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Research article

Existence of solutions to a generalized quasilinear Schrödinger equation with concave-convex nonlinearities and potentials vanishing at infinity

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Abstract: In this paper, we investigate the existence of solutions to a generalized quasilinear Schrödinger equation with concave-convex nonlinearities and potentials vanishing at infinity. Using the mountain pass theorem, we get the existence of a positive solution.

Keywords: quasilinear Schrödinger equation; concave-convex nonlinearities; vanishing potentials; positive solutions; mountain pass theorem
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1. Introduction and main results

This article is concerned with a class of generalized quasilinear Schrödinger equations

$$\begin{cases} -\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = K(x)f(u) + \lambda W(x)h(u), & \text{in } \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N \geq 3$, $\lambda > 0$, $f, h\colon \mathbb{R} \to \mathbb{R}$ and $V, K, W\colon \mathbb{R}^N \to \mathbb{R}$ are nonnegative continuous and $g(s) \in C^1(\mathbb{R}, \mathbb{R}^+)$, which is nondecreasing with respect to $|s|$.

These equations are related to the existence of solitary waves for the Schrödinger equation

$$i\partial_t z = -\Delta z + V(x)z - k(x,z)z - \Delta l(|z|^2)l'(|z|^2)z, \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $z\colon \mathbb{R}\times\mathbb{R}^N \to \mathbb{C}$, $V\colon \mathbb{R}^N \to \mathbb{R}$ is a given potential, $l\colon \mathbb{R} \to \mathbb{R}$ and $k\colon \mathbb{R}^N\times\mathbb{C} \to \mathbb{R}$ are fixed functions. Quasilinear equations of the form (1.2) appear naturally in mathematical physics and have been derived as models of several physical phenomena corresponding to various types of l. For instance, the case $l(s) = s$ appears in the superfluid film equation in plasma physics [18]. If $l(s) = \sqrt{1+s}$, the equation models the propagation of a high-irradiance laser in a plasma, as well as the self-channeling of a high-power ultrashort laser in matter [19]. For more physical motivations and more references dealing with various applications, we refer to [5, 16, 17, 26, 28].

If we set $z(t,x) = e^{-iEt}u(x)$ in (1.2), we obtain the corresponding equation of elliptic type

$$-\Delta u + V(x)u - \Delta(l(u^2))l'(u^2)u = k(x,u)u, \quad x \in \mathbb{R}^N. \tag{1.3}$$

Notice that if we let

$$g^2(u) = 1 + \frac{[(l(u)^2)']^2}{2},$$

we have the following equation

$$-\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = k(x,u)u. \tag{1.4}$$

One of the most interesting cases is that $g(s) = \sqrt{1+2s^2}$, and then (1.4) changes to

$$-\Delta u + V(x)u - [\Delta(u)^2]u = k(x,u)u. \tag{1.5}$$

The Schrödinger equation is quasilinear as the term $[\Delta(u)^2]u$ is linear about the second derivatives. Over the past decades, many interesting results about the existence of solutions to (1.5) have been established. It is difficult to give a complete reference, so we only refer to some early works [23, 24] for special $k(x,u)u$ and some papers [1, 6, 9, 13, 22, 35] closely related to our paper. Particularly, Wang and Yao [36] studied the existence of nontrivial solutions to (1.5) with concave-convex nonlinearities $\mu|u|^{\hat{p}-2}u + |u|^{\hat{q}-2}u$, $2 < \hat{p} < 4$, $4 < \hat{q} < 22^*$, and the potential $V(x)$ satisfied the following conditions:

(V_1') $V \in C(\mathbb{R}^N, \mathbb{R})$ and $0 < V_0 \leq \inf_{x\in\mathbb{R}^N} V(x)$;

(V_2') There exists $V_1 > 0$ such that $V(x) = V(|x|) \leq V_1$ for all $x \in \mathbb{R}^N$;

(V_3') $\nabla V(x)x \leq 0$ for all $x \in \mathbb{R}^N$.

In this paper we investigate the more general Eq (1.4) where the nonlinearity is like $\mu W(x)|u|^{\hat{p}-2}u + K(x)|u|^{\hat{q}-2}u$, $1 < \hat{p} < 2$, $4 < \hat{q} < 22^*$ and $V, K, W\colon \mathbb{R}^N \to \mathbb{R}$ satisfy some conditions listed below. There are also many works on the equation in the recent years, but we only mention those closely related to our paper, [7, 8, 10, 29, 30] and the references therein. Particularly, Furtado et al. [14] investigated solutions to (1.4) with a huge class of functions g satisfying the following condition (g_0).

(g_0) $g \in C^1(\mathbb{R}, (0, +\infty))$ is even, non-decreasing in $[0, +\infty)$, $g(0) = 1$ and satisfies

$$g_\infty := \lim_{t\to\infty} \frac{g(t)}{t} \in (0, \infty) \tag{1.6}$$

and

$$\beta := \sup_{t\in\mathbb{R}} \frac{tg'(t)}{g(t)} \leq 1. \tag{1.7}$$

When g satisfies (g_0), the existence of solutions to (1.4) has been investigated by several authors over the past years [15, 27] and the references therein. In particular, in [25] the authors considered the positive solutions to it when the nonlinearity is like $\mu|u|^{\hat{p}-2}u + |u|^{\hat{q}-2}u$, $1 < \hat{p} < 2$, $4 < \hat{q} < 22^*$ where the potential $V(x)$ satisfied (V_1') and the following condition:

(V_4') $[V(x)]^{-1} \in L^1(\mathbb{R}^N)$.

An important class of problems associated to (1.1) is the case when $V(x)$ vanishes at infinity

$$\lim_{|x|\to+\infty} V(x) = 0,$$

which has been extensively investigated for the corresponding second order nonlinear Schrödinger equations after the researches of e.g., [2,3]. See also [11,21,32–34] for some work about $V(x)$ vanishing at infinity. However, there are only few works in this case for the more general Eq (1.1). Motivated by the above articles, we investigate the existence of solutions to (1.1) when the potential V vanishes at infinity for a huge class of g (satisfying (g_0)).

In this paper, we consider the generalized quasilinear Schrödinger Eq (1.1) with vanishing potentials and concave-convex nonlinearity $K(x)f(u) + \lambda W(x)h(u)$. Since the problem is set on the whole space $\mathbb{R}^N$, we have to deal with the loss of compactness. In this respect we use the class of functions V, K introduced in [2] for second order Schrödinger equations, which is more general than those in [3].

As in [2], it is said that $(V, K) \in \mathcal{K}$ if the following conditions hold:

(I) $K(x),\ V(x) > 0,\ \forall x \in \mathbb{R}^N$ and $K \in L^\infty(\mathbb{R}^N)$.

(II) If $\{A_n\} \subset \mathbb{R}^N$ is a sequence of Borel sets, such that $|A_n| \le R$ for some $R > 0$ and for all $n \in \mathbb{N}$, then

$$\lim_{r\to+\infty} \int_{A_n\cap B_r^c(0)} K(x)dx = 0,\ \text{uniformly in } n \in \mathbb{N}. \tag{K_1}$$

(III) One of the below conditions satisfies:

$$\frac{K}{V} \in L^\infty(\mathbb{R}^N) \tag{K_2}$$

or there is $\sigma \in (2, 2^*)$ such that

$$\frac{K(x)}{[V(x)]^{\frac{2^*-\sigma}{2^*-2}}} \to 0,\ \text{as } |x| \to +\infty. \tag{K_3}$$

We also use the following conditions on V and W:

(V_1) $V(x) \in L^\infty(\mathbb{R}^N)$;

(W_0) $W(x) > 0$ for all $x \in \mathbb{R}^N$;

(W_1) $W(x) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$;

(W_2) $\frac{W(x)}{V(x)} \in L^\infty(\mathbb{R}^N)$.

We impose the following conditions on h and f:

(H_0) $h \in C(\mathbb{R}, \mathbb{R}^+)$ and $h(t) = 0$ for all $t \le 0$;

(H_1) There exists $b_1,\ b_2 > 0$ such that $h(t) \le b_1|t|^{\tau_1-1} + b_2|t|^{\tau_2-1}$, $\tau_1, \tau_2 \in (1, 2)$ for any $t \in \mathbb{R}$;

(F_0) $f \in C(\mathbb{R}, \mathbb{R}^+)$ and $f(t) = 0$ for all $t \le 0$;

(F_1) $\lim_{|t|\to+\infty} \frac{f(t)}{|t|^{22^*-1}} = 0$;

(F_2) $\lim\limits_{|t|\to 0}\frac{f(t)}{|t|}=0$ if (K_2) holds or $\lim\limits_{|t|\to 0}\frac{f(t)}{|t|^{\sigma-1}}=0$ if (K_3) holds;

(F_3) $\frac{F(t)}{t^4}\to+\infty$, as $t\to+\infty$;

(F_4) There exists $\mu>2+2\beta$ such that $\frac{1}{\mu}f(t)t\geq F(t)$, where β is in (1.7).

Observe that there are many natural functions $f(t),h(t)$ satisfying the above conditions. For example, $f(t)=|t|^{2^*+1}$ and $h(t)=|t|^{\frac{1}{2}}$ may serve as examples satisfying (F_1)–(F_4) and (H_1), respectively.

Our main theorem is stated as follows.

Theorem 1.1. *Assume that* $(V,K)\in\mathcal{K}$, (g_0), (V_1), (W_0)–(W_2), (F_0)–(F_4), (H_0) *and* (H_1) *hold. Then, there exists* $\lambda_0>0$ *such that* (1.1) *possesses a positive solution for any* $\lambda\in(0,\lambda_0)$.

Furthermore, for the case where (K_2) holds, we can prove that (1.1) possesses a ground state solution. To this end, we assume the following conditions on h and f:

(H_0') $h\in C(\mathbb{R},\mathbb{R})$, $h(t)$ is odd and $h(t)\geq 0$ for all $t\geq 0$.

(H_1') There exists $b_3>0$ and $\tau_3\in(1,2)$ such that $h(t)\leq b_3|t|^{\tau_3-1}$.

(H_2') There exists a constant $\tilde{C}>0$ such that $\lim\limits_{t\to 0}\frac{H(t)}{|t|^{\tau_3}}=\tilde{C}$.

(F_0') $f\in C(\mathbb{R},\mathbb{R})$, $f(t)$ is odd and $f(t)\geq 0$ for all $t\geq 0$.

Proposition 1.2. *Assume that* $(V,K)\in\mathcal{K}$ *where* (K_2) *holds and* (g_0), (V_1), (W_0)–(W_2), (F_0'), (F_1), (F_2), (F_4), (H_0')–(H_2') *hold. Then, there exists* $\lambda_1>0$ *such that* (1.1) *possesses a ground state solution for any* $\lambda\in(0,\lambda_1)$.

We emphasize that the main result in this paper is essentially different from the aforementioned works. Indeed, in [25, 36] the authors considered two kinds of quasilinear Schrödinger equations with concave-convex nonlinearities, but required that the potential $V(x)$ have a positive lower bound. In [11, 21] the authors showed the existence of nontrivial solutions for different problems with vanishing potentials. In this paper, we investigate a different class of generalized quasilinear Schrödinger equations with vanishing potentials and concave-convex nonlinearities. As far as we know, few works in this case seem to have appeared in the literature.

The paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we verify that the functional associated to the problem satisfies the geometric conditions of the mountain pass theorem, and the boundedness of the Cerami sequences associated with the corresponding minimax level is proved. Lastly, in Section 4, the existence of a positive solution and a ground state solution for (1.1) is established.

2. Preliminaries

As usual, we use the Sobolev space

$$X=\{u\in D^{1,2}(\mathbb{R}^N):\int_{\mathbb{R}^N}V(x)u^2dx<+\infty\}\tag{2.1}$$

endowed with the norm

$$\|u\| = (\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2)dx)^{\frac{1}{2}}. \tag{2.2}$$

The weighted Lebesgue space is defined as follows

$$L^q_K(\mathbb{R}^N) = \{u : \mathbb{R}^N \to \mathbb{R} \mid u \text{ is measurable and } \int_{\mathbb{R}^N} K(x)|u|^q dx < +\infty\}$$

endowed with the norm

$$\|u\|_{K,q} := (\int_{\mathbb{R}^N} K(x)|u|^q dx)^{\frac{1}{q}}.$$

The space $L^p_W(\mathbb{R}^N)$ with the norm $\|u\|_{W,p}$ is similarly defined.

The following proposition is proved in [2].

Proposition 2.1. *[2] Assume that $(V, K) \in \mathcal{K}$. Then, X is compactly embedded in $L^q_K(\mathbb{R}^N)$ for all $q \in (2, 2^*)$ if (K_2) holds. If (K_3) holds, X is compactly embedded in $L^\sigma_K(\mathbb{R}^N)$.*

To resolve (1.1), due to the appearance of the nonlocal term $\int_{\mathbb{R}^N} g^2(u)|\nabla u|^2 dx$, the right working space seems to be

$$X_0 = \{u \in X : \int_{\mathbb{R}^N} g^2(u)|\nabla u|^2 dx < \infty\}.$$

However, generally X_0 is not a linear space and the functional

$$I_\lambda(u) = \frac{1}{2}\int g(u)^2|\nabla u|^2 dx + \frac{1}{2}\int V(x)u^2 dx - \int K(x)F(u)dx - \lambda\int W(x)H(u)dx \tag{2.3}$$

may be not well defined on X_0, where

$$F(u) = \int_0^u f(s)ds, \quad H(u) = \int_0^u h(s)ds.$$

To avoid these drawbacks, following [20, 26, 30], we make a change of variables

$$v = G(u) = \int_0^u g(t)dt.$$

Then, it follows from the properties of g, G and G^{-1}, which will be listed in Lemma 2.4 that if $v \in X$, then $u = G^{-1}(v) \in X$ and

$$\int_{\mathbb{R}^N} g^2(u)|\nabla u|^2 dx = \int_{\mathbb{R}^N} g^2(G^{-1}(v))|\nabla G^{-1}(v)|^2 dx = \int_{\mathbb{R}^N} |\nabla v|^2 dx < \infty.$$

After the change of variables, (1.1) changes to

$$-\Delta v + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} - K(x)\frac{f(G^{-1}(v))}{g(G^{-1}(v))} - \lambda W(x)\frac{h(G^{-1}(v))}{g(G^{-1}(v))} = 0. \tag{2.4}$$

One can easily derive that if $v \in X$ is a classical solution to (2.4), then $u = G^{-1}(v) \in X$ is a classical solution to (1.1). Thus, we only need to seek weak solutions to (2.4). The associated function to (2.4) is

$$J_\lambda(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 dx - \int_{\mathbb{R}^N} K(x)F(G^{-1}(v))dx - \lambda\int_{\mathbb{R}^N} W(x)H(G^{-1}(v))dx. \tag{2.5}$$

By the conditions on g, f and h, it is easy to prove that J_λ is well defined and belongs to C^1 on X. Hence, X is a proper working space for the problem. Here, we say that $v \in X$ is a weak solution to (2.4) if

$$\langle J'_\lambda(v), \varphi\rangle = \int_{\mathbb{R}^N} [\nabla v \nabla \varphi + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\varphi - K(x)\frac{f(G^{-1}(v))}{g(G^{-1}(v))}\varphi - \lambda W(x)\frac{h(G^{-1}(v))}{g(G^{-1}(v))}\varphi]dx = 0 \quad (2.6)$$

for all $\varphi \in X$.

Before proving the main theorem, we show some technical embedding results for possiblely $p \le 2$, which can be used to deal with sublinear problems comparing with Proposition 2.1.

Lemma 2.2. *Assume that* (W_0)–(W_2) *hold. Then, X is continuously embedded in* $L^p_W(\mathbb{R}^N)$ *for all* $p \in (1, 2^*/2)$.

Proof. As mentioned in [2], $W(x)$ satisfies (K_1) and (K_2) since it satisfies (W_1) and (W_2). It is clearly $2p \in (2, 2^*)$ for $p \in (1, 2^*/2)$. Therefore, Proposition 2.1 shows that X is compactly embedded in $L^{2p}_W(\mathbb{R}^N)$ for every $p \in (1, 2^*/2)$, and, thus, there exists $\nu_{W,2p} > 0$ such that

$$\int_{\mathbb{R}^N} W(x)|u|^{2p}dx \le \nu^{2p}_{W,2p}\|u\|^{2p}$$

for every $p \in (1, 2^*/2)$. Moreover, since $W(x) \in L^1(\mathbb{R}^N)$, by Hölder's inequality and (W_0)–(W_2), we deduce for any $u \in X$

$$\begin{aligned}\int_{\mathbb{R}^N} W(x)|u|^p dx &= \int_{\mathbb{R}^N} W(x)^{\frac{1}{2}} W(x)^{\frac{1}{2}}|u|^p dx \\ &\le (\int_{\mathbb{R}^N} W(x)dx)^{\frac{1}{2}}(\int_{\mathbb{R}^N} W(x)|u|^{2p}dx)^{\frac{1}{2}} \\ &\le (\|W(x)\|_1^{\frac{1}{2p}}\nu_{W,2p})^p\|u\|^p\end{aligned} \quad (2.7)$$

for all $p \in (1, 2^*/2)$, implying that X is continuously embedded in $L^p_W(\mathbb{R}^N)$. □

Lemma 2.3. *Assume that* (W_0)–(W_2) *hold. Then, X is compactly embedded in* $L^p_W(\mathbb{R}^N)$ *for all* $p \in (1, 2)$, $N \ge 3$.

Proof. Lemma 2.2 shows that X is continuously embedded in $L^p_W(\mathbb{R}^N)$ for every $p \in (1, 2)$, and $N \le 4$ since $2 \le 2^*/2$ in this case. For every $p \in (1, 2)$, fix $p_0 \in (1, p)$ and $q_0 \in (2, 2^*)$. Then, it follows by Hölder's inequality that

$$\|u\|^p_{W,p} \le \|u\|^{\frac{p_0(q_0-p)}{q_0-p_0}}_{W,p_0}\|u\|^{\frac{q_0(p-p_0)}{q_0-p_0}}_{W,q_0} \quad \text{for all } u \in X, \quad (2.8)$$

which implies by Lemma 2.2 and Proposition 2.1 that X is compactly embedded in $L^p_W(\mathbb{R}^N)$ for all $p \in (1, 2)$ and $N \le 4$. Moreover, in the case $N \ge 5$, for every $p \in [2^*/2, 2)$, we fix $p_1 \in (1, 2^*/2)$ and $q_1 \in (2, 2^*)$. By a similar inequality, we obtain that X is compactly embedded in $L^p_W(\mathbb{R}^N)$ for all $p \in (1, 2)$, $N \ge 5$.

In conclusion, X is compactly embedded in $L^p_W(\mathbb{R}^N)$ for all $p \in (1, 2)$. □

Now we list the main properties of the function G^{-1} [14, 29].

Lemma 2.4. *Suppose that g satisfies (g_0). Then, the function $G^{-1} \in C^2(\mathbb{R}, \mathbb{R})$ satisfies the following properties:*

(g_1) *G^{-1} is increasing and G, G^{-1} are odd functions;*

(g_2) $0 < \frac{d}{dt}(G^{-1}(t)) = \frac{1}{g(G^{-1}(t))} \leq \frac{1}{g(0)}$ *for all $t \in \mathbb{R}$;*

(g_3) $|G^{-1}(t)| \leq \frac{|t|}{g(0)}$ *for all $t \in \mathbb{R}$;*

(g_4) $\lim_{t\to 0} \frac{G^{-1}(t)}{t} = \frac{1}{g(0)}$*;*

(g_5) $1 \leq \frac{tg(t)}{G(t)} \leq 2$ *and* $1 \leq \frac{G^{-1}(t)g(G^{-1}(t))}{t} \leq 2$ *for all $t \neq 0$;*

(g_6) $\frac{G^{-1}(t)}{\sqrt{t}}$ *is non-decreasing in $(0, +\infty)$ and $|G^{-1}(t)| \leq (2/g_\infty)^{1/2}\sqrt{|t|}$ for all $t \in \mathbb{R}$;*

(g_7) *The following inequalities hold*

$$\left|G^{-1}(t)\right| \geq \begin{cases} G^{-1}(1)|t| & \text{for all } |t| \leq 1, \\ G^{-1}(1)\sqrt{|t|} & \text{for all } |t| \geq 1; \end{cases}$$

(g_8) $\frac{t}{g(t)}$ *is increasing and* $\left|\frac{t}{g(t)}\right| \leq \frac{1}{g_\infty}$ *for all $t \in \mathbb{R}$;*

(g_9) $[G^{-1}(s-t)]^2 \leq 4([G^{-1}(s)]^2 + [G^{-1}(t)]^2)$ *for all $s, t \in \mathbb{R}$;*

(g_{10}) $\lim_{t\to+\infty} \frac{G^{-1}(t)}{\sqrt{t}} = (\frac{2}{g_\infty})^{1/2}$.

Remark 2.1. *Define the function Ψ: $X \to \mathbb{R}$ by*

$$\Psi(v) = \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)[G^{-1}(v)]^2)dx.$$

It is easy to verify that it is a C^1 function on X by the conditions on g. Moreover, by (g_3) and $V(x) > 0$ for all $x \in \mathbb{R}^N$, we have

$$\Psi(v) \leq \|v\|^2 \text{ for all } v \in X,$$

and as stated in [1], by (g_3), (g_7) and (V_1), there is a constant $\xi > 0$ such that

$$\xi\|v\|^2 \leq \Psi(v) + [\Psi(v)]^{2^*/2} \text{ for all } v \in X.$$

Throughout this paper, C denotes the various positive constant. $\nu_{K,q} > 0$ denotes the Sobolev embedding constant for $X \hookrightarrow L^q_K(\mathbb{R}^N)$, that is $\|u\|_{K,q} \leq \nu_{K,q}\|u\|$ for any $u \in X$, and the definition of Sobolev embedding constant for $X \hookrightarrow L^p_W(\mathbb{R}^N)$ is similar. Besides, it is well known that the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is continuous, i.e., there exists $\nu_1 > 0$ such that $\|u\|_{2^*} \leq \nu_1\|u\|_{D^{1,2}(\mathbb{R}^N)}$ for any $u \in D^{1,2}(\mathbb{R}^N)$.

3. The mountain pass geometry and the boundedness of the Cerami sequences

In this section, we first state a version of the mountain pass theorem due to Ambrosetti and Rabinowitz [4], which is an essential tool in this paper, then we show that the function associated to (2.4) possesses a Cerami sequence at the corresponding mountain pass level. Afterward, the boundedness of the Cerami sequence is established.

We recall the definition of Cerami sequence. Let X be a real Banach space and J_λ: $X \to \mathbb{R}$ a functional of class C^1. We say that $\{v_n\} \subset X$ is a Cerami sequence at c ($(Ce)_c$ for short) for J_λ if $\{v_n\}$ satisfies

$$J_\lambda(v_n) \to c \tag{3.1}$$

and

$$(1 + \|v_n\|)J'_\lambda(v_n) \to 0 \tag{3.2}$$

as $n \to \infty$. J_λ is said to satisfy the Cerami condition at c, if any Cerami sequence at c possesses a convergent subsequence.

Theorem 3.1. *[31] Let X be a real Banach space and* $J \in C^1(X, \mathbb{R})$*. Let* Σ *be a closed subset of X, which disconnects (arcwise) X into distinct connected* X_1 *and* X_2*. Suppose further that* $J(0) = 0$ *and*
(J_1) $0 \in X_1$*, and there is* $\alpha > 0$ *such that* $J|_\Sigma \geq \alpha > 0$*,*
(J_2) *there is* $e \in X_2$ *such that* $J(e) < 0$*.*
Then, J possesses a $(Ce)_c$ *sequence with* $c \geq \alpha > 0$ *given by*

$$c := \inf_{\gamma\in\Lambda} \max_{0\leq t\leq 1} J(\gamma(t)),$$

where

$$\Lambda = \{\gamma \in C([0, 1], X) : \gamma(0) = 0,\ J(\gamma(1)) < 0\}.$$

Lemma 3.2. *Assume that* $(V, K) \in \mathcal{K}$*.* (g_0), (F_0)–(F_3), (W_0)–(W_2), (H_0) *and* (H_1) *hold. Then, there exists* $\lambda_0, \alpha_0 > 0$ *such that for any* $\lambda \in (0, \lambda_0)$, J_λ *possesses a Cerami sequence at*

$$c_\lambda := \inf_{\gamma\in\Lambda_\lambda} \max_{0\leq t\leq 1} J_\lambda(\gamma(t)) \geq \alpha_0 > 0,$$

where

$$\Lambda_\lambda = \{\gamma \in C([0, 1], X) : \gamma(0) = 0,\ J_\lambda(\gamma(1)) < 0\}.$$

Proof. It is enough to prove that the function satisfies the mountain pass geometry. We only consider the case where (K_2) holds and the proof is similar if (K_3) holds.

First note that $J_\lambda(0) = 0$ for any $\lambda > 0$. For every $\rho > 0$, define

$$\Sigma_\rho := \{v \in X : \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)[G^{-1}(v)]^2)dx = \rho^2\}.$$

Since the function $\int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)[G^{-1}(v)]^2)dx$ is continuous on X, Σ_ρ is a closed subset in X which disconnects the space X.

(1) There exists $\lambda_0, \rho_0, \alpha_0 > 0$ such that $J_\lambda(v) \geq \alpha_0 > 0$ for any $\lambda \in (0, \lambda_0)$, $v \in \Sigma_{\rho_0}$. Indeed, for every $\rho > 0$, by (K_2), we have

$$\begin{aligned}\int_{\mathbb{R}^N} K(x)|G^{-1}(v)|^2 dx &\leq \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} \Big|\frac{K(x)}{V(x)}\Big| \int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 dx \\ &\leq \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} \Big|\frac{K(x)}{V(x)}\Big|\rho^2\end{aligned} \tag{3.3}$$

for any $v \in \Sigma_\rho$. Moreover, by $K(x) \in L^\infty(\mathbb{R}^N)$, (g_6) and Sobolev embedding, we conclude that

$$\begin{aligned}\int_{\mathbb{R}^N} K(x)|G^{-1}(v)|^{22^*} dx &\leq \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} |K(x)| \int_{\mathbb{R}^N} |\frac{2}{g_\infty} v|^{2^*} dx \\ &\leq \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} |K(x)|(\nu_1 \frac{2}{g_\infty})^{2^*} (\int_{\mathbb{R}^N} |\nabla v|^2 dx)^{\frac{2^*}{2}} \\ &\leq \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} |K(x)|(\nu_1 \frac{2}{g_\infty})^{2^*} \rho^{2^*}\end{aligned} \tag{3.4}$$

for any $v \in \Sigma_\rho$. Thus, by (F_0)–(F_2), (3.3) and (3.4), we obtain for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^N} K(x)F(G^{-1}(v))dx &\leq \varepsilon \int_{\mathbb{R}^N} K(x)|G^{-1}(v)|^2 dx + C_\varepsilon \int_{\mathbb{R}^N} K(x)|G^{-1}(v)|^{22^*} dx \\ &\leq \varepsilon \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} \Big|\frac{K(x)}{V(x)}\Big|\rho^2 + C_\varepsilon \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} |K(x)|(\nu_1 \frac{2}{g_\infty})^{2^*} \rho^{2^*}\end{aligned} \tag{3.5}$$

for any $v \in \Sigma_\rho$.

In addition, according to Lemma 2.3, (g_2) and (g_3), we deduce that

$$\int_{\mathbb{R}^N} W(x)|G^{-1}(v)|^{\tau_1} dx \leq \nu_{W,\tau_1}^{\tau_1} \|G^{-1}(v)\|^{\tau_1} \leq \nu_{W,\tau_1}^{\tau_1} \rho^{\tau_1} \tag{3.6}$$

and

$$\int_{\mathbb{R}^N} W(x)|G^{-1}(v)|^{\tau_2} dx \leq \nu_{W,\tau_2}^{\tau_2} \rho^{\tau_2} \tag{3.7}$$

for any $v \in \Sigma_\rho$.

Thus, by (H_0), (H_1), (3.6) and (3.7), it follows that

$$\begin{aligned}\int_{\mathbb{R}^N} W(x)H(G^{-1}(v))dx &\leq \frac{b_1}{\tau_1} \int_{\mathbb{R}^N} W(x)|G^{-1}(v)|^{\tau_1} dx + \frac{b_2}{\tau_2} \int_{\mathbb{R}^N} W(x)|G^{-1}(v)|^{\tau_2} dx \\ &\leq \frac{b_1}{\tau_1} \nu_{W,\tau_1}^{\tau_1} \rho^{\tau_1} + \frac{b_2}{\tau_2} \nu_{W,\tau_2}^{\tau_2} \rho^{\tau_2}\end{aligned} \tag{3.8}$$

for any $v \in \Sigma_\rho$.

Choose $\varepsilon_0 > 0$ such that $\operatorname{ess\,sup}_{x\in\mathbb{R}^N} \Big|\frac{K(x)}{V(x)}\Big|\varepsilon_0 < \frac{1}{2}$. By (3.5) and (3.8), we conclude that

$$J_\lambda(v) \geq \rho^2 (\frac{1}{2} - \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} \Big|\frac{K(x)}{V(x)}\Big|\varepsilon_0 - C_{\varepsilon_0} \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} |K(x)|(\nu_1 \frac{2}{g_\infty})^{2^*} \rho^{2^*-2}) - \lambda (\frac{b_1}{\tau_1} \nu_{W,\tau_1}^{\tau_1} \rho^{\tau_1} + \frac{b_2}{\tau_2} \nu_{W,\tau_2}^{\tau_2} \rho^{\tau_2})$$

for any $\lambda > 0$, $\rho > 0$, $v \in \Sigma_\rho$.

Choose $\rho_0 > 0$ such that

$$\frac{1}{2} - \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} \Big|\frac{K(x)}{V(x)}\Big|\varepsilon_0 - C_{\varepsilon_0}\operatorname*{ess\,sup}_{x\in\mathbb{R}^N} |K(x)|(\nu_1 \frac{2}{g_\infty})^{2^*}\rho_0^{2^*-2} > 0$$

and set

$$\lambda_0 := \frac{\rho_0^2(\frac{1}{2} - \operatorname{ess\,sup}_{x\in\mathbb{R}^N} \big|\frac{K(x)}{V(x)}\big|\varepsilon_0 - C_{\varepsilon_0}\operatorname{ess\,sup}_{x\in\mathbb{R}^N} |K(x)|(\nu_1 \frac{2}{g_\infty})^{2^*}\rho_0^{2^*-2})}{2(\frac{b_1}{\tau_1}\nu_{W,\tau_1}^{\tau_1}\rho_0^{\tau_1} + \frac{b_2}{\tau_2}\nu_{W,\tau_2}^{\tau_2}\rho_0^{\tau_2})} > 0,$$

$$\alpha_0 := \frac{\rho_0^2}{2}(\frac{1}{2} - \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} \Big|\frac{K(x)}{V(x)}\Big|\varepsilon_0 - C_{\varepsilon_0}\operatorname*{ess\,sup}_{x\in\mathbb{R}^N} |K(x)|(\nu_1 \frac{2}{g_\infty})^{2^*}\rho_0^{2^*-2}) > 0.$$

Then,

$$\begin{aligned} J_\lambda(v) &\geq \rho_0^2(\frac{1}{2} - \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} \Big|\frac{K(x)}{V(x)}\Big|\varepsilon_0 - C_{\varepsilon_0}\operatorname*{ess\,sup}_{x\in\mathbb{R}^N} |K(x)|(\nu_1 \frac{2}{g_\infty})^{2^*}\rho_0^{2^*-2}) - \lambda(\frac{b_1}{\tau_1}\nu_{W,\tau_1}^{\tau_1}\rho_0^{\tau_1} + \frac{b_2}{\tau_2}\nu_{W,\tau_2}^{\tau_2}\rho_0^{\tau_2}) \\ &\geq \alpha_0 \\ &> 0 \end{aligned}$$

for any $\lambda \in (0, \lambda_0)$, $v \in \Sigma_{\rho_0}$.

(2) For any $\lambda \in (0, \lambda_0)$, there exists $e \in X$ such that

$$\int_{\mathbb{R}^N} (|\nabla e|^2 + V(x)|G^{-1}(e)|^2)dx > \rho_0$$

and $J_\lambda(e) < 0$. To this end, for any $\lambda \in (0, \lambda_0)$, fixed $v \in X$ is a nonnegative smooth function with $m(\mathrm{supp} v) > 0$, where

$$\mathrm{supp} v = \overline{\{x \in \mathbb{R}^N | v(x) \neq 0\}}$$

is the support of v. We prove $J_\lambda(tv) < 0$ if $t > 0$ and $\int_{\mathbb{R}^N} (|\nabla(tv)|^2 + V(x)|G^{-1}(tv)|^2)dx$ is large enough. Suppose by contradiction that there exists a sequence $\{t_n\} \subset \mathbb{R}^+$ such that

$$\int_{\mathbb{R}^N} (|\nabla(t_n v)|^2 + V(x)[G^{-1}(t_n v)]^2)dx \to \infty \text{ as } n \to \infty$$

and $J_\lambda(t_n v) \geq 0$ for all $n \in \mathbb{N}$. By (g_3), we know

$$|t_n|^2 \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)|v|^2)dx \geq \int_{\mathbb{R}^N} (|\nabla(t_n v)|^2 + V(x)[G^{-1}(t_n v)]^2)dx,$$

which means that $t_n \to +\infty$. Set $\varpi = \frac{v}{\|v\|}$. Noticing that $K(x),\ W(x) > 0$, $\forall x \in \mathbb{R}^N$, by (H_0), (F_0) and (g_3) we get

$$0 \leq \frac{J_\lambda(t_n v)}{\int_{\mathbb{R}^N} (|\nabla(t_n v)|^2 + V(x)[G^{-1}(t_n v)]^2)dx} \leq \frac{1}{2} - \int_{\mathrm{supp} v} K(x)\frac{F(G^{-1}(t_n v))}{|G^{-1}(t_n v)|^4}\frac{|G^{-1}(t_n v)|^4}{|(t_n v)|^2}|\varpi|^2 dx. \tag{3.9}$$

Since $t_n v(x) \to +\infty$ as $n \to +\infty$, for $x \in \mathrm{supp} v$, it follows from (g_{10}), $K(x) > 0$, (F_0), (F_3) and Fatou's lemma that

$$\int_{\mathrm{supp} v} K(x)\frac{F(G^{-1}(t_n v))}{|G^{-1}(t_n v)|^4}\frac{|G^{-1}(t_n v)|^4}{|(t_n v)|^2}|\varpi|^2 dx \to +\infty$$

as $n \to +\infty$, which is a contradiction by inequality (3.9).

The proof is ended. □

We now show the boundedness of the Cerami sequence.

Lemma 3.3. *Assume that* (g_0), (V_1), (W_0)–(W_2), (H_0), (H_1) *and* (F_4) *hold, then any* $(Ce)_{c_\lambda}$ *sequence of* J_λ *is bounded in* X *for any* $\lambda \in (0, \lambda_0)$.

Proof. Let $\{v_n\}$ be the corresponding $(Ce)_{c_\lambda}$ sequence for J_λ. Denote $\omega_n = G^{-1}(v_n)g(G^{-1}(v_n))$. Then, it follows from (1.7) that

$$\begin{aligned}\langle J'_\lambda(v_n), \omega_n\rangle \leq &(1+\beta)\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 dx \\ &- \int_{\mathbb{R}^N} K(x) f(G^{-1}(v_n))G^{-1}(v_n)dx - \lambda \int_{\mathbb{R}^N} W(x)h(G^{-1}(v_n))G^{-1}(v_n)dx.\end{aligned} \tag{3.10}$$

By (1.7) and (g_5), we get

$$|\nabla \omega_n| \leq 2|\nabla v_n| \quad \text{and} \quad |\omega_n| \leq 2|v_n|.$$

Hence, $\omega_n \in X$ and $\|\omega_n\| \leq 4\|v_n\|$, which gives

$$|\langle J'_\lambda(v_n), \omega_n\rangle| \leq J'_\lambda(v_n)(1 + 4\|v_n\|) = o_n(1). \tag{3.11}$$

Therefore, taking into account (H_0), (H_1), (W_0), (F_4), (3.10) and (3.11), we conclude that

$$\begin{aligned}c_\lambda + o_n(1) \geq &J_\lambda(v_n) - \frac{1}{\mu}\langle J'_\lambda(v_n), \omega_n\rangle \\ \geq &(\frac{1}{2} - \frac{1+\beta}{\mu})\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + (\frac{1}{2} - \frac{1}{\mu})\int_{\mathbb{R}^N} V(x)[G^{-1}(v_n)]^2 dx \\ &- \lambda \int_{\mathbb{R}^N} W(x)[\frac{b_1}{\tau_1}|G^{-1}(v_n)|^{\tau_1} + \frac{b_2}{\tau_2}|G^{-1}(v_n)|^{\tau_2}]dx.\end{aligned} \tag{3.12}$$

Hence, combining with (W_0)–(W_2), Lemma 2.3, (3.12) and (g_2), we deduce that for any $\lambda > 0$,

$$\begin{aligned}(\frac{1}{2} - \frac{1+\beta}{\mu})\Psi(v_n) &\leq c_\lambda + \lambda\int_{\mathbb{R}^N} W(x)[\frac{b_1}{\tau_1}|G^{-1}(v_n)|^{\tau_1} + \frac{b_2}{\tau_2}|G^{-1}(v_n)|^{\tau_2}]dx + o_n(1) \\ &\leq c_\lambda + \frac{\lambda b_1 \nu_{W,\tau_1}^{\tau_1}}{\tau_1}\|G^{-1}(v_n)\|^{\tau_1} + \frac{\lambda b_2 \nu_{W,\tau_2}^{\tau_2}}{\tau_2}\|G^{-1}(v_n)\|^{\tau_2} + o_n(1) \\ &\leq c_\lambda + \frac{\lambda b_1 \nu_{W,\tau_1}^{\tau_1}}{\tau_1}\Psi(v_n)^{\frac{\tau_1}{2}} + \frac{\lambda b_2 \nu_{W,\tau_2}^{\tau_2}}{\tau_2}\Psi(v_n)^{\frac{\tau_2}{2}} + o_n(1).\end{aligned}$$

Since $\tau_1,\ \tau_2 \in (1,2)$, $\{\Psi(v_n)\}$ is bounded in X, by Remark 2.1 we obtain that $\{v_n\}$ is bounded in X. □

4. Proofs of main theorems

Under the hypotheses of Lemmas 3.2 and 3.3, for any fixed $\lambda \in (0, \lambda_0)$, let $\{v_n\}$ be the $(Ce)_{c_\lambda}$ sequence for J_λ. Then, by Lemma 3.3 we know that $\{v_n\}$ is bounded in X. Thus, there exists a subsequence still denoted by $\{v_n\}$, and $v \in X$ such that

$$v_n \rightharpoonup v \text{ in } X,\ v_n \to v \text{ in } L^s_{loc}(\mathbb{R}^N) \text{ for any } s \in [1, 2^*) \text{ and } v_n \to v \text{ a.e., on } \mathbb{R}^N, \tag{4.1}$$

and there is $L > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \int_{\mathbb{R}^N} V(x)|v_n|^2 dx \leq L \quad \text{and} \quad \int_{\mathbb{R}^N} |v_n|^{2^*} dx \leq L, \ \forall n \in \mathbb{N}. \tag{4.2}$$

We conclude this section showing that the weak limit v is a positive solution to (1.1).

Lemma 4.1. *Assume that* (g_0), (W_0)–(W_2), (H_0), (H_1) *hold and* $\{v_n\}$ *is a* $(Ce)_{c_\lambda}$ *sequence for* J_λ *given by Lemmas 3.2 and 3.3. Then, the following statements hold:*

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} W(x)H(G^{-1}(v_n))dx = \int_{\mathbb{R}^N} W(x)H(G^{-1}(v))dx, \tag{4.3}$$

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} W(x)\frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))}\varphi dx = \int_{\mathbb{R}^N} W(x)\frac{h(G^{-1}(v))}{g(G^{-1}(v))}\varphi dx, \text{ for any } \varphi \in X, \tag{4.4}$$

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} W(x)h(G^{-1}(v_n))G^{-1}(v_n)dx = \int_{\mathbb{R}^N} W(x)h(G^{-1}(v))G^{-1}(v)dx, \tag{4.5}$$

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} W(x)\frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))}v_n dx = \int_{\mathbb{R}^N} W(x)\frac{h(G^{-1}(v))}{g(G^{-1}(v))}v dx. \tag{4.6}$$

Proof. First, we give the proof of (4.3). Since $\tau_1, \ \tau_2 \in (1, 2)$, from (W_0)–(W_2) and Lemma 2.3, we have

$$\int_{\mathbb{R}^N} W(x)|v_n|^{\tau_1} dx \to \int_{\mathbb{R}^N} W(x)|v|^{\tau_1} dx \quad \text{and} \quad \int_{\mathbb{R}^N} W(x)|v_n|^{\tau_2} dx \to \int_{\mathbb{R}^N} W(x)|v|^{\tau_2} dx. \tag{4.7}$$

Then, given $\varepsilon > 0$, there is $r > 0$ such that

$$\int_{B_r^c} W(x)|v_n|^{\tau_1} dx < \varepsilon \quad \text{and} \quad \int_{B_r^c} W(x)|v_n|^{\tau_2} dx < \varepsilon \quad \text{for all} \quad n \in \mathbb{N}, \tag{4.8}$$

where $B_r^c := \{x \in \mathbb{R}^N : |x| > r\}$, which together with (H_0), (H_1) and (g_3) yields that

$$\begin{aligned}\int_{B_r^c} W(x)H(G^{-1}(v_n))dx \leq & \frac{b_1}{\tau_1}\int_{B_r^c} W(x)|G^{-1}(v_n)|^{\tau_1} dx + \frac{b_2}{\tau_2}\int_{B_r^c} W(x)|G^{-1}(v_n)|^{\tau_2} dx \\ \leq & \frac{b_1}{\tau_1}\int_{B_r^c} W(x)|v_n|^{\tau_1} dx + \frac{b_2}{\tau_2}\int_{B_r^c} W(x)|v_n|^{\tau_2} dx \\ < & (\frac{b_1}{\tau_1} + \frac{b_2}{\tau_2})\varepsilon\end{aligned}$$

for any $n \in \mathbb{N}$.

Moreover, for each fixed $r > 0$, it is easy to verify that

$$\lim_{n\to+\infty} \int_{B_r(0)} W(x)H(G^{-1}(v_n))dx = \int_{B_r(0)} W(x)H(G^{-1}(v))dx,$$

where $B_r(0) = \{x \in \mathbb{R}^N : |x| \leq r\}$. This completes the proof of (4.3). □

Proof. Now we are going to prove (4.4). Noticing (4.7), given $\varepsilon > 0$, there is $r > 0$ such that

$$\int_{B_r^c} W(x)|v_n|^{\tau_1}dx < \varepsilon^{\frac{\tau_1}{\tau_1-1}} \quad \text{and} \quad \int_{B_r^c} W(x)|v_n|^{\tau_2}dx < \varepsilon^{\frac{\tau_2}{\tau_2-1}} \quad \text{for all} \quad n \in \mathbb{N}. \tag{4.9}$$

By (H_0), (H_1), (W_0), (g_2), (g_3) and Hölder's inequality, we obtain that

$$\begin{aligned}
\Big|\int_{B_r^c} W(x)\frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))}\varphi dx\Big| \leq & b_1\int_{B_r^c} W(x)\frac{|G^{-1}(v_n)|^{\tau_1-1}}{g(G^{-1}(v_n))}|\varphi|dx + b_2\int_{B_r^c} W(x)\frac{|G^{-1}(v_n)|^{\tau_2-1}}{g(G^{-1}(v_n))}|\varphi|dx \\
\leq & b_1\int_{B_r^c} W(x)|v_n|^{\tau_1-1}|\varphi|dx + b_2\int_{B_r^c} W(x)|v_n|^{\tau_2-1}|\varphi|dx \\
\leq & b_1(\int_{B_r^c} W(x)|v_n|^{\tau_1}dx)^{\frac{\tau_1-1}{\tau_1}}(\int_{B_r^c} W(x)|\varphi|^{\tau_1}dx)^{\frac{1}{\tau_1}} \\
& + b_2(\int_{B_r^c} W(x)|v_n|^{\tau_2}dx)^{\frac{\tau_2-1}{\tau_2}}(\int_{B_r^c} W(x)|\varphi|^{\tau_2}dx)^{\frac{1}{\tau_2}}
\end{aligned} \tag{4.10}$$

for any $n \in \mathbb{N}$, $\varphi \in X$. Since $\tau_1, \tau_2 \in (1, 2)$, Lemma 2.3 implies that $\int_{B_r^c} W(x)|\varphi|^{\tau_1}dx < \infty$ and $\int_{B_r^c} W(x)|\varphi|^{\tau_2}dx < \infty$. Thus, combining with (4.9) and (4.10), we conclude that

$$\Big|\int_{B_r^c} W(x)\frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))}\varphi dx\Big| < C_1\varepsilon$$

for any $\varphi \in X$, where $C_1 = b_1\|\varphi\|_{W,\tau_1} + b_2\|\varphi\|_{W,\tau_2}$.

Moreover, for each fixed $r > 0$, it is easy to verify that

$$\lim_{n\to+\infty}\int_{B_r(0)} W(x)\frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))}\varphi dx = \int_{B_r(0)} W(x)\frac{h(G^{-1}(v))}{g(G^{-1}(v))}\varphi dx \quad \text{for any} \quad \varphi \in X.$$

This completes the proof of (4.4). □

Repeating the similar arguments used in the proofs of (4.3) and (4.4), we can obtain that (4.5) and (4.6) hold.

Lemma 4.2. *Assume that $(V, K) \in \mathcal{K}$, (g_0), (F_0)–(F_2) hold and $\{v_n\}$ is a $(Ce)_{c_\lambda}$ sequence for J_λ given by Lemmas 3.2 and 3.3. Then, the following statements hold:*

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} K(x)F(G^{-1}(v_n))dx = \int_{\mathbb{R}^N} K(x)F(G^{-1}(v))dx, \tag{4.11}$$

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} K(x)\frac{f(G^{-1}(v_n))}{g(G^{-1}(v_n))}\varphi dx = \int_{\mathbb{R}^N} K(x)\frac{f(G^{-1}(v))}{g(G^{-1}(v))}\varphi dx \text{ for all } \varphi \in X, \tag{4.12}$$

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} K(x)f(G^{-1}(v_n))G^{-1}(v_n)dx = \int_{\mathbb{R}^N} K(x)f(G^{-1}(v))G^{-1}(v)dx, \tag{4.13}$$

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} K(x)\frac{f(G^{-1}(v_n))}{g(G^{-1}(v_n))}v_n dx = \int_{\mathbb{R}^N} K(x)\frac{f(G^{-1}(v))}{g(G^{-1}(v))}v dx. \tag{4.14}$$

Proof. (1) We begin the proof of (4.11) by assuming that (K_2) holds. By (F_0)–(F_2), we obtain that there exists $C_2 > 0$ such that

$$F(G^{-1}(s)) \leq C_2|G^{-1}(s)|^2 + C_2|G^{-1}(s)|^{22^*} \text{ for all } s \in \mathbb{R},$$

which together with (F_0)–(F_2), (g_3) and (g_6) yields that, for any fixed $q \in (2, 2^*)$, given $\varepsilon > 0$ there exists $0 < s_0 < s_1$ such that

$$\begin{aligned}
|F(G^{-1}(s))| &\leq \frac{\varepsilon}{2}|G^{-1}(s)|^2 + \frac{\varepsilon}{22^*}|G^{-1}(s)|^{22^*} + \chi_{[s_0,s_1]}(|\theta|)(C_2|G^{-1}(s)|^2 + C_2|G^{-1}(s)|^{22^*}) \\
&\leq \frac{\varepsilon}{2}|G^{-1}(s)|^2 + \frac{\varepsilon}{22^*}|G^{-1}(s)|^{22^*} + \chi_{[s_0,s_1]}(|\theta|)C_2(\frac{1}{|s_0|^{q-2}} + |s_1|^{22^*-q})|G^{-1}(s)|^q \\
&\leq \varepsilon(\frac{1}{2} + \frac{(2/g_\infty)^{2^*}}{22^*})(|s|^2 + |s|^{2^*}) + C_2(\frac{1}{|s_0|^{q-2}} + |s_1|^{22^*-q})|s|^q
\end{aligned} \tag{4.15}$$

for all $s \in \mathbb{R}$, where $\theta = G^{-1}(s)$.

In addition, by $K(x),\ V(x) > 0$ for all $x \in \mathbb{R}^N$, (K_2) and $K(x) \in L^\infty(\mathbb{R}^N)$, we obtain that there exists $C_3 > 0$ such that

$$\begin{aligned}
\int_{\mathbb{R}^N} (K(x)|s|^2 + K(x)|s|^{2^*})dx &\leq \int_{\mathbb{R}^N} (\operatorname*{ess\,sup}_{x\in\mathbb{R}^N} \Big|\frac{K(x)}{V(x)}\Big|V(x)|s|^2 + \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} \big|K(x)\big||s|^{2^*})dx \\
&\leq C_3 \int_{\mathbb{R}^N} (V(x)|s|^2 + |s|^{2^*})dx
\end{aligned} \tag{4.16}$$

for any $s \in \mathbb{R}^N$.

Furthermore, noticing $q \in (2, 2^*)$, then from Proposition 2.1 we have

$$\int_{\mathbb{R}^N} K(x)|v_n|^q dx \to \int_{\mathbb{R}^N} K(x)|v|^q dx \ \ as \ \ n \to +\infty, \tag{4.17}$$

which gives that there is $r > 0$ such that

$$\int_{B_r^c} K(x)|v_n|^q dx < \frac{\varepsilon}{C_2(\frac{1}{|s_0|^{q-2}} + |s_1|^{22^*-q})},\ \forall n \in \mathbb{N}. \tag{4.18}$$

Therefore, combining with (4.2), (4.15), (4.16) and (4.18), we conclude that

$$\begin{aligned}
\Big|\int_{B_r^c} K(x)F(G^{-1}(v_n))dx\Big| \leq& \varepsilon(\frac{1}{2} + \frac{(2/g_\infty)^{2^*}}{22^*}) \int_{\mathbb{R}^N} (K(x)|v_n|^2 + K(x)|v_n|^{2^*})dx \\
&+ C_2(\frac{1}{|s_0|^{q-2}} + |s_1|^{22^*-q}) \int_{B_r^c} K(x)|v_n|^q dx \\
\leq& \varepsilon(\frac{1}{2} + \frac{(2/g_\infty)^{2^*}}{22^*})C_3 \int_{\mathbb{R}^N} (V(x)|v_n|^2 + |v_n|^{2^*})dx \\
&+ C_2(\frac{1}{|s_0|^{q-2}} + |s_1|^{22^*-q}) \int_{B_r^c} K(x)|v_n|^q dx \\
<& (\frac{C_3L}{2} + \frac{(2/g_\infty)^{2^*}}{22^*}C_3L + 1)\varepsilon
\end{aligned}$$

for all $n \in \mathbb{N}$.

Now, if (K_3) holds, by (F_0)–(F_2), we obtain there exists $C_4 > 0$ such that

$$F(G^{-1}(s)) \leq C_4|G^{-1}(s)|^\sigma + C_4|G^{-1}(s)|^{22^*},$$

which together with (F_0)–(F_2), (g_3) and (g_6) yields that, given $\varepsilon \in (0,1)$, there exists $0 < s_0 < s_1$ such that

$$\begin{aligned}
F(G^{-1}(s)) &\leq \frac{\varepsilon}{\sigma}|G^{-1}(s)|^\sigma + \frac{\varepsilon}{22^*}|G^{-1}(s)|^{22^*} + \chi_{[s_0,s_1]}(|\theta|)(C_4|G^{-1}(s)|^\sigma + C_4|G^{-1}(s)|^{22^*}) \\
&\leq \frac{\varepsilon}{\sigma}|G^{-1}(s)|^\sigma + \frac{\varepsilon}{22^*}|G^{-1}(s)|^{22^*} + \chi_{[s_0,s_1]}(|\theta|)C_4(1+|s_1|^{22^*-\sigma})|G^{-1}(s)|^\sigma \\
&\leq \frac{1}{\sigma}|s|^\sigma + \varepsilon\frac{(2/g_\infty)^{2^*}}{22^*}|s|^{2^*} + C_4(1+|s_1|^{22^*-\sigma})|s|^\sigma \\
&\leq (\frac{1}{\sigma} + C_4 + C_4|s_1|^{22^*-\sigma})|s|^\sigma + \varepsilon\frac{(2/g_\infty)^{2^*}}{22^*}|s|^{2^*}
\end{aligned} \tag{4.19}$$

for any $s \in \mathbb{R}$.

Furthermore, noticing $\sigma \in (2, 2^*)$, by Proposition 2.1 we have

$$\int_{\mathbb{R}^N} K(x)|v_n|^\sigma dx \to \int_{\mathbb{R}^N} K(x)|v|^\sigma dx \text{ as } n \to +\infty, \tag{4.20}$$

which gives that there is $r > 0$ such that

$$\int_{B_r^c} K(x)|v_n|^\sigma dx < \frac{\varepsilon}{\frac{1}{\sigma} + C_4 + C_4|s_1|^{22^*-\sigma}}, \ \forall n \in \mathbb{N}. \tag{4.21}$$

Therefore, by (4.2), (4.19), (4.21), $K(x) > 0$ for all $x \in \mathbb{R}^N$ and $K \in L^\infty(\mathbb{R}^N)$, we obtain that

$$\begin{aligned}
\int_{B_r^c} K(x)F(G^{-1}(v_n))dx &\leq (\frac{1}{\sigma} + C_4 + C_4|s_1|^{22^*-\sigma})\int_{B_r^c} K(x)|v_n|^\sigma dx + \varepsilon\frac{(2/g_\infty)^{2^*}}{22^*}\int_{B_r^c} K(x)|v_n|^{2^*}dx \\
&\leq (\frac{1}{\sigma} + C_4 + C_4|s_1|^{22^*-\sigma})\int_{B_r^c} K(x)|v_n|^\sigma dx + \varepsilon\frac{(2/g_\infty)^{2^*}}{22^*}\operatorname*{ess\,sup}_{x\in\mathbb{R}^N}|K(x)|\int_{B_r^c}|v_n|^{2^*}dx \\
&< C_5\varepsilon
\end{aligned}$$

for any $n \in \mathbb{N}$, where

$$C_5 = (1 + \frac{(2/g_\infty)^{2^*}}{22^*}\operatorname*{ess\,sup}_{x\in\mathbb{R}^N}|K(x)|L).$$

Furthermore, for each fixed $r > 0$, it is easy to verify that

$$\lim_{n\to+\infty}\int_{B_r(0)} K(x)F(G^{-1}(v_n))dx = \int_{B_r(0)} K(x)F(G^{-1}(v))dx.$$

This completes the proof of (4.11).

(2) We begin the proof of (4.12) if (K_2) holds. By (F_0)–(F_2), we have that there is $C_6 > 0$ such that

$$\frac{f(G^{-1}(s))}{g(G^{-1}(s))} \leq C_6\frac{|G^{-1}(s)|}{g(G^{-1}(s))} + C_6\frac{|G^{-1}(s)|^{22^*-1}}{g(G^{-1}(s))} \ \text{ for all } s \in \mathbb{R},$$

which together with (F_0)–(F_2), (g_2), (g_3), (g_6) and (g_8) yields that, for any fixed $q \in (2, 2^*)$, given $\varepsilon \in (0,1)$ there exists $0 < s_0 < s_1$ such that

$$\begin{aligned}\frac{f(G^{-1}(s))}{g(G^{-1}(s))} &\leq \varepsilon\frac{|G^{-1}(s)|}{g(G^{-1}(s))} + \varepsilon\frac{|G^{-1}(s)|^{22^*-1}}{g(G^{-1}(s))} + \chi_{[s_0,s_1]}(|\theta|)C_6(\frac{1}{|s_0|^{q-2}} + |s_1|^{22^*-q})|G^{-1}(s)|^{q-1} \\ &\leq \varepsilon|G^{-1}(s)| + \varepsilon\frac{1}{g_\infty}|G^{-1}(s)|^{22^*-2} + C_6(\frac{1}{|s_0|^{q-2}} + |s_1|^{22^*-q})|G^{-1}(s)|^{q-1} \\ &\leq \varepsilon(1 + (2/g_\infty)^{2^*})(|s| + |s|^{2^*-1}) + C_6(\frac{1}{|s_0|^{q-2}} + |s_1|^{22^*-q})|s|^{q-1}\end{aligned} \tag{4.22}$$

for any $s \in \mathbb{R}$.

On the other hand, noticing $q \in (2, 2^*)$, by (4.17) we have there is $r > 0$ such that

$$\int_{B_r^c} K(x)|v_n|^q dx < \frac{\varepsilon}{C_6(\frac{1}{|s_0|^{q-2}} + |s_1|^{22^*-q})}, \ \forall n \in \mathbb{N}. \tag{4.23}$$

Thus, taking into account (K_2), $K(x), V(x) > 0$ for all $x \in \mathbb{R}^N$, $K(x) \in L^\infty(\mathbb{R}^N)$, (4.22) and Hölder's inequality, we obtain that, for any $\varphi \in X$,

$$\begin{aligned}\Big|\int_{B_r^c} K(x)\frac{f(G^{-1}(v_n))}{g(G^{-1}(v_n))}\varphi dx\Big| \leq& \varepsilon(1 + (2/g_\infty)^{2^*})(\int_{B_r^c} K(x)|v_n||\varphi|dx + \int_{B_r^c} K(x)|v_n|^{2^*-1}|\varphi|dx) \\ &+ C_6(\frac{1}{|s_0|^{q-2}} + |s_1|^{22^*-q})\int_{B_r^c} K(x)|v_n|^{q-1}|\varphi|dx \\ \leq& \varepsilon(1 + (2/g_\infty)^{2^*})[(\int_{B_r^c} \operatorname{ess\,sup}_{x\in\mathbb{R}^N}\Big|\frac{K(x)}{V(x)}\Big|V(x)|v_n|^2 dx)^{\frac{1}{2}}(\int_{B_r^c} K(x)|\varphi|^2 dx)^{\frac{1}{2}} \\ &+ \operatorname{ess\,sup}_{x\in\mathbb{R}^N}|K(x)|(\int_{B_r^c}|v_n|^{2^*}dx)^{\frac{2^*-1}{2^*}}(\int_{B_r^c}|\varphi|^{2^*}dx)^{\frac{1}{2^*}}] \\ &+ C_6(\frac{1}{|s_0|^{q-2}} + |s_1|^{22^*-q})(\int_{B_r^c} K(x)|v_n|^q dx)^{\frac{q-1}{q}}(\int_{B_r^c} K(x)|\varphi|^q dx)^{\frac{1}{q}}\end{aligned} \tag{4.24}$$

for any $n \in \mathbb{N}$. Moreover, for any $\varphi \in X$, by Proposition 2.1, (K_2) and Sobolev embedding, we have

$$\int_{\mathbb{R}^N} K(x)|\varphi|^q dx < +\infty, \quad \int_{\mathbb{R}^N} K(x)|\varphi|^2 dx < \operatorname{ess\,sup}_{x\in\mathbb{R}^N}\Big|\frac{K(x)}{V(x)}\Big|\int_{\mathbb{R}^N} V(x)|\varphi|^2 dx < +\infty$$

and

$$\int_{\mathbb{R}^N}|\varphi|^{2^*}dx < \nu_1\|\varphi\|^{2^*} < +\infty,$$

respectively. Thus, it follows from (4.2), (4.23) and (4.24) that

$$\Big|\int_{B_r^c} K(x)\frac{f(G^{-1}(v_n))}{g(G^{-1}(v_n))}\varphi dx\Big| < C_7\varepsilon,$$

where

$$C_7 = (1 + (2/g_\infty)^{2^*})[(\operatorname{ess\,sup}_{x\in\mathbb{R}^N}\Big|\frac{K(x)}{V(x)}\Big|)L^{\frac{1}{2}}\|\varphi\| + \operatorname{ess\,sup}_{x\in\mathbb{R}^N}|K(x)|L^{\frac{2^*-1}{2^*}}\|\varphi\|_{2^*}] + \|\varphi\|_{K,q}.$$

Now, if (K_3) holds, by (F_0)–(F_2) we obtain there exists $C_8 > 0$ such that

$$\frac{f(G^{-1}(s))}{g(G^{-1}(s))} \le C_8 \frac{|G^{-1}(s)|^{\sigma-1}}{g(G^{-1}(s))} + C_8 \frac{|G^{-1}(s)|^{22^*-1}}{g(G^{-1}(s))},$$

which together with (F_0)–(F_2), (g_3), (g_6) and (g_8) yields that, given $\varepsilon \in (0,1)$ there exists $0 < s_0 < s_1$ such that

$$\begin{aligned}\frac{f(G^{-1}(s))}{g(G^{-1}(s))} &\le \varepsilon \frac{|G^{-1}(s)|^{\sigma-1}}{g(G^{-1}(s))} + \varepsilon \frac{|G^{-1}(s)|^{22^*-1}}{g(G^{-1}(s))} + \chi_{[s_0,s_1]}(|\theta|)(C_8 \frac{|G^{-1}(s)|^{\sigma-1}}{g(G^{-1}(s))} + C_8 \frac{|G^{-1}(s)|^{22^*-1}}{g(G^{-1}(s))}) \\ &\le |G^{-1}(s)|^{\sigma-1} + \varepsilon(1/g_\infty)|G^{-1}(s)|^{22^*-2} + \chi_{[s_0,s_1]}(|\theta|)C_8(1+|s_1|^{22^*-\sigma})|G^{-1}(s)|^{\sigma-1} \\ &\le (1 + C_8 + C_8|s_1|^{22^*-\sigma})|s|^{\sigma-1} + \varepsilon(2/g_\infty)^{2^*}|s|^{2^*-1}\end{aligned} \tag{4.25}$$

for any $s \in \mathbb{R}$. Furthermore, from $(V,K) \in \mathcal{K}$ and Proposition 2.1, we infer that there is $r > 0$ such that

$$\int_{B_r^c} K(x)|v_n|^\sigma dx < \frac{\varepsilon^{\frac{\sigma}{\sigma-1}}}{(1 + C_8 + C_8|s_1|^{22^*-\sigma})^{\frac{\sigma}{\sigma-1}}},\ \forall n \in \mathbb{N}. \tag{4.26}$$

Combining with Hölder's inequality, (4.2), (4.25), (4.26), $K(x), V(x) > 0$ for all $x \in \mathbb{R}^N$, $K(x) \in L^\infty(\mathbb{R}^N)$ and Proposition 2.1, it follows that

$$\begin{aligned}\Big|\int_{B_r^c} K(x)\frac{f(G^{-1}(v_n))}{g(G^{-1}(v_n))}\varphi dx\Big| \le&(1 + C_8 + C_8|s_1|^{22^*-\sigma})\int_{B_r^c} K(x)|v_n|^{\sigma-1}|\varphi|dx + \varepsilon(2/g_\infty)^{2^*}\int_{B_r^c} K(x)|v_n|^{2^*-1}|\varphi|dx \\ \le&(1 + C_8 + C_8|s_1|^{22^*-\sigma})(\int_{B_r^c} K(x)|v_n|^\sigma dx)^{\frac{\sigma-1}{\sigma}}(\int_{B_r^c} K(x)|\varphi|^\sigma dx)^{\frac{1}{\sigma}} \\ &+ \varepsilon(2/g_\infty)^{2^*}\operatorname*{ess\,sup}_{x\in\mathbb{R}^N}|K(x)|(\int_{B_r^c}|v_n|^{2^*}dx)^{\frac{2^*-1}{2^*}}(\int_{B_r^c}|\varphi|^{2^*}dx)^{\frac{1}{2^*}} \\ <&C_9\varepsilon,\end{aligned}$$

where

$$C_9 = \|\varphi\|_{K,\sigma} + (2/g_\infty)^{2^*} L^{\frac{2^*-1}{2^*}}\|\varphi\|_{2^*}\operatorname*{ess\,sup}_{x\in\mathbb{R}^N}|K(x)|$$

for all $\varphi \in X$, $n \in \mathbb{N}$.

Moreover, for each fixed $r > 0$, it is easy to verify that

$$\lim_{n\to+\infty}\int_{B_r(0)} K(x)\frac{f(G^{-1}(v_n))}{g(G^{-1}(v_n))}\varphi dx = \int_{B_r(0)} K(x)\frac{f(G^{-1}(v))}{g(G^{-1}(v))}\varphi dx.$$

This completes the proof of (4.12)

Repeating the similar arguments used in the proof of (4.11) and (4.12), we obtain that (4.13) and (4.14) hold. □

Lemma 4.3. *Assume that $(V,K) \in \mathcal{K}$, (g_0), (V_1), (W_0)–(W_2), (F_0)–(F_2), (H_0), (H_1) hold and $\{v_n\}$ is a $(Ce)_{c_\lambda}$ sequence for J_λ given by Lemmas 3.2 and 3.3. Then, the following statements hold:*
(i) For each $\varepsilon > 0$ there exists $r_0 > 1$, such that for any $r > r_0$

$$\limsup_{n\to+\infty}\int_{B_{2r}^c}(|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2)dx < (3+\lambda)\varepsilon, \tag{4.27}$$

$$\limsup_{n\to+\infty}\int_{B_{2r}^c}(|\nabla v_n|^2+V(x)\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}v_n)dx<(3+\lambda)\varepsilon \tag{4.28}$$

and

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}V(x)|G^{-1}(v_n)|^2dx=\int_{\mathbb{R}^N}V(x)|G^{-1}(v)|^2dx, \tag{4.29}$$

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}V(x)\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}v_ndx=\int_{\mathbb{R}^N}V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}vdx. \tag{4.30}$$

(ii) The weak limit v of $\{v_n\}$ is a critical point for the function J_λ on X.
(iii) The weak limit v is a nontrivial critical point of J_λ and $J_\lambda(v)=c_\lambda$. Moreover, the function J_λ satisfies the Cerami condition on X.

Proof. (i) For $r>1$, we choose a cut-off function $\eta=\eta_r\in C_0^\infty(B_r^c)$ such that

$$\eta\equiv 1 \text{ in } B_{2r}^c,\quad \eta\equiv 0 \text{ in } B_r,\quad 0\le\eta\le 1 \tag{4.31}$$

and

$$|\nabla\eta|\le\frac{2}{r}\ \text{ for all }\ x\in\mathbb{R}^N. \tag{4.32}$$

As $\{v_n\}$ is bounded in X, the sequence $\{\eta\omega_n\}$ where $\omega_n=G^{-1}(v_n)g(G^{-1}(v_n))$ is also bounded in X. Hence, from (3.11) we have

$$|\langle J_\lambda'(v_n),\eta\omega_n\rangle|=o_n(1),$$

that is

$$\begin{aligned}&\int_{\mathbb{R}^N}(1+\frac{g'(t)\big|_{t=G^{-1}(v_n)}G^{-1}(v_n)}{g(G^{-1}(v_n))})\eta|\nabla v_n|^2dx+\int_{\mathbb{R}^N}V(x)[G^{-1}(v_n)]^2\eta dx\\&=-\int_{\mathbb{R}^N}\nabla\eta\nabla v_n\omega_ndx+\int_{\mathbb{R}^N}K(x)f(G^{-1}(v_n))G^{-1}(v_n)\eta dx\\&\quad+\int_{\mathbb{R}^N}\lambda W(x)h(G^{-1}(v_n))G^{-1}(v_n)\eta dx+o_n(1).\end{aligned} \tag{4.33}$$

Then, by (g_0), (F_0), (H_0), (4.31) and (4.33) we infer that

$$\begin{aligned}\int_{B_r^c}(|\nabla v_n|^2+V(x)|G^{-1}(v_n)|^2)\eta dx\le& o_n(1)+\int_{B_r^c}|\nabla\eta||\nabla v_n||\omega_n|dx\\&+\int_{B_r^c}K(x)f(G^{-1}(v_n))G^{-1}(v_n)\eta dx\\&+\int_{B_r^c}\lambda W(x)h(G^{-1}(v_n))G^{-1}(v_n)\eta dx\end{aligned} \tag{4.34}$$

for any $r>1$.

By (4.2), (4.32), (g_5) and Hölder's inequality, we obtain

$$\begin{aligned}\int_{B_r^c}|\nabla\eta||\nabla v_n||\omega_n|dx&\le\frac{4}{r}\int_{\{r\le|x|\le 2r\}}|\nabla v_n||v_n|dx\\&\le\frac{4}{r}(\int_{\mathbb{R}^N}|\nabla v_n|^2dx)^{\frac{1}{2}}(\int_{\{r\le|x|\le 2r\}}|v_n|^2dx)^{\frac{1}{2}}\\&\le\frac{4}{r}L^{\frac{1}{2}}(\int_{\{r\le|x|\le 2r\}}|v_n|^2dx)^{\frac{1}{2}}\end{aligned} \tag{4.35}$$

for any $r > 1$, $n \in \mathbb{N}$. Noticing that $v_n \to v$ in $L^2(B_{2r} \setminus B_r)$ and $|B_{2r} \setminus B_r| \leq |B_{2r}| = \omega_N(2r)^N$ for any fixed $r > 1$, then (4.35) follows that

$$\begin{aligned}\limsup_{n\to+\infty} \int_{B_r^c} |\nabla\eta||\nabla v_n||\omega_n|dx &\leq \frac{4}{r} L^{\frac{1}{2}} \Big(\int_{\{r\leq|x|\leq 2r\}} |v|^2 dx\Big)^{\frac{1}{2}} \\ &\leq \frac{4}{r} L^{\frac{1}{2}} \Big(\int_{\{r\leq|x|\leq 2r\}} |v|^{2^*} dx\Big)^{\frac{1}{2^*}} |B_{2r} \setminus B_r|^{\frac{1}{N}}, \\ &\leq 8L^{\frac{1}{2}} \omega_N^{\frac{1}{N}} \Big(\int_{\{r\leq|x|\leq 2r\}} |v|^{2^*} dx\Big)^{\frac{1}{2^*}}\end{aligned} \tag{4.36}$$

for any $r > 1$. Furthermore, for any $\varepsilon > 0$ there exists $r_1 > 1$, and for any $r > r_1$

$$8L^{\frac{1}{2}} \omega_N^{\frac{1}{N}} \Big(\int_{\{r\leq|x|\leq 2r\}} v^{2^*} dx\Big)^{\frac{1}{2^*}} < \varepsilon. \tag{4.37}$$

Therefore, combining with (4.36) and (4.37), we have that for any $r > r_1$,

$$\limsup_{n\to+\infty} \int_{B_r^c} |\nabla\eta||\nabla v_n||\omega_n|dx < \varepsilon. \tag{4.38}$$

In addition, according to the (4.5) and (4.13), we infer that there is $r_2 > 1$ such that

$$\int_{B_r^c} \lambda W(x)h(G^{-1}(v_n))G^{-1}(v_n)\eta dx \leq \int_{B_r^c} \lambda W(x)h(G^{-1}(v_n))G^{-1}(v_n)dx < \lambda\varepsilon, \quad \text{for any } n \in \mathbb{N} \tag{4.39}$$

and

$$\int_{B_r^c} K(x)f(G^{-1}(v_n))G^{-1}(v_n)\eta dx \leq \int_{B_r^c} K(x)f(G^{-1}(v_n))G^{-1}(v_n)dx < \varepsilon, \quad \text{for any } n \in \mathbb{N} \tag{4.40}$$

for any $r > r_2$.

Set $r_0 = \max\{r_1, r_2\}$. Then, taking into account (4.34), (4.38), (4.39) and (4.40), we know that (4.27) is valid.

Noticing (g_5), we know that $\left|\frac{v_n}{g(G^{-1}(v_n))}\right| < \left|G^{-1}(v_n)\right|$ for any $n \in \mathbb{N}$. Then, (4.27) implies that (4.28) holds.

Moreover, the limit (4.27) gives that

$$\limsup_{n\to\infty} \int_{B_{2r}^c} V(x)|G^{-1}(v_n)|^2 dx < (3 + \lambda)\varepsilon \tag{4.41}$$

for any $r > r_0$ and consequently,

$$\int_{B_{2r}^c} V(x)|G^{-1}(v)|^2 dx < (3 + \lambda)\varepsilon \tag{4.42}$$

for any $r > r_0$. Since $v_n \to v$ in $L^2(B_{2r}(0))$ for any fixed $r \in (0, +\infty)$, then by (g_3) and the continuity of $V(x)$, using the Lebesgue dominated convergence theorem we know that

$$\lim_{n\to+\infty} \int_{B_{2r}(0)} V(x)|G^{-1}(v_n)|^2 dx = \int_{B_{2r}(0)} V(x)|G^{-1}(v)|^2 dx. \tag{4.43}$$

Then, (4.41)–(4.43) yield that

$$\limsup_{n\to+\infty}\Big|\int_{\mathbb{R}^N} V(x)[|G^{-1}(v_n)|^2-|G^{-1}(v)|^2]dx\Big|<2(3+\lambda)\varepsilon$$

and, hence, (4.29) holds. Similarly, it follows from (4.28) that (4.30) holds.

(ii) It is clear that

$$\sqrt{V(x)}\frac{G^{-1}(v_n(x))}{g(G^{-1}(v_n(x)))}\to\sqrt{V(x)}\frac{G^{-1}(v(x))}{g(G^{-1}(v(x)))}\text{ a.e., } x\in\mathbb{R}^N$$

as $n\to+\infty$. Noting that $\{\sqrt{V(x)}\frac{G^{-1}(v_n(x))}{g(G^{-1}(v_n(x)))}\}$ is bounded in $L^2(\mathbb{R}^N)$ and $\sqrt{V}\varphi\in L^2(\mathbb{R}^N)$ for any $\varphi\in X$, we have that

$$\sqrt{V(x)}\frac{G^{-1}(v_n(x))}{g(G^{-1}(v_n(x)))}\rightharpoonup\sqrt{V(x)}\frac{G^{-1}(v(x))}{g(G^{-1}(v(x)))}$$

in $L^2(\mathbb{R}^N)$, as $n\to+\infty$ and, hence, the following equality holds

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}V(x)[\frac{G^{-1}(v_n(x))}{g(G^{-1}(v_n(x)))}-\frac{G^{-1}(v(x))}{g(G^{-1}(v(x)))}]\varphi dx=0,\quad\text{for any }\varphi\in X. \tag{4.44}$$

Furthermore, since $v_n\rightharpoonup v$ in $D^{1,2}(\mathbb{R}^N)$ and $\varphi\in D^{1,2}(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N}\nabla v_n\nabla\varphi dx\to\int_{\mathbb{R}^N}\nabla v\nabla\varphi dx,\quad\text{for any }\varphi\in X. \tag{4.45}$$

Thus, by (4.4), (4.12), (4.44) and (4.45), we deduce that

$$\lim_{n\to+\infty}\langle J_\lambda'(v_n),\varphi\rangle=\langle J_\lambda'(v),\varphi\rangle,\ \forall\ \varphi\in X.$$

Thus, $J_\lambda'(v)=0$, which implies that (ii) holds.

(iii) We have proved that $J_\lambda'(v)=0$. Now, we show that $v\neq 0$. Suppose that $v\equiv 0$; because $\{v_n\}$ is a $(Ce)_{c_\lambda}$ sequence, according to (3.2), we know

$$\begin{aligned}(J_\lambda'(v_n),v_n)=&\int_{\mathbb{R}^N}|\nabla v_n|^2dx+\int_{\mathbb{R}^N}V(x)\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}v_ndx\\&-\int_{\mathbb{R}^N}K(x)\frac{f(G^{-1}(v_n))}{g(G^{-1}(v_n))}v_ndx-\lambda\int_{\mathbb{R}^N}W(x)\frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))}v_ndx\\&\to 0.\end{aligned} \tag{4.46}$$

Moreover, by (4.5), (4.13) and (4.30),we get

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}V(x)\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}v_ndx=0,$$

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}W(x)h(G^{-1}(v_n))G^{-1}(v_n)dx=0$$

and

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}K(x)f(G^{-1}(v_n))G^{-1}(v_n)dx=0.$$

Then, it follows from (4.46) that

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 dx \to 0 \ \ as\, n \to +\infty.$$

In addition, from (4.3), (4.11) and (4.29), we have

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 dx = 0,$$

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} W(x)H(G^{-1}(v_n))dx = 0$$

and

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} K(x)F(G^{-1}(v_n))dx = 0.$$

Hence,

$$\begin{aligned} J_\lambda(v_n) =& \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 dx - \int_{\mathbb{R}^N} K(x)F(G^{-1}(v_n))dx \\ &- \lambda \int_{\mathbb{R}^N} W(x)H(G^{-1}(v_n))dx \to 0, \end{aligned}$$

which is a contradiction to $J_\lambda(v_n) \to c_\lambda > \alpha_0 > 0$. Therefore, $v \neq 0$.

Now, we show that $J_\lambda(v) = c_\lambda$. By $\langle J'_\lambda(v_n), v_n\rangle = o_n(1)$, passing to the limit in the following expression

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx =& - \int_{\mathbb{R}^N} V(x)\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} v_n dx + \int_{\mathbb{R}^N} K(x)\frac{f(G^{-1}(v_n))}{g(G^{-1}(v_n))} v_n dx \\ &+ \lambda \int_{\mathbb{R}^N} W(x)\frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))} v_n dx + o_n(1) \end{aligned}$$

and using (4.6), (4.14) and (4.30) together with $(J'_\lambda(v), v) = 0$, we obtain that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx = \int_{\mathbb{R}^N} |\nabla v|^2 dx. \tag{4.47}$$

By (4.3), (4.11), (4.29) and (4.47), we conclude that

$$\begin{aligned} J_\lambda(v_n) =& \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 dx \\ &- \int_{\mathbb{R}^N} K(x)F(G^{-1}(v_n))dx - \lambda \int_{\mathbb{R}^N} W(x)H(G^{-1}(v_n))dx \\ \to& J_\lambda(v), \end{aligned}$$

which results that $J_\lambda(v) = c_\lambda$.

To show that the function J_λ satisfies the Cerami condition, we verify that $\|v_n - v\| \to 0$. By Remark 2.1, we have

$$\xi\|v_n - v\|^2 \leq \Psi(v_n - v) + [\Psi(v_n - v)]^{2^*/2},$$

where

$$\Psi(v_n - v) = \int_{\mathbb{R}^N} [|\nabla(v_n - v)|^2 + V(x)|G^{-1}(v_n - v)|^2]dx.$$

Combining with (g_9), (4.41) and (4.42), we obtain that given $\varepsilon > 0$

$$\begin{aligned} \limsup_{n\to+\infty} \int_{B_{2r}^c} V(x)[G^{-1}(v_n - v)]^2 dx &\leq \limsup_{n\to+\infty} \int_{B_{2r}^c} 4V(x)[|G^{-1}(v_n)|^2 + |G^{-1}(v)|^2]dx \\ &< 8(3 + \lambda)\varepsilon \end{aligned} \tag{4.48}$$

for any $r > r_0$. Furthermore, noticing $v_n \to v$ in $L^2(B_{2r})$ for any fixed $r > 0$, by (g_3) we infer that

$$0 \leq \lim_{n\to+\infty} \int_{B_{2r}} V(x)[G^{-1}(v_n - v)]^2 dx \leq \lim_{n\to+\infty} \int_{B_{2r}} V(x)|v_n - v|^2 dx = 0,$$

implying that

$$\lim_{n\to+\infty} \int_{B_{2r}} V(x)[G^{-1}(v_n - v)]^2 dx = 0. \tag{4.49}$$

Thus, (4.48) and (4.49) lead to

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} V(x)[G^{-1}(v_n - v)]^2 dx = 0,$$

which together with (4.47) yields that $\Psi(v_n - v) \to 0$. Then, $\|v_n - v\| \to 0$ holds, implying that J_λ satisfies the Cerami condition. □

Proof of Theorem 1.1. Combining all the results above, we get that for every $\lambda \in (0, \lambda_0)$, (2.4) possesses a nontrivial solution v. Furthermore, letting $v^- = \max\{-v, 0\}$, by $J'_\lambda(v) = 0$, (F_0) and (W_0), we have that

$$\langle J'_\lambda(v), v^-\rangle = \int_{\mathbb{R}^N} [|\nabla v^-|^2 + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}v^-]dx = 0.$$

Since $G^{-1}(v)v^- \geq 0$, $V(x) > 0$ and $g(G^{-1}(v)) > 0$, we have that

$$\int_{\mathbb{R}^N} |\nabla v^-|^2 dx = 0 \text{ and } \int_{\mathbb{R}^N} V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}v^- dx = 0.$$

Thus, $v^- = 0$ a.e., $x \in \mathbb{R}^N$. Therefore, v is a positive solution to (2.4); that is, $u = G^{-1}(v)$ is a positive solution to (1.1).

Now, we prove Proposition 1.2. At first we show the following lemma.

Set $S_r = \{v \in X : \|v\| = r\}$ and $\mathcal{B}_r = \{v \in X : \|v\| < r\}$.

Lemma 4.4. *Assume that $(V, K) \in \mathcal{K}$ where (K_2) holds and (g_0), (V_1), (W_0)–(W_2), (F'_0), (F_1), (F_2), (F_4), (H'_0)–(H'_2) hold. Then, there exists $r_1, \alpha_1, \lambda_1 > 0$ such that $J_\lambda|_{S_{r_1}} \geq \alpha_1$ and $\inf_{v\in\mathcal{B}_{r_1}} J_\lambda(v) < 0$ for any $\lambda \in (0, \lambda_1)$.*

Proof. Noticing Remark 2.1, we have

$$\Psi(v) \geq \xi\|v\|^2 - [\Psi(v)]^{2^*/2} \geq \xi\|v\|^2 - \|v\|^{2^*} \text{ for all } v \in X. \tag{4.50}$$

Choose $\varepsilon_1 > 0$ such that

$$\frac{1}{2}-\frac{\varepsilon_1}{2}\operatorname*{ess\,sup}_{x\in\mathbb{R}^N}\left|\frac{K(x)}{V(x)}\right|>0.$$

By (F_0'), (F_1), (F_2), (H_0'), (H_1'), (W_0)–(W_2), (K_2), $K(x)\in L^\infty(\mathbb{R}^N)$, $K(x),V(x)>0$ for all $x\in\mathbb{R}^N$, (g_6), (4.50) and Lemma 2.3, we obtain there exists $C_{\varepsilon_1}>0$ such that

$$\begin{aligned}
J_\lambda(v)\geq&\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x)|G^{-1}(v)|^2dx-\frac{\varepsilon_1}{2}\int_{\mathbb{R}^N}K(x)|G^{-1}(v)|^2dx-\frac{C_{\varepsilon_1}}{22^*}\int_{\mathbb{R}^N}K(x)|G^{-1}(v)|^{22^*}dx\\
&-\lambda\frac{b_3}{\tau_3}\int_{\mathbb{R}^N}W(x)\left|G^{-1}(v)\right|^{\tau_3}dx\\
\geq&(\frac{1}{2}-\frac{\varepsilon_1}{2}\operatorname*{ess\,sup}_{x\in\mathbb{R}^N}\left|\frac{K(x)}{V(x)}\right|)\Psi(v)-\operatorname*{ess\,sup}_{x\in\mathbb{R}^N}|K(x)|(\frac{2}{g_\infty})^{2^*}\frac{C_{\varepsilon_1}}{22^*}\int_{\mathbb{R}^N}|v|^{2^*}dx-\lambda\frac{b_3}{\tau_3}\int_{\mathbb{R}^N}W(x)\left|G^{-1}(v)\right|^{\tau_3}dx\\
\geq&(\frac{1}{2}-\frac{\varepsilon_1}{2}\operatorname*{ess\,sup}_{x\in\mathbb{R}^N}\left|\frac{K(x)}{V(x)}\right|)(\xi\|v\|^2-\|v\|^{2^*})-\operatorname*{ess\,sup}_{x\in\mathbb{R}^N}|K(x)|(\nu_1\frac{2}{g_\infty})^{2^*}\frac{C_{\varepsilon_1}}{22^*}\|v\|^{2^*}-\lambda\frac{b_3}{\tau_3}\nu_{W,\tau_3}^{\tau_3}\|v\|^{\tau_3}\\
=&(\frac{1}{2}-\frac{\varepsilon_1}{2}\operatorname*{ess\,sup}_{x\in\mathbb{R}^N}\left|\frac{K(x)}{V(x)}\right|)\xi\|v\|^2-(\frac{1}{2}-\frac{\varepsilon_1}{2}\operatorname*{ess\,sup}_{x\in\mathbb{R}^N}\left|\frac{K(x)}{V(x)}\right|+\operatorname*{ess\,sup}_{x\in\mathbb{R}^N}|K(x)|(\nu_1\frac{2}{g_\infty})^{2^*}\frac{C_{\varepsilon_1}}{22^*})\|v\|^{2^*}\\
&-\lambda\frac{b_3}{\tau_3}\nu_{W,\tau_3}^{\tau_3}\|v\|^{\tau_3}
\end{aligned}\tag{4.51}$$

for any $v\in X$, $\lambda>0$.

Consider

$$l_1(t)=(\frac{1}{2}-\frac{\varepsilon_1}{2}\operatorname*{ess\,sup}_{x\in\mathbb{R}^N}\left|\frac{K(x)}{V(x)}\right|)\xi t^2-(\frac{1}{2}-\frac{\varepsilon_1}{2}\operatorname*{ess\,sup}_{x\in\mathbb{R}^N}\left|\frac{K(x)}{V(x)}\right|+\operatorname*{ess\,sup}_{x\in\mathbb{R}^N}|K(x)|(\nu_1\frac{2}{g_\infty})^{2^*}\frac{C_{\varepsilon_1}}{22^*})t^{2^*}$$

for $t\geq 0$. Obviously, there exists $r_1>0$ such that $\max\limits_{t\geq 0}l_1(t)=l_1(r_1)\triangleq\Lambda_1>0$. Then, it follows from (4.51) that

$$J_\lambda(v)\geq\Lambda_1-\lambda\frac{b_3}{\tau}\nu_{W,\tau_3}^{\tau_3}r_1^{\tau_3}\ \text{ for any }\ v\in S_{r_1},\lambda>0.\tag{4.52}$$

Set

$$\lambda_1=\frac{\Lambda_1}{\frac{2b_3}{\tau_3}\nu_{W,\tau_3}^{\tau_3}r_1^{\tau_3}}$$

and $\alpha_1=\frac{\Lambda_1}{2}$, and it follows from (4.52) that

$$J_\lambda(v)\geq\Lambda_1-\lambda_1\frac{b_3}{\tau_3}\nu_{W,\tau_3}^{\tau_3}r_1^{\tau_3}\geq\alpha_1>0\ \ \textit{for any}\ \lambda\in(0,\lambda_1),\ v\in S_{r_1}.$$

On the other hand, (4.51) implies that $J_\lambda(v)$ is bounded blow in $\mathcal{B}_{r_1}$ for any $\lambda>0$. Taking $\varphi\in X$ and $\varphi\neq 0$, by (g_4), (H_2') and (W_0) and combining the Lebesgue dominated convergence theorem and Sobolev embedding theorem we have

$$\lim_{t\to 0^+}\frac{\int_{\mathbb{R}^N}W(x)H(G^{-1}(t\varphi))dx}{|t|^{\tau_3}}=\lim_{t\to 0^+}\int_{\mathbb{R}^N}W(x)\frac{H(G^{-1}(t\varphi))}{|G^{-1}(t\varphi)|^{\tau_3}}\frac{|G^{-1}(t\varphi)|^{\tau_3}}{|t\varphi|^{\tau_3}}|\varphi|^{\tau_3}dx=\tilde{C}\int_{\mathbb{R}^N}W(x)|\varphi|^{\tau_3}dx>0.$$

Then, there exists $\delta>0$ for any $0<t<\delta$,

$$\int_{\mathbb{R}^N}W(x)H(G^{-1}(t\varphi))dx>\frac{\tilde{C}\int_{\mathbb{R}^N}W(x)|\varphi|^{\tau_3}dx}{2}t^{\tau_3}.\tag{4.53}$$

Then, by (F_0'), $K(x) > 0$ for all $x \in \mathbb{R}^N$, (g_3) and (4.53), we obtain that

$$\begin{aligned} J_\lambda(t\varphi) \leq& \frac{1}{2}\int_{\mathbb{R}^N} |\nabla(t\varphi)|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(t\varphi)|^2 dx - \lambda \int_{\mathbb{R}^N} W(x)H(G^{-1}(t\varphi))dx \\ \leq& \frac{t^2}{2}(\int_{\mathbb{R}^N} |\nabla\varphi|^2 dx + \int_{\mathbb{R}^N} V(x)|\varphi|^2 dx) - \lambda t^{\tau_3} \frac{\tilde{C}\int_{\mathbb{R}^N} W(x)|\varphi|^{\tau_3} dx}{2} \end{aligned} \tag{4.54}$$

for any $0 < t < \delta$, $\lambda > 0$. Since $\tau_3 \in (1, 2)$, there exists small $t > 0$ such that $t\varphi \in \mathcal{B}_{r_1}$ and $J_\lambda(t\varphi) < 0$ for any $\lambda > 0$. Therefore, we complete the proof of this lemma. □

Proof of Proposition 1.2. By Lemma 4.4 and Ekeland's variational principle [12], we infer that, for any $\lambda \in (0, \lambda_1)$, there is a minimizing sequence $\{v_n\} \subset \bar{\mathcal{B}}_{r_1}$ of the infimum $c_0 = \inf_{v\in\bar{\mathcal{B}}_{r_1}} J_\lambda(v) < 0$, such that

$$c_0 \leq J_\lambda(v_n) \leq c_0 + \frac{1}{n} \tag{4.55}$$

and

$$J_\lambda(\varphi) \geq J_\lambda(v_n) - \frac{1}{n}\|\varphi - v_n\|, \text{ for all } \varphi \in \bar{\mathcal{B}}_{r_1}. \tag{4.56}$$

First, we claim that $\|v_n\| < r_1$ for large $n \in \mathbb{N}$. Otherwise, we may assume that $\|v_n\| = r_1$. Up to a subsequence, by Lemma 4.4 we get $J_\lambda(v_n) \geq \alpha_1 > 0$, which and (4.55) imply that $0 > c_0 \geq \alpha_1 > 0$, which is a contradiction. In general, we suppose that $\|v_n\| < r_1$ for all $n \in \mathbb{N}$. Next, we will show that $J_\lambda'(v_n) \to 0$ in X^*. For any $n \in \mathbb{N}$ and $\varphi \in X$ with $\|\varphi\| = 1$, we choose sufficiently small $\delta_n > 0$ such that $\|v_n + t\varphi\| < r_1$ for all $0 < t < \delta_n$. It follows from (4.56) that

$$\frac{J_\lambda(v_n + t\varphi) - J_\lambda(v_n)}{t} \geq -\frac{1}{n}. \tag{4.57}$$

Letting $t \to 0^+$, we get

$$\langle J_\lambda'(v_n), \varphi\rangle \geq -\frac{1}{n}$$

for any $n \in \mathbb{N}$. Similarly, replacing φ with $-\varphi$ in the above arguments, we have

$$\langle J_\lambda'(v_n), \varphi\rangle \leq \frac{1}{n}$$

for any $n \in \mathbb{N}$. Therefore, we conclude that, for all $\varphi \in X$ with

$$\|\varphi\| = 1, \ \ \langle J_\lambda'(v_n), \varphi\rangle \to 0 \text{ as } n \to \infty.$$

Thus, we obtain that $J_\lambda(v_n) \to c_0$ and $J_\lambda'(v_n) \to 0$ as $n \to \infty$. Noticing that $\|v_n\| < r_1$, we get that $\{v_n\}$ is a $(Ce)_{c_0}$ sequence for $J_\lambda(v)$ in X, and there exists $v_* \in \bar{\mathcal{B}}_{r_1}$ such that $v_n \rightharpoonup v_*$ in X. Using the same type of arguments in Lemmas 4.1–4.3, we obtain that v_* is a critical point for $J_\lambda(v)$ in X satisfying $v_* \neq 0$ and $J_\lambda(v_*) = c_0 < 0$.

Next, we investigate the existence of ground state solutions for (1.1). For any $\lambda \in (0, \lambda_1)$, define

$$\mathcal{S} = \{v \in X : J_\lambda'(v) = 0, \ v \neq 0\} \text{ and } \mathcal{M}_0 = \inf_{v\in\mathcal{S}} J_\lambda(v).$$

Clearly, $\mathcal{S}$ is nonempty and $\mathcal{M}_0 < 0$. For all $v \in \mathcal{S}$, set

$$\varpi = G^{-1}(v)g(G^{-1}(v)).$$

Then, we deduce from (1.7), $K(x) > 0$ for all $x \in \mathbb{R}^N$, (F_4), (H_0'), (H_1'), (g_2), (W_0)–(W_2) and Lemma 2.3 that for any $\lambda \in (0, \lambda_1)$,

$$\begin{aligned}
J_\lambda(v) =& J_\lambda(v) - \frac{1}{\mu}\langle J_\lambda'(v), \varpi\rangle \\
\geq& (\frac{1}{2} - \frac{1+\beta}{\mu}) \int_{\mathbb{R}^N} |\nabla v|^2 dx + (\frac{1}{2} - \frac{1}{\mu}) \int_{\mathbb{R}^N} V(x)[G^{-1}(v)]^2 dx - \lambda \frac{b_3}{\tau_3} \int_{\mathbb{R}^N} W(x)|G^{-1}(v)|^{\tau_3} dx \\
\geq& (\frac{1}{2} - \frac{1+\beta}{\mu}) \int_{\mathbb{R}^N} |\nabla G^{-1}(v)|^2 dx + (\frac{1}{2} - \frac{1}{\mu}) \int_{\mathbb{R}^N} V(x)[G^{-1}(v)]^2 dx - \lambda \frac{b_3}{\tau_3} \int_{\mathbb{R}^N} W(x)|G^{-1}(v)|^{\tau_3} dx \\
\geq& (\frac{1}{2} - \frac{1+\beta}{\mu})\|G^{-1}(v)\|^2 - \lambda \frac{b_3}{\tau_3} \nu_{W,\tau_3}^{\tau_3} \|G^{-1}(v)\|^{\tau_3}.
\end{aligned} \tag{4.58}$$

Consider the function

$$l_2(t) = (\frac{1}{2} - \frac{1+\beta}{\mu})t^2 - \lambda \frac{b_3}{\tau_3} \nu_{W,\tau_3}^{\tau_3} t^{\tau_3}$$

for $t \geq 0$. Since $\tau_3 \in (1, 2)$, for any fixed $\lambda \in (0, \lambda_1)$ there exists $t_2 > 0$ such that

$$-\infty < \min_{t \geq 0} l_2(t) = l_2(t_2) < 0.$$

Then, it follows from (4.58) that $J_\lambda(v) \geq l_2(t_2) > -\infty$ for any $v \in \mathcal{S}$, which implies $\mathcal{M}_0 > -\infty$. Letting $\{v_n\} \subset \mathcal{S}$ be a minimizing sequence of $\mathcal{M}_0$ such that $J_\lambda(v_n) \to \mathcal{M}_0$, set

$$\varpi_n = G^{-1}(v_n)g(G^{-1}(v_n)).$$

Since $v_n \in \mathcal{S}$ for any $n \in \mathbb{N}$, then $\langle J_\lambda'(v_n), \varpi_n\rangle = 0$ for any $n \in \mathbb{N}$. Repeating the ideas explored in the proof of Lemma 3.3, we have that $\{v_n\}$ is bounded in X. Thus, $\{v_n\}$ is a $(Ce)_{\mathcal{M}_0}$ sequence and there exists $v^* \in X$ such that $v_n \rightharpoonup v^*$ in X. Using the same type of arguments in Lemmas 4.1–4.3, we obtain that v^* is a critical point for $J_\lambda(v)$ satisfying $v^* \neq 0$ and $J_\lambda(v^*) = \mathcal{M}_0 < 0$. Thus, v^* is a ground state solution of (2.4). This ends the proof of Proposition 1.2.

5. Conclusions

By using the variational method, this paper studies a kind of generalized quasilinear Schrodinger equation with concave-convex nonlinearities and potentials vanishing at infinity. We use the mountain pass theorem to prove that this problem has a positive solution. In addition, the existence of a ground state solution is also proved by Ekeland's variational principle. To the best of our knowledge, few works in this case seem to have appeared in the literature.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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