabla n \mathrm{d}x + \frac{P'(\rho_\infty)}{\rho_\infty} \int_{\mathbb{R}^3} (\partial_x^\alpha \nabla n)^2 \mathrm{d}x + \kappa \int_{\mathbb{R}^3} (\partial_x^\alpha \triangle n)^2 \mathrm{d}x \\
= \int_{\mathbb{R}^3} \partial_x^\alpha u \partial_x^\alpha \nabla n_t \mathrm{d}x + \int_{\mathbb{R}^3} \partial_x^\alpha \Big(\frac{\mu}{n+\tilde{\rho}} \triangle u\Big) \partial_x^\alpha \nabla n \mathrm{d}x \\
+ \int_{\mathbb{R}^3} \partial_x^\alpha \Big(\frac{\mu+\nu}{n+\tilde{\rho}} \nabla(\nabla \cdot u)\Big) \partial_x^\alpha \nabla n \mathrm{d}x + \int_{\mathbb{R}^3} \partial_x^\alpha f \cdot \partial_x^\alpha \nabla n \mathrm{d}x \\
:= \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4.
\end{aligned} \tag{3.36}
$$

Based on Lemma 2.1, noticing that $|\alpha| = 1$, it follows from Hölder inequality, $(3.2)_1$, (3.4) and (3.6) that

$$
\begin{aligned}
\mathcal{M}_1 =& - \int_{\mathbb{R}^3} \partial_x^\alpha u \partial_x^\alpha \nabla \Big(\nabla(n + \tilde{\rho}) u + (n + \tilde{\rho}) \nabla \cdot u \Big) \mathrm{d}x \\
=& \int_{\mathbb{R}^3} \partial_x^\alpha \nabla \cdot u \partial_x^\alpha \Big(\nabla(n + \tilde{\rho}) u + (n + \tilde{\rho}) \nabla \cdot u \Big) \mathrm{d}x \\
\le& \|\partial_x^\alpha \nabla \cdot u\| \Big(\|\partial_x^\alpha \nabla(n + \tilde{\rho})\| \|u\|_{L^\infty} + \|\nabla(n + \tilde{\rho})\|_{L^6} \|\partial_x^\alpha u\|_{L^3} \\
&+ \|\partial_x^\alpha (n + \tilde{\rho})\|_{L^6} \|\nabla \cdot u\|_{L^3} + \|n + \tilde{\rho}\|_{L^\infty} \|\partial_x^\alpha \nabla \cdot u\| \Big) \\
\le& C(\delta + \varepsilon_1 + \rho_\infty) \|\nabla u\|_1^2.
\end{aligned} \tag{3.37}
$$

Similarly, there exists a positive constant γ_2 such that

$$
\begin{aligned}
\mathcal{M}_2 + \mathcal{M}_3 =& - \int_{\mathbb{R}^3} \Big(\frac{\mu}{n + \tilde{\rho}} \triangle u \Big) (\partial_x^\alpha)^2 \nabla n \mathrm{d}x - \int_{\mathbb{R}^3} \Big(\frac{\mu + \nu}{n + \tilde{\rho}} \nabla(\nabla \cdot u) \Big) (\partial_x^\alpha)^2 \nabla n \mathrm{d}x \\
\le& \|\frac{\mu}{n + \tilde{\rho}}\|_{L^\infty} \|\triangle u\| \|(\partial_x^\alpha)^2 \nabla n\| + \|\frac{\mu + \nu}{n + \tilde{\rho}}\|_{L^\infty} \|\nabla(\nabla \cdot u)\| \|(\partial_x^\alpha)^2 \nabla n\| \\
\le& \frac{\kappa}{2} \|(\partial_x^\alpha)^2 \nabla n\|^2 + \gamma_2 \|\nabla^2 u\|^2,
\end{aligned} \tag{3.38}
$$

and

$$
\begin{aligned}
\mathcal{M}_4 =& \int_{\mathbb{R}^3} \partial_x^\alpha f \cdot \partial_x^\alpha \nabla n \mathrm{d}x \\
\sim& -\int_{\mathbb{R}^3} \partial_x^\alpha \Big((u\cdot\nabla)u + n\nabla\tilde{\rho} + n\nabla n + (\tilde{\rho}-\rho_\infty)\nabla n \Big) \cdot \partial_x^\alpha \nabla n \mathrm{d}x \\
=& \int_{\mathbb{R}^3} \Big((u\cdot\nabla)u + n\nabla\tilde{\rho} + n\nabla n \Big) \cdot (\partial_x^\alpha)^2 \nabla n \mathrm{d}x \\
&+ \frac{1}{2}\int_{\mathbb{R}^3} (\partial_x^\alpha)^2(\tilde{\rho}-\rho_\infty)(\nabla n)^2 \mathrm{d}x - \int_{\mathbb{R}^3} (\tilde{\rho}-\rho_\infty)(\partial_x^\alpha \nabla n)^2 \mathrm{d}x \\
\le& \Big(\|u\|_{L^6}\|\nabla u\|_{L^3} + \|n\|_{L^6}\|\nabla\tilde{\rho}\|_{L^3} + \|n\|_{L^6}\|\nabla n\|_{L^3} \Big) \|(\partial_x^\alpha)^2 \nabla n\| \\
&+ \frac{1}{2}\|(\partial_x^\alpha)^2(\tilde{\rho}-\rho_\infty)\|_{L^3}\|\nabla n\|\|\nabla n\|_{L^6} + \|\tilde{\rho}-\rho_\infty\|_{L^\infty}\|\partial_x^\alpha \nabla n\|^2 \\
\le& C(\delta+\varepsilon_1)(\|\nabla u\|_1^2 + \|\triangle\nabla n\|^2 + \|\nabla n\|_1^2).
\end{aligned} \tag{3.39}
$$

Finally, (3.30) can be concluded by taking (3.36)–(3.39) and the smallness of δ and ε_1 into account. The proof is complete. □

With the above Lemmas at hand, we have the following conclusion.

Proposition 3.2. *Assume that (3.4) hold and let $(n, u)(x, t)$ be a solution of system (3.2) in $[0, T]$, then under the conditions of Theorem 1.1, the following a-priori estimate holds*

$$
\|n(t)\|_2^2 + \|u(t)\|_1^2 + \int_0^t (\|\nabla n\|_2^2 + \|\nabla u\|_1^2)\mathrm{d}s \le C_0(\|n_0\|_2^2 + \|u_0\|_1^2), \tag{3.40}
$$

where C_0 is a positive constant independent of t.

Proof. Adding (3.8) and (3.14), we can conclude that there exist positive constants C_1 and C_2, such that

$$
\frac{\mathrm{d}}{\mathrm{d}t}(\|n(t)\|_2^2 + \|u(t)\|_1^2) + C_1\|\nabla u(t)\|_1^2 \le C_2(\delta+\varepsilon_1)(\|\nabla n(t)\|_1^2 + \|\nabla u(t)\|_1^2 + \|\nabla\triangle n(t)\|^2). \tag{3.41}
$$

Similarly, adding (3.29) and (3.30), since δ and ε_1 are small, we conclude that there exists a positive constant C_3, such that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} (u\nabla n + \nabla u\nabla^2 n)\mathrm{d}x + \frac{P'(\rho_\infty)}{\rho_\infty}\|\nabla n\|_1^2 + \kappa\|\triangle n(t)\|_1^2 \le C_3\|\nabla u\|_1^2. \tag{3.42}
$$

Multiplying (3.42) by $C_4 := \min\{1, \frac{C_1}{2C_3}\}$, then adding (3.41), noticing that the smallness of δ and ε_1, we conclude there exist positive constants C_4 and C_5, such that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|n(t)\|_2^2 + \|u(t)\|_1^2 + C_4\int_{\mathbb{R}^3} (u\nabla n + \nabla u\nabla^2 n)\mathrm{d}x \Big) + C_5\Big(\|\nabla u(t)\|_1^2 + \|\nabla n\|_1^2 + \|\triangle n(t)\|_1^2 \Big) \\
=&\frac{\mathrm{d}}{\mathrm{d}t}\Big\{ \|\nabla n(t)\|_1^2 + (1-\frac{C_4}{4})\|\nabla n(t)\|_1^2 + \|u(t)\|_1^2 + C_4\int_{\mathbb{R}^3} \Big[(\frac{u}{2}+\frac{\nabla n}{2})^2 + (\frac{\nabla u}{2}+\frac{\nabla^2 n}{2})^2 \Big]\mathrm{d}x \Big\} \\
&+ C_5\Big(\|\nabla u(t)\|_1^2 + \|\nabla n\|_1^2 + \|\triangle n(t)\|_1^2 \Big) \le 0.
\end{aligned} \tag{3.43}
$$

Integrating the above inequality from 0 to t, we can conclude there exists a positive constant C_0, such that

$$\|n(t)\|_2^2 + \|u(t)\|_1^2 + \int_0^t (\|\nabla n\|_2^2 + \|\nabla u\|_1^2)\mathrm{d}s \le C_0(\|n_0\|_2^2 + \|u_0\|_1^2), \quad \forall\, t \in [0, T],$$

i.e., (3.40) holds. The proof is complete. □

The proof of Theorem 1.1. Based on the method of continuity, the global existence of solution $(n(x,t), \rho(x,t))$ to problem (3.2) follows from Propositions 3.1 and 3.2. Since $(n, u) = (\rho - \tilde{\rho}, u)$, $(\rho(x,t), u(x,t))$ is the unique global strong solution of problem (1.2). Moreover, (1.5) follows from (3.40). The proof is complete. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interests regarding this paper.

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